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The Dirichlet Problem for the Total Variation Flow

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We introduce a new concept of solution for the Dirichlet problem for the total variational flow named entropy solution. Using Kruzhkov's method of doubling variables both in space and in time we prove uniqueness and a comparison principle in L^1 for entropy solutions. To prove the existence we use the nonlinear semigroup theory and we show that when the initial and boundary data are nonnegative the semigroup solutions are strong solutions. © 2001 Academic Press

1. INTRODUCTION

Suppose that Ω is an open bounded domain with a Lipschitz boundary and $\varphi \in L^\infty(\partial\Omega)$. Let $\theta\colon \Omega \to \mathbb{R}^N$ be a vector field (whose smoothness will be made precise below) with $|\theta| \leqslant 1$. Recently, in [7], a variational method was proposed to extend the data φ from $\partial\Omega$ to a function u in Ω along the integral curves of $\theta^\perp$, the vector orthogonal to θ, so that u is constant along the integral curves of $\theta^\perp$. Formally, we think of θ as the vector field made by the normals to the level sets of u, i.e., the sets $\{x \in \Omega : u(x) \geqslant \lambda\}$, $\lambda \in \mathbb{R}$. In that case we would have that $\theta \cdot Du = |Du|$. In the case that u is a function of bounded variation, almost all levels sets are of finite perimeter and, therefore, one can compute the normal along the boundary of the level sets (modulo a set of H^{N-1} null measure). Moreover, to get φ as a trace of a function u in Ω, the right function space is $BV(\Omega)$, the space of

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functions of bounded variation in Ω. Thus, to extend φ from $\partial\Omega$ to Ω, it was proposed in [7] to minimize the functional $F(u) = \int_\Omega |\nabla u| - \int_\Omega \theta \cdot \nabla u$ defined in the set of functions of bounded variation $BV(\Omega)$ whose trace at the boundary is given by φ. Formally, if we integrate by parts in the second term of $F(u)$ we obtain

$$F(u) = \int_\Omega |\nabla u| + \int_\Omega div(\theta) \cdot u - \int_{\partial\Omega} \theta \cdot \vec{n} u.$$

Since u, θ are known at the boundary, minimizing F amounts to minimizing

$$E(u) = \int_\Omega |\nabla u| + \int_\Omega div(\theta) \cdot u.$$

Let us comment on the class of admissible functions where E has to be minimized. We assume that $div(\theta) \in L^1(\Omega)$ and $\varphi \in L^\infty(\partial\Omega)$. It seems reasonable to impose that the solution u is a bounded function with an L^∞ bound given by $\|\varphi\|_\infty$. Then the second integral in the definition of $E(u)$ is well defined. The first integral requires the use of the space of bounded variation functions. Thus our admissible class is $\mathscr{A} = \{u \in BV(\Omega) : |u(x)| \leqslant \|\varphi\|_\infty \text{ a.e. } u|_{\partial\Omega} = \varphi\}$. The final model is [7]

$$\underset{u \in \mathscr{A}}{\text{Minimize}} \int_\Omega |\nabla u| + \int_\Omega div(\theta) \cdot u. \tag{1.1}$$

As is well know [16, 19] the solution of this problem has to be understood in a weak sense as the solution of the problem

$$\underset{\substack{u \in BV(\Omega) \\ |u| \leqslant \|\varphi\|_\infty}}{\text{Minimize}} \int_\Omega |\nabla u| + \int_\Omega div(\theta) \cdot u + \int_{\partial\Omega} |u - \varphi| \, dH^1. \tag{1.2}$$

Existence for this variational problem was proved in [19, Theorem 1.4] when $\theta \in L^1_{loc}(\Omega)^2$, $div(\theta) \in L^1(\mathbb{R}^2)$, $\varphi \in L^\infty(\partial\Omega)$.

This is one of our motivations to study the Dirichlet problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = div\left(\frac{Du}{|Du|}\right) + f(t, x) && \text{in} \quad Q = (0, \infty) \times \Omega \\ &u(t, x) = \varphi(x) && \text{on} \quad S = (0, \infty) \times \partial\Omega \qquad (1.3) \\ &u(0, x) = u_0(x) && \text{in} \quad x \in \Omega, \end{aligned}$$

where $u_0 \in L^1(\Omega)$ and $\varphi \in L^\infty(\partial\Omega)$. This evolution equation is related to the gradient descent method used to minimize the functional (1.2), if we forget about the constraint $|u| \leqslant \|\varphi\|_\infty$. The constraint would introduce a further term in (1.3) but will not change the nature of the difficulties related to the solution of the PDE. We shall even make a further simplification, since we shall consider $f(t,x)=0$. Hence, our aim is to study existence and uniqueness of solutions of the Dirichlet problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = div\left(\frac{Du}{|Du|}\right) && \text{in} \quad Q=(0,\infty)\times\Omega \\ &u(t,x)=\varphi(x) && \text{on} \quad S=(0,\infty)\times\partial\Omega \\ &u(0,x)=u_0(x) && \text{in} \quad x\in\Omega, \end{aligned} \tag{1.4}$$

where Ω is an open bounded domain with a Lipschitz boundary, $u_0 \in L^1(\Omega)$, and $\varphi \in L^\infty(\partial\Omega)$.

The other motivation for the study of (1.4) comes from [2], [3], and [8]. The general purpose of the works [8] and [3] is the study of elliptic and parabolic problems in divergence form with initial data in L^1. Existence and uniqueness results of entropy solutions when the associated variational energy has a growth at infinity of order p with $p>1$ are proved (see also [4, 11]). In [2], the authors consider the equation

$$u_t = div\left(\frac{Du}{|Du|}\right) \tag{1.5}$$

in an open bounded Lipschitz domain with Neumann boundary conditions, proving existence and uniqueness of entropy (or renormalized) solutions (called weak solutions in [2]). Let us recall that this PDE appears when one uses the steepest descent method to minimize the total variation, a method introduced by Rudin and Osher [24, 25] in the context of image denoising and reconstruction. The main point is that, in the case of Neumann boundary conditions, this equation generates a nonlinear contraction semigroup in $L^1(\Omega)$ which is homogeneous of degree 0, a fact related to the regularity in time of the solutions on (1.5). Indeed, the homogeneity of the operator permits one to conclude that $u_t(t) \in L^1(\Omega)$ a.e. for $t>0$. This was used to prove uniqueness of solutions of (1.5) in the case of Neumann boundary conditions. This property is loosed when we consider the case of Dirichlet boundary conditions. Thus, a different approach is needed and we believe it to be helpful with a view to the general case of energy functionals with linear growth in $|Du|$. A result about existence and uniqueness of solutions (named pseudosolutions) for the Dirichlet problem in the case of energy functionals with linear growth in $|Du|$, concretely for the Dirichlet problem for the time-dependent minimal surface equation, is studied in [22].

The aim of this paper is to introduce a new concept of solution of the problem (1.4), for which existence and uniqueness for initial data in $L^1(\Omega)$ are proved.

The paper is organized as follows: in Section 2 the results we need about functions of bounded variation are summarized. In the next section we give the definition of entropy solution and we state the main result. In Sections 4 and 5 we study the problem from the point of view of nonlinear semi-group theory, showing that for initial data in $L^2(\Omega)$, the semigroup solution is a strong solution. The next section is devoted to prove the existence and uniqueness of entropy solutions. Finally, in the last section we obtain that the time derivative of the entropy solution is an L^1_{loc} function when the initial data are nonnegative.

2. DEFINITIONS AND PRELIMINARY FACTS

To make precise our notion of solution let us recall several facts concerning functions of bounded variation.

A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, ..., \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi \, dx = -\int_\Omega \varphi \, d\mu_i \tag{2.1}$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, ..., N$. Thus the gradient of u is a vector valued measure with finite total variation

$$\|Du\| = \sup \left\{ \int_\Omega u \, div \, \varphi \, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N, |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega.) \right\} \tag{2.2}$$

The space $BV(\Omega)$ is endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + \|Du\|. \tag{2.3}$$

For further information concerning functions of bounded variation we refer to [1, 17, and 28].

We shall need several results from [5]. Following [5], let

$$X(\Omega) = \{ z \in L^\infty(\Omega, \mathbb{R}^N) : div(z) \in L^1(\Omega) \}. \tag{2.4}$$

If $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ we define the functional (z, Dw): $C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z, Dw), \varphi \rangle = -\int_\Omega w\varphi \, div(z) \, dx - \int_\Omega wz \cdot \nabla\varphi \, dx. \tag{2.5}$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \tag{2.6}$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, and

$$\left| \int_B (z, Dw) \right| \leqslant \int_B |(z, Dw)| \leqslant \|z\|_\infty \int_B \|Dw\| \tag{2.7}$$

for any Borel set $B \subseteq \Omega$. Moreover, (z, Dw) is absolutely continuous with respect to $\|Dw\|$ with the Radon–Nikodym derivative $\theta(z, Dw, x)$ which is a $\|Dw\|$ measurable function from Ω to $\mathbb{R}$ such that

$$\int_B (z, Dw) = \int_B \theta(z, Dw, x) \, \|Dw\| \tag{2.8}$$

for any Borel set $B \subseteq \Omega$. We also have that

$$\|\theta(z, Dw, \cdot)\|_{L^\infty(\Omega, \|Dw\|)} \leqslant \|z\|_{L^\infty(\Omega, \mathbb{R}^N)}. \tag{2.9}$$

In [5], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator γ: $X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leqslant \|z\|_\infty$$

$$\gamma(z)(x) = z(x) \cdot v(x) \qquad \text{for all} \quad x \in \partial\Omega \quad \text{if } z \in C^1(\bar{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(z)(x)$ by $[z, v](x)$. Moreover, the following *Green's formula* relating the function $[z, v]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$\int_\Omega w \, div(z) \, dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, v] \, w \, dH^{N-1}. \tag{2.10}$$

We also need to introduce, as in [5], a weak trace on $\partial\Omega$ of the normal component of certain vector fields in Ω. We define the space

$$Z(\Omega) := \{(z, \xi) \in L^\infty(\Omega, \mathbb{R}^N) \times BV(\Omega)^* : div(z) = \xi \text{ in } \mathscr{D}'(\Omega)\}.$$

We denote $R(\Omega) := W^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$. For $(z, \xi) \in Z(\Omega)$ and $w \in R(\Omega)$ we define

$$\langle (z, \xi), w \rangle_{\partial\Omega} := \langle \xi, w \rangle_{BV(\Omega)^*, BV(\Omega)} + \int_\Omega z \cdot \nabla w.$$

Then, working as in the proof of Theorem 1.1 of [5], we obtain that if w, $v \in R(\Omega)$ and $w = v$ on $\partial\Omega$ one has

$$\langle (z, \xi), w \rangle_{\partial\Omega} = \langle (z, \xi), v \rangle_{\partial\Omega} \qquad \forall (z, \xi) \in Z(\Omega). \tag{2.11}$$

As a consequence of (2.11), we can give the following definition: Given $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $(z, \xi) \in Z(\Omega)$, we define $\langle (z, \xi), u \rangle_{\partial\Omega}$ by setting

$$\langle (z, \xi), u \rangle_{\partial\Omega} := \langle (z, \xi), w \rangle_{\partial\Omega},$$

where w is any function in $R(\Omega)$ such that $w = u$ on $\partial\Omega$. Again, working as in the proof of Theorem 1.1. of [5], we can prove that for every $(z, \xi) \in Z(\Omega)$ there exists $M_{z,\xi} > 0$ such that

$$|\langle (z, \xi), u \rangle_{\partial\Omega}| \leqslant M_{z,\xi} \, \|u\|_{L^1(\partial\Omega)} \qquad \forall u \in BV(\Omega) \cap L^\infty(\Omega). \tag{2.12}$$

Now, taking a fixed $(z, \xi) \in Z(\Omega)$, we consider the linear functional F: $L^\infty(\partial\Omega) \to \mathbb{R}$ defined by

$$F(v) := \langle (z, \xi), w \rangle_{\partial\Omega},$$

where $v \in L^\infty(\partial\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ is such that $w_{|\partial\Omega} = v$. By estimate (2.12), there exists $\gamma_{z,\xi} \in L^\infty(\partial\Omega)$ such that

$$F(v) = \int_{\partial\Omega} \gamma_{z,\xi}(x) \, v(x) \, dH^{N-1}.$$

Consequently there exists a linear operator γ: $Z(\Omega) \to L^\infty(\partial\Omega)$, with $\gamma(z, \xi) := \gamma_{z,\xi}$, satisfying

$$\langle (z, \xi), w \rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma_{z,\xi}(x) \, w(x) \, dH^{N-1} \qquad \forall w \in BV(\Omega) \cap L^\infty(\Omega).$$

In the case $z \in C^1(\bar{\Omega}, \mathbb{R}^N)$, we have $\gamma_z(x) = z(x) \cdot \nu(x)$ for all $x \in \partial\Omega$. Hence, the function $\gamma_{z,\xi}(x)$ is the weak trace of the normal component of (z, ξ). For simplicity of notation, we shall denote $\gamma_{z,\xi}(x)$ by $[z, \nu](x)$.

We need to consider the space $BV(\Omega)_2$, defined as $BV(\Omega) \cap L^2(\Omega)$ endowed with the norm

$$\|w\|_{BV(\Omega)_2} := \|w\|_{L^2(\Omega)} + \|Dw\|.$$

It easy to see that $L^2(\Omega) \subset BV(\Omega)_2^*$ and

$$\|w\|_{BV(\Omega)_2^*} \leqslant \|w\|_{L^2(\Omega)} \qquad \forall w \in L^2(\Omega). \tag{2.13}$$

Now, it is well known (see, for instance, [26]) that the dual space $(L^1(0, T; BV(\Omega)_2))^*$ is isometric to the space $L^\infty(0, T; BV(\Omega)_2^*, BV(\Omega)_2)$ of all weakly* measurable functions $f: [0, T] \to BV(\Omega)_2^*$, such that $v(f) \in L^\infty([0, T])$, where $v(f)$ denotes the supremum of the set $\{|\langle w, f \rangle| : \|w\|_{BV(\Omega)_2} \leqslant 1\}$ in the vector lattice of measurable real functions. Moreover, the dual paring of the isometric is defined by

$$\langle w, f \rangle = \int_0^T \langle w(t), f(t) \rangle \, dt,$$

for $w \in L^1(0, T; BV(\Omega)_2)$ and $f \in L^\infty(0, T; BV(\Omega)_2^*, BV(\Omega)_2)$.

By $L^1_w(0, T, BV(\Omega))$ we denote the space of weakly measurable functions $w: [0, T] \to BV(\Omega)$ (i.e., $t \in [0, T] \to \langle w(t), \phi \rangle$ is measurable for every $\phi \in BV(\Omega)^*$) such that $\int_0^T \|w(t)\| < \infty$. Observe that, since $BV(\Omega)$ has a separable predual (see [1]), it follows easily that the map $t \in [0, T] \to \|w(t)\|$ is measurable.

To make precise our notion of solution we need the following definitions.

Definition 1. Let $\Psi \in L^1(0, T, BV(\Omega))$. We say Ψ admits a *weak derivative* in $L^1_w(0, T, BV(\Omega)) \cap L^\infty(Q_T)$ if there is a function $\Theta \in L^1_w(0, T, BV(\Omega)) \cap L^\infty(Q_T)$ such that $\Psi(t) = \int_0^t \Theta(s) \, ds$, the integral being taken as a Pettis integral.

Definition 2. Let $\xi \in (L^1(0, T, BV(\Omega)_2))^*$. We say that ξ is the *time derivative* in the space $(L^1(0, T, BV(\Omega)_2))^*$ of a function $u \in L^1((0, T) \times \Omega)$ if

$$\int_0^T \langle \xi(t), \Psi(t) \rangle \, dt = -\int_0^T \int_\Omega u(t, x) \, \Theta(t, x) \, dx \, dt$$

for all test functions $\Psi \in L^1(0, T, BV(\Omega))$ which admit a weak derivative $\Theta \in L^1_w(0, T, BV(\Omega)) \cap L^\infty(Q_T)$ and have compact support in time.

Observe that if $w \in L^1(0, T, BV(\Omega)) \cap L^\infty(Q_T)$ and $z \in L^\infty(Q_T, \mathbb{R}^N)$ is such that there exists $\xi \in (L^1(0, T, BV(\Omega)))^*$ with $div(z) = \xi$ in $\mathcal{D}'(Q_T)$, we can define, associated to the pair (z, ξ), the distribution (z, Dw) in Q_T by

$$\langle (z, Dw), \phi \rangle := -\int_0^T \langle \xi(t), w(t)\, \phi(t) \rangle - \int_0^T \int_\Omega z(t, x)\, w(t, x)\, \nabla_x \phi(t, x) \tag{2.14}$$

for all $\phi \in \mathcal{D}(Q_T)$.

DEFINTION 3. Let $\xi \in (L^1(0, T, BV(\Omega)_2))^*$, $z \in L^\infty(Q_T, \mathbb{R}^N)$. We say that $\xi = div(z)$ in $(L^1(0, T, BV(\Omega)_2))^*$ if (z, Dw) is a Radon measure in Q_T with normal boundary values $[z, \nu] \in L^\infty((0, T) \times \partial\Omega)$, such that

$$\int_{Q_T} (z, Dw) + \int_0^T \langle \xi(t), w(t) \rangle\, dt = \int_0^T \int_{\partial\Omega} [z(t, x), \nu]\, w(t, x)\, dH^{N-1}\, dt,$$

for all $w \in L^1(0, T, BV(\Omega)) \cap L^\infty(Q_T)$.

We shall denote by

$$sign_0(r) := \begin{cases} 1 & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ -1 & \text{if } r < 0 \end{cases}$$

and by

$$sign(r) := \begin{cases} 1 & \text{if } r > 0 \\ a \in [-1, 1] & \text{if } r = 0 \\ -1 & \text{if } r < 0. \end{cases}$$

Let $T_k(r) = [k - (k - |r|)^+]\, sign_0(r)$, $k \geq 0$, $r \in \mathbb{R}$. We consider the set $\mathcal{T} = \{T_k, T_k^+, T_k^- : k > 0\}$. We need to consider a more general set of truncature functions, concretely, the set $\mathcal{P}$ of all nondecreasing continuous functions $p\colon \mathbb{R} \to \mathbb{R}$, such that there exists p' except a finite set and $\mathrm{supp}(p')$ is compact. Obviously, $\mathcal{T} \subset \mathcal{P}$.

3. THE MAIN RESULT

In this section we give the concept of solution for the Dirichlet problem (1.4) and we state the existence and uniqueness result for this type of solution.

DEFINITION 4. A measurable function $u: (0, T) \times \Omega \to \mathbb{R}$ is an *entropy solution* of (1.4) in $Q_T = (0, T) \times \Omega$ if $u \in C([0, T]; L^1(\Omega))$, $p(u(\cdot)) \in L^1_w(0, T, BV(\Omega))$ $\forall p \in \mathcal{T}$ and there exist $(z(t), \xi(t)) \in Z(\Omega)$ with $\|z(t)\|_\infty \leqslant 1$, and $\xi \in (L^1(0, T, BV(\Omega)_2))^*$ such that ξ is the time derivative of u in $(L^1(0, T, BV(\Omega)_2))^*$, $\xi = div(z)$ in $(L^1(0, T, BV(\Omega)))^*$ and $[z(t), v] \in \operatorname{sign}(p(\varphi) - p(u(t)))$ a.e. in $t \in [0, T]$, satisfying

$$-\int_0^T \int_\Omega j(u(t) - l)\, \eta_t + \int_0^T \int_\Omega \eta(t)\, \|Dp(u(t) - l)\|$$

$$+ z(t) \cdot D\eta(t)\, p(u(t) - l) \leqslant \int_0^T \int_{\partial\Omega} [z(t), v]\, \eta(t)\, p(u(t) - l),$$

for all $l \in \mathbb{R}$, for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t, x) = \phi(t)\, \psi(x)$, being $\phi \in \mathcal{D}(]0, T[)$, $\psi \in C^\infty(\bar{\Omega})$, and $p \in \mathcal{T}$, where $j(r) = \int_0^r p(s)\, ds$.

Our main result is:

THEOREM 1. *Let $u_0 \in L^1(\Omega)$, and $\varphi \in L^1(\partial\Omega)$. Then there exists a unique entropy solution of* (1.4) *in* $(0, T) \times \Omega$ *for every* $T > 0$ *such that* $u(0) = u_0$. *Moreover, if $u(t)$, $\hat{u}(t)$ are the entropy solutions corresponding to initial data u_0, $\hat{u}_0$, respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_1 \leqslant \|(u_0 - \hat{u}_0)^+\|_1 \quad \textit{and} \quad \|u(t) - \hat{u}(t)\|_1 \leqslant \|u_0 - \hat{u}_0\| \tag{3.1}$$

for all $t \geqslant 0$.

4. THE SEMIGROUP SOLUTION

To prove Theorem 1 we shall use the techniques of completely accretive operators and the Crandall–Liggett semigroup generation theorem [14]. Let us recall the notion of completely accretive operators introduces in [9]. Let $\mathcal{M}(\Omega)$ be the space of measurable functions in Ω. Given $u, v \in \mathcal{M}(\Omega)$, we shall write

$$u \ll v \qquad \text{if and only if} \quad \int_\Omega j(u)\, dx \leqslant \int_\Omega j(v)\, dx \tag{4.1}$$

for all $j \in J_0$ where

$$J_0 = \{ j \in \mathbb{R} \to [0, \infty], \text{ convex, l.s.c., } j(0) = 0 \} \tag{4.2}$$

(l.s.c. is an abbreviation for lower semicontinuous function). Let A be a operator (possibly multivalued) in $\mathcal{M}(\Omega)$, i.e., $A \subseteq \mathcal{M}(\Omega) \times \mathcal{M}(\Omega)$. We shall say that A is *completely accretive* if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \qquad \text{for all} \quad \lambda > 0 \quad \text{and all} \quad (u, v), (\hat{u}, \hat{v}) \in A. \tag{4.3}$$

Let

$$P_0 = \{ p \in C^\infty(\mathbb{R}) : 0 \leqslant p' \leqslant 1, \, supp(p') \text{ is compact and } 0 \notin supp(p) \}.$$

If $A \subseteq L^1(\Omega) \times L^1(\Omega)$, then A is completely accretive if and only if

$$\int_\Omega p(u - \hat{u})(v - \hat{v}) \geqslant 0 \qquad \text{for any} \quad p \in P_0, \quad (u, v), (\hat{u}, \hat{v}) \in A. \tag{4.4}$$

A completely accretive operator in $L^1(\Omega)$ is said to be m-completely accretive if $R(I + \lambda A) = L^1(\Omega)$ for any $\lambda > 0$. In that case, by Crandall–Liggett's theorem, A generates a contraction semigroup in $L^1(\Omega)$ given by the exponential formula

$$e^{-tA} u_0 = \lim_{n \to \infty} \left(I + \frac{t}{n} A \right)^{-n} u_0 \qquad \text{for any} \quad u_0 \in L^1(\Omega).$$

Let us write $u(t) = e^{-tA} u_0$. Then $u \in C([0, T], L^1(\Omega))$, for any $T > 0$, and is a *mild solution* (a solution in the sense of semigroups [10]) of

$$\frac{du}{dt} + Au \ni 0 \tag{4.5}$$

such that $u(0) = u_0$.

We shall use a stronger notion of solution of (4.5). We say that $v \in C([0, T], L^1(\Omega))$ is a *strong solution* of (4.5) on $[0, T]$ if $v \in W^{1,1}_{loc}((0, T), L^1(\Omega))$ and $v'(t) + Av(t) \ni 0$ for almost all $t \in (0, T)$. If $u_0 \in D(A) = \{\bar{u} \in L^1(\Omega) \colon (\bar{u}, \bar{v}) \in A, \text{ for some } \bar{v} \in L^1(\Omega)\}$ (the domain of A) and A is m-completely accretive, then $u \in W^{1,1}_{loc}((0, T), L^1(\Omega))$ and $u(t)$ is a strong solution of (4.5) on $(0, T)$, for all $T > 0$.

To prove Theorem 1 we shall associate a completely accretive operator $\mathcal{A}_\varphi$ to the formal differential expression $-div(\frac{Du}{|Du|})$ together with the Dirichlet boundary condition.

Let us introduce the following operator $\mathcal{A}_\varphi$ in $L^1(\Omega)$.

$(u, v) \in \mathcal{A}_\varphi$ if and only if $u, v \in L^1(\Omega)$, $p(u) \in BV(\Omega)$ for all $p \in \mathcal{P}$

and there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -div(z)$ in $\mathcal{D}'(\Omega)$ such that

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega z \cdot \nabla w - \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)|$$

$$- \int_{\partial\Omega} |p(u) - p(\varphi)|,$$

$\forall w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, and $\forall p \in \mathcal{P}$.

THEOREM 2. *Let $\varphi \in L^1(\partial\Omega)$. The operator $\mathcal{A}_\varphi$ is m-completely accretive in $L^1(\Omega)$ with dense domain.*

To prove this theorem, we need first to consider the following operator, which is related with the p-Laplacian operator with Dirichlet boundary condition. For $p > 1$, let $\varphi \in W^{1-1/p,\, p}(\partial\Omega)$, and

$$W^{1,\,p}_\varphi(\Omega) := \{u \in W^{1,\,p}(\Omega) : u_{|\partial\Omega} = \varphi \ H^{N-1}\text{-}a.e. \text{ on } \partial\Omega\}.$$

We define the operator $A_{\varphi,\,p}$ in $L^1(\Omega)$ as

$(u, v) \in A_{\varphi,\,p}$ if and only if $u \in W^{1,\,p}_\varphi(\Omega) \cap L^\infty(\Omega)$, $v \in L^1(\Omega)$ and

$$\int_\Omega (w - u)\, v \leqslant \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla(w - u)$$

for every $w \in W^{1,\,p}_\varphi(\Omega) \cap L^\infty(\Omega)$.

PROPOSITION 1. *Let $\varphi \in L^\infty(\partial\Omega) \cap W^{1-1/p,\,p}(\partial\Omega)$. The operator $A_{\varphi,\,p}$ is completely accretive and $L^\infty(\Omega) \subseteq R(I + A_{\varphi,\,p})$.*

Proof. Let $p \in P_0$ and $(u, v), (\hat{u}, \hat{v}) \in A_{\varphi,\,p}$. Since $(u, v) \in A_{\varphi,\,p}$, taking $w = u - p(u - \hat{u})$ as a test function in the definition of the operator $A_{\varphi,\,p}$ we get

$$\int_\Omega p(u - \hat{u})\, v \geqslant \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla p(u - \hat{u}).$$

Similarly, since $(\hat{u}, \hat{v}) \in A_{\varphi,\,p}$, taking $w = \hat{u} + p(u - \hat{u})$ as a test function in the definition of the operator $A_{\varphi,\,p}$ we get

$$\int_\Omega p(u - \hat{u})\, \hat{v} \leqslant \int_\Omega |\nabla \hat{u}|^{p-2}\, \nabla \hat{u} \cdot \nabla p(u - \hat{u}).$$

Hence

$$\int_{\Omega} (v - \hat{v})\, p(u - \hat{u}) \geqslant \int_{\Omega} (|\nabla u|^{p-2}\, \nabla u - |\nabla \hat{u}|^{p-2}\, \nabla \hat{u}) \cdot \nabla p(u - \hat{u}) \geqslant 0.$$

Therefore, $A_{\varphi, p}$ is completely accretive.

Let us now see that $L^{\infty}(\Omega) \subseteq R(I + A_{\varphi, p})$. Let $v \in L^{\infty}(\Omega)$. We need to prove that there exists $u \in W^{1,p}_{\varphi}(\Omega) \cap L^{\infty}(\Omega)$ such that $(u, v - u) \in A_{\varphi, p}$; i.e.,

$$\int_{\Omega} (w - u)(v - u) \leqslant \int_{\Omega} |\nabla u|^{p-2}\, \nabla u \cdot \nabla(w - u)$$
$$\forall w \in W^{1,p}_{\varphi}(\Omega) \cap L^{\infty}(\Omega). \tag{4.6}$$

For $n \in \mathbb{N}$, let $\gamma_n(s) := T_n(s) + \frac{1}{n}\, |s|^{p-2}\, s$, and consider the operators A_n: $W^{1,p}_{\varphi}(\Omega) \to (W^{1,p}(\Omega))^*$, defined by

$$\langle A_n u, w \rangle := \int_{\Omega} |\nabla u|^{p-2}\, \nabla u \cdot \nabla w + \int_{\Omega} \gamma_n(u)\, w.$$

It is easy to see that A_n is monotone, coercive, and continuous on finite dimensional subspaces. Then, by classical results (see, for instance, [20]), given v there exists $u_n \in W^{1,p}_{\varphi}(\Omega)$ such that

$$\langle A_n u_n, u_n - w \rangle \leqslant \int_{\Omega} v(u_n - w) \qquad \forall w \in W^{1,p}_{\varphi}(\Omega).$$

That is

$$\int_{\Omega} (w - u_n)(v - \gamma_n(u_n)) \leqslant \int_{\Omega} |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla(w - u_n)$$
$$\forall w \in W^{1,p}_{\varphi}(\Omega). \tag{4.7}$$

Let $k > 0$ be such that $\|\varphi\|_{\infty} \leqslant k$. If we take $w = T_k(u_n)$ in (4.7), we get

$$\int_{\Omega} (T_k(u_n) - u_n)(v - \gamma_n(u_n)) \leqslant \int_{\Omega} |\nabla u_n|^{p-2}\, \nabla u_n \cdot \nabla(T_k(u_n) - u_n).$$

Hence, if

$$A_n(k) := \{x \in \Omega : |u_n(x)| > k\},$$

we have that

$$\int_\Omega |\nabla(u_n - T_k(u_n))|^p = \int_{A_n(k)} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n$$

$$= \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla(u_n - T_k(u_n))$$

$$\leqslant \int_\Omega v(u_n - T_k(u_n)) - \int_\Omega \gamma_n(u_n)(u_n - T_k(u_n))$$

$$\leqslant \int_\Omega v(u_n - T_k(u_n)).$$

Now, by Young's inequality

$$\int_\Omega v(u_n - T_k(u_n)) \leqslant C_\varepsilon \|v\|_\infty^{p'} \lambda_N(A_n(k)) + \varepsilon C \int_\Omega |u_n - T_k(u_n)|^p.$$

From here, since $u_n - T_k(u_n) \in W_0^{1,p}(\Omega)$, using Poincaré's inequality, we obtain that

$$\|u_n - T_k(u_n)\|_{1,p} \leqslant R\lambda_N(A_n(k))^{1/p},$$

from which it follows, applying the classical Stampacchia methods (see for instance, Appendix B in [20]), that there exists a constant $M_1 = M_1(\|v\|_\infty, \|\varphi\|_\infty)$ such that

$$\|u_n\|_\infty \leqslant M_1 \qquad \forall n \in \mathbb{N}. \tag{4.8}$$

On the other hand, taking w_0 as a test function in (4.7) and applying Young's inequality, we obtain

$$\int_\Omega |\nabla u_n|^p \leqslant \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w_0 + \int_\Omega v(u_n - w_0)$$

$$+ \int_\Omega \gamma_n(u_n)(w_0 - u_n)$$

$$\leqslant \varepsilon C \int_\Omega |\nabla u_n|^p + C_\varepsilon \int_\Omega |\nabla w_0|^p + \int_\Omega v u_n$$

$$+ \int_\Omega w_0(\gamma_n(u_n) - v).$$

From this it follows that there exists a constant $M_2 = M_2(\lambda_N(\Omega), \|v\|_\infty, \|\varphi\|_\infty, \|w_0\|_{1,p})$ such that

$$\int_\Omega |\nabla u_n|^p \leqslant M_2 \qquad \forall n \in \mathbb{N}. \tag{4.9}$$

As a consequence of (4.8) and (4.9), $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. Hence there exists a subsequence, still denoted u_n, such that $u_n \to u \in W^{1,p}(\Omega)$ weakly in $W^{1,p}(\Omega)$. Moreover, by the Rellich–Kondrachov theorem, $u_n \to u$ in $L^p(\Omega)$, and by Theorem 3.4.5 in [23], $u_n \to u$ in $L^p(\partial\Omega)$. After passing to a suitable subsequence, we can assume that $u_n \to u$ a.e. in Ω. So, by (4.8), $\|u\|_\infty \leqslant M_1$.Therefore we have that $u \in W^{1,p}_\varphi(\Omega) \cap L^\infty(\Omega)$.

Proceeding as in the proof of step 3 of Theorem 2.1 in [3], we obtain that

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \qquad \text{in measure, and a.e.}$$

Now, by (4.9), we have that $\{|\nabla u_n|^{p-2}\nabla u_n\}_{n\in\mathbb{N}}$ is bounded in $(L^{p'}(\Omega))^N$. Hence

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \qquad \text{weakly in } (L^{p'}(\Omega))^N. \tag{4.10}$$

Given $w \in W^{1,p}_\varphi(\Omega) \cap L^\infty(\Omega)$, by (4.10), we get

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla w \to \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w, \tag{4.11}$$

and by Fatou's lemma, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u \leqslant \liminf_{n\to\infty} \int_\Omega \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n. \tag{4.12}$$

On the other hand, since $u_n \to u$ in $L^p(\Omega)$ we have

$$\lim_{n\to\infty} \int_\Omega (w-u_n)(v-\gamma_n(u_n)) = \int_\Omega (w-u)(v-u). \tag{4.13}$$

From (4.11), (4.12), and (4.13), passing to the limit in (4.7) we get (4.6), and the proof concludes. ∎

To prove Theorem 2, we need to give the following characterization of the operator $\mathcal{A}_\varphi$.

PROPOSITION 2. *The following assertions are equivalent*:

(a) $(u, v) \in \mathcal{A}_\varphi$

(b) $u, v \in L^1(\Omega)$, $p(u) \in BV(\Omega)$ *for all* $p \in \mathcal{P}$, *and there exists* $z \in X(\Omega)$, *with* $\|z\|_\infty \leqslant 1$, $v = -div(z)$ *in* $\mathcal{D}'(\Omega)$ *such that*

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega (z, Dw) = \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)| - \int_{\partial\Omega} |p(u) - p(\varphi)| \tag{4.14}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ *and* $p \in \mathcal{P}$.

(c) $u, v \in L^1(\Omega)$, $p(u) \in BV(\Omega)$ *for all* $p \in \mathcal{P}$, *and there exists* $z \in X(\Omega)$, *with* $\|z\|_\infty \leqslant 1$, $v = -div(z)$ *in* $\mathcal{D}'(\Omega)$ *such that*

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega (z, Dw) - \|Dp(u)\| - \int_{\partial\Omega} [z, \nu](w - p(\varphi)) - \int_{\partial\Omega} |p(u) - p(\varphi)| \tag{4.15}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ *and* $p \in \mathcal{P}$.

(d) $u, v \in L^1(\Omega)$, $p(u) \in BV(\Omega)$ *for all* $p \in \mathcal{P}$, *and there exists* $z \in X(\Omega)$, *with* $\|z\|_\infty \leqslant 1$, $v = -div(z)$ *in* $\mathcal{D}'(\Omega)$ *such that*

$$\int_\Omega (z, Dp(u)) = \|Dp(u)\| \qquad \forall p \in \mathcal{P} \tag{4.16}$$

$$[z, \nu] \in sign(p(\varphi) - p(u)) \qquad H^{N-1}\text{-a.e.} \qquad \text{on } \partial\Omega, \quad \forall p \in \mathcal{P}. \tag{4.17}$$

Proof. Let $(u, v) \in \mathcal{A}_\varphi$. Then, there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -div(z)$ in $\mathcal{D}'(\Omega)$, such that

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega z \cdot Dw - \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)| - \int_{\partial\Omega} |p(u) - p(\varphi)| \tag{4.18}$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and every $p \in \mathcal{P}$. Let $w \in BV(\Omega) \cap L^\infty(\Omega)$, $p \in \mathcal{P}$. Using Lemmas 5.2 and 1.8 in [5] we know that there exists a sequence $w_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that

$$w_n \to w \qquad \text{in } L^1(\Omega),$$

$$\int_\Omega |\nabla w_n| \to \|Dw\|, \tag{4.19}$$

$$\int_\Omega z \cdot \nabla w_n = \int_\Omega (z, Dw_n) \to \int_\Omega (z, Dw).$$

and $w_{n\,|\partial\Omega} = w_{|\partial\Omega}$, $\|w_n\|_\infty \leqslant \|w\|_\infty$, $\forall n \in \mathbb{N}$. Then taking w_n as a test function in (4.18) and letting $n \to \infty$ we get that (4.18) holds for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and all $p \in \mathcal{P}$. Thus (a) and (b) are equivalent.

Since

$$-\int_{\partial\Omega} [z, \nu](w - p(\varphi)) \leqslant \int_{\partial\Omega} |w - p(\varphi)|,$$

to prove the equivalence between (b) and (c), it is enough to show that if $(u, v) \in \mathcal{A}_\varphi$, then (4.15) is satisfied. In fact, since $(u, v) \in \mathcal{A}_\varphi$, there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -div(z)$ in $\mathcal{D}'(\Omega)$, such that

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega (z, Dw) - \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)| - \int_{\partial\Omega} |p(u) - p(\varphi)| \tag{4.20}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and every $p \in \mathcal{P}$. Now, given $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $p \in \mathcal{P}$, by Lemmas 5.2 and 5.5 of [5], there exists $w_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that $w_n \to w$ in $L^1(\Omega)$, $w_{n\,|\partial\Omega} = p(\varphi)$, and $\|w_n\|_\infty \leqslant \|w\|_\infty + \|p(\varphi)\|_\infty$, $\forall n \in \mathbb{N}$. Then taking w_n as a test function in (4.18) and using Green's formula, we get

$$\begin{aligned}\int_\Omega (w_n - p(u))\, v &\leqslant \int_\Omega (z, Dw_n) - \|Dp(u)\| - \int_{\partial\Omega} |p(u) - p(\varphi)| \\ &= -\int_\Omega div(z)\, w_n + \int_{\partial\Omega} [z, \nu]\, p(\varphi) - \|Dp(u)\| \\ &\quad - \int_{\partial\Omega} |p(u) - p(\varphi)|.\end{aligned}$$

Letting $n \to \infty$, it follows that

$$\int_\Omega (w - p(u))\, v \leqslant - \int_\Omega div(z)\, w + \int_{\partial\Omega} [z, \nu]\, p(\varphi) - \|Dp(u)\| - \int_{\partial\Omega} |p(u) - p(\varphi)|.$$

Therefore, applying Green's formula again, we obtain (4.15).

Suppose now that (b) or, equivalent, (c) is satisfied. Taking $w = p(u)$ in (4.18) we obtain

$$0 \leqslant \int_\Omega (z, Dp(u)) - \|Dp(u)\|.$$

Thus,

$$\int_\Omega (z, Dp(u)) \leqslant \|z\|_\infty \|Dp(u)\| \leqslant \|Dp(u)\| \leqslant \int_\Omega (z, Dp(u)),$$

and (4.16) holds. Let us prove (4.17). Since $p(\varphi) \in L^\infty(\partial\Omega)$, by Lemma 5.5 in [5], there exist $w_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ satisfying:

$$w_{n\,|\,\partial\Omega} = p(\varphi) \qquad \forall n \in \mathbb{N},$$

$$\int_\Omega |\nabla w_n| \leqslant \int_{\partial\Omega} |p(\varphi)| + \frac{1}{n} \qquad \forall n \in \mathbb{N},$$

$$\|w_n\|_1 \leqslant \frac{1}{n}, \qquad \|w_n\|_\infty \leqslant \|p(\varphi)\|_\infty \qquad \forall n \in \mathbb{N}.$$

Taking $w = w_n$ in (4.18) and using Green's formula (2.10), we get

$$\int_\Omega (w_n - p(u))\, v \leqslant - \int_\Omega div(z)\, w_n + \int_{\partial\Omega} [z, \nu]\, p(\varphi) - \|Dp(u)\| - \int_{\partial\Omega} |p(u) - p(\varphi)|. \tag{4.21}$$

Then, letting $n \to \infty$ in (4.21), we obtain

$$-\int_{\Omega} p(u)\, v \leqslant \int_{\partial\Omega} [z, \nu]\, p(\varphi) - \|Dp(u)\| - \int_{\partial\Omega} |p(u) - p(\varphi)|.$$

Now, by (4.16), and applying Green's formula, we have that

$$\|Dp(u)\| = \int_{\Omega} (z, Dp(u)) = \int_{\Omega} vp(u) + \int_{\partial\Omega} [z, \nu]\, p(u).$$

Hence,

$$0 \leqslant \int_{\partial\Omega} ([z, \nu](p(\varphi) - p(u)) - |p(u) - p(\varphi)|).$$

Since

$$[z, \nu](p(\varphi) - p(u)) - |p(u) - p(\varphi)| \leqslant 0,$$

we have that

$$[z, \nu](p(\varphi) - p(u)) = |p(u) - p(\varphi)| \qquad H^{N-1}\text{-}a.e. \qquad \text{on } \partial\Omega,$$

and we obtain (4.17). Finally, to prove that (d) implies (c), we only need to apply Green's formula. ∎

Remark 1. (1) As a consequence of the proof of the above proposition we can put equality in the definition of the operator; that is, the following characterization of the operator $\mathscr{A}_\varphi$ holds.

$$(u, v) \in \mathscr{A}_\varphi \qquad \text{if and only if } u, v \in L^1(\Omega),\ p(u) \in BV(\Omega) \text{ for all } p \in \mathscr{P}$$

and there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -div(z)$ in $\mathscr{D}'(\Omega)$ such that

$$\int_{\Omega} (w - p(u))\, v = \int_{\Omega} (z, Dw) - \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)|$$

$$- \int_{\partial\Omega} |p(u) - p(\varphi)|,$$

$\forall w \in BV(\Omega) \cap L^\infty(\Omega)$ and $\forall p \in \mathscr{P}$.

(2) As a consequence of the above proposition, if $(u, v) \in \mathcal{A}_\varphi$, we have that $\theta(z, DT_k(u), x) = 1$ a.e. with respect to the measure $\|DT_k(u)\|$. In the case that $z \in C(\Omega, \mathbb{R}^N)$, this implies that

$$z(x) \cdot \frac{DT_k(u)}{\|DT_k(u)\|} = 1, \qquad \|DT_k(u)\|\text{-a.e.},$$

where $DT_k(u)/\|DT_k(u)\|$ denotes the density of $DT_k(u)$ with respect to $\|DT_k(u)\|$. Heuristically, this amounts to saying that $z = \frac{Du}{\|Du\|}$. When z is not continuous we have that

$$z(x) \cdot \frac{DT_k(u)}{\|DT_k(u)\|} = 1, \qquad \|\nabla T_k(u)\|\text{-a.e.},$$

where $\|\nabla T_k(u)\|$ denotes the absolutely continuous part of $\|DT_k(u)\|$ with respect to the Lebesgue measure in $\mathbb{R}^N$ [5]. In particular, if $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ we have that

$$z(x) \cdot \frac{\nabla u}{\|\nabla u\|} = 1, \qquad \|\nabla u\|\text{-a.e.}$$

(3) Observe that by (d) in the above proposition, if $u \in L^\infty(\Omega)$, then the truncatures are redundant in the definition of $\mathcal{A}_\varphi$.

To prove the following result, we need to introduce the function $\Phi\colon L^1(\Omega) \to (-\infty, +\infty]$ defined by

$$\Phi(u) = \begin{cases} \|Du\| + \int_{\partial\Omega} |u - \varphi| & \text{if} \quad u \in BV(\Omega) \\ +\infty & \text{if} \quad u \in L^1(\Omega) \backslash BV(\Omega). \end{cases} \tag{4.22}$$

The functional Φ is convex and lower semicontinuous in $L^1(\Omega)$ (see [6] or [27]).

PROPOSITION 3. *Let $\varphi \in L^1(\partial\Omega)$. Then $L^\infty(\Omega) \subset R(I + \mathcal{A}_\varphi)$ and $D(\mathcal{A}_\varphi)$ is dense in $L^1(\Omega)$.*

Proof. Suppose first that $\varphi \in W^{1/2, 2}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Let $v \in L^\infty(\Omega)$. We shall find $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that $(u, v - u) \in \mathcal{A}_\varphi$, i.e., there is $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$ such that $v - u = -div(z)$ and

$$\int_\Omega (w - u)(v - u) \leqslant \int_\Omega z \cdot \nabla w \, dx - \|Du\| + \int_{\partial\Omega} |w - \varphi| - \int_{\partial\Omega} |u - \varphi| \tag{4.23}$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$.

Since $\varphi \in W^{1-1/p,\,p}(\partial\Omega)$ for all $p>1$, by Proposition 1, we know that for any $1<p\leqslant 2$ there is $u_p \in W^{1,\,p}_{\varphi}(\Omega) \cap L^\infty(\Omega)$ such that $(u_p, v-u_p) \in A_{\varphi,\,p}$. Hence

$$\int_\Omega (w-u_p)(v-u_p) \leqslant \int_\Omega |\nabla u_p|^{p-2}\, \nabla u_p \cdot \nabla(w-u_p), \tag{4.24}$$

for every $w \in W^{1,\,p}_{\varphi}(\Omega) \cap L^\infty(\Omega)$.

Let $M := \sup\{\|\varphi\|_\infty, \|v\|_\infty\}$. Then, taking $w = u_p - (u_p - M)^+$ as a test function in (4.24), we obtain

$$\int_\Omega (u_p - M)^+\,(u_p - v) \leqslant 0.$$

Hence,

$$\int_{\{u_p > M\}} (u_p - M)^2 \leqslant \int_{\{u_p > M\}} (u_p - M)(u_p - v)$$

$$= \int_\Omega (u_p - M)^+\,(u_p - v) \leqslant 0.$$

Consequently, $u_p \leqslant M$ a.e. in Ω. Analogously, taking $w = u_p + (u_p + M)^-$ as a test function, we get $-M \leqslant u_p$ a.e. in Ω. Therefore,

$$\|u_p\|_\infty \leqslant M \qquad \text{for all} \quad 1<p\leqslant 2. \tag{4.25}$$

Taking $w = w_0 \in W^{1,\,p}_{\varphi}(\Omega) \cap L^\infty(\Omega)$ in (4.24) and applying Young's inequality we obtain

$$\int_\Omega |\nabla u_p|^p \leqslant \int_\Omega |\nabla u_p|^{p-2}\,\nabla u_p \cdot \nabla w_0 - \int_\Omega (w_0 - u_p)(v-u_p)$$

$$\leqslant \varepsilon \int_\Omega |\nabla u_p|^p + C_\varepsilon \int_\Omega |\nabla w_0|^p + C(\|v\|_\infty, \|w_0\|_\infty).$$

Thus

$$\int_\Omega |\nabla u_p|^p \leqslant M_1 \qquad \forall 1<p\leqslant 2, \tag{4.26}$$

where M_1 depends on $\|v\|_\infty$, $\|w_0\|_\infty$, and $\|w_0\|_{1,2}$. Using Hölder's inequality we also have that

$$\int_\Omega |\nabla u_p| \leqslant M_2 \qquad \forall 1 < p \leqslant 2, \tag{4.27}$$

where M_2 does not depend on p. Thus, $\{u_p\}_{p>1}$ is bounded in $W^{1,1}(\Omega)$ and we may extract a subsequence such that u_p converges in $L^1(\Omega)$ and almost everywhere to some $u \in L^1(\Omega)$ as $p \to 1+$. Now, by (4.25) and (4.27), we have that $u \in BV(\Omega) \cap L^\infty(\Omega)$.

Let us prove that $\{|\nabla u_p|^{p-2} \nabla u_p\}_{p>1}$ is weakly relatively compact in $L^1(\Omega, \mathbb{R}^N)$. For that, using (4.26), we observe that

$$\int_\Omega |\nabla u_p|^{p-1} \leqslant \left(\int_\Omega |\nabla u_p|^p \right)^{(p-1)/p} \lambda_N(\Omega)^{1/p} \leqslant M_3,$$

where M_3 does not depend on p. On the other hand, for any measurable subset $E \subseteq \Omega$ such that $\lambda_N(E) < 1$, we have

$$\left| \int_E |\nabla u_p|^{p-2} \nabla u_p \right| \leqslant \int_E |\nabla u_p|^{p-1} \leqslant M_1^{(p-1)/p} \lambda_N(E)^{1/p} \leqslant M_4 \lambda_N(E)^{1/2}$$

Thus, $\{|\nabla u_p|^{p-2} \nabla u_p\}_{p>1}$, being bounded and equiintegrable in $L^1(\Omega, \mathbb{R}^N)$, is weakly relatively compact in $L^1(\Omega, \mathbb{R}^N)$. We may assume that

$$|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z \qquad \text{as} \quad p \to 1^+, \text{ weakly in } L^1(\Omega, \mathbb{R}^N). \tag{4.28}$$

Given $\psi \in C_0^\infty(\Omega)$, taking $w = u_p \pm \psi$ in (4.24) and letting $p \to 1^+$, we obtain

$$\int_\Omega (v - u)\, \psi = \int_\Omega z \cdot \nabla \psi,$$

that is, $v - u = -div(z)$ in $\mathcal{D}'(\Omega)$. Moreover, the same technique that we use in the proof of Lemma 1 in [2] shows that $\|z\|_\infty \leqslant 1$.

For every $w \in W^{1,2}_\varphi(\Omega) \cap L^\infty(\Omega)$, by (4.24) and Young's inequality, we get

$$\int_\Omega |\nabla u_p| + \int_{\partial\Omega} |u_p - \varphi| \leqslant \frac{p-1}{p} \lambda_N(\Omega) - \frac{1}{p} \int_\Omega (w - u_p)(v - u_p)$$
$$+ \frac{1}{p} \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w.$$

Then, using the lower semicontinuity of the functional Φ defined by (4.22), letting $p \to 1^+$, we obtain

$$\|Du\| + \int_{\partial\Omega} |u-\varphi| \leqslant -\int_{\Omega} (w-u)(v-u) + \int_{\Omega} z \cdot \nabla w, \tag{4.29}$$

for every $w \in W^{1,2}_{\varphi}(\Omega) \cap L^\infty(\Omega)$.

Now, to prove (4.23), we assume first that there exists $w_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi = w_{0|\partial\Omega}$ (i.e., φ is the trace of w_0). Let $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and let $w_n \in W^{1,2}_{\varphi}(\Omega) \cap L^\infty(\Omega)$ be such that $w_n \to w$ in $L^1(\Omega)$ as $n \to \infty$ and $\|w_n\|_\infty \leqslant \|w\|_\infty$. Using w_n as a test function in (4.29) and applying Green's formula (2.10), we may write

$$\int_{\Omega} (w_n-u)(v-u) \leqslant \int_{\Omega} z \cdot \nabla w_n - \|Du\| - \int_{\partial\Omega} |u-\varphi|$$

$$= -\int_{\Omega} div(z)\, w_n + \int_{\partial\Omega} [z, \nu]\, \varphi - \|Du\| - \int_{\partial\Omega} |u-\varphi|.$$

From here, letting $n \to \infty$ and applying again the Green's formula, we get

$$\int_{\Omega} (w-u)(v-u)$$

$$\leqslant -\int_{\Omega} div(z)\, w + \int_{\partial\Omega} [z, \nu]\, \varphi - \|Du\| - \int_{\partial\Omega} |u-\varphi|$$

$$= \int_{\Omega} z \cdot \nabla w - \int_{\partial\Omega} [z, \nu]\, w + \int_{\partial\Omega} [z, \nu]\, \varphi - \|Du\| - \int_{\partial\Omega} |u-\varphi|$$

$$\leqslant \int_{\Omega} z \cdot \nabla w - \|Du\| + \int_{\partial\Omega} |w-\varphi| - \int_{\partial\Omega} |u-\varphi|,$$

and the proof of (4.23), in this particular case, concludes.

Suppose now we are in the general case, that is, $\varphi \in L^1(\partial\Omega)$. Take $v_n \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi_n := v_{n|\partial\Omega} \to \varphi$ in $L^1(\partial\Omega)$. From the above, there exists $u_n \in BV(\Omega) \cap L^\infty(\Omega)$ and $z \in X(\Omega)$ with $\|z_n\|_\infty \leqslant 1$ such that $v - u_n = -div(z_n)$ and

$$\int_{\Omega} (w-u_n)(v-u_n) \leqslant \int_{\Omega} z_n \cdot \nabla w\, dx - \|Du_n\| + \int_{\partial\Omega} |w-\varphi_n|$$

$$-\int_{\partial\Omega} |u_n - \varphi_n| \tag{4.30}$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Moreover, by (4.25), we have $\|u_n\|_\infty \leqslant \max\{\|v\|_\infty, \|\varphi_n\|_\infty\}$. We can assume that $z_n \to z$ weakly* in $L^\infty(\Omega)$. Now, taking $w = 0$ in (4.30), we get

$$-\int_\Omega u_n v + \int_\Omega (u_n)^2 + \|Du_n\| + \int_{\partial\Omega} |u_n - \varphi_n| \leqslant \int_{\partial\Omega} |\varphi_n|.$$

Hence,

$$\|u_n\|_2^2 + \|Du_n\| \leqslant \int_\Omega u_n v + \int_{\partial\Omega} |\varphi_n| \leqslant \tfrac{1}{2}\|u_n\|_2^2 + \tfrac{1}{2}\|v\|_2^2 + \int_{\partial\Omega} |\varphi_n|.$$

Thus, $\{u_n\}$ is a bounded sequence in $BV(\Omega) \cap L^2(\Omega)$. Then, since $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$ (see [28] or [17]), there is a subsequence, still denoted by $\{u_n\}$ such that $u_n \to u$ in $L^1(\Omega)$. Finally, taking limits in (4.30), we obtain that $(u, v - u) \in \mathscr{A}_\varphi$.

To prove the density of $D(\mathscr{A}_\varphi)$ in $L^1(\Omega)$, we prove that $C_0^\infty(\Omega) \subseteq \overline{D(\mathscr{A}_\varphi)}^{L^1(\Omega)}$. Let $v \in C_0^\infty(\Omega)$. By the above, $v \in R(I + \frac{1}{n}\mathscr{A}_\varphi)$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$ there exists $u_n \in D(\mathscr{A}_\varphi)$ such that $(u_n, n(v - u_n)) \in \mathscr{A}_\varphi$ and, therefore, there exists some $z_n \in X(\Omega)$ with $\|z_n\|_\infty \leqslant 1$, $n(v - u_n) = -div(z_n)$ in $\mathscr{D}'(\Omega)$ such that

$$\int_\Omega (w - T_k(u_n))\, n(v - u_n) \leqslant \int_\Omega z_n \cdot \nabla w - \|DT_k(u_n)\| + \int_{\partial\Omega} |w - T_k(\varphi)| - \int_{\partial\Omega} |T_k(u_n) - T_k(\varphi)|$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Taking $w = T_k(v)$ and applying Fatou's lemma we have that

$$\int_\Omega (v - u_n)^2 \leqslant \frac{1}{n}\left(\int_\Omega |\nabla u| + \int_{\partial\Omega} |\varphi|\right).$$

Letting $n \to \infty$, it follows that $u_n \to v$ in $L^2(\Omega)$. Therefore $v \in \overline{D(\mathscr{A}_\varphi)}^{L^1(\Omega)}$. ∎

Proof of Theorem 2. Let $(u, v), (\hat{u}, \hat{v}) \in \mathscr{A}_\varphi$, $p \in P_0$. We have to prove that

$$\int_\Omega p(u - \hat{u})(v - \hat{u}) \geqslant 0. \tag{4.31}$$

Let $z, \hat{z} \in X(\Omega)$, $\|z\|_\infty \leqslant 1$, $\|\hat{z}\|_\infty \leqslant 1$, be such that $v = -div(z)$, $\hat{v} = -div(\hat{z})$ and

$$\int_\Omega (w - T_k(u))\, v \leqslant \int_\Omega (z, Dw) - \|DT_k(u)\| - \int_{\partial\Omega} [z, \nu](w - T_k(\varphi)) - \int_{\partial\Omega} |T_k(u) - T_k(\varphi)|, \tag{4.32}$$

$$\int_\Omega (w - T_k(\hat{u}))\, \hat{v} \leqslant \int_\Omega (\hat{z}, Dw) - \|DT_k(\hat{u})\| - \int_{\partial\Omega} [\hat{z}, \nu](w - T_k(\varphi)) - \int_{\partial\Omega} |T_k(\hat{u}) - T_k(\varphi)|, \tag{4.33}$$

for any $w \in BV(\Omega) \cap L^\infty(\Omega)$ and any $k > 0$. As observed in the previous remark, $\theta(z, DT_k(u), x) = 1$ $\|DT_k(u)\|$-*a.e.*, and, using Corollary 1.6 in [5], we obtain that

$$\int_B (z, DT_k(u)) = \int_B \theta(z, DT_k(u), x)\, \|DT_k(u)\| = \int_B \|DT_k(u)\|,$$

$$\left| \int_B (\hat{z}, DT_k(u)) \right| \leqslant \int_B \|DT_k(u)\|$$

for any Borel set $B \subseteq \Omega$. Similarly,

$$\int_B (\hat{z}, DT_k(\hat{u})) = \int_B \|DT_k(\hat{u})\|,$$

$$\left| \int_B (z, DT_k(\hat{u})) \right| \leqslant \int_B \|DT_k(\hat{u})\|$$

for any Borel set $B \subseteq \Omega$. It follows that

$$\int_B (z - \hat{z}, D(T_k(u) - T_k(\hat{u}))) \geqslant 0$$

for any Borel set $B \subseteq \Omega$. This implies that

$$\theta(z - \hat{z}, D(T_k(u) - T_k(\hat{u})), x) \geqslant 0, \qquad \|D(T_k(u) - T_k(\hat{u}))\|\text{-a.e.}$$

Since, according to Proposition 2.8 in [5], we have that

$$\theta(z - \hat{z}, Dp(T_k(u) - T_k(\hat{u})), x) = \theta(z - \hat{z}, D(T_k(u) - T_k(\hat{u})), x)$$

a.e. with respect to the measures $\|D(T_k(u)-T_k(\hat{u}))\|$ and $\|Dp(T_k(u)-T_k(\hat{u}))\|$. We conclude that

$$\theta(z-\hat{z}, Dp(T_k(u)-T_k(\hat{u})), x) \geqslant 0, \qquad \|Dp(T_k(u)-T_k(\hat{u}))\|\text{-a.e.} \tag{4.34}$$

Taking $w=T_k(u)-p(T_k(u)-T_k(\hat{u}))$ in (4.32) and $w=T_k(\hat{u})+p(T_k(u)-T_k(\hat{u}))$ in (4.33), adding both terms, and using (4.17) and (4.34), we obtain

$$\begin{aligned}
&\int_\Omega p(T_k(u)-T_k(\hat{u}))(\hat{v}-v)\\
&\quad\leqslant \int_\Omega (\hat{z}-z, Dp(T_k(u)-T_k(\hat{u})))\\
&\qquad+\int_{\partial\Omega} ([z,\nu]-[\hat{z},\nu])\, p(T_k(u)-T_k(\hat{u}))\\
&\quad= -\int_\Omega \theta(z-\hat{z}, Dp(T_k(u)-T_k(\hat{u})), x)\, \|Dp(T_k(u)-T_k(\hat{u}))\|\\
&\qquad+\int_{\partial\Omega} ([z,\nu]-[\hat{z},\nu])\, p(T_k(u)-T_k(\hat{u})) \leqslant 0.
\end{aligned}$$

The inequality (4.31) follows by letting $k\to\infty$. Therefore $\mathscr{A}_\varphi$ is completely accretive.

In view of Proposition 3, to prove that $\mathscr{A}_\varphi$ satisfies the range condition, it is enough to prove that $\mathscr{A}_\varphi$ is closed. Let $(u_n, v_n)\in\mathscr{A}_\varphi$, such that $(u_n, v_n)\to(u, v)$ in $L^1(\Omega)\times L^1(\Omega)$. Let us see that $(u, v)\in\mathscr{A}_\varphi$. Since $(u_n, v_n)\in\mathscr{A}_\varphi$, there exists $z_n\in X(\Omega)$, $\|z_n\|_\infty\leqslant 1$ with $v_n=-div(z_n)$ in $\mathscr{D}'(\Omega)$ such that

$$\begin{aligned}
\int_\Omega (w-p(u_n))\, v_n \leqslant \int_\Omega (z_n, Dw) - \|Dp(u_n)\| + \int_{\partial\Omega} |w-p(\varphi)|&\\
-\int_{\partial\Omega} |p(u_n)-p(\varphi)|&
\end{aligned} \tag{4.35}$$

for every $w\in BV(\Omega)\cap L^\infty(\Omega)$ and all $p\in\mathscr{P}$. Since $\|z_n\|_\infty\leqslant 1$ we may assume that $z_n\rightharpoonup z$ in the weak* topology of $L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty\leqslant 1$. Moreover, since $v_n\to v$ in $L^1(\Omega)$, we have $v=-div(z)$ in $\mathscr{D}'(\Omega)$, and

$$\lim_{n\to\infty}\int_\Omega (z_n, Dw)=\int_\Omega (z, Dw).$$

Now, letting $n \to \infty$ in (4.35), and having in mind the lower semicontinuity of the function Φ, defined in (4.22), we obtain that

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega (z, Dw) - \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)| - \int_{\partial\Omega} |p(u) - p(\varphi)|.$$

Consequently, $(u, v) \in \mathscr{A}_\varphi$. ∎

5. STRONG SOLUTIONS FOR DATA IN $L^2(\Omega)$

In this section we are going to see that when the initial datum is in $L^2(\Omega)$, then the semigroup solution is a strong solution.

Let $\{S(t)\}_{t \geqslant 0}$ be the contraction semigroup in $L^1(\Omega)$ generated by the operator $\mathscr{A}_\varphi$ via Crandall–Liggett's exponential formula. Since $\mathscr{A}_\varphi$ is an m-completely accretive operator, $S(t)(L^2(\Omega)) \subset L^2(\Omega)$. Let $\Psi_\varphi \colon L^2(\Omega) \to]-\infty, +\infty]$, the restriction to $L^2(\Omega)$ of the functional Φ defined by (4.22), i.e.,

$$\Psi_\varphi(u) = \begin{cases} \|Du\| + \int_{\partial\Omega} |u - \varphi| & \text{if} \quad u \in BV(\Omega) \cap L^2(\Omega) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \backslash BV(\Omega) \cap L^2(\Omega). \end{cases} \tag{5.1}$$

Since the function Ψ_φ is convex and lower semicontinuous in $L^2(\Omega)$, we have that $\partial\Psi_\varphi$ is a maximal monotone operator in $L^2(\Omega)$, and consequently (see [12]), if $\{T(t)\}_{t \geqslant 0}$ is the semigroup in $L^2(\Omega)$ generated by $\partial\Psi_\varphi$, for every $u_0 \in L^2(\Omega)$, $u(t) := T(t)\, u_0$ is a strong solution of the problem

$$\begin{aligned} &\frac{du}{dt} + \partial\Psi_\varphi u(t) \ni 0 \\ &u(0) = u_0. \end{aligned} \tag{5.2}$$

Recall that the operator $\partial\Psi_\varphi$ is defined by

$$(u, v) \in \partial\Psi_\varphi \quad \text{if and only if} \quad u, v \in L^2(\Omega), \quad \text{and}$$

$$\Psi_\varphi(w) \geqslant \Psi_\varphi(u) + \int_\Omega (w - u)\, v, \qquad \forall w \in L^2(\Omega).$$

LEMMA 1. *Let* $B_\varphi := \mathscr{A}_\varphi \cap (L^2(\Omega) \times L^2(\Omega))$. *Then* $B_\varphi = \partial \Psi_\varphi$.

Proof. Let $(u, v) \in B_\varphi$. Then, $u, v \in L^2(\Omega)$, $p(u) \in BV(\Omega)$ for all $p \in \mathscr{P}$, and there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -div(z)$ in $\mathscr{D}'(\Omega)$ such that

$$\int_\Omega (w - p(u))\, v \leqslant \int_\Omega (z, Dw) - \|Dp(u)\| + \int_{\partial\Omega} |w - p(\varphi)| - \int_{\partial\Omega} |p(u) - p(\varphi)|,$$

$\forall w \in BV(\Omega) \cap L^\infty(\Omega)$ and $\forall p \in \mathscr{P}$. Letting $p = T_k$ and $k \to \infty$ we obtain that

$$\int_\Omega (w - u)\, v \leqslant \int_\Omega (z, Dw) - \|Du\| + \int_{\partial\Omega} |w - \varphi| - \int_{\partial\Omega} |u - \varphi|,$$

$\forall w \in BV(\Omega) \cap L^\infty(\Omega)$. To prove that $(u, v) \in \partial \Psi_\varphi$, we have to prove that

$$\int_\Omega (w - u)\, v \leqslant \|Dw\| - \|Du\| + \int_{\partial\Omega} |w - \varphi| - \int_{\partial\Omega} |u - \varphi| \tag{5.3}$$

for every $w \in L^2(\Omega) \cap BV(\Omega)$. Now, given $w \in L^2(\Omega) \cap BV(\Omega)$, since $(u, v) \in B_\varphi$, by the first observation of the lemma, there exists $z \in X(\Omega)$, with $\|z\|_\infty \leqslant 1$, $v = -div(z)$ in $\mathscr{D}'(\Omega)$ such that

$$\int_\Omega (T_k(w) - u)\, v \leqslant \int_\Omega (z, DT_k(w)) - \|Du\| + \int_{\partial\Omega} |T_k(w) - \varphi| - \int_{\partial\Omega} |u - \varphi|,$$

for every $k > 0$. From this, it follows that

$$\int_\Omega (T_k(w) - u)\, v \leqslant \|DT_k(w)\| - \|Du\| + \int_{\partial\Omega} |T_k(w) - \varphi| - \int_{\partial\Omega} |u - \varphi|. \tag{5.4}$$

Now, since $\lim_{k \to \infty} T_k(w) = w$ in $L^2(\Omega)$,

$$\|Dw\| \leqslant \liminf_{k \to \infty} \|DT_k(w)\|.$$

Moreover, since $\|DT_k(w)\| \leqslant \|w\|$, we also have that

$$\limsup_{k\to\infty} \|DT_k(w)\| \leqslant \|Dw\|.$$

Thus

$$\lim_{k\to\infty} \|DT_k(w)\| = \|Dw\|.$$

Therefore, letting $k \to \infty$ in (5.4), we obtain (5.3). We have proved that $B_\varphi \subset \partial\Psi_\varphi$.

By Proposition 3, we have that $L^\infty(\Omega) \subset R(I+B_\varphi)$. Hence, $\partial\Psi_\varphi = \overline{B}_\varphi^{L^2(\Omega)}$. It follows that $\partial\Psi_\varphi = \mathcal{A}_\varphi \cap (L^2(\Omega) \times L^2(\Omega))$. ∎

Using this lemma and having in mind Proposition 2, we have the following result.

THEOREM 3. *Let $\varphi \in L^1(\partial\Omega)$. Given $u_0 \in L^2(\Omega)$, $u(t) = S(t)\, u_0$ is a strong solution of* (5.2). *Moreover, $u'(t) \in L^2(\Omega)$, $p(u(t)) \in BV(\Omega)$ for all $p \in \mathcal{P}$, and there exists $z(t) \in X(\Omega)$, $\|z(t)\|_\infty \leqslant 1$, and $u'(t) = \mathrm{div}(z(t))$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, +\infty[$, satisfying*

$$\begin{aligned} &\int_\Omega (w - p(u(t)))\, u'(t) \\ &\qquad \leqslant \int_\Omega (z(t), Dw) - \|Dp(u(t))\| \\ &\qquad\quad - \int_{\partial\Omega} [z(t), \nu](w - p(\varphi)) - \int_{\partial\Omega} |p(u(t)) - p(\varphi)| \end{aligned} \tag{5.5}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $p \in \mathcal{P}$.

Moreover, $u(t)$ is also characterized as follows: there exists $z(t) \in X(\Omega)$, $\|z(t)\|_\infty \leqslant 1$, and $u'(t) = \mathrm{div}(z(t))$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, +\infty[$, satisfying

$$\int_\Omega (z(t), Dp(u(t))) = \|Dp(u(t))\| \qquad \forall p \in \mathcal{P} \tag{5.6}$$

$$[z(t), \nu] \in \mathrm{sign}(p(\phi) - p(u(t))) \qquad H^{N-1}\text{-a.e.} \quad \text{on} \quad \partial\Omega, \forall p \in \mathcal{P}. \tag{5.7}$$

Remark 2. Note that under the assumptions of Theorem 3, since $u(t) \in BV(\Omega)$, applying the lower semicontinuity of Ψ_φ, if we take $p = T_k$ and take limits when $k \to \infty$, we obtain that (5.5), (5.6), and (5.7) are true when p is the identity map.

We have the following weak form of the maximum principle.

THEOREM 4. *Let u_1 and u_2 be two strong solutions of*

$$\begin{aligned} &\frac{du_i}{dt} + \partial \Psi_{\varphi_i} u_i(t) \ni 0 \\ &u_i(0) = u_{i,0}, \qquad i = 1, 2, \end{aligned} \tag{5.8}$$

where $u_{i,0} \in L^2(\Omega)$ and $\varphi_i \in L^1(\partial\Omega)$. Suppose that $u_{1,0} \geqslant u_{2,0}$ and $\varphi_1 \geqslant \varphi_2$. Then we have $u_1 \geqslant u_2$.

Proof. By Theorem 3 and the above remark, we have that $u_i(t)$, $u_i'(t) \in L^2(\Omega)$, and there exist $z_i(t) \in X(\Omega)$, $\|z_i(t)\|_\infty \leqslant 1$, and $u_i'(t) = div(z_i(t))$ in $\mathcal{D}'(\Omega)$, satisfying:

$$\int_\Omega (z_i(t), D(u_i(t))) = \|D(u_i(t))\| \tag{5.9}$$

$$[z_i(t), \nu] \in sign(\varphi_i - u_i(t)) \qquad H^{N-1}\text{-a.e.} \quad \text{on } \partial\Omega. \tag{5.10}$$

Since $\frac{d}{dt}(u_2(t) - u_1(t)) = div(z_2(t) - z_1(t))$ in $L^2(\Omega)$, multiplying by $(u_2(t) - u_1(t))^+$, integrating, and using Green's formula, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \frac{d}{dt}[(u_2(t) - u_1(t))^+]^2 \\ &\quad = \int_\Omega div(z_2(t) - z_1(t))(u_2(t) - u_1(t))^+ \\ &\quad = -\int_\Omega (z_2(t) - z_1(t), D((u_2(t) - u_1(t))^+)) \\ &\qquad + \int_{\partial\Omega} [z_2(t) - z_1(t), \nu](u_2(t) - u_1(t))^+. \end{aligned} \tag{5.11}$$

Now, by (5.9) it follows that

$$\theta(z_2(t) - z_1(t), D(u_2(t) - u_1(t)), x) \geqslant 0 \qquad \|D(u_2(t) - u_1(t))\|\text{-a.e.}$$

According to Proposition 2.8 in [5], we have

$$\begin{aligned} &\theta(z_2(t) - z_1(t), D(u_2(t) - u_1(t)), x) \\ &\quad = \theta(z_2(t) - z_1(t), D(u_2(t) - u_1(t))^+, x) \end{aligned}$$

a.e. with respect to $\|D(u_2(t)-u_1(t))\|$ and $\|D(u_2(t)-u_1(t))^+\|$. Hence we can conclude that

$$\theta(z_2(t)-z_1(t), D(u_2(t)-u_1(t))^+, x)\geqslant 0, \qquad \|D(u_2(t)-u_1(t))^+\|\text{-a.e.}$$

Consequently, we have

$$\begin{aligned}\int_\Omega &(z_2(t)-z_1(t), D((u_2(t)-u_1(t))^+))\\ &=\int_\Omega \theta(z_2(t)-z_1(t), D(u_2(t)-u_1(t))^+, x)\, \|D(u_2(t)-u_1(t))^+\|\\ &\geqslant 0.\end{aligned} \tag{5.12}$$

On the other hand, since $\varphi_1 \geqslant \varphi_2$, from (5.10), it is easy to see that

$$\int_{\partial\Omega} [z_2(t)-z_1(t), \nu](u_2(t)-u_1(t))^+ \leqslant 0. \tag{5.13}$$

From (5.11), (5.12), and (5.13), we obtain that

$$\frac{1}{2}\int_\Omega \frac{d}{dt}[(u_2(t)-u_1(t))^+]^2 \leqslant 0.$$

Hence the initial condition $u_{1,0}\geqslant u_{2,0}$ gives $u_1 \geqslant u_2$, and the proof concludes. ∎

PROPOSITION 4. *Let* $0\leqslant u_0 \in L^2(\Omega)$ *and* $0 \leqslant \varphi \in L^1(\partial\Omega)$. *Then, if* u *is the strong solution of the problem* (5.1), *we have*

$$u'(t) \leqslant \frac{u(t)}{t} \qquad \textit{for} \quad t>0.$$

The opposite inequality holds if u_0, $\varphi \leqslant 0$.

Proof. We shall prove the proposition only when u_0, $\varphi \geqslant 0$, the other case being similar. First, let us see that for $\lambda>0$, we have

$$\lambda^{-1}u(\lambda t) = e^{-t\mathscr{A}_{\lambda^{-1}\varphi}}(\lambda^{-1}u_0). \tag{5.14}$$

By Crandall–Liggett's exponential formula, it is enough to prove that for all $\mu > 0$,

$$(I+\mu\mathscr{A}_{\lambda^{-1}\varphi})^{-1}(\lambda^{-1}u_0)=\lambda^{-1}(I+\lambda\mu\mathscr{A}_{\varphi})^{-1}(u_0). \tag{5.15}$$

In fact, $v_\mu := (I+\mu\mathscr{A}_{\lambda^{-1}\varphi})^{-1}(\lambda^{-1}u_0)$ if and only if $(v_\mu, (\lambda^{-1}u_0 - v_\lambda)/\mu) \in \mathscr{A}_{\lambda^{-1}\varphi}$, which is equivalent to the existence of $z_\mu \in X(\Omega)$, such that

$$-div(z_\mu)=\frac{\lambda^{-1}u_0-v_\lambda}{\mu},$$

$$\int_\Omega (z_\mu, Dv_\mu) \quad = \|Dv_\mu\|,$$

$$[z_\mu, \nu] \in sign(\lambda^{-1}\varphi - v_\mu).$$

Then, we have

$$-div(z_\mu)=\frac{u_0-\lambda v_\lambda}{\lambda\mu},$$

$$\int_\Omega (z_\mu, D\lambda v_\mu) = \|D\lambda v_\mu\|,$$

$$[z_\mu, \nu] \in sign(\varphi - \lambda v_\mu),$$

which is equivalent to saying that $(\lambda v_\mu, (u_0 - \lambda v_\mu)/\lambda\mu) \in \mathscr{A}_\varphi$, that is, $v_\mu = \lambda^{-1}(I+\lambda\mu\mathscr{A}_\varphi)^{-1}(\lambda^{-1}u_0)$, and (5.15) holds.

Fix $t > 0$. For $h > 0$, if $\lambda t = t + h$, applying (5.14), we obtain

$$u(t+h)-u(t)=u(\lambda t)-u(t)=(1-\lambda^{-1})\,u(\lambda t)+\lambda^{-1}u(\lambda t)-u(t)$$

$$=\frac{h}{t+h}\,u(t+h)+e^{-t\mathscr{A}_{\lambda^{-1}\varphi}}(\lambda^{-1}u_0)-u(t).$$

Now, since $\lambda^{-1}u_0 \leqslant u_0$ and $\lambda^{-1}\varphi \leqslant \varphi$, by Theorem 4, we get

$$e^{-t\mathscr{A}_{\lambda^{-1}\varphi}}(\lambda^{-1}u_0) \leqslant u(t).$$

Consequently,

$$u(t+h)-u(t) \leqslant \frac{h}{t+h}\,u(t+h),$$

and the result follows. ∎

6. EXISTENCE AND UNIQUENESS FOR DATA IN $L^1(\Omega)$

In this section we are going to prove Theorem 1.

Proof of Theorem 1 (Existence).

Let $u_0 \in L^1(\Omega)$ and $\{S(t)\}_{t \geqslant 0}$ the contraction semigroup in $L^1(\Omega)$ generated by $\mathscr{A}_\varphi$. We shall prove that $u(t) := S(t)\, u_0$ is an entropy solution of problem (1.4). We divide the proof in different steps.

Step 1. Since $\mathscr{D}(\mathscr{A}_\varphi) \cap L^\infty(\Omega)$ is dense in $L^1(\Omega)$, given $u_0 \in L^1(\Omega)$ there exists a sequence $u_{0,n} \in \mathscr{D}(\mathscr{A}_\varphi) \cap L^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$. Then, if $u_n(t) := S(t)\, u_{0,n}$, we have that $u_n \to u$ in $C([0, T]; L^1(\Omega))$ for every $T > 0$. As a consequence of Theorem 3, $u_n(t), u_n'(t) \in L^2(\Omega)$, $p(u_n(t)) \in BV(\Omega)$ for all $p \in \mathscr{P}$ and there exists $z_n(t) \in X(\Omega)$, $\|z_n(t)\|_\infty \leqslant 1$, and $u_n'(t) = div(z_n(t))$ in $\mathscr{D}'(\Omega)$ a.e. $t \in [0, +\infty[$, satisfying

$$
\begin{aligned}
-\int_\Omega (w - p(u_n(t)))\, u_n'(t)
&\leqslant \int_\Omega (z_n(t), Dw) - \|Dp(u_n(t))\| \\
&\quad - \int_{\partial\Omega} [z_n(t), \nu](w - p(\varphi)) - \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \qquad (6.1)
\end{aligned}
$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $p \in \mathscr{P}$. Moreover,

$$
\int_\Omega (z_n(t), Dp(u_n(t))) = \|Dp(u_n(t))\| \qquad \forall p \in \mathscr{P} \qquad (6.2)
$$

and

$$
[z_n(t), \nu] \in sign(p(\varphi) - p(u_n(t))) \qquad H^{N-1}\text{-a.e. on } \partial\Omega, \forall p \in \mathscr{P}. \qquad (6.3)
$$

Since $\|[z_n(t), \nu]\|_\infty \leqslant \|z_n(t)\|_\infty \leqslant 1$, we can suppose (up to extraction of a subsequence, if necessary) that

$$
[z_n(\cdot), \nu] \to \rho \qquad \sigma(L^\infty(S_T), L^1(S_t)).
$$

Step 2. *Convergence of the derivatives and identification of the limit.* Since the map $t \mapsto u_n'(t)$ is strongly measurable from $[0, T]$ into $L^2(\Omega)$, and by (2.13),

$$
\|u_n'(t)\|_{BV(\Omega)_2^*} \leqslant \|u_n'(t)\|_{L^2(\Omega)},
$$

it follows that this map is strongly measurable from $[0, T]$ into $BV(\Omega)_2^*$. Moreover, for every $w \in BV(\Omega)_2$, by Green's formula we have

$$\int_\Omega u_n'(t)\, w = \int_\Omega div(z_n(t))\, w = -\int_\Omega (z_n(t), Dw) + \int_{\partial\Omega} [z_n(t), \nu]\, w.$$

Hence

$$\left| \int_\Omega u_n'(t)\, w \right| \leqslant \|Dw\| + \int_{\partial\Omega} |w| \leqslant M\, \|w\|_{BV(\Omega)_2} \qquad \forall n \in \mathbb{N}.$$

Thus,

$$\|u_n'(t)\|_{BV(\Omega)_2^*} \leqslant M \qquad \forall n \in \mathbb{N} \quad \text{and} \quad t \in [0, T].$$

Consequently, $\{u_n'\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(0, T; BV(\Omega)_2^*)$. Since $L^\infty(0, T; BV(\Omega)_2^*)$ is a vector subspace of the dual space $(L^1(0, T; BV(\Omega)_2))^*$, we can find a net $\{u_\alpha'\}$ such that

$$u_\alpha' \to \xi \in (L^1(0, T; BV(\Omega)_2))^* \qquad \text{weakly*}. \tag{6.4}$$

Since $\|z_n(t)\|_\infty \leqslant 1$ for all $n \in \mathbb{N}$ and a.e. $t \in [0, T]$, we can suppose that

$$z_n \to z \in L^\infty(Q_T, \mathbb{R}^N) \qquad \text{weakly*}. \tag{6.5}$$

Given $\eta \in \mathscr{D}(Q_T)$, since $\eta \in L^1(0, T; BV(\Omega)_2)$, we have

$$\begin{aligned}
\langle \xi, \eta \rangle &= \lim_\alpha \langle u_\alpha', \eta \rangle = \lim_\alpha \int_0^T \langle u_\alpha'(t), \eta(t) \rangle\, dt \\
&= \lim_\alpha \int_0^T \int_\Omega u_\alpha'(t)\, \eta(t)\, dx\, dt \\
&= \lim_\alpha \int_0^T \int_\Omega div(z_\alpha(t))\, \eta(t)\, dx\, dt \\
&= -\lim_\alpha \int_0^T \int_\Omega z_\alpha(t) \cdot \nabla \eta(t)\, dx\, dt \\
&= -\int_{Q_T} z \cdot \nabla \eta = \langle div_x(z), \eta \rangle.
\end{aligned}$$

Hence,

$$\xi = div_x(z) \qquad \text{in } \mathscr{D}'(Q_T). \tag{6.6}$$

On the other hand, if we take $\eta(t,x)=\phi(t)\,\psi(x)$ with $\phi\in\mathscr{D}(]0,T[)$ and $\psi\in\mathscr{D}(\Omega)$, the same calculation as above shows that

$$\xi(t)=div_x(z(t)) \qquad \text{in} \quad \mathscr{D}'(\Omega) \text{ a.e.} \quad t\in[0,T]. \tag{6.7}$$

Consequently, $(z(t),\xi(t))\in Z(\Omega)$ for almost all $t\in[0,T]$; therefore we can consider $[z(t),v]$ defined as in Section 2.

LEMMA 2. *ξ is the time derivative of u in the sense of Definition* 2.

Proof. Let $\Psi\in L^1(0,T,BV(\Omega))$ be the weak derivative of $\Theta\in L^1_w(0,T,BV(\Omega))\cap L^\infty(Q_T)$, i.e., $\Psi(t)=\int_0^t\Theta(s)\,ds$, the integral being taken as a Pettis integral. By (6.4) we have that

$$\int_0^T\langle\xi(t),\Psi(t)\rangle\,dt=\lim_\alpha\int_0^T\langle u'_\alpha(t),\Psi(t)\rangle.$$

Now,

$$\begin{aligned}\int_0^T\langle u'_\alpha(t),\Psi(t)\rangle&=\lim_h\int_0^T\int_\Omega\Psi(t)\,\frac{u_\alpha(t+h)-u(t)}{h}\,dx\,dt\\&=\lim_h\int_0^T\int_\Omega\frac{\Psi(t-h)-\Psi(t)}{h}\,u_\alpha(t)\,dx\,dt\\&=-\lim_h\int_0^T\int_\Omega\frac{1}{h}\int_{t-h}^t\Theta(s)\,ds\,u_\alpha(t)\,dx\,dt\\&=-\int_0^T\int_\Omega\Theta(t,x)\,u_\alpha(t,x)\,dx\,dt.\end{aligned}$$

Passing to the limit in α in the above expression, we obtain

$$\int_0^T\langle\xi(t),\Psi(t)\rangle\,dt=-\int_0^T\int_\Omega\Theta(t,x)\,u(t,x)\,dx\,ds. \quad ∎ \tag{6.8}$$

Step 3. *Convergence of the energy*. In this step we shall prove that for any $p\in\mathscr{P}$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\|Dp(u_n(t))\|+\int_0^T\int_{\partial\Omega}|p(u_n(t))-p(\varphi)|\\=\int_0^T\|Dp(u(t))\|+\int_0^T\int_{\partial\Omega}|p(u(t))-p(\varphi)|.\end{aligned} \tag{6.9}$$

Taking $w=0$ in (6.1) we get

$$\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \leqslant -\int_{\Omega} p(u_n(t))\, u_n'(t) + \int_{\partial\Omega} [z_n(t), \nu]\, p(\varphi).$$

If we denote $J_p(r) := \int_0^r p(s)\, ds$, it follows that

$$\begin{aligned}\int_0^t \|Dp(u_n(t))\| &+ \int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \\ &\leqslant -\int_0^T \frac{d}{dt} \int_{\Omega} J_p(u_n(t)) + \int_0^T \int_{\partial\Omega} |p(\varphi)| \\ &= \int_{\Omega} (J_p(u_{0,n}) - J_p(u_n(T))) + \int_0^T \int_{\partial\Omega} |p(\varphi)| \leqslant M_p.\end{aligned}$$

Since the functional $\Phi_p\colon L^1(\Omega) \to \,]-\infty, +\infty]$, defined by

$$\Phi_p(w) = \begin{cases} \|Du\| + \int_{\partial\Omega} |u - p(\varphi)| & \text{if} \quad w \in BV(\Omega) \\ +\infty & \text{if} \quad u \in L^1(\Omega) \backslash BV(\Omega), \end{cases} \tag{6.10}$$

is lower semicontinuous in $L^1(\Omega)$, we have

$$\begin{aligned}\Phi_p(p(u(t))) &\leqslant \liminf_{n \to \infty} \Phi_p(p(u_n(t))) \\ &= \liminf_{n \to \infty} \left(\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \right).\end{aligned} \tag{6.11}$$

On the other hand, by the Fatou's lemma, it follows that

$$\begin{aligned}\int_0^T \liminf_{n \to \infty} &\left(\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \right) \\ &\leqslant \liminf_{n \to \infty} \int_0^T \left(\|Dp(u_n(t))\| + \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \right) \leqslant M_p.\end{aligned} \tag{6.12}$$

As a consequence of (6.11) and (6.12), we obtain that $p(u(t)) \in BV(\Omega)$ for almost all $t \in [0, T]$.

LEMMA 3. *The map $t \mapsto p(u(t))$ from $[0, T]$ into $BV(\Omega)$ is weakly measurable.*

Proof. Let $E := C_c(\Omega)^{N+1}$ and $S: BV(\Omega) \to E^*$ be the map defined by

$$S(w) := \left(w\, dx, \frac{\partial w}{\partial x_1}, ..., \frac{\partial w}{\partial x_N} \right).$$

Then, $\|w\|_{BV(\Omega)} \leqslant \|S(w)\|_{E^*} \leqslant N\, \|w\|_{BV(\Omega)}$. If we denote by F the closure in E of the set

$$\{(\phi_0, \phi_1, ..., \phi_N): \phi_i \in \mathscr{D}(\Omega), \text{ and } \phi_0 = div(\phi_1, ..., \phi_N)\},$$

in [1] it is shown that $S(BV(\Omega))$ is isomorphic to $(\frac{E}{F})^*$; that is, $G := \frac{E}{F}$ is the predual of the space $BV(\Omega)$. Now, if $\phi = (\phi_0, \phi_1, ..., \phi_N) \in \mathscr{D}(\Omega)^{N+1}$,

$$\langle S(p(u(t))), \phi \rangle = \int p(u(t))\, \phi_0 - \sum_{i=1}^{N} \int_\Omega p(u(t))\, \frac{\partial \phi}{\partial x_i}.$$

Hence, the map $t \mapsto \langle S(p(u(t))), \phi \rangle$ is measurable. From here, approximating the functions of $C_c(\Omega)^{N+1}$ by functions of $\mathscr{D}(\Omega)^{N+1}$, we get that for every $\phi \in G$, the function $t \mapsto \langle S(p(u(t))), \phi \rangle$ is measurable. Thus, since G is separable, it follows that the map

$$t \mapsto \|p(u(t))\|_{BV(\Omega)} = \sup_{\phi \in G,\ \|\phi\| \leqslant 1} \langle S(p(u(t))), \phi \rangle$$

is measurable.

Given $w \in BV(\Omega)^*$, let $g(t) := \langle p(u(t)), w \rangle$. To see that g is measurable, consider $w_\alpha \in G$ such that $w_\alpha \to w$ with respect to $\sigma(G^{**}, G^*) = \sigma(BV(\Omega)^*, BV(\Omega))$. From the above, we know that if $g_\alpha(t) := \langle S(p(u(t))), w_\alpha \rangle$, g_α is measurable, and $g_\alpha(t) \to g(t)$. Now, since

$$\begin{aligned} |g_\alpha(t)| &\leqslant \|p(u(t))\|_{BV(\Omega)}\, \|w_\alpha\|_{BV(\Omega)^*} \\ &\leqslant R\, \|p(u(t))\|_{BV(\Omega)} = F(t) \in L^1(0, T), \end{aligned}$$

and the order interval $[-F, F]$ in $L^1(0, T)$ is $\sigma(L^1(0, T), L^\infty(0, T))$-relatively compact, there exists a sequence g_{α_n} such that

$$g_{\alpha_n} \to g \qquad \text{in } \sigma(L^2(0, T), L^\infty(0, T)).$$

Hence, g is measurable. ∎

From the above, if $0 \leqslant \eta \in \mathscr{D}(]0, T[)$, the map $t \mapsto p(u(t))\, \eta(t)$ from $[0, T]$ into $BV(\Omega)$ is weakly measurable.

LEMMA 4. *For any* $\tau > 0$, *we define the function* ψ^τ *as the Dunford integral* (*see* [15])

$$\psi^\tau := \frac{1}{\tau} \int_{t-\tau}^{t} \eta(s)\, p(u(s))\, ds \in BV(\Omega)^{**},$$

that is,

$$\langle \psi^\tau(t), w \rangle = \frac{1}{\tau} \int_{t-\tau}^{t} \langle \eta(s)\, p(u(s)), w \rangle\, ds,$$

for any $w \in BV(\Omega)^*$. *Then* $\psi^\tau \in C([0, T]; BV(\Omega))$. *Moreover,* $\psi^\tau(t) \in L^2(\Omega)$, *and, thus,* $\psi^\tau(t) \in BV(\Omega)_2$.

Proof. Given $\phi \in \mathscr{D}(\Omega)$,

$$\begin{aligned}|\langle \psi^\tau(t), \phi \rangle| &\leqslant \frac{1}{\tau} \int_{t-\tau}^{t} |\eta(s)|\, |\langle p(u(s)), \phi \rangle|\, ds \\ &= \frac{1}{\tau} \int_{t-\tau}^{t} |\eta(s)| \left(\int_\Omega |p(u(s))|\, |\phi|\, dx \right) ds \leqslant C\, \|\phi\|_\infty.\end{aligned}$$

Consequently, $\psi^\tau(t)$ is a finite Radon measure in Ω. Moreover, a similar calculation shows that for every $i = 1, 2, ..., N$, $\frac{\partial \psi^\tau(t)}{\partial x_i}$ is also a finite Radon measure in Ω. Hence, we have $\psi^\tau(t) \in BV(\Omega)$ (see Exercise 3.2 in [1]), and the Dunford integral of the definition of $\psi^\tau(t)$ is a Pettis integral. Moreover, if $a_n \to 0$ (for simplicity suppose that $a_n > 0$), given $w \in BV(\Omega)^*$ with $\|w\| \leqslant 1$, we have

$$\begin{aligned}&|\langle \psi^\tau(t + a_n) - \psi^\tau(t), w \rangle| \\ &\quad = \left| \frac{1}{\tau} \int_{t+a_n-\tau}^{t+a_n} \eta(s) \langle p(u(s)), w \rangle\, ds - \frac{1}{\tau} \int_{t-\tau}^{t} \eta(s) \langle p(u(s)), w \rangle\, ds \right| \\ &\quad \leqslant \left| \frac{1}{\tau} \int_{t}^{t+a_n} \eta(s) \langle p(u(s)), w \rangle\, ds - \frac{1}{\tau} \int_{t-\tau}^{t-\tau+a_n} \eta(s) \langle p(u(s)), w \rangle\, ds \right| \\ &\quad \leqslant \frac{1}{\tau} \int_{t}^{t+a_n} |\eta(s)|\, \|p(u(s))\|_{BV(\Omega)}\, ds \\ &\qquad + \frac{1}{\tau} \int_{t-\tau}^{t-\tau+a_n} |\eta(s)|\, \|p(u(s))\|_{BV(\Omega)}\, ds.\end{aligned}$$

Since the function $s \mapsto |\eta(s)| \, \|p(u(s))\|_{BV(\Omega)}$ is in $L^1([0, T])$,

$$\lim_{n\to\infty} \|\psi^\tau(t+a_n) - \psi^\tau(t)\|_{BV(\Omega)} = 0.$$

Thus, $\psi^\tau \in C([0, T]; BV(\Omega))$.

Moreover, $\psi^\tau(t) \in L^2(\Omega)$. In fact, given $g \in L^\infty(\Omega)$, with $\|g\|_2 \leqslant 1$, since $g \in BV(\Omega)^*$, we have

$$\begin{aligned} |\langle \psi^\tau(t), g\rangle| &= \left| \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \langle p(u(s)), g\rangle \, ds \right| \\ &= \left| \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \left(\int_\Omega p(u(s)) \, g \, dx \right) ds \right| \\ &\leqslant \frac{1}{\tau} \int_{t-\tau}^t |\eta(s)| \, \|p(u(s))\|_2 \, \|g\|_2 \leqslant M. \end{aligned}$$

From the density of $L^\infty(\Omega)$ in $L^2(\Omega)$, we obtain that $\psi^\tau(t) \in L^2(\Omega)$. ∎

LEMMA 5. *For $\tau > 0$ small enough, we have*

$$\int_0^T \langle \psi^\tau(t), \xi(t)\rangle \, dt \leqslant -\int_0^T \int_\Omega \frac{\eta(t-\tau) - \eta(t)}{-\tau} J_p(u(t)). \tag{6.13}$$

Proof. Since $\psi^\tau \in C([0, T], BV(\Omega))$ admits a weak derivative in $L^1_w(0, T, BV(\Omega)) \cap L^\infty(Q_T)$, using (6.8) we have for $\tau > 0$ small enough that

$$\int_0^T \langle \psi^\tau(t), \xi(t)\rangle \, dt = \int_0^T \int_\Omega \frac{u(t+\tau) - u(t)}{\tau} \eta(t) \, p(u(t)).$$

Now, since p is nondecreasing, we have

$$J_p(u(t)) - J_p(u(t+\tau)) \leqslant (u(t) - u(t+\tau)) \, p(u(t))$$

and consequently, for $\tau > 0$ small enough, we obtain

$$\begin{aligned} \int_0^T \int_\Omega \frac{u(t+\tau) - u(t)}{\tau} \eta(t) \, p(u(t)) &\leqslant \int_0^T \int_\Omega \frac{J_p(u(t+\tau)) - J_p(u(t))}{\tau} \eta(t) \\ &= \int_0^T \int_\Omega \frac{\eta(s-\tau) - \eta(s)}{\tau} J_p(u(s)), \end{aligned}$$

and we finish the proof of (6.13). ∎

Now, we can conclude the proof of Step 3. As a consequence of (6.13), using Green's formula, we have

$$\begin{aligned}
\int_0^T \int_\Omega \frac{\eta(t-\tau)-\eta(t)}{-\tau} J_p(u(t)) &\leqslant -\int_0^T \int_0^T \langle \psi^\tau(t), \xi(t)\rangle \, dt \\
&= -\lim_\alpha \int_0^T \langle \psi^\tau(t), u'_\alpha(t)\rangle \, dt \\
&= -\lim_\alpha \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^t \eta(s)\langle p(u(s)), u'_\alpha(t)\rangle \, ds\right) dt \\
&= -\lim_\alpha \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^t \eta(s)\left(\int_\Omega p(u(s)) \, div \, z_\alpha(t)\right) ds\right) dt \\
&= \lim_\alpha \left[\int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^t \eta(s)\left(\int_\Omega (z_\alpha(t), Dp(u(s)))\right) ds\right) dt\right. \\
&\quad \left. -\int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^t \eta(s)\left(\int_{\partial\Omega} [z_\alpha(t), \nu] \, p(u(s))\right) ds\right) dt\right] \\
&\leqslant \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^t \eta(s) \, \|Dp(u(s))\| \, ds\right) dt \\
&\quad -\int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^t \eta(s)\left(\int_{\partial\Omega} \rho(t) \, p(u(s))\right) ds\right) dt.
\end{aligned}$$

Then, taking the limit as $\tau \to 0^+$, we get

$$\int_0^T \int_\Omega \eta'(t) \, J_p(u(t)) \leqslant \int_0^T \eta(t) \, \|Dp(u(t))\| - \int_0^T \eta(t) \int_{\partial\Omega} \rho(t) \, p(u(t)).$$

Now, since this is true for all $0 \leqslant \eta \in \mathscr{D}(\,]0, T[\,)$, it follows that

$$-\frac{d}{dt}\int_\Omega J_p(u(t)) \leqslant \|Dp(u(t))\| - \int_{\partial\Omega} \rho(t) \, p(u(t)),$$

and consequently

$$\int_\Omega (J_p(u_0) - J_p(u(T))) \leqslant \int_0^T \|Dp(u(t))\| - \int_0^T \int_{\partial\Omega} \rho(t)\, p(u(t)). \tag{6.14}$$

Finally, using (6.14), we obtain

$$\begin{aligned}
&\int_0^T \|Dp(u(t))\| + \int_0^T \int_{\partial\Omega} |p(u(t)) - p(\varphi)| \\
&\quad \leqslant \liminf_{n\to\infty} \int_0^T \|Dp(u_n(t))\| + \int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)| \\
&\quad \leqslant \liminf_{n\to\infty} \left(\int_0^T \int_\Omega p(u_n(t))\, u_n'(t) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\, p(\varphi) \right) \\
&\quad = \int_\Omega J_p(u_0) - J_p(u(T)) + \int_0^T \int_{\partial\Omega} \rho(t)\, p(\varphi) \\
&\quad \leqslant \int_0^T \|Dp(u(t))\| + \int_0^T \int_{\partial\Omega} \rho(t)(p(\varphi) - p(u(t))) \\
&\quad \leqslant \int_0^T \|Dp(u(t))\| + \int_0^T \int_{\partial\Omega} |p(u(t)) - p(\varphi)|,
\end{aligned}$$

which concludes the proof of (6.9). Moreover, we get that

$$\rho(t) \in \operatorname{sign}(p(\varphi) - p(u(t))) \qquad H^{N-1}\text{-a.e.} \quad \text{on } \partial\Omega, \quad \text{a.e. } t \in [0, T]. \tag{6.15}$$

Step 4. *The boundary condition.* Let us now see that

$$\rho(t) = [z(t), \nu] \qquad H^{N-1}\text{-a.e.} \quad \text{on } \partial\Omega, \quad \text{a.e. } t \in [0, T]. \tag{6.16}$$

In fact, if $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $v \in R(\Omega)$ such that $v_{|\partial\Omega} = w_{|\partial\Omega}$, we have that

$$\int_0^t \langle z_\alpha(s), w \rangle_{\partial\Omega} = \int_0^t \langle div(z_\alpha(s)), v \rangle + \int_0^t \int_\Omega z_\alpha(s) \cdot \nabla v.$$

Hence

$$\begin{aligned}
\lim_\alpha \int_0^t \langle z_\alpha(s), w \rangle_{\partial\Omega} &= \int_0^t \langle \xi(s), v \rangle + \int_0^t \int_\Omega z(s) \cdot \nabla v \\
&= \int_0^t \langle z(s), w \rangle_{\partial\Omega} = \int_0^t \int_{\partial\Omega} [z(s), \nu]\, dH^{N-1}.
\end{aligned} \tag{6.17}$$

On the other hand, since $z_\alpha(s) \in X(\Omega)$, if we apply Green's formula we have that

$$\int_0^t \langle div(z_\alpha(s)), v \rangle = -\int_0^t \int_\Omega z_\alpha(s) \cdot \nabla v + \int_0^t \int_{\partial\Omega} [z_\alpha(s), \nu]\, w.$$

Consequently,

$$\int_0^t \langle z_\alpha(s), w \rangle_{\partial\Omega} = \int_0^t \int_{\partial\Omega} [z_\alpha(s), \nu]\, w.$$

From here, taking limits in α, we get

$$\int_0^t \int_{\partial\Omega} \rho(s)\, w = \int_0^t \int_{\partial\Omega} [z(s), \nu]\, w$$

$$\forall w \in BV(\Omega) \cap L^\infty(\Omega), \qquad \text{and} \quad t \in [0, T]. \tag{6.18}$$

Now, if $w \in L^1(\partial\Omega)$, we take $w_k \in BV(\Omega) \cap L^\infty(\Omega)$ such that $w_{k_{|\partial\Omega}} = T_k(w)$. By (6.18), we have

$$\int_0^t \int_{\partial\Omega} \rho(s)\, w_k = \int_{\partial\Omega} [z(s), \nu]\, w_k.$$

Letting $k \to \infty$, it follows that

$$\int_0^t \int_{\partial\Omega} \rho(s)\, w = \int_0^t \int_{\partial\Omega} [z(s), \nu]\, w \qquad \forall w \in L^1(\partial\Omega), \quad \text{and} \quad t \in [0, T],$$

and consequently (6.16) holds.

Step 5. Next, we prove that $\xi = div(z)$ in $(L^1(0, T, BV(\Omega)_2))^*$ in the sense of Definition 3. To do that let us first observe that (z, Dw), defined by (2.14), is a Radon measure in Q_T for all $w \in L^1_w(0, T, BV(\Omega)) \cap L^\infty(Q_T)$. Let $\phi \in \mathcal{D}(Q_T)$, then

$$\langle (z, Dw), \phi \rangle = -\int_0^T \langle \xi(t) - u'_\alpha(t), w(t)\, \phi(t) \rangle - \int_{Q_T} w(z - z_\alpha) \cdot \nabla_x \phi$$

$$+ \int_0^T \langle (z_\alpha(t), Dw(t))\, \phi(t) \rangle.$$

Then by (6.4), taking limits in α, we get

$$\langle (z, Dw), \phi \rangle = \lim_{\alpha} \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t) \rangle. \tag{6.19}$$

Therefore

$$|\langle (z, Dw), \phi \rangle| \leqslant \|\phi\|_\infty \int_0^T \|Dw(t)\| \, dt,$$

from which it follows that (z, Dw) is a Radon measure in Q_T. Moreover, from (6.19), applying Green's formula we obtain that

$$\begin{aligned} \int_{Q_T} (z, Dw) &= \lim_{\alpha} \int_0^T (z_\alpha(t), Dw(t)) \\ &= \lim_{\alpha} \left(-\int_0^T \int_\Omega div(z_\alpha(t)) \, w(t) + \int_0^T \int_{\partial\Omega} [z_\alpha(t), \nu] \, w(t) \right) \\ &= -\int_0^T \langle \xi(t), w(t) \rangle + \int_0^T \int_{\partial\Omega} [z(t), \nu] \, w(t). \end{aligned}$$

Consequently

$$\int_{Q_T} (z, Dw) + \int_0^T \langle \xi(t), w(t) \rangle = \int_0^T \int_{\partial\Omega} [z(t), \nu] \, w(t). \tag{6.20}$$

Step 6. *Conclusion*. Finally, we are going to prove that u verifies:

$$\begin{aligned} -\int_0^T \int_\Omega j(u(t) - l) \, \eta_t &+ \int_0^T \int_\Omega \eta(t) \, \|Dp(u(t) - l)\| \\ &+ \int_0^T \int_\Omega z(t) \cdot D\eta(t) \, p(u(t) - l) \\ &\leqslant \int_0^T \int_{\partial\Omega} [z(t), \nu] \, \eta(t) \, p(u(t) - l), \end{aligned} \tag{6.21}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t, x) = \phi(t) \, \psi(x)$, being $\phi \in \mathcal{D}(]0, T[)$, $\psi \in C^\infty(\overline{\Omega})$, and $p \in \mathcal{P}$, where $j(r) = \int_0^r p(s) \, ds$.

Let $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t, x) = \phi(t) \, \psi(x)$, being $\phi \in \mathcal{D}(]0, T[)$, $\psi \in C^\infty(\overline{\Omega})$, and $p \in \mathcal{P}$, $a \in \mathbb{R}$. Let $H_p(r) := \int_a^r p(s) \, ds$. Since $u_n'(t) = div(z_n(t))$, multiplying by $p(u_n(t)) \, \eta(t)$ and integrating, we obtain that

$$\int_0^T \int_\Omega \frac{d}{dt} H_p(u_n(t))\, \eta(t)$$

$$= \int_0^T \int_\Omega p(u_n(t))\, u_n'(t)\, \eta(t)$$

$$= \int_0^T \int_\Omega div(z_n(t))\; p(u_n(t))\, \eta(t)$$

$$= -\int_0^T \int_\Omega (z_n(t), D(p(u_n(t))\, \eta(t))) + \int_0^t \int_{\partial\Omega} [z_n(t), \nu]\; p(u_n(t))\, \eta(t)$$

$$= -\int_0^T \int_\Omega \eta(t)\, \|Dp(u_n(t))\| - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t)\; p(u_n(t))$$

$$+ \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\; p(u_n(t))\, \eta(t)$$

$$= -\int_0^T \int_\Omega \eta(t)\, \|Dp(u_n(t))\| - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t)\; p(u_n(t))$$

$$- \int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)|\, \eta(t) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\; p(\varphi)\, \eta(t).$$

Hence, having in mind that $\eta(0) = \eta(T) = 0$, we get

$$\int_0^T \int_\Omega \eta(t)\, \|Dp(u_n(t))\| + \int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)|\, \eta(t)$$

$$= -\int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t)\; p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\; p(\varphi)\, \eta(t)$$

$$- \int_0^T \int_\Omega \frac{d}{dt} H_p(u_n(t))\, \eta(t)$$

$$= -\int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t)\; p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\; p(\varphi)\, \eta(t)$$

$$- \int_0^T \int_\Omega \frac{d}{dt} (H_p(u_n(t))\, \eta(t)) + \int_0^T \int_\Omega H_p(u_n(t))\, \eta_t$$

$$= -\int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t)\; p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\; p(\varphi)\, \eta(t)$$

$$+ \int_0^T \int_\Omega H_p(u_n(t))\, \eta_t.$$

Letting $n \to \infty$, it follows that

$$\int_0^T \int_\Omega \eta(t)\, \|Dp(u(t))\| + \int_0^T \int_{\partial\Omega} |p(u(t)) - p(\varphi)|\, \eta(t)$$

$$\leqslant \liminf_{n\to\infty} \left[\int_0^T \int_\Omega \eta(t)\, \|Dp(u_n(t))\| + \int_0^T \int_{\partial\Omega} |p(u_n(t)) - p(\varphi)|\, \eta(t) \right]$$

$$= \liminf_{n\to\infty} \left[-\int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t)\, p(u_n(t)) \right.$$

$$\left. + \int_0^T \int_{\partial\Omega} [z_n(t), \nu]\, p(\varphi)\, \eta(t) + \int_0^T \int_\Omega H_p(u_n(t))\, \eta_t \right]$$

$$= -\int_0^T \int_\Omega z(t) \cdot \nabla\eta(t)\, p(u(t)) + \int_0^T \int_{\partial\Omega} [z(t), \nu]\, p(\varphi)\, \eta(t)$$

$$+ \int_0^T \int_\Omega H_p(u(t))\, \eta_t.$$

Now, using that $|p(u(t)) - p(\varphi)| = [z(t), \nu](p(\varphi) - p(u(t)))$, we have

$$-\int_0^T \int_\Omega H_p(u(t))\, \eta_t + \int_0^T \int_\Omega \eta(t)\, \|Dp(u(t))\|$$

$$+ \int_0^T \int_\Omega z(t) \cdot \nabla\eta(t)\, p(u(t)) \leqslant \int_0^T \int_{\partial\Omega} [z(t), \nu]\, p(u(t))\, \eta(t). \qquad (6.22)$$

Finally, given $l \in \mathbb{R}$ and $p \in \mathscr{P}$, since $q(r) := p(r - l)$ is an element of $\mathscr{P}$, and taking $a = l$, we obtain (6.21) as a consequence of (6.22) and the proof of the existence is finished.

Proof of Theorem 1 (Uniqueness)

To prove uniqueness we shall show that the entropy solutions and semigroup solutions coincide. As a consequence of the semigroup theory (3.1) is satisfied. Our technique is inspired by a method introduced by Kruzhkov [21] to prove L^1-contraction for entropy solutions for scalar conservation laws: the doubling of variables. Carrillo [13] probably was the first to apply Kruzhkov's method to second order equations (see also [18]).

Let $u(t)$ be an entropy solution with initial datum $u_0 \in L^1(\Omega)$ and $\bar{u}(t) = S(t)\,\bar{u}_0$ the semigroup solution with initial datum $\bar{u}_0 \in L^\infty(\Omega)$. Then, there exist $z(t), \bar{z}(t) \in Z(\Omega)$ with $\|z(t)\|_\infty \leqslant 1$, $\|\bar{z}(t)\|_\infty \leqslant 1$, and

$$[z(t), v] \in \operatorname{sign}(T_k^+(\varphi) - T_k^+(u(t))) \qquad \text{a.e. in} \quad t \in [0, T], \tag{6.23}$$

$$[\bar{z}(t), v] \in \operatorname{sign}(T_k^+(\varphi) - T_k^+(\bar{u}(t))) \qquad \text{a.e. in} \quad t \in [0, T], \tag{6.24}$$

and such that, if $r, \bar{r} \in \mathbb{R}^N$, with $\|r\| \leqslant 1$, $\|\bar{r}\| \leqslant 1$, and $l_1, l_2 \in \mathbb{R}$, then

$$\begin{aligned} -\int_0^T \int_\Omega j_k^+(u(t)-l_1)\,\eta_t + \int_0^T \int_\Omega \eta(t)\,\|DT_k^+(u(t)-l_1)\| \\ + \int_0^T \int_\Omega (z(t)-r)\cdot D\eta(t)\,T_k^+(u(t)-l_1) \\ + \int_0^T \int_\Omega r\cdot D\eta(t)\,T_k^+(u(t)-l_1) \\ \leqslant \int_0^T \int_{\partial\Omega} [z(t), v]\,\eta(t)\,T_k^+(u(t)-l_1), \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} -\int_0^T \int_\Omega j_k^-(\bar{u}(t)-l_2)\,\eta_t + \int_0^T \int_\Omega \eta(t)\,\|DT_k^-(\bar{u}(t)-l_2)\| \\ + \int_0^T \int_\Omega (\bar{z}(t)-\bar{r})\cdot D\eta(t)\,T_k^-(\bar{u}(t)-l_2) \\ + \int_0^T \int_\Omega \bar{r}\cdot D\eta(t)\,T_k^-(\bar{u}(t)-l_2) \\ \leqslant \int_0^T \int_{\partial\Omega} [\bar{z}(t), v]\,\eta(t)\,T_k^-(\bar{u}(t)-l_2), \end{aligned} \tag{6.26}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\,\psi(x)$, being $\phi \in \mathscr{D}(]0,T[)$, $\psi \in C^\infty(\bar{\Omega})$, and $j_k^+(r) = \int_0^r T_k^+(s)\,ds$, $j_k^-(r) = \int_0^r T_k^-(s)\,ds$.

We choose two different pairs of variables (t, x), (s, y) and consider u, z as functions in (t, x), $\bar{u}$, $\bar{z}$ in (s, y). Let $0 \leqslant \phi \in \mathscr{D}(]0, T[)$, $0 \leqslant \psi \in \mathscr{D}(\Omega)$,

ρ_n a classical sequence of mollifiers in $\mathbb{R}^N$ and $\tilde{\rho}_n$ a sequence of mollifiers in $\mathbb{R}$. Define

$$\eta_n(t, x, s, y) := \rho_n(x-y)\, \tilde{\rho}_n(t-s)\, \phi\left(\frac{t+s}{2}\right) \psi\left(\frac{x+y}{2}\right).$$

Note that for n sufficiently large,

$$(t, x) \mapsto \eta_n(t, x, s, y) \in \mathcal{D}(]0, T[\times \Omega) \qquad \forall (s, y) \in Q_T,$$

$$(s, y) \mapsto \eta_n(t, x, s, y) \in \mathcal{D}(]0, T[\times \Omega) \qquad \forall (t, x) \in Q_T.$$

Hence, for (s, y) fixed, if we take in (6.25) $l_1 = \bar{u}(s, y)$ and $r = \bar{z}(s, y)$, we get

$$\begin{aligned} &-\int_0^T \int_\Omega j_k^+(u(t, x) - \bar{u}(s, y))(\eta_n)_t + \int_0^T \int_\Omega \eta_n \, \|D_x T_k^+(u(t, x) - \bar{u}(s, y))\| \\ &\quad + \int_0^T \int_\Omega (z(t, x) - \bar{z}(s, y)) \cdot \nabla_x \eta_n T_k^+(u(t, x) - \bar{u}(s, y)) \\ &\quad + \int_0^T \int_\Omega \bar{z}(s, y) \cdot \nabla_x \eta_n T_k^+(u(t, x) - \bar{u}(s, y)) \leqslant 0. \end{aligned} \tag{6.27}$$

Similarly, for (t, x) fixed, if we take in (6.26) $l_2 = u(t, x)$ and $\bar{r} = z(t, x)$, we get

$$\begin{aligned} &-\int_0^T \int_\Omega j_k^-(\bar{u}(s, y) - u(t, x))(\eta_n)_s + \int_0^T \int_\Omega \eta_n \, \|D_y T_k^-(\bar{u}(s, y) - u(t, x))\| \\ &\quad + \int_0^T \int_\Omega (\bar{z}(s, y) - z(t, x)) \cdot \nabla_y \eta_n T_k^-(\bar{u}(s, y) - u(t, x)) \\ &\quad + \int_0^T \int_\Omega z(t, x) \cdot \nabla_y \eta_n T_k^-(\bar{u}(s, y) - u(t, x)) \leqslant 0. \end{aligned} \tag{6.28}$$

Now, since $T_k^-(r) = -T_k^+(-r)$ and $j_k^-(r) = j_k^+(-r)$, we can rewrite (6.28) as

$$\begin{aligned} &-\int_0^T \int_\Omega j_k^+(u(t, x) - \bar{u}(s, y))(\eta_n)_s + \int_0^T \int_\Omega \eta_n \, \|D_y T_k^+(u(t, x) - \bar{u}(s, y))\| \\ &\quad + \int_0^T \int_\Omega (z(t, x) - \bar{z}(s, y)) \cdot \nabla_y \eta_n T_k^+(u(t, x) - \bar{u}(s, y)) \\ &\quad - \int_0^T \int_\Omega z(s, y) \cdot \nabla_y \eta_n T_k^+(u(t, x) - \bar{u}(s, y)) \leqslant 0. \end{aligned} \tag{6.29}$$

Integrating (6.27) in (s, y) and (6.29) in (t, x) and taking their sum yields

$$
\begin{aligned}
&-\int_{Q_T\times Q_T} j_k^+(u(t,x)-\bar{u}(s,y))((\eta_n)_t+(\eta_n)_s)\\
&\quad+\int_{Q_T\times Q_T}\eta_n\,\|D_xT_k^+(u(t,x)-\bar{u}(s,y))\|\\
&\quad+\int_{Q_T\times Q_T}\eta_n\,\|D_yT_k^+(u(t,x)-\bar{u}(s,y))\|\\
&\quad+\int_{Q_T\times Q_T}(z(t,x)-\bar{z}(s,y))\cdot(\nabla_x\eta_n+\nabla_y\eta_n)\,T_k^+(u(t,x)-\bar{u}(s,y))\\
&\quad+\int_{Q_T\times Q_T}\bar{z}(s,y)\cdot\nabla_x\eta T_k^+(u(t,x)-\bar{u}(s,y))\\
&\quad-\int_{Q_T\times Q_T}z(t,x)\cdot\nabla_y\eta_n T_k^+(u(t,x)-\bar{u}(s,y))\leqslant 0.
\end{aligned}
\tag{6.30}
$$

Now, by Green's formula we have

$$
\begin{aligned}
&\int_{Q_T\times Q_T}\bar{z}(s,y)\cdot\nabla_x\eta_n T_k^+(u(t,x)-\bar{u}(s,y))\\
&\qquad+\int_{Q_T\times Q_T}\eta_n\,\|D_xT_k^+(u(t,x)-\bar{u}(s,y))\|\\
&\quad=-\int_{Q_T\times Q_T}\eta_n\bar{z}(s,y)\cdot D_xT_k^+(u(t,x)-\bar{u}(s,y))\\
&\qquad+\int_{Q_T\times Q_T}\eta_n\,\|D_xT_k^+(u(t,x)-\bar{u}(s,y))\|\geqslant 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{Q_T\times Q_T}z(t,x)\cdot\nabla_y\eta_n T_k^+(u(t,x)-\bar{u}(s,y))\\
&\qquad+\int_{Q_T\times Q_T}\eta_n\,\|D_yT_k^+(u(t,x)-\bar{u}(s,y))\|\\
&\quad=\int_{Q_T\times Q_T}\eta_n z(t,x)\cdot D_yT_k^+(u(t,x)-\bar{u}(x,y))\\
&\qquad+\int_{Q_T\times Q_T}\eta_n\,\|D_yT_k^+(u(t,x)-\bar{u}(s,y))\|\geqslant 0.
\end{aligned}
$$

Hence, from (6.30), it follows that

$$-\int_{Q_T\times Q_T} j_k^+(u(t,x)-\bar{u}(s,y))((\eta_n)_t+(\eta_n)_s)$$
$$+\int_{Q_T\times Q_T}(z(t,x)-\bar{z}(s,y))$$
$$\times(\nabla_x\eta_n+\nabla_y\eta_n)\,T_k^+(u(t,x)-\bar{u}(s,y))\leqslant 0. \tag{6.31}$$

Since,

$$(\eta_n)_t+(\eta_n)_s=\rho_n(x-y)\,\tilde{\rho}_n(t-s)\,\phi'\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right)$$

and

$$\nabla_x\eta_n+\nabla_y\eta_n=\rho_n(x-y)\,\tilde{\rho}_n(t-s)\,\phi\left(\frac{t+s}{2}\right)\nabla\psi\left(\frac{x+y}{2}\right),$$

passing to the limit in (6.31), it yields

$$-\int_{Q_T} j_k^+(u(t,x)-\bar{u}(t,x))\,\varphi'(t)\,\psi(x)$$
$$+\int_{Q_T}(z(t,x)-\bar{z}(t,x))\cdot\nabla\psi(x)\,\phi(t)\,T_k^+(u(t,x)-\bar{u}(t,x))\leqslant 0. \tag{6.32}$$

We have to prove that

$$\lim_n\int_{Q_T}(z(t,x)-\bar{z}(t,x))\cdot\nabla\psi_n(x)\,\phi(t)\,T_k^+(u(t,x)-\bar{u}(t,x))\geqslant 0$$

for any sequence $\psi_n\uparrow \mathbb{1}_\Omega$. Since $\xi=div(z)$, $\bar{\xi}=div(\bar{z})$ in $(L^1(0,T,BV(\Omega)_2))^*$, the following integration by parts formula holds

$$\int_{Q_T}(z-\bar{z},Dw)+\int_0^T\langle\xi(t)-\bar{\xi}(t),w(t)\rangle\,dt$$
$$=\int_0^T\int_{\partial\Omega}[z(t,x)-\bar{z}(t,x),\nu]\,w(t,x)\,dH^{N-1}\,dt,$$

for all $w \in L^1(0, T, BV(\Omega)) \cap L^\infty(Q_T)$. Set

$$w = ((\psi - 1)\, \phi T_k^+(u - \bar{u}))^\tau\, (t, x) = (\psi(x) - 1)(\phi T_k^+(u - \bar{u}))^\tau\, (t, x),$$

where

$$(\phi T_k^+(u - \bar{u}))^\tau\, (t, x) = \frac{1}{\tau} \int_t^{t+\tau} \phi(s)\, T_k^+(u(s, x) - \bar{u}(s, x))\, dt,$$

in the above formula to obtain

$$\int_{Q_T} (z(t, x) - \bar{z}(t, x)) \cdot \nabla(\psi(x) - 1)(\phi T_k^+(u - \bar{u}))^\tau\, (t, x)\, dx\, dt$$

$$= -\int_{Q_T} (z(t, x) - \bar{z}(t, x)) \cdot (\psi(x) - 1)\, D(\phi T_k^+(u - \bar{u}))^\tau\, (t, x)\, dx\, dt$$

$$- \int_{Q_T} (\xi(t) - \bar{\xi}(t))(\psi(x) - 1)(\phi T_k^+(u - \bar{u}))^\tau\, (t, x)$$

$$+ \int_0^T \int_{\partial\Omega} [z(t, x) - \bar{z}(t, x), \nu](\psi(x) - 1)$$

$$\times (\phi T_k^+(u - \bar{u}))^\tau\, (t, x)\, dH^{N-1}\, dt.$$

Since

$$\int_{Q_T} (z - \bar{z}) \cdot \nabla(\psi - 1)\, \phi T_k^+(u - \bar{u})\, dx\, dt$$

$$= \lim_{\tau \to 0+} \int_{Q_T} (z - \bar{z}) \cdot \nabla(\psi - 1)(\phi T_k^+(u - \bar{u}))^\tau\, dx\, dt,$$

and, using that $\psi|_{\partial\Omega} = 0$, also

$$- \int_0^T \int_{\partial\Omega} [z - \bar{z}, \nu]\, \phi T_k^+(u - \bar{u})\, dH^{N-1}\, dt$$

$$= \lim_{\tau \to 0+} \int_0^T \int_{\partial\Omega} [z - \bar{z}, \nu]\, (\psi - 1)(\phi T_k^+(u - \bar{u}))^\tau\, dH^{N-1}\, dt$$

we may write

$$\int_{Q_T} (z-\bar{z})\, \nabla\psi \phi T_k^+(u-\bar{u})$$

$$= \int_{Q_T} (z-\bar{z})\, \nabla(\psi-1)\, \phi T_k^+(u-\bar{u})$$

$$= \lim_{\tau} \int_{Q_T} (z-\bar{z}) \cdot (1-\psi)\, D(\phi T_k^+(u-\bar{u}))^\tau\, dx\, dt$$

$$+ \int_{Q_T} (\xi-\bar{\xi})(1-\psi)(\phi T_k^+(u-\bar{u}))^\tau$$

$$- \int_0^T \int_{\partial\Omega} [z-\bar{z}, v]\, \phi T_k^+(u-\bar{u})\, dH^{N-1}\, dt.$$

Now, since ξ, $\bar{\xi}$ are the time derivatives of u, resp. $\bar{u}$, in $(L^1(0, T, BV(\Omega)_2))^*$, we have that

$$\int_0^T \int_\Omega (\xi-\bar{\xi})(1-\psi)(\phi T_k^+(u-\bar{u}))^\tau$$

$$= \int_0^T \int_\Omega (\xi-\bar{\xi})((1-\psi)\, \phi T_k^+(u-\bar{u}))^\tau$$

$$= \int_0^T \int_\Omega (1-\psi)\, \phi T_k^+(u-\bar{u})\, \frac{1}{\tau}\, \Delta_\tau^-(u-\bar{u}),$$

where $\Delta_\tau^-(u-\bar{u}) = (u-\bar{u})(t) - (u-\bar{u})(t-\tau)$. Let $v = u-\bar{u}$. Since

$$T_k^+(v(t))(v(t)-v(t-\tau)) \geqslant J_{T_k^+}(v(t)) - J_{T_k^+}(v(t-\tau))$$

($J_{T_k^+}$ being the primitive of T_k^+), and ϕ, $(1-\psi) \geqslant 0$, we have for τ small enough that

$$\int_0^T \int_\Omega (\xi-\bar{\xi})(1-\psi)(\phi T_k^+(u-\bar{u}))^\tau$$

$$\geqslant \int_0^T \int_\Omega (1-\psi)\, \phi\, \frac{J_{T^k_+}(v(t)) - J_{T_k^+}(v(t-\tau))}{\tau}$$

$$= -\int_0^T \int_\Omega \frac{\phi(t+\tau)-\phi(t)}{\tau}\, (1-\psi)\, J_{T_k^+}(u-\bar{u}).$$

Thus, we have

$$\int_{Q_T} (z-\bar{z})\, \nabla\psi\phi T_k^+(u-\bar{u})$$

$$\geqslant \lim_{\tau} \left(\int_{Q_T} (z-\bar{z}) \cdot (1-\psi)\, D(\phi T_k^+(u-\bar{u}))^{\tau}\, dx\, dt - \int_0^T \int_{\Omega} \frac{\phi(t+\tau)-\phi(t)}{\tau} (1-\psi)\, J_{T_k^+}(u-\bar{u}) \right) - \int_0^T \int_{\partial\Omega} [z-\bar{z}, \nu]\, \phi T_k^+(u-\bar{u})\, dH^{N-1}\, dt.$$

Finally, we observe that

$$\lim_{\tau} \left| \int_{Q_T} (z-\bar{z})(1-\psi)\, D(\phi T_k^+(u-\bar{u}))^{\tau}\, dx\, dt \right| \leqslant 2 \int_{Q_T} (1-\psi)\, \phi\, \|DT_k^+(u-\bar{u})\|\, dx\, dt,$$

which enables us to write that

$$\int_{Q_T} (z-\bar{z})\, \nabla\psi\phi T_k^+(u-\bar{u}) \geqslant -2 \int_{Q_T} (1-\psi)\, \phi\, \|DT_k^+(u-\bar{u})\|\, dx\, dt - \int_0^T \int_{\Omega} \phi'(t)(1-\psi)\, J_{T_k^+}(u-\bar{u}) - \int_0^T \int_{\partial\Omega} [z-\bar{z}, \nu]\, \phi T_k^+(u-\bar{u})\, dH^{N-1}\, dt.$$

Let $\psi = \psi_n$ where $\psi_n \uparrow \mathbb{1}_{\Omega}$ in the last expression. Using that $\|DT_k^+(u(t)-\bar{u}(t))\|$ is a Radon measure a.e. in t with $\|DT_k^+(u(t)-\bar{u}(t))\| \in L^1(0, T)$, letting $n \to \infty$, we obtain

$$\lim_{n} \int_{Q_T} (z-\bar{z})\, \nabla\psi_n\phi T_k^+(u-\bar{u}) \geqslant -\int_0^T \int_{\partial\Omega} [z-\bar{z}, \nu]\, \phi T_k^+(u-\bar{u})\, dH^{N-1}\, dt.$$

Thus

$$\int_{Q_T} j_k^+(u(t,x)-\bar{u}(t,x))\,\phi'(t)$$

$$\geqslant -\int_0^T \int_{\partial\Omega} [z-\bar{z}, \nu]\,\phi T_k^+(u-\bar{u})\,dH^{N-1}\,dt \geqslant 0. \qquad (6.33)$$

Since this is true for all $0 \leqslant \phi \in \mathcal{D}(\,]0, T[\,)$, we get

$$\frac{d}{dt}\int_\Omega j_k^+(u(t,x)-\bar{u}(t,x)) \leqslant 0.$$

Hence

$$\int_\Omega j_k^+(u(t,x)-\bar{u}(t,x)) \leqslant \int_\Omega j_k^+(u_0-\bar{u}_0).$$

Then, letting $k \to \infty$, we obtain

$$\int_\Omega (u(t,x)-\bar{u}(t,x))^+ \leqslant \int_\Omega (u_0-\bar{u}_0)^+.$$

From here we deduce that

$$\|u(t)-\bar{u}(t)\|_1 \leqslant \|u_0-\bar{u}_0\|_1, \qquad \forall t \geqslant 0.$$

Hence, taking $u_n(t) = S(t)\,u_{0,n}$, $u_{0,n} \in L^\infty(\Omega)$, and $u_{0,n} \to u_0$ in $L^1(\Omega)$, we have

$$\|u(t)-u_n(t)\|_1 \leqslant \|u_0-u_{0,n}\|_1, \qquad \forall t \geqslant 0.$$

Consequently, letting $n \to \infty$, $u(t) = S(t)\,u_0$, and the proof of the uniqueness concludes. ∎

7. REGULARITY FOR POSITIVE INITIAL DATA

In this section we shall see that when the initial data are nonnegative, the semigroup solutions are strong solution.

We need to consider truncatures $T_{a,b}$, $a<b$, defined by

$$T_{a,b}(r) := \begin{cases} a & \text{if } \ r<a \\ r & \text{if } \ a \leqslant r \leqslant b \\ b & \text{if } \ r>b. \end{cases}$$

PROPOSITION 5. *Let $u_0 \in L^1(\Omega)$, $\varphi \in L^1(\partial\Omega)$. Let $(S(t))_{t\geqslant 0}$ be the semi-group generated by $\mathcal{A}_\varphi$. Then, if $u(t) = S(t)\, u_0$,*

(i)

$$\int_\Omega j(u(t)) + \int_0^t \Phi(p(u(s))) \leqslant \int_0^t \int_{\partial\Omega} |p(\varphi)| + \int_\Omega j(u_0),$$

where p is a truncature ($p = T_{a,b}$), j is the primitive of p, and Φ is the functional defined by (4.22).

(ii) *$p(u)_t \in L^2_{loc}(0, \tau, L^2(\Omega))$, for every truncature p as above. Moreover, we have the estimate*

$$\Phi(p(u(t))) + \int_s^t \int_\Omega |p(u)_t|^2 \leqslant C,$$

where $C > 0$ depends on s, $\|u_0\|_{L^1}$, $\|\varphi\|_{L^1}$, and p.

Proof. (i) First, assume that $u_0 \in L^2(\Omega)$

$$\begin{aligned}\frac{d}{dt}\int_\Omega j(u) &= \int_\Omega p(u)\, u_t = \int_\Omega p(u)\, div(z)\\ &= -\int_\Omega z\cdot Dp(u) + \int_{\partial\Omega} [z, \nu]\, p(u)\\ &= -\int_\Omega |Dp(u)| + \int_{\partial\Omega} [z, \nu](p(u) - p(\varphi) + p(\varphi))\\ &= -\int_\Omega |Dp(u)| - \int_{\partial\Omega} |p(u) - p(\varphi)| + \int_{\partial\Omega} [z, \nu]\, p(\varphi).\end{aligned}$$

Integrating this expression, we obtain

$$\int_\Omega j(u(t)) + \int_0^T \Phi(p(u(t))) \leqslant \int_0^T \int_{\partial\Omega} |p(\varphi)| + \int_\Omega j(u_0). \tag{7.1}$$

Since j has linear growth at infinity, if $u_0 \in L^1(\Omega)$, the estimate in (i) follows by approximating u_0 by functions $u_{0n} \in L^2(\Omega)$ and passing to the limit.

(ii) Assume first that $u_0 \in L^2(\Omega)$. Let $\delta > 0$ and t, $s \geqslant \delta$ such that $(u(t), -u_t(t))$, $(u(s), -u_t(s)) \in \mathscr{A}_\varphi$. We know that

$$\int_\Omega (p(u(t)) - w)\, u_t(t) \leqslant \int_\Omega (z(t), Dw) - \|Dp(u(t))\| + \int_{\partial\Omega} |w - p(\varphi)| - \int_{\partial\Omega} |p(u(t)) - p(\varphi)|$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$. Setting $w = p(u(s))$ in the above expression we have

$$\Phi(p(u(t))) - \Phi(p(u(s))) \leqslant \int_\Omega u_t(t)(p(u(s)) - p(u(t))).$$

Using the estimate for semigroups generated by subdifferentials in L^2 (see for instance [12, Theorem 3.2]) we have

$$\Phi(p(u(t))) - \Phi(p(u(s))) \leqslant C(\delta)\, \|u_0\|_2\, \|p(u(s)) - p(u(t))\|_2.$$

Since a similar estimate holds with s and t interchanged, we have

$$|\Phi(p(u(t))) - \Phi(p(u(s)))| \leqslant C(\delta)\, \|u_0\|_2\, \|p(u(s)) - p(u(t))\|_2. \qquad (7.2)$$

Since $u \in W^{1,1}_{loc}((0, \tau), L^2(\Omega))$, i.e., is a locally absolutely continuous function of time, then $p(u)$ is also, and, from (7.2), we deduce that $\Phi(p(u))$ is absolutely continuous in $[0, \tau]$ for all $\tau > 0$. Let $t \in [0, \infty)$ be such that u, $p(u)$, $\Phi(p(u))$ are differentiable at t and $(u(t), -u_t(t)) \in \mathscr{A}_\varphi$. Set $w = p(u(t+\varepsilon))$, $w = p(u(t-\varepsilon))$ in the above expression to obtain

$$\int_\Omega (p(u(t)) - p(u(t \pm \varepsilon)))\, u_t(t) \leqslant \Phi(p(u(t \pm \varepsilon))) - \Phi(p(u(t))).$$

Letting $\varepsilon \to 0+$ we have

$$\frac{d}{dt} \Phi(p(u(t))) + \int_\Omega p'(u(t))\, u_t(t)^2 = 0.$$

In particular, since p' is either 0 or 1, we have

$$\frac{d}{dt} \Phi(p(u(t))) + \int_\Omega |p(u)_t(t)|^2 \leqslant 0.$$

In particular $\Phi(p(u(t))$ is a decreasing function of t. If $u_0 \in BV(\Omega) \cap L^2(\Omega)$, integrating from 0 to t we get

$$\Phi(p(u(t))) + \int_0^t \int_\Omega |p(u)_t|^2 \leqslant \Phi(u_0).$$

Observe that by the estimate in (i), if $u_0 \in L^2(\Omega)$, then $u(s) \in BV(\Omega) \cap L^2(\Omega)$ for almost all $s > 0$ and we have

$$\Phi(p(u)(t)) + \int_s^t \int_\Omega |p(u)_t|^2 \leqslant \Phi(p(u)(s)),$$

for almost all $0 < s < t$. Now, let $u_0 \in L^1(\Omega)$ and $u_{0n} \in L^2(\Omega)$ be such that $u_{0n} \to u_0$ in $L^1(\Omega)$. Then, if $u_n(t, x)$ denotes the solution corresponding to initial datum u_{0n} we have

$$\Phi(p(u_n)(t)) + \int_s^t \int_\Omega |p(u_n)_t|^2 \leqslant \Phi(p(u_n)(s)), \tag{7.3}$$

for almost all $0 < s < t$ and all n. Now, observe that by the estimate in (i),

$$\int_0^t \Phi(p(u_n)(\tau)) \leqslant C$$

for some constant $C > 0$. Let $\delta > 0$. Then

$$\int_0^t \Phi(p(u_n)(\tau)) \geqslant \int_0^\delta \Phi(p(u_n)(\tau)) \geqslant \Phi(p(u_n)(\delta))\, \delta.$$

Consequently, $\Phi(p(u_n)(s))$ is a bounded sequence for almost all $s > 0$. Thus, for a.e. $s > 0$, the left hand side of (7.3) is bounded. Hence, we may assume that $p(u_n(t)) \to p(u(t))$ in $L^1(\Omega)$ for a.e. $t > 0$. Now, we may pass to the limit in (7.3) and use the lower semicontinuity of the left hand side to obtain that

$$\Phi(p(u)(t)) + \int_s^t \int_\Omega |p(u)_t|^2 \leqslant C, \tag{7.4}$$

where C depends on s, $\|u_0\|_{L^1}$, $\|\varphi\|_{L^1}$, p. ∎

THEOREM 5. *Let $u_0 \in L^1(\Omega)$, $\varphi \in L^1(\partial\Omega)$. Suppose that $u_0 + M \geqslant 0$, $\varphi + M \geqslant 0$ (or $u_0 - M \leqslant 0$, $\varphi - M \leqslant 0$) for some $M \geqslant 0$. Let $(S(t))_{t \geqslant 0}$ be the semigroup generated by $\mathscr{A}_\varphi$. Then, if $u(t) = S(t)\, u_0$, $u_t \in L^1_{loc}(0, T, L^1(\Omega))$.*

Proof. It is easy to check via the resolvents that the semigroup solution corresponding to the data $u_0 \pm M$, $\varphi \pm M$ coincides with the semigroup solution corresponding to the data u_0, φ plus the constant $\pm M$; i.e., $S(t)(u_0 \pm M, \varphi \pm M) = S(t)(u_0, \varphi) \pm M$. Thus, without loss of generality we may assume that $M = 0$. Let us prove the theorem in case u_0, $\varphi \geqslant 0$, the other case being analogous. We know, by the homogeneity estimate, Proposition 4, that u_t is a Randon measure in $(s, t) \times \Omega$, for all $0 < s < t$. Thus, its mass is bounded; i.e.,

$$\int_s^t \int_\Omega |u_t| < \infty.$$

Now, taking $p = T_{a,b}$, the estimate in (ii) of the previous proposition says that u_t is a function in $L^2(Q_{a,b})$, for all $a < b$, where $Q_{a,b} = \{(t, x) \in Q : a < u(t, x) < b\}$. Thus, this, with the last integral bound, proves that $u_t \in L^1_{loc}(0, \tau, L^1(\Omega))$. ∎

Remark 3. Under the assumption of the above theorem, since $u_t \in L^1_{loc}(0, T, L^1(\Omega))$, working as in [2] we can prove that u is a strong solution. Consequently, existence and uniqueness can be obtained in an easier way than in the general case using the same technique as in [2].

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