Construction of Low Complexity Regular Quantizers for Overcomplete Expansions in $\mathcal{R}^N$

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Abstract

In this paper, we study the construction of structured regular quantizers for overcomplete expansions in $\mathcal{R}^N$. Our goal is to design structured quantizers allowing simple reconstruction algorithms with low (memory and computational) complexity and having good performance in terms of accuracy. Most related work to date in quantized redundant expansions has assumed that uniform scalar quantization with the same stepsize was used on the redundant expansion and then has dealt with more complex methods to improve the reconstruction. Instead, we consider the design of scalar quantizers with different stepsizes for each coefficient of an overcomplete expansion in such a way as to produce an equivalent vector quantizer with periodic structure. The periodicity makes it possible to achieve good accuracy using simple reconstruction algorithms from the quantized coefficients of the overcomplete expansion.

1 Introduction and Motivation

Quantized redundant expansions are useful in different applications such as oversampled A/D conversion of band-limited signals [1, 2, 7, 4] and multiple description quantization [9]. An equivalent vector quantizer can be defined given a quantized redundant expansion, where the quantized vector is given by the reconstruction obtained from the quantized coefficients of the redundant expansion.

The accuracy that can be attained with quantized overcomplete expansions depends on two things: the reconstruction algorithm and the quantization scheme. Simple reconstructions (e.g. linear) are normally prefered for practical reasons. Alternative reconstruction algorithms have been proposed that improve the accuracy over linear reconstruction [7, 4, 2]. Although these reconstruction algorithms can achieve very good accuracy for high enough redundancies r, their computational complexity is too high for some practical applications. One of the reasons that improvements can be obtained is that linear reconstruction may not be optimal in some cases because a coded signal may not be reproduced by the nearest among all the possible output

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signals. In quantization terms this means that the equivalent vector quantizer that has been designed is not regular. In the quantization of overcomplete representations literature this is described as inconsistency.

Unlike previous work, which assumes a known quantization of the expansion coefficients and focuses on improving the reconstruction, in our work we assume that a simple reconstruction will be used (e.g. linear or look-up table). Our approach has focused on providing the tools to design the overcomplete expansions and the corresponding quantization system so that the equivalent vector quantizer is regular under simple reconstruction algorithms. We restrict ourselves to using scalar quantizers for each of the components of the expansion, but allow the stepsizes to be different in each component. Thus, the complexity of our encoder is similar to that of standard systems, while simple reconstruction algorithms can be used without a loss in accuracy with respect to more sophisticated reconstruction techniques.

This paper is organized as follows. First in section 2 we describe the equivalence between a quantized overcomplete expansion and a VQ system. Then, based on this equivalence, in section 3, we introduce the concept of periodic quantizer, show how to construct periodic quantizers, and in section 3.2 we explain the advantages that are provided by this periodic structure. Finally, several examples of periodic quantizer designs and some numerical results are shown in section 4.

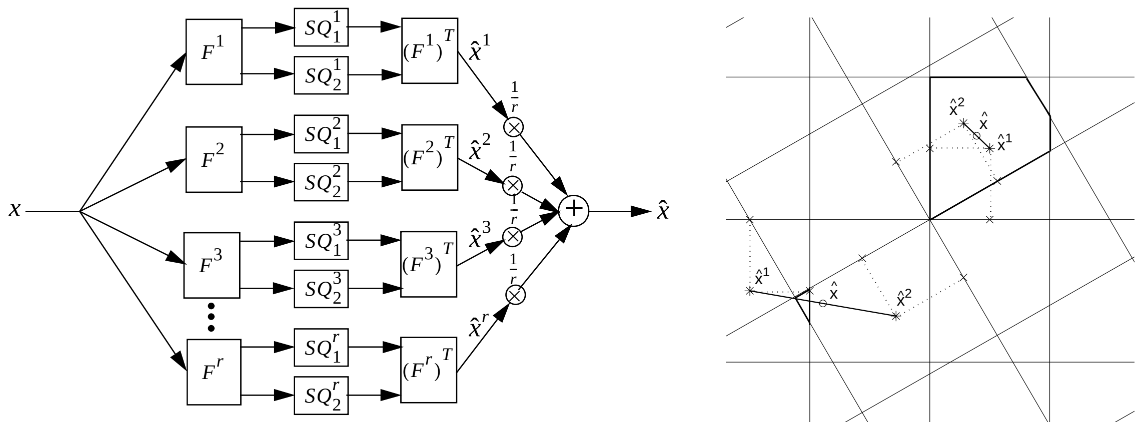
2 Linear Reconstruction, Consistency and Equivalent VQ

Let $\boldsymbol{x} \in \mathcal{R}^N$ and let $\Phi = \{\boldsymbol{\varphi}_i\}_{i=1}^M$ be a tight frame in $\mathcal{R}^N$ with $\|\boldsymbol{\varphi}_i\| = 1 \ \forall\ i = 1, \cdots, M$. Then, for all $\boldsymbol{x}$, the expansion with respect to the frame $\Phi = \{\boldsymbol{\varphi}_i\}_{i=1}^M$ whose coefficients have the minimum possible norm is given by the minimal dual frame $\widetilde{\boldsymbol{\Phi}} = \{\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{i}}\}_{i=1}^M$, where $\widetilde{\boldsymbol{\varphi}}_{\boldsymbol{i}} = \frac{1}{r}\boldsymbol{\varphi}_i$, $i = 1, \cdots, M$ [8]:

$$\boldsymbol{x} = \sum_{i=1}^{M} \langle \boldsymbol{x}, \widetilde{\boldsymbol{\varphi}}_i \rangle \boldsymbol{\varphi}_i = \frac{1}{r} \sum_{i=1}^{M} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \boldsymbol{\varphi}_i \tag{1}$$

where $y_i = \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle$ is the i-th coefficient of the expansion. Although there is an infinite number of possible dual frames, when discussing linear reconstruction in this paper, we assume that the minimal dual frame is always used. We provide in section 3.2 a necessary condition to achieve consistency using linear reconstruction in tight frames in $\mathcal{R}^N$ with integer redundancy r and which are composed by a set of orthogonal bases. We restrict most of our attention in this paper to tight frames that are composed by a set of orthogonal bases because the geometric analysis is much simpler. With this restriction, we can group the vectors $\{\boldsymbol{\varphi}_i\}_{i=1}^M$ that compose the tight frame as $\{\{\boldsymbol{\varphi}_i^j\}_{i=1}^N\}_{j=1}^r$, where $\{\boldsymbol{\varphi}_i^j\}_{i=1}^N$ is the j-th basis. For the sake of simplicity, we restrict most of the equations, without any loss of generality, to $\mathcal{R}^2$ and explain later the extension to $\mathcal{R}^N$. For $N = 2$, we define each orthogonal matrix $\boldsymbol{F}^j$ as $\boldsymbol{F}^j = [\boldsymbol{\varphi}_1^j \boldsymbol{\varphi}_2^j]^T$ and we call $\boldsymbol{y}^j = [y_1^j, y_2^j]^T$ the 2-dimensional vector of coefficients associated with the j-th basis, which is given by $\boldsymbol{y}^j = \boldsymbol{F}^j \boldsymbol{x}$.

Let SQ_i^j be a uniform scalar quantizer with stepsize Δ_i^j and decision points $\{m\Delta_i^j\}_{m \in \mathcal{Z}}$. Let $SQ_1^1 \times SQ_2^1 \times \cdots \times SQ_1^r \times SQ_2^r$ be an M-dimensional product scalar quantizer (PSQ) applied to the M-dimensional vector of coefficients $\boldsymbol{y}$. Given a frame



(a) (b)

Figure 1: a) Definition of a quantizer Q in $\mathcal{R}^2$ based on the linear reconstruction of a tight frame, b) Reconstructions for the quantizers Q^1, Q^2 and Q when linear reconstruction is used. The partial reconstructions $\hat{\boldsymbol{x}}^j$, $j = 1, 2$ are represented by $'*'$ and the final reconstruction $\hat{\boldsymbol{x}}$ is represented by $'\circ'$. The final reconstructions are obtained by taking the halfway point between $\hat{\boldsymbol{x}}^1$ and $\hat{\boldsymbol{x}}^2$, that is, $\hat{\boldsymbol{x}} = \frac{1}{2}(\hat{\boldsymbol{x}}^1 + \hat{\boldsymbol{x}}^2)$.

Φ and a PSQ, an equivalent vector quantizer $Q : \mathcal{R}^2 \longrightarrow \mathcal{R}^2$ can be defined such that $Q(\boldsymbol{x}) = \hat{\boldsymbol{x}} = \frac{1}{r}\sum_{j=1}^{r} \hat{\boldsymbol{x}}^j$ (see Fig. 1(a)) is the average taken over the r partial reconstructions $Q^j(\boldsymbol{x}) = \hat{\boldsymbol{x}}^j$ $j = 1, \cdots, r$, where Q^j is a quantizer with rectangular Voronoi cells and associated with the j-th orthogonal basis, as illustrated in Fig. 1(b).

The scalar uniform quantizers $\{SQ_1^j, SQ_2^j\}$ induce rectangular Voronoi cells $\{V_i^{Q^j}\}$ for each quantizer Q^j, whose sides are parallel to the axes of a rotated coordinate system and the vertices of these cells are the points defined by a real lattice Λ^j with generator matrix $\boldsymbol{M}_{\Lambda^j} = (\Delta_1^j \boldsymbol{\varphi}_1^j | \Delta_2^j \boldsymbol{\varphi}_2^j)^T$. The final partition defined by Q is induced by the application of the PSQ on the coefficients and the Voronoi cells $\{V_i^Q\}$ are determined by the *intersection of the rectangular Voronoi cells of the quantizers* $\{Q^j\}_{j=1}^r$. Thus, the form of these intersections depend totally on the lattices Λ^j $j = 1, \cdots, r$, which depend on the stepsizes and vectors of the frame that are chosen. Notice that the same happens for any dimension N.

Given a generic (not necessarily linear) reconstruction algorithm, a Voronoi cell V_i^Q is said to be consistent iff $\forall \boldsymbol{x} \in V_i^Q$ the reconstructed vector $\hat{\boldsymbol{x}}$ is consistent, i.e. $\hat{\boldsymbol{x}} \in V_i^Q$. For the particular case of linear reconstruction, we say that the cell V_i^Q is linearly consistent. This property is desirable because a consistent reconstruction gives a lower MSE than an inconsistent reconstruction, for the same Voronoi cells. A quantizer Q is a consistent quantizer iff all its cells $\{V_i^Q\}$ are consistent. The consistency property in a quantizer Q is actually equivalent to the property of regularity in a general vector quantizer. Fig. 2 shows an example of a quantizer Q for $r = 2$ that is linearly consistent. On the contrary, Fig. 1(b) shows an example of a cell (lower cell) that is not consistent linearly.

A consistent reconstruction could in principle be implemented using a look-up table, but in practice this is not feasible for moderate to high rates, as the table would be prohibitively large. As explained in the next section, by choosing lattices Λ^j $j = 1, \cdots, r$ with certain properties it is possible a) to reconstruct consistently

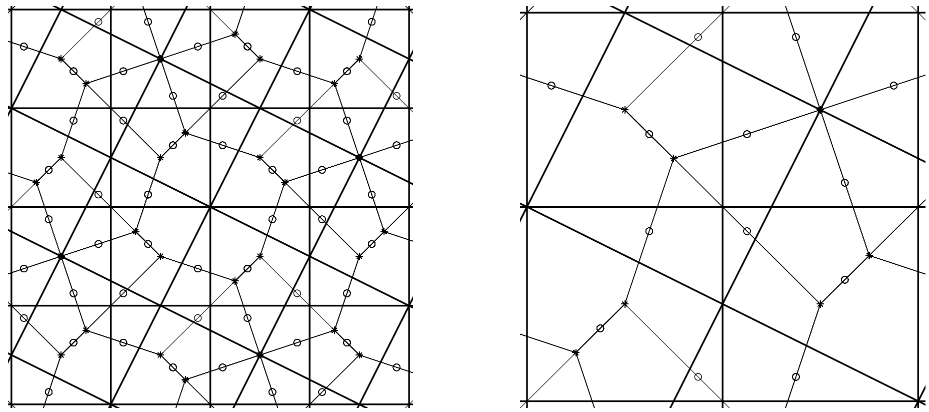


Figure 2: a)Example of a linearly consistent quantizer Q. The reconstructions $\hat{\boldsymbol{x}}^j$, $j = 1, 2$ are represented by $'*'$ and the final reconstruction $\hat{\boldsymbol{x}}$ is represented by $'\circ'$. Parameters: $\Delta_2^1 = \Delta_1^1$, $\Delta_2^2 = \Delta_1^2 = \frac{\sqrt{5}}{2}\Delta_1^1$, $\tan(\theta) = \frac{1}{2}$, b) Zoom.

under linear reconstruction and b) to reconstruct consistently using a small look-up table.

3 Quantizers with Periodic Structure

A periodic structure in the partition of the quantizer Q can be induced by choosing certain lattices, and this property of periodicity is necessary to solve efficiently the problem of reconstructing consistently.

3.1 Definition and Construction

A "periodic quantizer" Q is a quantizer where there is only a finite number of distinct Voronoi cells $\{V_i^Q\}$ that are repeated periodically (see how the pattern in Fig. 2(a) is periodic). Let us first assume to facilitate the understanding that Q is a quantizer in $\mathcal{R}^2$. In order to impose a periodic structure in Q, it is necessary to have a sublattice structure [6]. A sublattice $S\Lambda \subset \Lambda$ of a given lattice Λ is a subset of the elements of Λ that is itself a lattice. Given a real lattice Λ with generator matrix $\boldsymbol{M}_\Lambda$, a sublattice[1] $S\Lambda$ is completely specified by an integer matrix $\boldsymbol{B}_{S\Lambda}$ that maps a basis of Λ into a basis of $S\Lambda$, that is, $\boldsymbol{M}_{S\Lambda} = \boldsymbol{B}_{S\Lambda}\boldsymbol{M}_\Lambda$.

Definition 1 *Given a real lattice Λ in $\mathcal{R}^2$ with generator matrix $\boldsymbol{M}_\Lambda$, a lattice Λ' is geometrically scaled-similar to Λ iff:*

$$\boldsymbol{M}_{\Lambda'} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \boldsymbol{U}\boldsymbol{M}_\Lambda\boldsymbol{R} \tag{2}$$

where $\boldsymbol{R}$ is a 2×2 orthogonal matrix, that is, a rotation and/or a reflection in $\mathcal{R}^2$, $\boldsymbol{U}$ is a 2×2 unimodular matrix, that is, a matrix with integer components satisfying

[1] We assume that both Λ and $S\Lambda$ are full rank lattices, that is, the matrices $\boldsymbol{M}_\Lambda$ and $\boldsymbol{M}_{S\Lambda}$ are full rank

that $|det(\boldsymbol{U})| = 1$, and $c_1, c_2 \in \mathcal{R}_+$. If $\Lambda' = S\Lambda \subset \Lambda$, then $S\Lambda$ is a geometrically scaled-similar sublattice of Λ and c_1, c_2 and $\boldsymbol{R}$ are constrained.

It can be seen in (2) that a scaled-similar sublattice $S\Lambda$ is obtained by simply rotating and/or reflecting the lattice Λ and then scaling each of the new rotated axes by a certain factor, which is allowed to be different in each of the axes. Notice that in the particular case of having $c_1 = c_2$, $S\Lambda$ is a geometrically similar (or equivalent) sublattice of Λ, as defined by Conway et al. [5][6]. Fig. 3(a) shows an example

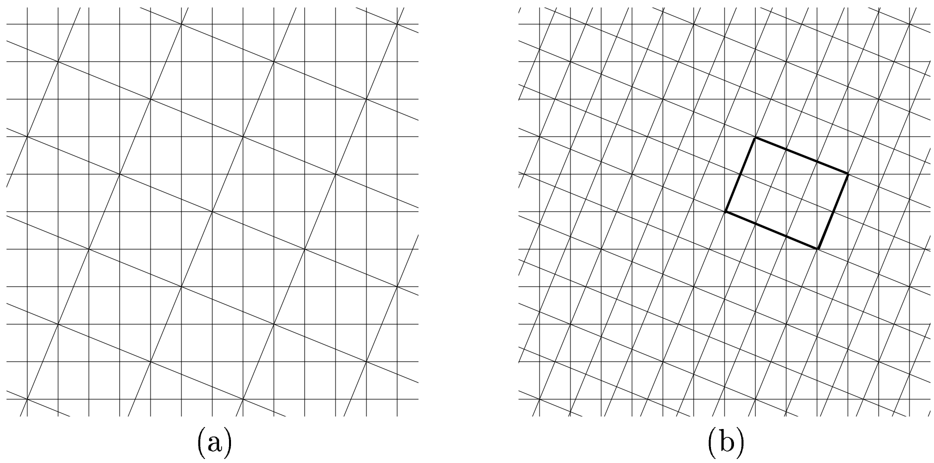


Figure 3: Example 1: a) Sublattice structure b) Voronoi cells $\{V_i^Q\}$. Parameters: $\beta = \sqrt{\frac{3}{2}}$, $\Delta_2^1 = \beta\Delta_1^1$, $\Delta_1^2 = \frac{1}{2\cos(\theta)}\Delta_1^1$, $\Delta_2^2 = \frac{1}{3\cos(\theta)}\beta\Delta_1^1$, $\tan(\theta) = \sqrt{6}$.

of sublattice for $r = 2$, where the cell indicated with bold line is the fundamental polytope of $S\Lambda$, which we denote by $V_o^{S\Lambda}$.

Let Λ^1 be a rectangular lattice with $\boldsymbol{M}_{\Lambda^1} = diag[\Delta_1^1, \Delta_2^1]$, which defines a quantizer Q^1. If there are r sublattices of Λ^1, we will denote them by $S\Lambda^1$, $S\Lambda^2$, ..., $S\Lambda^r$, and for notational convenience, we take $S\Lambda^1 = \Lambda^1$. We will always take $U = I$ in (2) so that the basis vectors of the j-th geometrically scaled-similar sublattice are orthogonal and can be associated with the j-th orthogonal basis of a tight frame.

In $\mathcal{R}^2$ it is easy to parameterize all the geometrically scaled-similar sublattices of the rectangular lattices Λ^1 where $\boldsymbol{M}_{\Lambda^1} = diag[\Delta_1^1, \Delta_2^1]$, but in general in $\mathcal{R}^N$, it is much simpler to restrict ourselves to finding geometrically scaled-similar sublattices of the hypercubic lattice Λ^1 where $\boldsymbol{M}_{\Lambda^1} = \boldsymbol{I}\Delta_1^1$, that is, if the dimension is N, then $\Delta_1^1 = \Delta_2^1 = \cdots = \Delta_N^1$. In this case, we can construct matrices $\boldsymbol{B}_{S\Lambda^j}$, which give rise to scaled-similar sublattices of Λ^1 in $\mathcal{R}^N$, in the following way:

$$\boldsymbol{B}_{S\Lambda^j} = \begin{pmatrix} a_1^j & 0 & \cdots & 0 \\ 0 & a_2^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_N^j \end{pmatrix} \boldsymbol{H}_{S\Lambda^j}, \quad a_1^j, \cdots, a_N^j \in \mathcal{Z}, \quad \boldsymbol{H}_{S\Lambda^j}\boldsymbol{H}_{S\Lambda^j}^T = k^j\boldsymbol{I}, \quad k^j \in \mathcal{Z}_+$$

where $\boldsymbol{H}_{S\Lambda^j}$ has integer entries. The problem of finding matrices $\boldsymbol{H}_{S\Lambda}$ satisfying the above property has been studied extensively [10]. In order to design periodic quantizers Q in $\mathcal{R}^N$ with good properties it is however necessary to search for good solutions.

Periodicity Property: Since $S\Lambda$ is a sublattice of Λ^1, $S\Lambda$ is a subgroup of the additive group Λ^1, and it follows by group theory that the partition defined by $\{V_i^{\Lambda^1}\} \cap \{V_i^{S\Lambda}\}$ has a periodic structure (tesselation) with the basic unit cell being $\{V_i^{\Lambda^1}\} \cap V_o^{S\Lambda}$ (see Fig. 3(a)). Since the subgroup structure is the same for any dimension N, the periodicity property is true for any dimension N.

In order to construct a general periodic quantizer Q for a redundancy r, we have to construct $r-1$ quantizers $\{Q^j\}_{j=2}^r$, which will be defined by lattices $\Lambda^j \supset S\Lambda^j$ ($S\Lambda^j$ being a geometrically scaled-similar sublattice of Λ^1) with generator matrices of the form $\boldsymbol{M}_{\Lambda^j} = diag[1/d_1^j, 1/d_2^j]\boldsymbol{M}_{S\Lambda^j}$, where $d_1^j, d_2^j \in \mathcal{Z}_+$. The important property is that when all the lattices $\{S\Lambda^j\}_{j=1}^r$ are sublattices of Λ^1, the intersection of all the lattices Λ^j, $j = 1, \cdots, r$ is not empty, and therefore, by group theory, is a lattice.

The division by the integers $\{d_1^j, d_2^j\}$ ensures that the Voronoi cells $\{V_i^Q\}$ will keep a periodic structure, which is still determined by $V_o^{S\Lambda^j}$ (see Fig. 3(b)).

Definition 2 *Given a set of lattices Λ^j $j = 1, \cdots, r$ (as defined in (4)), we define the coincidence site lattice (CSL) Λ^{CSL} as:*

$$\Lambda^{CSL} = \Lambda^1 \cap \Lambda^2 \cap \cdots \cap \Lambda^r \tag{3}$$

and thus, it is the finest common sublattice of all the lattices Λ^j, $j = 1, \cdots, r$.

The generator matrix $\boldsymbol{M}_{\Lambda^{CSL}}$ can be calculated making use of the concept of dual lattices [3].

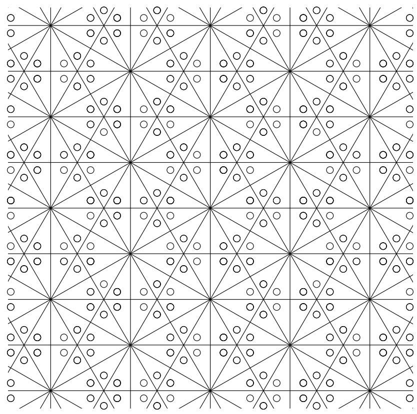
The importance of calculating the coincidence site lattice Λ^{CSL} comes from the fact that its fundamental polytope $V_o^{\Lambda^{CSL}}$ is the smallest unit cell that is repeated in the periodic structure of the resulting quantizer Q, as stated in the following Lemma. The Lemma follows directly from group theory because Λ^{CSL} is the finest common sublattice (subgroup) of all the lattices Λ^j, $j = 1, \cdots, r$.

Lemma 1 *Given r quantizers Q^j, $j = 1, \cdots, r$, defined by the lattices Λ^j $j = 1, \cdots, r$ (as given in (4)), the partition of Voronoi cells $\{V_i^Q\}$ of the final quantizer Q has a periodic structure, with the unit cell that is repeated periodically being $V_o^{\Lambda^{CSL}}$, the fundamental polytope of the coincidence site lattice Λ^{CSL}.*

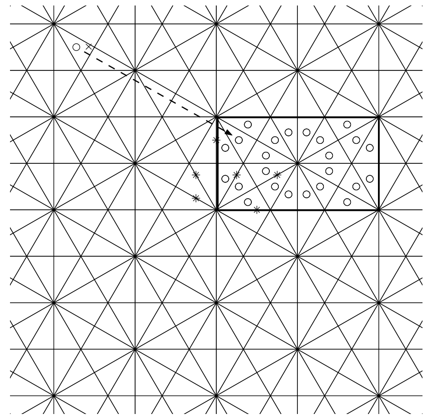
3.2 Consistent Reconstruction in Periodic Quantizers

Let $\Phi = \{\{\boldsymbol{\varphi}_i^j\}_{i=1}^N\}_{j=1}^r$ be a tight frame of redundancy r which is composed of r orthogonal bases. It turns out that a necessary condition to have consistency under linear reconstruction for this tight frame is that the quantizer Q has to be periodic. This result follows from the fact that when there is no periodicity in the partition defined by a quantizer Q, the vertices of any two lattices Λ^{j_1} and Λ^{j_2} ($j_1 \neq j_2$) can have arbitrary relative positions, at least in one of the components, which makes always possible to find linearly inconsistent cells. On the contrary, when there is periodicity, there is only a finite number of relative positions (see Fig. 4(a)) and linearly consistency may be held. A formal proof is not given here for reasons of space, but can be found in [3].

Theorem 1 *If Q is a non-periodic quantizer in $\mathcal{R}^N$, then it is always possible to find a linearly inconsistent cell, and hence, Q is a quantizer which is not consistent linearly. Therefore, periodicity in a quantizer Q is a necessary condition to achieve consistency under linear reconstruction.*



(a)



(b)

Figure 4: a) Example of linear consistency for $r = 3$; b) Efficient reconstruction based on look-up table: $'\circ'$ represents reconstruction vectors, $'*'$ represents the values of the quantized coefficients which define the equivalent cell in the unit cell P_o, 'x' represents the input vector. All the information is first translated to the unit cell P_o, then the reconstruction vector of the equivalent cell is read, and finally it is translated back to the proper cell.

Notice that since in a quantizer Q which is periodic there are only a finite number of distinct Voronoi cells $\{V_i^Q\}$, to check if consistency is satisfied linearly, we only need to check on the Voronoi cells contained inside the fundamental polytope of the coincidence site lattice Λ^{CSL}. An example of linear consistency for $r = 2$ has been already shown in Fig. 2. An example for $r = 3$ is given in Fig. 4(a).

Given a periodic quantizer Q, it is also possible to reconstruct efficiently and accurately by using a look-up table scheme of small size, which also ensures consistency. Assume, for simplicity in the discussion, that $N = 2$ and let P_o be the smallest rectangular polytope, which is a basic unit cell for the partition defined by Q. There are several choices for P_o (Fig. 4(b) indicates a possible P_o with bold lines). The basic idea is that given any Voronoi cell V_i^Q it is possible to find very fast (floor operation) the equivalent cell $V_{i_o}^Q$ which is inside P_o. Given an input signal $\boldsymbol{x}$, a reconstruction vector $\hat{\boldsymbol{x}}_{\boldsymbol{o}} \in V_{i_o}^Q$ is read from a look-up table and finally this reconstruction vector is translated back into the proper cell V_i^Q. The fundamental advantage provided by the periodicity is that if the periodic quantizer Q is well designed, the size of the look-up table can be made small, and does not increase with the rate of the quantizer Q.

4 Design examples and numerical results in $\mathcal{R}^2$

Let Λ^1 be a rectangular lattice in $\mathcal{R}^2$ whose generator matrix is diagonal, all the geometrically scaled-similar sublattices $S\Lambda \subset \Lambda^1$ (with the matrix $\boldsymbol{R}$ in (2) constrained to be a rotation), have generator matrices of the form:

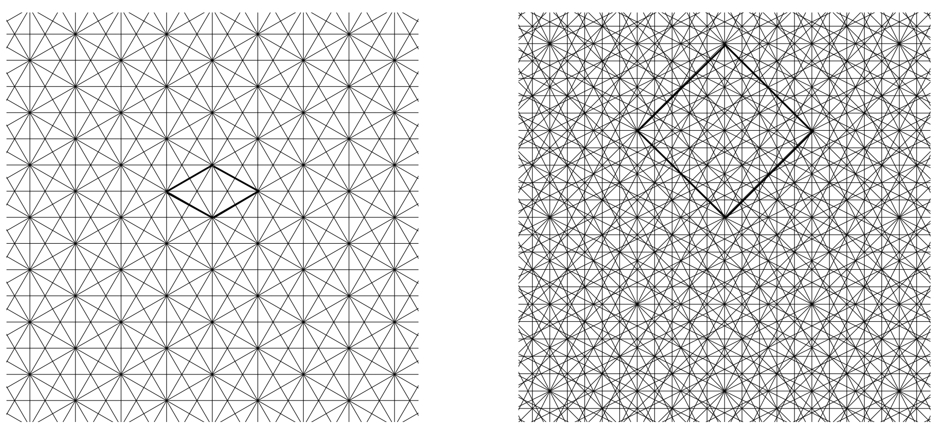
$$\boldsymbol{M}_{S\Lambda} = \begin{pmatrix} c_1\Delta_1^1 & 0 \\ 0 & c_2\beta\Delta_1^1 \end{pmatrix} \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ -k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \Delta_1^1$$

$$\text{where} \quad \beta = \frac{\Delta_2^1}{\Delta_1^1} = \sqrt{\frac{k_{11}k_{21}}{k_{12}k_{22}}}, \quad \tan(\theta) = \sqrt{\frac{k_{12}k_{21}}{k_{11}k_{22}}} = \frac{k_{12}}{k_{11}}\beta, \quad c_1 = \frac{k_{11}}{cos(\theta)}, \quad c_2 = \frac{k_{22}}{cos(\theta)},$$
$$k_{11}, k_{12}, k_{21}, k_{22} \in \mathcal{Z}_+, \quad 0 < \theta < \frac{\pi}{2}$$

where the integer matrix with entries $\{k_{lm}\}$ is denoted by $\boldsymbol{B}_{S\Lambda}$. Notice that only those angles θ, whose tangent is the square root of two integers lead to a geometrically scaled-similar sublattice. The lattice Λ^j and corresponding stepsizes are given by:

$$M_{\Lambda^j} = \begin{pmatrix} \frac{k_{11}^j}{d_1^j} & \frac{k_{12}^j}{d_1^j}\beta \\ \frac{-k_{21}^2}{d_2^j} & \frac{k_{22}^j}{d_2^j}\beta \end{pmatrix} \Delta_1^1 \qquad \begin{aligned} \Delta_1^j &= \frac{\Delta_1^1}{d_1^j}\sqrt{\frac{k_{11}^j}{k_{22}^j}(k_{11}^j k_{22}^j + k_{12}^j k_{21}^j)} \\ \Delta_2^j &= \frac{\Delta_1^1}{d_2^j}\sqrt{\frac{k_{21}^j}{k_{12}^j}(k_{11}^j k_{22}^j + k_{12}^j k_{21}^j)} \end{aligned} \tag{4}$$

In practice, the integers d_1^j and d_2^j are constrained to some intervals in order to get a final quantizer Q with Voronoi cells $\{V_i^Q\}$ having similar sizes, which obviously is useful in achieving good coding performance.



(a) (b)

Figure 5: a) Example for $r = 3$: Structure of the quantizer Q and unit cell of the structure; b) Example for $r = 4$: Structure of the quantizer Q and unit cell.

Example 1 *An example for $r = 3$ is shown in Fig. 5(a), which is composed by:*

$$\boldsymbol{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \cos(\frac{\pi}{6}) & \sin(\frac{\pi}{6}) \\ -\sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \\ \cos(\frac{\pi}{3}) & \sin(\frac{\pi}{3}) \\ -\sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{pmatrix} \qquad \begin{aligned} &\beta = \tfrac{1}{\sqrt{3}}, && \Delta_2^1 = \beta\Delta_1^1 = \tfrac{1}{\sqrt{3}}\Delta_1^1 \\ &\Delta_1^2 = \tfrac{1}{2}\left(\tfrac{1}{\cos(\frac{\pi}{6})}\right)\Delta_1^1, && \Delta_2^2 = \tfrac{1}{2}\left(\tfrac{3}{\cos(\frac{\pi}{6})}\right)\left(\tfrac{1}{\sqrt{3}}\right)\Delta_1^1 \\ &\Delta_1^3 = \tfrac{1}{2}\left(\tfrac{1}{\cos(\frac{\pi}{3})}\right)\Delta_1^1, && \Delta_2^3 = \tfrac{1}{2}\left(\tfrac{1}{\cos(\frac{\pi}{3})}\right)\left(\tfrac{1}{\sqrt{3}}\right)\Delta_1^1 \end{aligned}$$

$$\boldsymbol{M}_{\Lambda^{CSL}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \Delta_1^1$$

Example 2 *An example for $r = 4$ is shown in Fig. 5(b), which is composed by:*

$$
\boldsymbol{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}
\qquad
\begin{array}{ll}
\beta = 1, & \Delta_2^1 = \beta\Delta_1^1 = \Delta_1^1 \\
\Delta_1^2 = \sqrt{2}\Delta_1^1, & \Delta_2^2 = \sqrt{2}\Delta_1^1 \\
\Delta_1^3 = \frac{\sqrt{5}}{2}\Delta_1^1, & \Delta_2^3 = \frac{\sqrt{5}}{2}\Delta_1^1 \\
\Delta_1^4 = \frac{\sqrt{5}}{2}\Delta_1^1, & \Delta_2^4 = \frac{\sqrt{5}}{2}\Delta_1^1 \\
\Delta_1^5 = \frac{\sqrt{5}}{2}\Delta_1^1, & \Delta_2^5 = \frac{\sqrt{5}}{2}\Delta_1^1 \\
\Delta_1^6 = \frac{\sqrt{5}}{2}\Delta_1^1, & \Delta_2^6 = \frac{\sqrt{5}}{2}\Delta_1^1
\end{array}
\qquad
\boldsymbol{M}_{\Lambda^{CSL}} = \begin{pmatrix} -5 & 5 \\ 5 & 5 \end{pmatrix} \Delta_1^1
$$

Our work focuses on absolute values of the MSE for a given redundancy r, instead of looking at the asymptotic function $MSE = O(f(r))$ for high enough redundancies, as has been done in [1, 7, 4]. The reconstruction algorithms used in all the previous work consist of either a POCS (projection on convex sets) or linear programming based algorithm, and their performance is much better than linear reconstruction only asymptotically, that is, for large values of the redundancy. Besides, the complexity of the algorithms is much higher than the simple linear reconstruction algorithm. On the other hand, our designs are more suitable to be used for small redundancies and have a complexity similar to the linear reconstruction. At high redundancies, it is always possible to find designs but they may not be very efficients in terms of coding due to the number of constraints that have to be met. However, for some very important applications such as those involving very high-bandwidth analog signals, it is not feasible to use redundancies higher than $r = 3$ or $r = 4$.

We have compared linear reconstruction (with equal stepsizes) and reconstruction based on periodic quantizers (with different stepsizes) using the look-up table scheme, in terms of accuracy (MSE), with an input source being a 2-dimensional Gaussian distribution $\mathcal{N}(\boldsymbol{0}, \sigma^2\boldsymbol{I})$ with $\sigma = 0.3$. Although we could have also compared linear reconstruction using both a periodic quantizer and a non-periodic quantizer, we have used a look-up table scheme for 2 reasons: a) the complexity is the same as for the linear reconstruction and b) for the specific designs that have been used, it is possible to reconstruct linearly with the centroids (as in the look-up table) by using to calculate the reconstruction a frame different from the minimal. The comparison has been made by fixing the total rate, which is calculated assuming that all the coefficients are encoded independently with a fixed-length encoding. Actually, this can be seen as being equivalent to making the comparison when the density of points in the space is the same. Let $S = \{\Delta_i^j\}$ be the set of stepsizes used by a periodic quantizer Q and Δ the stepsize used (to quantize all the coefficients of the expansion) by another non-periodic quantizer Q'. Notice that each stepsize Δ_i^j of Q can be expressed as $\Delta_i^j = \alpha_i^j \Delta_1^1$, for some $\alpha_i^j \in \mathcal{R}$. The (fixed-length) rate corresponding to each stepsize Δ_i^j can be measured (associated with the density implied by Δ_i^j) as $\log_2(1/\Delta_i^j)$. In order to have the same total rate in both quantizers Q and Q', we need the following condition:

$$
\sum_{j=1}^{r}\sum_{i=1}^{N} \log_2\left(\frac{1}{\Delta_i^j}\right) = rN\log_2\left(\frac{1}{\Delta}\right) \Rightarrow \Delta_1^1 = \left(\frac{1}{\prod_{i,j}\alpha_i^j}\right)^{\frac{1}{rN}} \Delta
$$

In this way, we can perform a comparison at each value of the stepsize Δ. For each

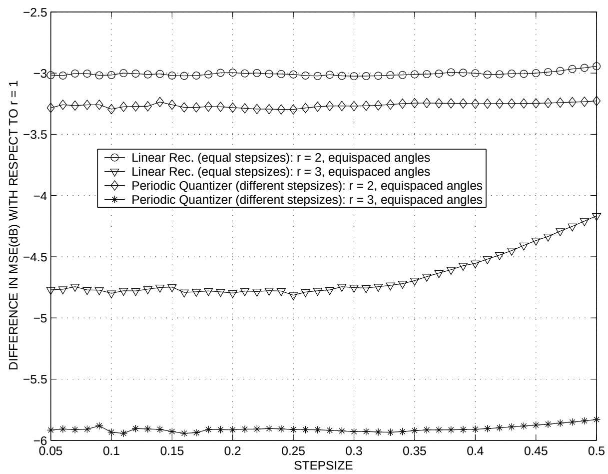


Figure 6: Comparison between Linear reconstruction with equal stepsizes and Reconstruction based on look-up table for a Periodic Quantizer. The values of MSE are given in dB and relative to $r = 1$.

value of Δ, the set $S = \{\Delta_i^j\}$ is calculated and the MSE is measured. Fig. 6 represents 2 comparisons, for $r = 2$ and $r = 3$. The periodic quantizer for $r = 3$ corresponds to the quantizer shown in Fig. 4(b)). For this quantizer in order to illustrate how the equivalent set of stepsizes is calculated, as an example, for $\Delta = 1$, we have that $\Delta_1^1 = 1.316, \Delta_2^1 = 0.759, \Delta_1^2 = 0.759, \Delta_2^2 = 1.316, \Delta_1^3 = 1.316, \Delta_2^3 = 0.759$. The periodic quantizer that has been used for $r = 2$ corresponds to $\beta = 1$, $\tan(\theta) = 1$ and $\Delta_1^2 = \Delta_2^2 = \sqrt{2}\Delta_1^1$. It can be seen that there is a gain for both quantizers.

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