DISCRETE MULTI-DIRECTIONAL WAVELET BASES

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ABSTRACT

The application of the wavelet transform in image processing is most frequently based on a separable construction. While simple, such an approach is not capable of capturing properly all 2D properties in images. In this paper, a new truly separable multi-directional transform is proposed with a subsampling method based on lattice theory. Applications are possible in many areas of image processing. Some promising improvements are achieved in non-linear approximation and denoising of images.

1. INTRODUCTION

The standard 2D wavelet transform in image processing is based on a separable construction. 2D filtering is performed through the outer product of two independent 1D filterings in horizontal and vertical directions. This approach is very simple and reduces importantly the computational complexity with respect to non-separable filtering. However, 2D phenomena present in images contain information in many more orientations other than only horizontal and vertical directions. Thus, the ability to capture information in more orientations is desirable.

Several approaches based on non-separable filterings have been proposed showing promising results [1, 5, 8]. However, the filter design is much more complex and the computational complexity is substantially higher. In addition, some of these designs are based on the continuous domain [12] or interpolates pixels in the discrete space [9] which may cause problems when applying them in the image discrete domain.

A new approach proposed in [10], the directional wavelet transform, borrows the simplicity of the normal separable wavelet transform while adding more directions for analysis, and this directly on discrete data. While the design of multi-directional frames is pretty straightforward using solutions from computer graphics [2], the construction of multi-directional bases having good representation properties presents a more difficult challenge due to the subsampling.

In this paper, we use a lattice theory based method to define a new convenient directional subsampling method. The proposed method considers each intersection of the digital lines and cosets produced by shifted versions of a lattice as independent transform lines (called *co-lines*). The transform is still separable and it allows many (much more than only two) transform directions. Moreover, the subsampling issue is solved simply and clearly for a general combination of different angles. It contributes also in an improvement of non-linear approximation of images.

The outline of the paper is as follows. Section 2 reviews the definition of the multi-directional frames using digital lines. Section 3 describes the specific construction based on lattice theory used in the construction of appropriate multi-directional basis functions. It also introduces the idea of a multi-directional wavelet decomposition tree convenient for non-linear approximation of images. Finally, Section 4 shows some improved results in denoising and promising results in non-linear approximation of images.

2. MULTI-DIRECTIONAL FRAMES

In [10], we used the definition of discrete lines given in [2]:

$$y[n] = \lfloor rx[n] \rfloor + \lfloor b \rfloor,\ r = \tan\theta,\ \frac{-\pi}{2} < \theta \leq \frac{\pi}{2},\ b \in \mathbb{R}. \quad (1)$$

This definition guarantees a complete partition of the discrete space $\mathbb{Z}^2$.

The digital lines build a useful background structure for applying a 1D directional filtering along a certain digital direction resulting in a directional transform. The directional transform consists of a set of 1D basic transforms applied along parallel lines for a fixed angle. If the basic transforms are orthogonal or tight frames, it was shown [10] that the directional transform (and an iterated multi-directional transform as well) leads to a tight frame or an orthogonal transform as a special case. The bound is simply the sum of all basic transform bounds [6].

In particular, a one-directional wavelet transform produces two subbands, where the high-pass subband does not contain any smooth object along the transform direction. Thus, the multi-directional wavelet transform can be used as a directional discriminator very convenient for non-linear approximation. Figure 1 shows an example with three transform directions applied on an image that contains directional objects along the same directions. Some directions are omitted in the resulting subbands depending on the order of the high-pass directional filtering.

3. MULTI-DIRECTIONAL BASES

In designing the multi-directional bases, we need to use the concept of integer lattices. The subsampling procedure will be ex-

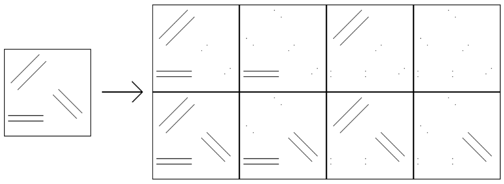


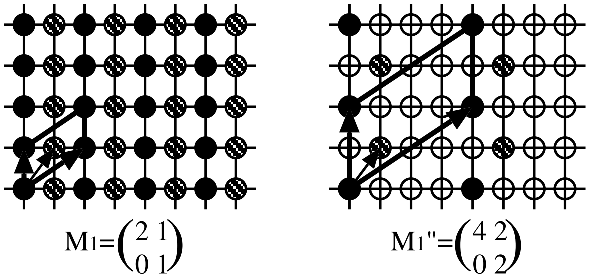
Fig. 1. An example of an iterated directional transform with three directions equal to 0^0, 45^0 and 135^0.

plained in terms of these lattices.

A full rank integer lattice Λ consists of the set of points obtained by taking linear combinations of two linearly independent vectors where both the components of the vectors and the coefficients are integers. Any integer lattice Λ is always a sublattice of the ordinary cubic integer lattice $\mathbb{Z}^2$, that is $\Lambda \subset \mathbb{Z}^2$, and can be represented by a (non-unique) generator matrix:

$$M_\Lambda = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}, \tag{2}$$

where $\Lambda = \{x : x = u_1\vec{v}_1 + u_2\vec{v}_2, u_i \in \mathbb{Z}, i = 1,2\}$. An example with $M_\Lambda = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ is shown in Figure 2(a).



(a) (b)

Fig. 2. (a) The initial cubic lattice is partitioned in two cosets of the lattice defined by M_1 corresponding to shift vectors $\vec{c}_0 = (0,0)$ and $\vec{c}_1 = (1,1)$, (b) the subsampled version is given by two cosets of the lattice defined by M_1'' (sublattice of M_1) with the same shift vectors.

It can be shown from lattice theory [4] that given an integer Λ with a generator matrix having determinant $\det(M_\Lambda)$, the cubic lattice $\mathbb{Z}^2$ is partitioned into $\det(M_\Lambda)$ cosets $B_{\Lambda,c_j} = \Lambda + c_j$, where $j = 0, \dots, |\det(M_\Lambda)| - 1$, which are shifted versions of the sublattice Λ.

In addition, call a set of pixels that belong to a digital line described by (1) as $L_{r,b}$. A co-line is defined as the intersection of a coset and a digital line:

$$\tilde{L}_{\Lambda,c_j,r,b} = L_{r,b} \cap B_{\Lambda,c_j},\ r \in \mathbb{Q}. \tag{3}$$

If M_Λ is a generator matrix for Λ, then all the possible generator matrices for Λ are given by UM_Λ where U is a unimodular matrix, that is, a matrix with integer components and $|\det(U)| = 1$.

Although in terms of lattice representation all generator matrices are equivalent, in terms of the way the sampling procedure is performed, the row vectors of each particular generator matrix can be associated with the digital directions along which the 1D filterings and subsampling will take place. Consider a lattice Λ_1 generated by a matrix $M_{\Lambda_1} = \begin{pmatrix} s_{11}^{(1)} & s_{12}^{(1)} \\ s_{21}^{(1)} & s_{22}^{(1)} \end{pmatrix}$. The tangent coefficients of the angles are defined as $r_1 = s_{12}^{(1)}/s_{11}^{(1)}$ and $r_2 = s_{22}^{(1)}/s_{21}^{(1)}$.

Now a wavelet transform can be applied along co-lines with the first slope r_1 for all cosets: $\{\tilde{L}_{\Lambda_1,c_j,r_1,b}\}$, where $j = 0, \dots, |\det M_{\Lambda_1}| - 1, b \in \mathbb{Z}$. Subsampling along that direction is done independently in each coset and the set of points obtained after the first subsampling consists of a lattice with generator matrix $M'_{\Lambda_1} = \begin{pmatrix} 2\vec{v_1} \\ \vec{v_2} \end{pmatrix}$. Discarding each second sample along each transform co-line secures a valid subsampling in the sense that perfect reconstruction condition is satisfied. The process is continued in a similar way along the second slope r_2 and the final generator matrix is simply given by $M''_{\Lambda_1} = \begin{pmatrix} 2\vec{v_1} \\ 2\vec{v_2} \end{pmatrix} = 2M_{\Lambda_1}$. The corresponding lattice Λ_1'' is clearly a sublattice of the initial one containing a quarter of the samples. The corresponding example is shown in Figure 2(b).

The matrix $2M_{\Lambda_1}$ produces four times more cosets. However, only the initial cosets should be kept. The other three-quarters of the cosets represent the discarded samples.

The next iteration is obtained by using a lattice Λ_2 which is in general a sublattice of Λ_1'', that is $\Lambda_2 \subseteq \Lambda_1''$. This means that any generator matrix for Λ_2 is given by:

$$M_{\Lambda_2} = T_1 \cdot M''_{\Lambda_1},\ T_1 = \begin{pmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{21}^{(1)} & t_{22}^{(1)} \end{pmatrix},\ t_{ij}^{(1)} \in \mathbb{Z}. \tag{4}$$

The new pair of tangent coefficients is: $r_3 = \frac{t_{11}^{(1)}s_{12}^{(1)} + t_{12}^{(1)}s_{22}^{(1)}}{t_{11}^{(1)}s_{11}^{(1)} + t_{12}^{(1)}s_{21}^{(1)}}$ and $r_4 = \frac{t_{21}^{(1)}s_{12}^{(1)} + t_{22}^{(1)}s_{22}^{(1)}}{t_{21}^{(1)}s_{11}^{(1)} + t_{22}^{(1)}s_{21}^{(1)}}$.

If $|\det(T_1)| = 1$, then $\Lambda_2 = \Lambda_1''$ and sampling along the digital lines with slopes r_3 and r_4 results simply in a different resampling of Λ_1'', which we call redirection step. After the new subsampling, we will obtain simply $\Lambda_2'' = 2\Lambda_1''$.

If the resampling is not unimodular, (4) yields more cosets. Then, each coset that survived the previous subsampling step is divided into $|\det T_1|$ new cosets.

In total there are $|\det M_{\Lambda_2}|/4$ cosets that are to be processed independently in the second step. The step involves two 1D filterings and subsamplings equivalently as in the first step, but along the new angles.

The iteration can be continued in a similar way as many times as desired. The redirection step can always be applied on a subsampled version of the initial generator matrix M_{Λ_1}. Indeed, consider the change from M''_{Λ_2} to M_{Λ_3}. We have that:

$$M_{\Lambda_3} = C \cdot M''_{\Lambda_2} = 2C \cdot M_{\Lambda_2} = 4C \cdot T_1 \cdot M_{\Lambda_1} = T_2 \cdot 4M_{\Lambda_1},$$

where C is an integer matrix and $T_2 = C \cdot T_1$. In general the following holds:

$$M_{\Lambda_{i+1}} = T_i \cdot 2^i M_{\Lambda_1}. \tag{5}$$

The tangent coefficients after each change are given by:

$$r_{2i-1} = \frac{t_{11}^{(i)} s_{12}^{(1)} + t_{12}^{(i)} s_{22}^{(1)}}{t_{11}^{(i)} s_{11}^{(1)} + t_{12}^{(i)} s_{21}^{(1)}}, \; r_{2i} = \frac{t_{21}^{(i)} s_{12}^{(1)} + t_{22}^{(i)} s_{22}^{(1)}}{t_{21}^{(i)} s_{11}^{(1)} + t_{22}^{(i)} s_{21}^{(1)}}. \tag{6}$$

Using a concatenation of unimodular redirection steps is desirable in non-linear approximation of images as described in Section 4.2, but this constrains the choice of directions because we need to have that:

$$|\det T_i| = |t_{11}^{(i)} t_{22}^{(i)} - t_{12}^{(i)} t_{21}^{(i)}| = 1. \tag{7}$$

However (6) and (7) together still allow a number of directions to be used. The following proposition gives the good approximation property of our construction.

Proposition: *Let $\{M_{\Lambda_i}\}_{i=1}^{K}$ be generator matrices of a set of lattices $\{\Lambda_i\}_{i=1}^{K}$ which satisfy the nesting property explained above. If 1D filterings and subsamplings are applied along the directions contained in the matrices $\{M_{\Lambda_i}\}_{i=1}^{K}$, then: a) perfect reconstruction can be achieved and b) concatenated operations of filtering and subsampling along the different directions do not create directional interaction, that is, the samples that are kept at each iteration are aligned along the next filtering and subsampling directions.*

This avoidance of directional interaction is the crucial property to achieve good non-linear approximation.

4. APPLICATIONS

4.1. Denoising of Images

Thresholding of the multi-directional frame coefficients led to promising results in denoising of images [10]. We present a comparison of the two standard denoising algorithms with their multi-directional counterparts in Figure 3 applied on several test images.

The first standard denoising method uses the undecimated wavelet transform [7]. The multi-directional undecimated wavelet transform is an extension of the standard transform using the multi-directional frame expansion along a set of 30 uniformly chosen directions. Thresholding of the multi-directional frame coefficients provides outperforming results comparing with the standard method.

The second standard method is shift-invariant cycle-spinning denoising introduced by Coifman and Donoho [3]. The multi-direct-ional shift-invariant method uses the standard but along the same set of 30 directions. The results are again improved comparing with the previous ones.

4.2. Non-linear Approximation of Images

Non-linear approximation is more efficient if a smaller number of significant coefficients is generated, i.e. if the energy of a signal is concentrated in a smaller number of coefficients. The standard wavelet transform along horizontal and vertical directions is not efficient if the analyzed image contains other directions. However, the multi-directional bases may adapt their orientation to match the dominant directions in an image. The multi-directional non-linear approximation uses the multi-directional transform of an image and approximates it by keeping a certain fixed number of the largest coefficients.

A well known fact from 1D wavelet theory [11] is that a smooth object can be represented by $O(\log_2 N)$ significant coefficients, where N is the length of the signal. In the multi-directional analysis this holds for each transform line or co-line if more cosets are used. Therefore distributing a line on more co-lines decreases the efficiency of approximation because more significant coefficients are produced. That is why a unimodular chain is preferable.

In order to reach the best approximation power we propose an iterative decomposition tree that involves multi-directionality. Consider an image containing a set of directions. Each directional wavelet filtering step results in a low-pass subband that contains the same directions as in the input and the high-pass one that annihilates the information along the transform direction. The size of both subbands is a half of the input size. The target depth of the iteration is either to eliminate all present directions or to decrease maximally the size of the subbands. It can be shown that the order of the approximation is close to $O\left(\log_2^D N\right)$, where N is a characteristic size of the image and D is the number of dominant directions in the image. A possible realization of a tree is shown in Figure 4 where three directions are used and the maximal depth of the tree is four.

A comparison between the multi-directional approximation and the standard one is made on three test images. The multi-directional method uses a wavelet decomposition along 0^0, 45^0, 90^0 and 135^0 with the corresponding multi-directional bases. Some of them are shown in Figure 5. The results of the approximation are shown in Figure 6 proving the superiority of our approach.

5. CONCLUSION AND FUTURE WORK

In this paper, we presented a new approach of multi-directional subsampling based on lattice theory that leads to the construction of good discrete multi-directional wavelet bases. The method is simple yet effective and provides a sparse representation of piecewise smooth images containing information along a set of orientations. Applications of the method are possible in many areas of image processing and we show promising results in non-linear approximation and denoising of images. Future research will be focused on denoising based on local thresholding of the wavelet coefficients rather than global and on design of non-separable filters that annihilate directional information along certain directions.

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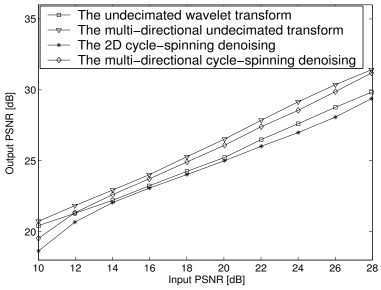

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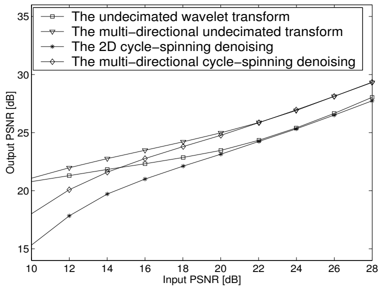

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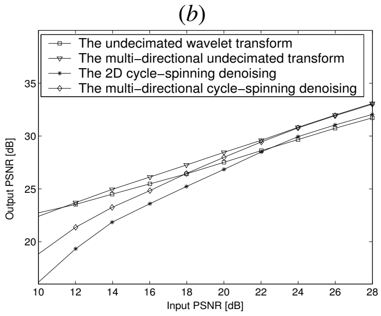

(*c*)

Fig. 3. Comparison of the two standard denoising methods (using the undecimated wavelet transform and the shift-invariant cycle-spinning method) with their multi-directional extensions using 30 uniformly placed directions. The comparison is made on the following test images: (*a*) Cameraman, (*b*) Barbara, (*c*) Lena. The multi-directional methods shows outperforming results.

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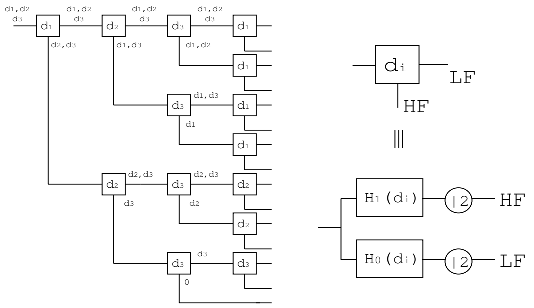
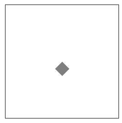


Fig. 4. A multi-directional decomposition tree with 3 directions and 4 steps at most.



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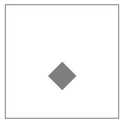

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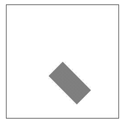

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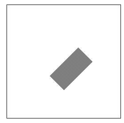
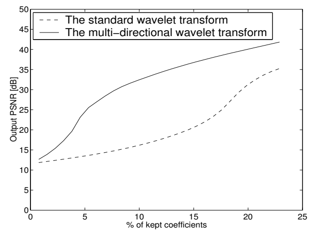


(*d*)

Fig. 5. Some multi-directional scaling functions for an iteration of 45^0 and 135^0. (*a*,*b*) An equal number of 45^0 and 135^0 steps, (*c*) more steps along 45^0, (*d*) more steps along 135^0.



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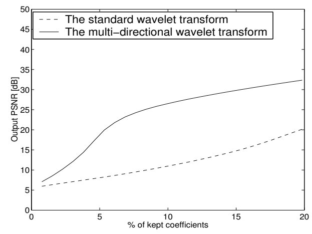

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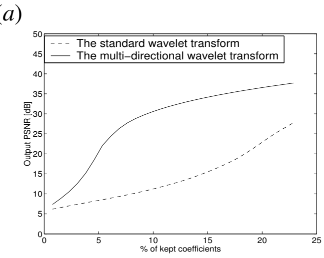

(*c*)

Fig. 6. Comparison between the standard (the dotted line) and the multi-directional (the full line) non-linear approximation of images applied on three test images: (*a*) Cameraman, (*b*) Barbara, (*c*) Lena.