Error-Rate Dependence of Non-Bandlimited Signals with Finite Rate of Innovation

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Abstract — **Recent results in sampling theory [1] showed that perfect reconstruction of non-bandlimited signals with finite rate of innovation can be achieved performing uniform sampling at or above the rate of innovation. We study analog-to-digital (A/D) conversion of these signals, introducing two types of ovrsampling and consistent reconstruction.**

In this work, we consider periodic streams of K Diracs, that is, $x(t) = \sum_{k\in\mathcal{Z}} c_k\delta(t-t_k) = \sum_{m\in\mathcal{Z}} X[m]e^{j(2\pi mt)/\tau}$ with $X[m] = \frac{1}{\tau}\sum_{k=0}^{K-1} c_k e^{-j(2\pi mt_k)/\tau}$ and period τ, where $t_{k+K} = t_k + \tau$, $c_{k+K} = c_k$, $\forall k \in \mathcal{Z}$ and $\delta(t)$ denotes a Dirac delta function. The signal has $2K/\tau$ degrees of freedom per unit of time. Time positions $\{t_k\}_{k=0}^{K-1}$ and weights $\{c_k\}_{k=0}^{K-1}$ can be perfectly reconstructed by first applying a sinc sampling kernel $h_B(t) = Bsinc(Bt)$ with bandwidth $[-B\pi, B\pi]$, thus obtaining $y(t) = x(t) * h_B(t)$, and then taking the $N \geq 2M+1$ uniform samples $y_n = \sum_{m=-M}^{M} X[m]e^{j2\pi mnT/\tau}$ with $T = \tau/N$, such that $B\tau = 2M+1 \geq 2K+1$. After computing $2K+1$ Fourier coefficients from y_n, we apply annihilating filter method. From the roots of annihilating filter $\{u_k = e^{-j2\pi t_k/\tau}\}_{k=0}^{K-1}$ we get the K time positions $\{t_k\}_{k=0}^{K-1}$, while the weights $\{c_k\}_{k=0}^{K-1}$ can then be directly computed.

We overcome the error in amplitude of the samples $\{y_n\}_{k=0}^{K-1}$, introduced due to the quantization, by performing two types of oversampling. The first one, *oversampling in time*, consists of taking more samples of $y(t)$ than necessary, so that $N > 2M+1$, with oversampling ratio $R_t = N/2M+1$. In the second one, *oversampling in frequency*, we extend the bandwidth $B = 2M+1$ so that it is greater than the rate of innovation, that is, $M > K$, with oversampling ratio $R_f = (2M+1)/(2K+1)$.

We also introduce the concept of consistent reconstruction for these types of signals. The idea is to exploit all the *a priori* knowledge of the original signal and the quantization process itself. We first define the three sets of constraints on which we have to project. **Set S_1** is defined by the quantization operation and consists of the quantization bins in which the samples $\{y_n\}_{n=0}^{N-1}$ lie. **Set S_2** is the set of continuous-time periodic signals bandlimited to $[-B\pi, B\pi]$ to which $y(t)$ belongs.

Based on this, satisfying these two sets we provide a first level of accuracy, *weak consistency*, which we achieve by iterating projections $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.

Def. 1 *A reconstruction $\hat{x}(t)$ satisfies weak consistency (WC) iff it is obtained from a signal $\hat{y}(t)$ such that: a) the samples $\{\hat{y}_n\}_{n=0}^{N-1}$ lie in the same quantization bins as the original ones, $\{\hat{y}_n\}_{n=0}^{N-1} \in \boldsymbol{S_2}$, b) $\hat{y}(t) \in \boldsymbol{S_1}$.*

Proj.$\boldsymbol{P_1}$: For every estimate $\hat{y}_n^i$, $\hat{y}_n^{i+1} = \boldsymbol{P_1}(\hat{y}_n^i)$ is given by: a) $\hat{y}_n^{i+1} = \hat{y}_n^i$ if $\hat{y}_n^i \in \boldsymbol{S_1}$, 2) else, $\hat{y}_n^{i+1}$ is set to the bound of the quantization interval in $\boldsymbol{S_1}$ closest to $\hat{y}_n^i$.

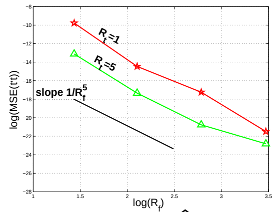

Figure 1: *Dependence of $MSE(\boldsymbol{t}, \hat{\boldsymbol{t}})$ on the factors R_t and R_f.*

Proj.$\boldsymbol{P_2}$: Given an estimate $\hat{y}^i(t)$, the new estimate $\hat{y}^{i+1}(t) = \boldsymbol{P_2}(\hat{y}^i(t))$ is obtained by low-pass filtering $\hat{y}^i(t)$, that is $\hat{y}^{i+1}(t) = \hat{y}^i(t) * h_B(t)$. The particular structure of the signal $x(t)$ defines the third set which, together with previous two sets, is used to enforce a stronger sense of consistency. The **Set $\boldsymbol{S_3}$** is the set of Fourier coefficients that originate from a periodic stream of Diracs, $X[m] = \frac{1}{\tau}\sum_{k=0}^{K-1} c_k e^{-j2\pi mt_k/\tau}$.

Def. 2 *A reconstruction $\hat{x}(t)$ satisfies strong consistency (SC) iff: a) it satisfies weak consistency, b) $\hat{y}_n = h_b(t) * \hat{x}(t)|_{nT}$ where $\hat{x}(t)$ is a periodic stream of K Diracs.*

Proj.$\boldsymbol{P_3}$: Given a set of estimated Fourier coefficients $\hat{\boldsymbol{X}}^i$, the projection $\boldsymbol{P_3}$ provides $\{(\hat{t}_k^{i+1}, \hat{c}_k^{i+1})\}_{k=0}^{K-1}$ and a set of Fourier coefficients $\hat{\boldsymbol{X}}^{i+1}$ such that $\hat{X}^{i+1}[m] = \frac{1}{\tau}\sum_{k=0}^{K-1} \hat{c}_k^{i+1} e^{-j2\pi m\hat{t}_k^{i+1}/\tau}$.

Theorem 1 *Given $x(t)$, for any reconstruction $\hat{x}(t)$ obtained using $\boldsymbol{P_3}$ and which satisfies WC, there exist $\xi \geq 1$ such that if $R_t, R_f \geq \xi$, there is a constant $c > 0$ which depends only on $x(t)$ and not on R_t and R_f, and $MSE(\boldsymbol{t}, \hat{\boldsymbol{t}}) \leq \frac{c}{R_t^3 R_f^2}$.* (see [3])

For the method that achieve *SC* the experimental results show, a performance of $MSE(\boldsymbol{t}, \hat{\boldsymbol{t}}) = O(1/R_t^2 R_f^5)$ for time positions (Fig. 1), with parameters: $K = 2$, $\tau = 10$, $t_k \in (0, \tau]$, $c_k \in [-1, 1]$.

We also compare two types of encoding, the traditional one, pulse-code modulation encoding (PCM) and the alternative one, based on threshold crossing encoding (TC) [2], and investigate in the dependence of the bit rate on the oversampling factors R_t and R_f, and the quantization step size Δ. The following table, shows the theoretical results for the bit rate and also both theoretical and experimental results for the MSE of time positions.

	Bit rate (b)	*MSE-WC*	*MSE-SC*
TC	$O(\log_2 R_t)$	$O(1/R_t^2)$	$O(1/R_t^2)$
	$O(R_f \log_2 R_f)$	$O(1/R_f^3)$	$O(1/R_f^5)$
	$O(1/\Delta)$	$O(\Delta^2)$	$O(\Delta^2)$
PCM	$O(R_t)$	$O(1/R_t^2)$	$O(1/R_t^2)$
	$O(R_f \log_2 R_f)$	$O(1/R_f^3)$	$O(1/R_f^5)$
	$O(\log_2(1/\Delta))$	$O(\Delta^2)$	$O(\Delta^2)$

Notice that oversampling in time provide the error-rate dependence ($O(2^{-2\alpha b})$) that can be obtain by decreasing the step size ($O(2^{-2\beta b})$).

References

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[2] N. T. Thao and M. Vetterli, "Deterministic Analysis of Oversampled A/D Conversion and Decoding Improvement Based on Consistent Estimates" *IEEE Trans. on Signal Proc.* , vol. 42, no. 3, pp. 519–531, 1994.

[3] I. Jovanović and B. Beferull-Lozano, "Oversampled A/D Conversion of Non-Bandlimited Signals with Finite Rate of Innovation", *To be submitted to IEEE Trans. on Signal Proc.*

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