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THE MINIMIZING TOTAL VARIATION FLOW WITH MEASURE INITIAL CONDITIONS

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In this paper we obtain existence and uniqueness of solutions for the Cauchy problem for the minimizing total variation flow when the initial condition is a Radon measure in $\mathbb{R}^N$. We study limit solutions obtained by weakly approximating the initial measure μ by functions in $L^1(\mathbb{R}^N)$. We are able to characterize limit solutions when the initial condition $\mu = h + \mu_s$, where $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $\mu_s = \alpha\mathcal{H}^k \llcorner S$, $\alpha \geq 0$, k is an integer and S is a k-dimensional manifold with bounded curvatures. In case $k < N-1$ we prove that the singular part of the solution does not move, it remains equal to μ_s for all $t \geq 0$. In particular, $u(t) = \delta_0$ when $u(0) = \delta_0$. In case $k = N-1$ we prove that the singular part of the limit solution is $(1 - \frac{2}{\alpha}t)^+\mu_s$ and we also characterize its absolutely continuous part. This explicit behaviour permits to characterize limit solutions. We also give an entropy condition characterization of the solution which is more satisfactory when $k < N-1$. Finally, we describe some distributional solutions which do not have the behaviour characteristic of limit solutions.

Keywords: Total variation; nonlinear parabolic equations; strong solutions, Radon measures.

1. Introduction

The purpose of this paper is to prove existence and uniqueness of the minimizing total variation flow in $\mathbb{R}^N$

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) \text{ in } Q_T =]0, T[\times\mathbb{R}^N\,, \tag{1.1}$$

coupled with the initial condition

$$u(0) = \mu , \quad \mu \text{ being a Radon measure in } \mathbb{R}^N . \tag{1.2}$$

This PDE appears (in a bounded domain D) in the steepest descent method for minimizing the total variation, a method introduced by L. Rudin, S. Osher and E. Fatemi [25] in the context of image denoising and reconstruction. When dealing with the denoising problem one minimizes the total variation functional

$$\int_D |Du| \tag{1.3}$$

with the constraint $z = u + n$ where n represents the noise. Then one minimizes (1.3) under the above constraint [25]. Numerical experiments show that the model is adapted to restore the discontinuities of the image [12, 16, 19, 25]. Indeed, the underlying functional model is the space of BV functions, i.e., functions of bounded variation, which admit a discontinuity set which is countably rectifiable ([2, 17, 26]).

To solve (1.3) (with the specified constraint) one formally computes the Euler-Lagrange equation and solves it with Neumann boundary conditions, which amounts to a reflection of the image across the boundary of D. Many numerical methods have been proposed to solve this equation in practice, see for instance [12, 16, 19, 25] (see also [24] for an interesting analysis of the features of most numerical methods explaining, in particular, the staircasing effect). This leads to an iterative process which, in some sense, can be understood as a gradient descent. This gradient descent flow (1.1) was initially studied in a bounded domain under Neumann boundary conditions in [3] where the authors proved existence and uniqueness of solutions with initial data in L^1, and constructed some particular explicit solutions of the equation. The corresponding results for the Dirichlet problem where proved in [4]. This study was completed in [5] where the authors proved that the solution reaches its asymptotic state in finite time and studied its extinction profile, given in terms of the eigenvalue problem

$$-\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = v . \tag{1.4}$$

In [8] the authors constructed many explicit solutions of the eigenvalue problem (1.4) and, as a consequence, they obtained explicit solutions of the evolution problem (1.1) and of the denoising problem in image processing [8]. All together, this gives a picture of how the flow (1.1) behaves to minimize the total variation of a function in L^1 under Neumann or Dirichlet boundary conditions.

In this paper we continue the study of the flow (1.1) when the initial conditions are Radon measures in $\mathbb{R}^N$. In other words, we study the well-posedness of the 1-Laplacian diffusion equation when the initial data are measures. Recall that, as it is mentioned in [13], the existence for the p-Laplacian heat equation can be proved as in [13], while uniqueness is mentioned as an open question.

Let us explain the plan of the paper and its main results. In Sec. 2 we recall some definitions concerning measures, functions of bounded variation, a generalized

Green's formula and the concept of strong solution of the Dirichlet problem for equation (1.1). Section 3 is devoted to the construction of limit solutions for equation (1.1) when the initial condition is a bounded Radon measure μ. Indeed, since (1.1) is well posed in $L^1(\mathbb{R}^N)$ we can approximate μ by functions in $u_n(0) \in L^1(\mathbb{R}^N)$, compute the corresponding solutions $u_n(t)$ and pass to the limit to obtain a function $u(t)$ taking values in the space of Radon measures. For later purpose let us denote $u(t) = u(t)_{ac} + u(t)_s$, where $u(t)_{ac}$ and $u(t)_s$ denote the absolutely continuous and singular parts of $u(t)$ with respect to Lebesgue measure in $\mathbb{R}^N$. In this paper we shall not consider general measures, instead we shall restrict ourselves to the case of measures

$$\mu = h + \alpha \mathcal{H}^k \llcorner S \tag{1.5}$$

where $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\alpha \geq 0$, and $\mathcal{H}^k$ being the k-dimensional Hausdorff measure in $\mathbb{R}^N$ and S is a k-manifold in $\mathbb{R}^N$ of class $W^{3,\infty}$. We also note that we may use many different approximations $u_n(0)$ to the measure μ. In the following sections we shall first stress the role of one of them, the one in which we approximate the singular part of μ, i.e., the measure $\alpha \mathcal{H}^k \llcorner S$ by constant functions. Indeed, using essentially the ideas of Minkowski's content we know that

$$\alpha \frac{\mathcal{H}^k(S)}{|I_n(S)|} \chi_{I_n(S)} \rightharpoonup \alpha \mathcal{H}^k \llcorner S \quad \text{weakly}^* \text{ as measures as } n \to \infty\,, \tag{1.6}$$

where $I_n(S) = \{x \in \mathbb{R}^N : d(x, S) \leq \frac{1}{n}\}$. This result is essentially contained in [2] and we recall the proof in the Appendix. In Sec. 4 we compute some explicit limit solutions for initial measures which have some radial symmetry, in particular for sums of Dirac measures concentrated at points or circles. These explicit solutions exhibit some curious behaviour, namely, Dirac measures concentrated at a finite number of points do not move, while the measure $\alpha \mathcal{H}^{N-1} \llcorner \partial B(0, R)$ has a more complex evolution described in (4.1). In particular, we note that there is no regularizing effect for (1.1) when the initial condition is a measure. On the other hand, this makes explicit that solutions have a very different behaviour according to the Hausdorff dimension of the support of the measure. If this dimension is $k < N - 1$ it seems that the singular part of the measure does not move, while it moves when $k = N - 1$. Our purpose will be to explore this behaviour. Indeed, we shall be able to prove it for the particular case of measures of the form (1.5). In Sec. 5 we characterize the behaviour of limit solutions. Let us consider first the case $k = N - 1$. Let C_2 denote the unbounded connected component of $\mathbb{R}^N \setminus S$ and C_1 its complement in $\mathbb{R}^N \setminus S$. When $k = N - 1$, in the time interval $[0, \frac{\alpha}{2}]$ we have $u(t) = u(t)_{ac} + u(t)_s$ with $u(t)_s = (1 - \frac{2}{\alpha} t)\mu_s$ and $u(t)_{ac}|_{C_i}$, $i = 1, 2$, is the strong solution of the Dirichlet problem (5.26). Note that $u(\frac{\alpha}{2})_s = 0$. In the time interval $[\frac{\alpha}{2}, \infty)$, $u(t) = u(t)_{ac}$ is the strong solution of (1.1) with initial condition $u(\frac{\alpha}{2})_{ac}$. In case $k < N - 1$, we prove that $u(t)_s = \mu_s$ for all $t \geq 0$, and $u(t)_{ac}$ is the strong solution of (1.1) with initial condition μ_{ac}. Furthermore, we observe that $u(t)$ satisfies some entropy condition which characterizes in some way the solution of (1.1). In Sec. 6 we study limit solutions when the initial measure μ is approximated by functions

$$u_{n0} = \mu_{ac} + \rho_n * \mu_s \,,$$

where $\rho_n(x) = n^N \rho(nx)$ and ρ is a radial, smooth, positive convolution kernel with compact support and $\mu_s = \alpha\mathcal{H}^k \llcorner S$ with $k < N-1$. We prove that the limit solution obtained in this case coincides with the limit solution obtained for the approximation (1.6) and studied in Sec. 5. At this moment we do not know if the analogous result holds when $k = N-1$. Related to the behaviour of limit solutions when $k = N-1$, we remark in Sec. 7 that, if μ is a measure in $BV(\mathbb{R}^N)^*$, then $u(t)$ is also a measure in $BV(\mathbb{R}^N)^*$. Finally, in Sec. 8 we construct some distributional solutions of (1.1) which do not coincide with the limit solutions constructed in previous sections. Finally, for the sake of completeness, the Appendix contains the proof of (1.6). We s

2. Preliminaries

2.1. *Measures, functions of a measure*

We denote by $C_c(\mathbb{R}^N)$ the space of all real continuous functions in $\mathbb{R}^N$ with compact support and by $C_0(\mathbb{R}^N)$ its completion with respect to the sup-norm. If we denote by $\mathcal{M}(\mathbb{R}^N)$ (resp. $\mathcal{M}_b(\mathbb{R}^N)$) the space of the scalar Radon (resp. finite scalar Radon) measures on $\mathbb{R}^N$, by Riesz Theorem, $\mathcal{M}(\mathbb{R}^N)$ (resp. $\mathcal{M}_b(\mathbb{R}^N)$) can be identified with the dual of $C_c(\mathbb{R}^N)$ endowed with its natural l.c. topology (resp. with the dual of the Banach space $C_0(\mathbb{R}^N)$).

Let μ, $\mu_n \in \mathcal{M}(\mathbb{R}^N)$, we say that (μ_n) *locally weakly* converges* to μ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f d\mu_n = \int_{\mathbb{R}^N} f d\mu \quad \forall\, f \in C_c(\mathbb{R}^N)\,;$$

if $\mu_n \in \mathcal{M}_b(\mathbb{R}^N)$, we say that (μ_n) *weakly* converges* to μ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f d\mu_n = \int_{\mathbb{R}^N} f d\mu \quad \forall\, f \in C_0(\mathbb{R}^N)\,.$$

We will denote this type of convergence by

$$\mu_n \rightharpoonup \mu \quad \text{weakly}^* \text{ as measures}\,.$$

Given a measure $\mu \in \mathcal{M}(\mathbb{R}^N)$ we denote by μ_{ac} and μ_s its absolutely continuous part and its singular part with respect to the Lebesgue measure $\mathcal{L}^N$, respectively. We denote by $\mu_{ac}(x)$ the density of the measure μ_{ac} with respect to $\mathcal{L}^N$ and by $\frac{d\mu_s}{d|\mu_s|}(x)$ the density of μ_s with respect to $|\mu_s|$.

We denote by $C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))$ the space of all weakly* continuous functions from $[0,T]$ to $\mathcal{M}_b(\mathbb{R}^N)$. In this space we consider the weakly* uniform convergence topology, that is, the topology defined by the family of seminorms

$$\|u\|_\varphi := \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^N} \varphi du(t) \right| ,$$

for each $u \in C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))$, $\varphi \in C_0(\mathbb{R}^N)$.

Recall the concept of function of a measure [14]. Given a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ which has at most a linear growth at infinity, i.e.,

$$|f(\xi)| \leq M(1 + \|\xi\|) , \quad \forall\, \xi \in \mathbb{R}^k ,$$

and such that f possesses an asymptotic function, i.e., such that the following limit exists

$$f_\infty(\xi) := \lim_{t\to\infty} \frac{f(t\xi)}{t} , \quad \forall\, \xi \in \mathbb{R}^k ,$$

for every $\mu \in \mathcal{M}_b(\mathbb{R}^N, \mathbb{R}^k)$, we may define the measure $f(\mu)$ by writing

$$\int_B f(\mu) := \int_B f(\mu_{ac}(x))dx + \int_B f_\infty \left(\frac{d\mu_s}{d|\mu_s|}(x) \right) d|\mu_s|(x) ,$$

for every Borel set $B \subset \mathbb{R}^N$.

2.2. *BV functions, measures in BV^**

The natural energy space to study problem (1.1) is the space of functions of bounded variation. Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du in Ω is defined by the formula

$$|Du|(\Omega) = \sup \left\{ \int_\Omega u \ \mathrm{div}(\phi) : \phi \in C_0^\infty(\Omega, \mathbb{R}^N), \|\phi\| \leq 1 \right\} .$$

The space $BV(\Omega)$ is endowed with norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega) .$$

If $\Omega = \mathbb{R}^N$, we consider $BV(\mathbb{R}^N)$ endowed with norm

$$\|u\|_{BV(\mathbb{R}^N)} = |Du|(\mathbb{R}^N) .$$

Recall that an $\mathcal{L}^N$-measurable subset E of $\mathbb{R}^N$ has *finite perimeter* if $\chi_E \in BV(\mathbb{R}^N)$. The perimeter of E is defined by $\mathrm{Per}(E) = |D\chi_E|(\mathbb{R}^N)$.

If $E \subseteq \mathbb{R}^N$ is $\mathcal{L}^N$-measurable and $x \in \mathbb{R}^N$, the upper and lower densities of x in E are defined by

$$\bar{D}(E,x) := \limsup_{\rho\to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} ,$$

$$\underline{D}(E,x) := \liminf_{\rho\to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} .$$

In case that the upper and lower limits are equal, we denote their common value by $D(E,x)$ and we call it the density of E at x. We denote by $E^i := \{x \in \mathbb{R}^N : D(E,x) = 1\}$ the measure theoretic interior of E, by $E^e := \{x \in \mathbb{R}^N : D(E,x) = 0\}$ the measure theoretic exterior of E and by $\partial_M E := \mathbb{R}^N \setminus (E^i \cup E^e)$ the measure theoretic boundary of E. We also use the notation $\bar{E}^M := E^i \cup \partial_M E$. Recall that, by definition,

$$u^+(x) = \inf\{t : D([u > t], x) = 0\},$$
$$u^-(x) = \sup\{t : D([u < t], x) = 0\}.$$

If $t < u^+(x)$ then $D([u > t], x) > 0$ and it follows that $x \in \overline{[u > t]}^M$. In that case, $\chi_{\overline{[u>t]}^M}(x) = 1$. Since $\chi_{\overline{[u>t]}^M}(x) = (\chi_{[u>t]})^+(x)$ we have

$$u^+(x) = \int_0^{u^+(x)} dt = \int_0^{u^+(x)} \chi_{\overline{[u>t]}^M}(x)dt$$
$$= \int_0^{\infty} \chi_{\overline{[u>t]}^M}(x)dt = \int_0^{\infty} (\chi_{[u>t]})^+(x)dt\,.$$

Now, since

$$u^-(x) = \inf\{t : x \in [u > t]^i\},$$

observing that $\chi_{[u>t]^i}(x) = (\chi_{[u>t]})^-(x)$, we have

$$u^-(x) = \int_0^{u^-(x)} dt = \int_0^{u^-(x)} \chi_{[u>t]^i}(x)dt$$
$$= \int_0^{\infty} \chi_{[u>t]^i}(x)dt = \int_0^{\infty} (\chi_{[u>t]})^-(x)dt\,.$$

The above equalities imply that

$$u^*(x) := \frac{u^+(x) + u^-(x)}{2} = \int_0^{\infty} \frac{(\chi_{[u>t]})^+(x) + (\chi_{[u>t]})^-(x)}{2} dt$$
$$= \int_0^{\infty} (\chi_{[u>t]})^*(x)dt\,. \tag{2.1}$$

The symbol $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^N$, $k \in \{0, 1, \ldots, N\}$, and ω_k denotes the Lebesgue measure of the unit ball of $\mathbb{R}^k$. For a $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$, we will use frecuently the notation $|A| := \mathcal{L}^N(A)$.

The following characterization of the positive Radon measures belonging to $BV(\mathbb{R}^N)^*$ is given by N. G. Meyer and W. P. Ziemer in [23] (see also [22, 26]).

Theorem 2.1. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. The following statements are equivalent.*

(i) $\mathcal{H}^{N-1}(A) = 0$ *implies that* $\mu(A) = 0$ *for all Borel sets* $A \subset \mathbb{R}^N$ *and there is a constant* M_1 *such that*

$$\left| \int_{\mathbb{R}^N} u^* d\mu \right| \le M_1 |Du|(\mathbb{R}^N) \quad \textit{forall } u \in BV(\mathbb{R}^N)\,.$$

(ii) *There is a constant* M_2 *such that* $\mu(A) \leq M_2 \mathrm{Per}(A)$ *for all Borel sets* $A \subset \mathbb{R}^N$ *with* $\mathcal{L}^N(A) < \infty$.

(iii) *There is a constant* M_3 *such that* $\mu(B_r(x)) \leq M_3 r^{N-1}$ *whenever* $x \in \mathbb{R}^N$ *and* $r \in \mathbb{R}$.

A positive Radon measure μ satisfying one of the conditions of the above theorem can be identified with an element of $BV(\mathbb{R}^N)^*$. Y. Meyer in [22] called these measures Guy David measures. Let us note that if μ is a Guy David measure, we have

$$\langle \mu, u \rangle_{BV^*, BV} = \int_{\mathbb{R}^N} u^* d\mu \,, \quad \forall\, u \in BV(\mathbb{R}^N)\,.$$

2.3. *A generalized Green's formula*

Let Ω be an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary. Following [6], for $1 \leq p < \infty$ let

$$X_p(\Omega) = \{z \in L^\infty(\Omega, \mathbb{R}^N) : \mathrm{div}(z) \in L^p(\Omega)\}\,. \tag{2.2}$$

If $z \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z, Dw), \varphi \rangle = -\int_\Omega w\varphi \, \mathrm{div}(z) dx - \int_\Omega wz \cdot \nabla\varphi dx\,. \tag{2.3}$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w dx \quad \forall\, w \in W^{1,1}(\Omega) \cap L^{p'}(\Omega)\,, \tag{2.4}$$

and

$$\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw|\,, \tag{2.5}$$

for any Borel set $B \subseteq \Omega$. Moreover, when $z \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, we have the following integration by parts formula

$$\int_\Omega w \, \mathrm{div}(z) dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w d\mathcal{H}^{N-1}\,, \tag{2.6}$$

where $[z, \nu]$ is the weak trace on $\partial\Omega$ of the normal component of z (see [6])

2.4. *Strong solutions of the Dirichlet problem*

Let Ω be an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary. We need to recall the concept of strong solution introduced in [4] for the Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{div}\left(\dfrac{Du}{|Du|}\right), & \text{in } Q = (0,\infty) \times \Omega\,, \\ u(t,x) = \varphi(x)\,, & \text{on } S = (0,\infty) \times \partial\Omega\,, \\ u(0,x) = u_0(x)\,, & \text{in } x \in \Omega\,, \end{cases} \tag{2.7}$$

where $u_0 \in L^2(\Omega)$ and $\varphi \in L^\infty(\partial\Omega)$.

By $L^1_w(0,T,BV(\Omega))$ we denote the space of weakly measurable functions $w : [0,T] \to BV(\Omega)$ (i.e., $t \in [0,T] \to \langle w(t), \phi\rangle$ is measurable for every $\phi \in BV(\Omega)^*$) such that $\int_0^T \|w(t)\| < \infty$. Observe that, since $BV(\Omega)$ has a separable predual [2], it follows easily that the map $t \in [0,T] \to \|w(t)\|$ is measurable.

We shall denote by

$$\operatorname{sign}_0(r) := \begin{cases} 1\,, & \text{if } r > 0\,, \\ 0\,, & \text{if } r = 0\,, \\ -1\,, & \text{if } r < 0\,, \end{cases}$$

and by

$$\operatorname{sign}(r) := \begin{cases} 1\,, & \text{if } r > 0\,, \\ a \in [-1,1]\,, & \text{if } r = 0 \\ -1\,, & \text{if } r < 0\,. \end{cases}$$

Let $T_k(r) = [k - (k - |r|)^+]\operatorname{sign}_0(r)$, $k \geq 0$, $r \in \mathbb{R}$. We consider the set $\mathcal{T} = \{T_k, T_k^+, T_k^- : k > 0\}$. We need to consider a more general set of truncature functions, concretely, the set $\mathcal{P}$ of all nondecreasing continuous fuctions $p : \mathbb{R} \to \mathbb{R}$, such that $p'(r)$ exists with the possible exception of a finite set of values of $r \in \mathbb{R}$ and $\operatorname{supp}(p')$ is compact. Obviously, $\mathcal{T} \subset \mathcal{P}$.

Definition 2.1. Let $u_0 \in L^2(\Omega)$, $\varphi \in L^1(\partial\Omega)$. A measurable function $u : (0,T) \times \Omega \to \mathbb{R}$ is a *strong solution* of problem (2.7) in $(0,T) \times \Omega$ if $u \in C([0,T], L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$, $u \in L^1_w(0,T; BV(\Omega))$ and there exists $z \in L^\infty((0,T) \times \Omega)$ with $\|z\|_\infty \leq 1$, $u_t = \operatorname{div}(z)$ in $\mathcal{D}'((0,T) \times \Omega)$ such that

$$\int_\Omega (u(t) - w)u_t(t) \leq \int_\Omega (z(t), Dw) - \int_\Omega |Du(t)| + \int_{\partial\Omega} |w - \varphi| - \int_{\partial\Omega} |u(t) - \varphi| \tag{2.8}$$

for every $w \in BV(\Omega) \cap L^2(\Omega)$ and a.e. on $[0,T]$.

The following result was proved in [4].

Theorem 2.2. *Let $u_0 \in L^2(\Omega)$, $\varphi \in L^1(\Omega)$. Then for every $T > 0$ there exists a unique strong solution of* (2.7) *in $(0,T) \times \Omega$. Moreover, the solution $u(t)$ of* (2.7) *is also characterized as follows: $u \in C([0,T], L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega))$, $u \in L^1_w(0,T; BV(\Omega))$ and there exists $z(t) \in X_2(\Omega)$, such that $\|z(t)\|_\infty \leq 1$, $u'(t) = \operatorname{div}(z(t))$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0,+\infty[$ and*

$$\int_\Omega (z(t), Du(t)) = \int_\Omega |Du(t)|\,, \tag{2.9}$$

$$[z(t), \nu] \in \operatorname{sign}(\varphi - u(t)) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega\,. \tag{2.10}$$

Finally, we have the following comparison principle: if $u(t)$, $\hat{u}(t)$ *are solutions of* (2.7) *corresponding to initial data* $u_0, \hat{u}_0 \in L^2(\Omega) \cap L^p(\Omega)$, *respectively,* $p \geq 1$, *and the same boundary data* φ, *then*

$$\|(u(t) - \hat{u}(t))^+\|_p \leq \|(u_0 - \hat{u}_0)^+\|_p \quad \text{and} \quad \|u(t) - \hat{u}(t)\|_p \leq \|u_0 - \hat{u}_0\|_p \,, \tag{2.11}$$

for all $t \geq 0$.

Let us make some comments on the proof which shall be useful in the sequel. Let $\Psi_\varphi : L^2(\Omega) \to]-\infty, +\infty]$ be defined by

$$\Psi_\varphi(u) = \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} |u - \varphi| & \text{if } u \in BV(\Omega) \cap L^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega) \cap L^2(\Omega) \,. \end{cases} \tag{2.12}$$

Since the functional Ψ_φ is convex and lower semicontinuous in $L^2(\Omega)$, we have that $\partial\Psi_\varphi$ is a maximal monotone operator in $L^2(\Omega)$, and consequently (see [11]), if $\{T(t)\}_{t\geq 0}$ is the semigroup in $L^2(\Omega)$ generated by $\partial\Psi_\varphi$, for every $u_0 \in L^2(\Omega)$, $u(t) := T(t)u_0$ is strong solution of the problem

$$\begin{cases} \dfrac{du}{dt} + \partial\Psi_\varphi u(t) \ni 0 \,, \\ u(0) = u_0 \,. \end{cases} \tag{2.13}$$

Recall that the operator $\partial\Psi_\varphi$ is defined by

$$(u, v) \in \partial\Psi_\varphi \text{ if and only if } u, v \in L^2(\Omega) \,, \quad \text{and}$$

$$\Psi_\varphi(w) \geq \Psi_\varphi(u) + \int_\Omega (w - u)v \,, \quad \forall\, w \in L^2(\Omega) \,.$$

Theorem 2.2 follows from a "distributional" characterization of $\partial\Psi_\varphi$. For that we define the operator $\mathcal{B}_\varphi$ in $L^2(\Omega)$ associated with problem (2.7) by

$$(u, v) \in \mathcal{B}_\varphi \text{ if and only if } u, v \in L^2(\Omega) \text{ and}$$

$$\text{there exists } z \in X_2(\Omega) \text{ with } \|z\|_\infty \leq 1, v = -\mathrm{div}(z) \text{ in } \mathcal{D}'(\Omega) \text{ such that}$$

$$\int_\Omega (w - u)v \leq \int_\Omega (z, Dw) - \int_\Omega |Du| + \int_{\partial\Omega} |w - \varphi| - \int_{\partial\Omega} |u - \varphi| \,,$$

$$\text{for all } w \in BV(\Omega) \cap L^2(\Omega) \,.$$

The following result was proved in [4]. Theorem 2.2 is a consequence of it and the fact that $u(t) = T(t)u_0$ is strong solution of (2.13).

Proposition 2.3. *The operator* $\mathcal{B}_\varphi$ *is maximal monotone with dense domain in* $L^2(\Omega)$. *Moreover* $\mathcal{B}_\varphi = \partial\Psi_\varphi$.

We also note that $\mathcal{B}_\varphi$ is completely accretive, i.e., the semigroup solution is in $L^p(\Omega)$ if $u(0) \in L^p(\Omega)$ and we have the contraction estimates described in Theorem 2.2.

Let us finally recall the following estimate on u_t which holds in general for strong solutions of equations like [11, (2.13)].

Proposition 2.4. *Let $u_0 \in L^2(\Omega)$, $\varphi \in L^1(\Omega)$. Let $u(t)$ be the strong solution of (2.7). Then $u(t)$ is a Lipschitz function on $[\delta, \infty)$ for any $\delta > 0$. More precisely, given $\delta > 0$, there is a constant C depending on u_0, φ and δ such that*

$$\int_\Omega |u_t(t,x)|^2 dx \le C \quad \text{a.e. } t \in [\delta, \infty)\,. \tag{2.14}$$

Remark 2.5. Theorem 2.2 and Proposition 2.4 also hold when Ω is an exterior domain, i.e., when $\Omega = \mathbb{R}^N \setminus \bar{U}$, U being an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary. The proof of Theorem 2.2 for exterior domains follows as a consequence of Proposition 2.3 for the same domains. Let us make some remarks about the proof. The proof of the monotonicity and the closedness of $\mathcal{B}_\varphi$ follows as in [4] or [8]. Now, if $\lambda > 0$, for any $f \in L^2(\Omega)$ there is a solution u of

$$u + \lambda \mathcal{B}_\varphi u = f\,. \tag{2.15}$$

Indeed, if $f \in L^2(\Omega) \cap L^\infty(\Omega)$ has compact support, $\mathrm{supp}(f) \subset\subset B(0,R)$, the solution of

$$\begin{cases} u - \mathrm{div}\left(\dfrac{Du}{|Du|}\right) = f\,, & \text{in } \Omega \cap B(0,R)\,, \\ u = \varphi\,, & \text{on } \partial\Omega\,, \\ u = 0\,, & \text{on } \partial B(0,R)\,, \end{cases} \tag{2.16}$$

is also a solution of (2.15). The closedness of $\mathcal{B}_\varphi$ implies that (2.15) can be solved for any $f \in L^2(\Omega)$. It follows that the range of $I + \lambda\mathcal{B}_\varphi$ is $L^2(\Omega)$, and therefore $\mathcal{B}_\varphi$ is maximal monotone. The density of the domain of $\mathcal{B}_\varphi$ can be proved as in [3]. The proof of $\mathcal{B} = \partial\Psi_\varphi$ is similar to the proof of Lemma 1 in [4]. The estimate of Proposition 2.4 holds for any semigroup evolution generated by the subdifferential of a convex, lower semicontinuous and proper functional [11].

2.5. *Strong solutions of the Cauchy problem in $L^2(\mathbb{R}^N)$*

Definition 2.6. A function $u \in C([0,T]; L^2(\mathbb{R}^N))$ is called a *strong solution* of (1.1) if

$$u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{R}^N)) \cap L^1_w(]0,T[; BV(\mathbb{R}^N))\,,$$

and there exists $z \in L^\infty(]0,T[\times\mathbb{R}^N; \mathbb{R}^N)$ with $\|z\|_\infty \le 1$ such that

$$u_t = \mathrm{div}(z) \text{ in } \mathcal{D}'(]0,T[\times\mathbb{R}^N)\,,$$

and

$$\begin{gathered} \int_{\mathbb{R}^N} (u(t) - w) u_t(t) = \int_{\mathbb{R}^N} (z(t), Dw) - \int_{\mathbb{R}^N} |Du(t)|\,, \\ \forall\ w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \text{a.e. } t \in [0,T]\,. \end{gathered} \tag{2.17}$$

We collect some results in the following theorem in [8].

Theorem 2.7. *Let $u_0 \in L^2(\mathbb{R}^N)$. Then there exists a unique strong solution u of* (1.1) *with $u(0) = u_0$ in $[0,T] \times \mathbb{R}^N$ for every $T > 0$. The solution $u(t)$ of* (1.1) *is also characterized as follows: $u \in C([0,T], L^2(\mathbb{R}^N)) \cap W^{1,2}_{\rm loc}(0,T; L^2(\mathbb{R}^N))$, $u \in L^1_w(0,T; BV(\mathbb{R}^N))$ and there exists $z(t) \in X_2(\mathbb{R}^N)$, such that $\|z(t)\|_\infty \le 1$, $u'(t) = \operatorname{div}(z(t))$ in $\mathcal{D}'(\mathbb{R}^N)$ a.e. $t \in [0,T]$ and*

$$\int_\Omega (z(t), Du(t)) = \int_{\mathbb{R}^N} |Du(t)| \quad \text{a.e. in } (0,T)\,. \tag{2.18}$$

Moreover, if $u_0 \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, then also $u(t) \in L^p(\mathbb{R}^N)$ for all $t > 0$. Finally, the contractivity estimates of Theorem 2.2 *also hold in this case.*

More general results concerning existence and uniqueness of entropy solutions of (1.1) for general data in $L^1_{\rm loc}(\mathbb{R}^N)$ were proved in [8].

3. Limit Solutions for Measure Initial Data with Singular Part Supported in Compact k-Manifold of $\mathbb{R}^N$

In this section we consider measure initial data whose singular part is supported in a set S which is an orientable compact k-manifold (k being the dimension) in $\mathbb{R}^N$ without boundary satisfying (A.4). From now on, to simplify, by a compact k-manifold we mean an orientable compact k-manifold in $\mathbb{R}^N$ without boundary. It is known that (A.4) holds if the compact k-manifold S is of class $W^{3,\infty}$.

Given $S \subset \mathbb{R}^N$, we denote by $I_n(S) := \{x \in \mathbb{R}^N : \operatorname{dist}(x,S) \le \frac{1}{n}\}$. We approximate the initial datum $\mu = \mu_{ac} + \alpha\mathcal{H}^k \llcorner S$ in the following way: for every $n \in \mathbb{N}$, let $u_{0,n}(\mu)$ be the L^1-function defined by

$$u_{0,n}(\mu) := \mu_{ac} + \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\chi_{I_n(S)}\,. \tag{3.1}$$

Lemma 3.1. *Let $\mu = \alpha\mathcal{H}^k \llcorner S$ where S is a compact k-manifold satisfying* (A.4), *and $\alpha \in \mathbb{R}$. Then, if $u_{0,n}(\mu) = \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\chi_{I_n(S)}$, we have that $u_{0,n} \rightharpoonup \mu$ weakly* as measures.*

Proof. Working as in the proof of [2, Theorem 2.106] (see appendix), it is possible to prove that

$$\lim_{n\to\infty} \frac{\int_{I_n(S)} \varphi(x)dx}{\omega_{N-k}\left(\frac{1}{n}\right)^{N-k}} = \int_S \varphi(x) d\mathcal{H}^k(x)\,, \quad \forall\, \varphi \in C_c(\mathbb{R}^N)\,. \tag{3.2}$$

Then, applying [2, Theorem 2.104] and (3.2), for every $\varphi \in C_c(\mathbb{R}^N)$ we get

$$\begin{aligned}\lim_{n\to\infty}\langle u_{0,n},\varphi\rangle &= \lim_{n\to\infty}\frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\int_{I_n(S)}\varphi(x)dx\\ &= \lim_{n\to\infty}\alpha\mathcal{H}^k(S)\left(\frac{w_{N-k}\left(\frac{1}{n}\right)^{N-k}}{|I_n(S)|}\right)\frac{\int_{I_n(S)}\varphi(x)dx}{\omega_{N-k}(\frac{1}{n})^{N-k}}\\ &= \alpha\int_S\varphi(x)d\mathcal{H}^k(x) = \langle\mu,\varphi\rangle\,.\end{aligned}$$

□

Given $\mu\in\mathcal{M}_b(\mathbb{R}^N)$ with $\mu_s=\alpha\mathcal{H}^k\llcorner S$, where S is a compact k-manifold satisfying (A.4), and $\alpha\in\mathbb{R}$, by the above lemma we have

$$u_{0,n}(\mu)\rightharpoonup\mu\quad\text{weakly}^*\text{ as measures}\,.$$

Now, since $u_{0,n}(\mu)\in L^1(\mathbb{R}^N)$, we know [8] there exists a unique strong solution u_n of the problem (1.1) with initial datum $u_{0,n}(\mu)$, that is

$$u_n\in C([0,T],L^1(\mathbb{R}^N))\cap W^{1,1}_{loc}(0,T;L^1(\mathbb{R}^N))\,,\quad p(u_n)\in L^1_w(0,T;BV(\mathbb{R}^N))\,,$$

for all $p\in\mathcal{P}$ and there exists $z_n\in L^\infty(]0,T[\times\mathbb{R}^N;\mathbb{R}^N)$ with $\|z_n\|_\infty\le 1$ such that

$$(u_n)_t=\operatorname{div}(z_n)\text{ in }\mathcal{D}'(]0,T[\times\mathbb{R}^N)\,,$$

and

$$\int_{\mathbb{R}^N}(p(u_n(t))-w)u_n'(t)\le\int_{\mathbb{R}^N}(z_n(t),Dw)-\int_{\mathbb{R}^N}|Dp(u_n(t))|\,,\tag{3.3}$$

for all $w\in L^\infty(\mathbb{R}^N)\cap BV(\mathbb{R}^N)$, a.e. $t\in[0,T]$ and $p\in\mathcal{P}$.

Moreover, from the homogeneity of the operator we have the following estimates:

$$|u_n(t)|<<|u_{0,n}(\mu)|\,,\tag{3.4}$$

$$\frac{|u_n(t+h)-u_n(t)|}{h}<<\frac{2}{t}|u_{0,n}(\mu)|\,,\tag{3.5}$$

where $u<<v$ means

$$\int_{\mathbb{R}^N}j(u)dx\le\int_{\mathbb{R}^N}j(v)dx\quad\forall\,j\in J_0\,,$$

with $J_0:=\{$convex l.s.c. maps $j:\mathbb{R}\to[0,+\infty]$ satisfying $j(0)=0\}$ ([10]).

We have $u_n\in C([0,T],L^1(\mathbb{R}^N))\subset C([0,T],\mathcal{M}_b(\mathbb{R}^N))$. Now, from (3.4) we get

$$\int_{\mathbb{R}^N}|u_n(t)|dx\le\int_{\mathbb{R}^N}|u_{0,n}(\mu)|dx\le|\mu|(\mathbb{R}^N)\quad\forall\,n\in\mathbb{N},0<t\le T\,,\tag{3.6}$$

and from (3.5), it follows that

$$\int_{\mathbb{R}^N}|u_n(t+h)-u_n(t)|dx\le\frac{2h}{t}|\mu|(\mathbb{R}^N)\quad\forall\,n\in\mathbb{N},0<t\le T\,.\tag{3.7}$$

From the estimates (3.6) and (3.7), using the standard technique to prove Ascoli-Arzela's Theorem, we deduce that, by extracting a subsequence, if necessary, there exists $u \in C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))$ such that

$$u_n \rightharpoonup u \text{ in } C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))\,.$$

Moreover, from (3.7) and Reshetnyak's semicontinuity Theorem [2], we have

$$\int_{\mathbb{R}^N} |u(t+h) - u(t)| dx \le \frac{2h}{t} |\mu|(\mathbb{R}^N) \quad 0 < t \le T\,, \tag{3.8}$$

and we obtain that

$$u \in C([\tau, T], \mathcal{M}_b(\mathbb{R}^N)) \quad \text{for all } 0 < \tau < T\,. \tag{3.9}$$

Remark 3.2. Taking $p = T_k$ and $w = 0$ in (3.3), we get

$$\int_{\mathbb{R}^N} T_k(u_n(t))(u_n)_t(t) + \int_{\mathbb{R}^N} |DT_k(u_n(t))| \le 0\,.$$

If we denote $J_k(r) := \int_0^r T_k(s)ds$, it follows that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} |DT_k(u_n(t))| &+ \int_{\mathbb{R}^N} J_k(u_n(T))dx \\ &\le \int_{\mathbb{R}^N} J_k(u_{0,n}(\mu))dx \le M_k |\mu|(\mathbb{R}^N)\,. \end{aligned} \tag{3.10}$$

for some constant $M_k > 0$. Then, having in mind (3.7), we obtain that

$$\{T_k(u_n) : n \in \mathbb{N}\} \text{ is a bounded sequence in } BV([\tau, T] \times \mathbb{R}^N)\,, \tag{3.11}$$

for all $0 < \tau < T$ and $k > 0$.

We shall say that $u(t)$ is a *limit solution* of (1.1) corresponding to the initial condition μ.

Remark 3.3. The above estimates for $u_n(t)$ and $u(t)$ also hold for any approximation $u_n(0)$ converging to μ weakly* in $\mathcal{M}_b(\mathbb{R}^N)$.

4. Some Explicit Limit Solutions

By the results in [8] we know the evolution of some step functions by the total variation flow. Let us recall the evolution of balls and annulus in $\mathbb{R}^N$.

Lemma 4.1. *For $0 < r < R$ and $x_0 \in \mathbb{R}^N$, take $\Omega_{R,r}(x_0) := B_R(x_0) \setminus \overline{B_r(x_0)}$. Let $\alpha \ge 0$ and $\beta > 0$. Then we have*

(i) *If $u_0 = \alpha \chi_{B_r(x_0)}$, the strong solution of* (1.1) *for the initial datum u_0 is given by*

$$u(t) = \left(\alpha - \frac{N}{r} t\right)^+ \chi_{B_r(x_0)}\,.$$

(ii) *Let* $u_0 = \alpha\chi_{B_r(x_0)} + \beta\chi_{\Omega_{R,r}(x_0)}$ *and* $u(t)$ *be the strong solution of* (1.1) *for the initial datum* u_0. *Then, if* $\alpha < \beta$, $u(t)$ *is given by*

$$u(t) = \left(\beta - \frac{\operatorname{Per}(\Omega_{R,r}(x_0))}{|\Omega_{R,r}(x_0)|}t\right)^+ \chi_{\Omega_{R,r}(x_0)} + \left(\alpha + \frac{N}{r}t\right)\chi_{B_r(x_0)}$$

for $t \in [0, T_r]$, *where* T_r *is such that*

$$T_r\left(\frac{\operatorname{Per}(\Omega_{R,r}(x_0))}{|\Omega_{R,r}(x_0)|} + \frac{N}{r}\right) = \beta - \alpha\,.$$

For times $t \geq T_r$, *the solution* $u(t)$ *is given by the evolution of* $u(T_r)$ *according to the solution model described in* (i).

In the case $\beta < \alpha$, $u(t)$ *is given by*

$$u(t) = \left(\beta - \frac{\operatorname{Per}(B_R(x_0)) - \operatorname{Per}(B_r(x_0))}{|\Omega_{R,r}(x_0)|}t\right)^+ \chi_{\Omega_{R,r}(x_0)} + \left(\alpha - \frac{N}{r}t\right)^+ \chi_{B_r(x_0)}\,,$$

for $t \in [0, T_r]$, *where* T_r *is such that*

$$T_r\left(\frac{N}{r} - \frac{\operatorname{Per}(B_R(x_0)) - \operatorname{Per}(B_r(x_0))}{|\Omega_{R,r}(x_0)|}\right) = \alpha - \beta\,,$$

and, for later times, it evolves as the solution given in (i) *until its extinction.*

Using Lemma 4.1 we may compute some explicit limit solutions:

(i) Let $u(0) = \delta_0$. Then, if $u_{0,n} = \frac{\chi_{B_{1/n}(0)}}{|B_{1/n}(0)|}$, we have $u_{0,n} \rightharpoonup \delta_0$. Now, by Lemma 4.1, the strong solution of (1.1) for the initial datum $u_{0,n}$ is given by

$$u_n(t) = \left(\frac{1}{|B_{1/n}(0)|} - \frac{\operatorname{Per}(B_{1/n}(0))}{|B_{1/n}(0)|}t\right)^+ \chi_{B_{1/n}(0)}\,.$$

Hence, for every $t > 0$,

$$u_n(t) \rightharpoonup \delta_0 \quad \text{locally weakly}^* \text{ as measures}\,.$$

Therefore, $u(t) = \delta_0$ for all $t \geq 0$ is the limit solution of (1.1) for the initial datum δ_0.

(ii) The above example can be extended to $u(0) = \sum_{i=1}^k \delta_{p_i}$ where $\{p_1, \ldots, p_k\}$ are a finite set of points of $\mathbb{R}^N$. Then again by approximating explicit solutions and passing to the limit we get $u(t) = u(0)$ for every $t > 0$.

(iii) For $0 < r < R$, we denote $\Omega_{R,r} := B_R(0) \setminus \overline{B_r(0)}$ and $\Gamma_R = \partial B_R(0)$. We are going to compute the limit solution of (1.1) for the initial datum $\mu = \alpha\mathcal{H}^{N-1} \llcorner \Gamma_R$, with $\alpha > 0$. For every $n \in \mathbb{N}$, let

$$u_{0,n}(\mu) = \frac{\alpha\mathcal{H}^{N-1}(\Gamma_R)}{|I_n(\Gamma_R)|}\chi_{I_n(\Gamma_R)} = \frac{\alpha\mathcal{H}^{N-1}(\Gamma_R)}{|I_n(\Gamma_R)|}\chi_{\Omega_{R+\frac{1}{n}, R-\frac{1}{n}}}\,.$$

Let $u_n(t)$ be the unique strong solution of (1.1) for the initial datum $u_{0,n}(\mu)$. Then, if

$$T_n = \frac{\alpha \mathcal{H}^{N-1}(\Gamma_R)|B_{R-\frac{1}{n}}(0)|}{|B_{R-\frac{1}{n}}(0)|\mathrm{Per}(I_n(\Gamma_R)) + |I_n(\Gamma_R)|\mathrm{Per}(B_{R-\frac{1}{n}}(0))}\,,$$

we know by Lemma 4.1, that $u_n(t)$ is given by

$$u_n(t) = \begin{cases} \left(\dfrac{\alpha\mathcal{H}^{N-1}(\Gamma_R)}{|I_n(\Gamma_R)|} - \dfrac{\mathrm{Per}(I_n(\Gamma_R))}{|I_n(\Gamma_R)|}t\right)^+ \chi_{I_n(\Gamma_R)} & \\ \quad + \dfrac{\mathrm{Per}(B_{R-\frac{1}{n}}(0))}{|B_{R-\frac{1}{n}}(0)|} t\chi_{B_{R-\frac{1}{n}}(0)}\,, & 0 < t \leq T_r\,, \\ \left(\alpha_n - \dfrac{\mathrm{Per}(B_{R+\frac{1}{n}})}{|B_{R+\frac{1}{n}}(0)|}t\right)^+ \chi_{B_{R+\frac{1}{n}}(0)}\,, & t \geq T_n\,, \end{cases}$$

with

$$\alpha_n = T_n\left(\frac{\mathrm{Per}(B_{R-\frac{1}{n}}(0))}{|B_{R-\frac{1}{n}}(0)|} + \frac{\mathrm{Per}(B_{R+\frac{1}{n}}(0))}{|B_{R+\frac{1}{n}}(0)|}\right).$$

Since

$$\lim_{n\to\infty} T_n = \frac{\alpha}{2}\,, \quad \text{and} \quad \lim_{n\to\infty}\alpha_n = \alpha\frac{\mathrm{Per}(B_R(0))}{|B_R(0)|}\,,$$

by Lemma 3.1, we have that

$$u_n(t) \rightharpoonup u(t) \quad \text{locally weakly}^* \text{ as measures when } n \to \infty\,,$$

where $u(t)$ is the Radon measure in $\mathbb{R}^N$ defined by

$$u(t) = \begin{cases} \left(1 - \dfrac{2}{\alpha}t\right)\mu + \dfrac{N}{R}t\chi_{B_R(0)}\,, & 0 < t \leq \dfrac{\alpha}{2}\,, \\ \left(\alpha\dfrac{N}{R} - \dfrac{N}{R}t\right)^+ \chi_{B_R(0)}\,, & t \geq \dfrac{\alpha}{2}\,. \end{cases} \tag{4.1}$$

In the particular case $\alpha = \frac{1}{\mathrm{Per}(B_R(0))}$, the initial datum coincides with the delta of unit mass supported on $\Gamma_R = \partial B_R(0)$, that is, the distrubution δ_{Γ_R} defined by

$$\langle \delta_{\Gamma_R}, \varphi\rangle = \frac{1}{\mathrm{Per}(B_R(0))}\int_{\Gamma_R}\varphi d\mathcal{H}^{N-1}\,.$$

Then, if we denote by $T(t)$ the solution flow, we have

$$T(t)(\delta_{\Gamma_R}) = (1 - 2\mathrm{Per}(B_R(0))t)\delta_{\Gamma_R} + \frac{N}{R}t\chi_{B_R(0)}\,, \quad 0 < t \leq \frac{1}{2\mathrm{Per}(B_R(0))}\,,$$

$$T(t)(\delta_{\Gamma_R}) = \left(\frac{1}{|B_R(0)|} - \frac{\mathrm{Per}(B_R(0))}{|B_R(0)|}t\right)^+ \chi_{B_R(0)}\,, \quad t \geq \frac{1}{2\mathrm{Per}(B_R(0))}\,.$$

(iv) Consider now the case where the initial datum is the measure $\mu = \delta_0 + \chi_{B_R(0)}$. Take $\frac{1}{n} < R$, and consider

$$u_{0,n}(\mu) = \chi_{B_R(0)} + \frac{\chi_{B_{1/n}(0)}}{|B_{1/n}(0)|} = \left(1 + \frac{1}{|B_{1/n}(0)|}\right)\chi_{B_{1/n}(0)} + \chi_{\Omega_{R,\frac{1}{n}}}.$$

Let $u_n(t)$ be the unique strong solution of (1.1) for the initial datum $u_{0,n}(\mu)$. Then, if

$$T_n = \frac{|\Omega_{R,\frac{1}{n}}|}{|\Omega_{R,\frac{1}{n}}|\mathrm{Per}(B_{\frac{1}{n}}(0)) - |B_{\frac{1}{n}}(0)|(\mathrm{Per}(B_R(0)) - \mathrm{Per}(B_{\frac{1}{n}}(0))},$$

by Lemma 4.1, we know that, for $0 < t \le T_n$, $u_n(t)$ is given by

$$u_n(t) = \left(\frac{1}{|B_{\frac{1}{n}}(0)|} - \frac{\mathrm{Per}(B_{\frac{1}{n}}(0))}{|B_{\frac{1}{n}}(0)|}t\right)^+ \chi_{B_{\frac{1}{n}}(0)}$$
$$+ \left(1 - \frac{\mathrm{Per}(B_R(0)) - \mathrm{Per}(B_{\frac{1}{n}}(0))}{|\Omega_{R,\frac{1}{n}}|}t\right)^+ \chi_{\Omega_{R,\frac{1}{n}}}.$$

At $t = T_n$, the two evolving sets reach the same height and $u_n(T_n) = \alpha_n \chi_{B_R(0)}$ with

$$\alpha_n = \frac{1}{|B_{\frac{1}{n}}(0)|} - \frac{\mathrm{Per}(B_{\frac{1}{n}}(0))}{|B_{\frac{1}{n}}(0)|}T_n.$$

Hence, for $t > T_n$ the solution $u_n(t)$ is equal to the solution starting from $\alpha_n \chi_{B_R(0)}$ (at time T_n), i.e.,

$$\begin{aligned} u_n(t) &= \left(\alpha_n - \frac{\mathrm{Per}(B_R(0))}{|B_R(0)|}(t - T_n)\right)\chi_{B_R(0)} \\ &= \left(\frac{1}{|B_{\frac{1}{n}}(0)|} + T_n\left(\frac{\mathrm{Per}(B_R(0))}{|B_R(0)|} - \frac{\mathrm{Per}(B_{\frac{1}{n}}(0))}{|B_{\frac{1}{n}}(0)|}\right)\right. \\ &\qquad \left. - \frac{\mathrm{Per}(B_R(0))}{|B_R(0)|}t\right)\chi_{B_R(0)}. \end{aligned}$$

Since $T_n \to +\infty$ as $n \to \infty$, we have that

$$u_n(t) \rightharpoonup u(t) \quad \text{locally weakly}^* \text{ as measures when } n \to \infty,$$

where $u(t)$ is the Radon measure in $\mathbb{R}^N$ defined by

$$u(t) = \delta_0 + \left(1 - \frac{N}{R}t\right)^+ \chi_{B_R(0)}.$$

Observe that, in this particular case, if $T(t)$ is the solution flow, we have

$$u(t) = T(t)(\delta_0 + \chi_{B_R(0)}) = T(t)(\delta_0) + T(t)(\chi_{B_R(0)}).$$

Lemma 4.2. *For* $0 < R_1 < R_2 < R_3$, *we denote* $B_i := B_{R_i}(0)$ $(i = 1, 2, 3)$, $\Omega_1 := B_{R_2} \setminus \overline{B_{R_1}}$ *and* $\Omega_2 := B_{R_3} \setminus \overline{B_{R_2}}$. *Let* $u_0 := a\chi_{B_{R_1}} + b\chi_{\Omega_1} + c\chi_{\Omega_2}$, *with* a, b, $c \geq 0$. *Then, if* $u(t)$ *is the strong solution of* (1.1) *for the initial datum* u_0, *we have*:

(i) *If* $b = 0 < a < c$, $u(t)$ *is given by*

$$u(t) = \begin{cases} \left(a - \dfrac{\operatorname{Per}(B_{R_1})}{|B_{R_1}|}t\right)\chi_{B_{R_1}} + \dfrac{\operatorname{Per}(\Omega_1)}{|\Omega_1|}t\chi_{\Omega_1}\,, & \\ \quad + \left(c - \dfrac{\operatorname{Per}(\Omega_2)}{|\Omega_2|}t\right)\chi_{\Omega_2}\,, & 0 \leq t \leq T_1\,, \\ \left(a_1 + \dfrac{\operatorname{Per}(B_{R_2})}{|B_{R_2}|}t\right)\chi_{B_{R_2}} + \left(c - \dfrac{\operatorname{Per}(\Omega_2)}{|\Omega_2|}t\right)\chi_{\Omega_2}\,, & T_1 \leq t \leq S_1\,, \\ \left(c_1 - \dfrac{\operatorname{Per}(B_{R_3})}{|B_{R_3}|}t\right)\chi_{B_{R_3}}\,, & t \geq S_1\,, \end{cases}$$

where

$$T_1 = \frac{a|B_{R_1}||\Omega_1|}{\operatorname{Per}(B_{R_1})|\Omega_1| + \operatorname{Per}(\Omega_1)|B_{R_1}|}\,,$$

$$a_1 = T_1\left(\frac{\operatorname{Per}(\Omega_1)}{|\Omega_1|} - \frac{\operatorname{Per}(B_{R_2})}{|B_{R_2}|}\right)$$

$$S_1 = (c - a_1)\frac{|\Omega_2||B_{R_2}|}{\operatorname{Per}(B_{R_2})|\Omega_2| + \operatorname{Per}(\Omega_2)|B_{R_2}|}\,,$$

$$c_1 = c + S_1\left(\frac{\operatorname{Per}(B_{R_3})}{|B_{R_3}|} - \frac{\operatorname{Per}(\Omega_2)}{|\Omega_2|}\right)\;;$$

if we assume that $T_1 \leq S_1$.

(ii) *If* $a < c < b$, $u(t)$ *is given by*

$$u(t) = \begin{cases} \left(a + \dfrac{\operatorname{Per}(B_{R_1})}{|B_{R_1}|}t\right)\chi_{B_{R_1}} + \left(b - \dfrac{\operatorname{Per}(\Omega_1)}{|\Omega_1|}t\right)\chi_{\Omega_1} & \\ \quad + \left(c - \dfrac{\operatorname{Per}(B_{R_3}) - \operatorname{Per}(B_{R_2})}{|\Omega_2|}t\right)\chi_{\Omega_2}\,, & 0 \leq t \leq T_1\,, \\ \left(a + \dfrac{\operatorname{Per}(B_{R_1})}{|B_{R_1}|}t\right)\chi_{B_{R_1}} + \left(b_1 - \dfrac{\operatorname{Per}(\Omega_3)}{|\Omega_3|}t\right)\chi_{\Omega_3}\,, & T_1 \leq t \leq S_1\,, \\ \left(a_1 - \dfrac{\operatorname{Per}(B_{R_3})}{|B_{R_3}|}t\right)\chi_{B_{R_3}}\,, \quad t \geq S_1\,, & \end{cases}$$

where

$$T_1 = (b - c)\frac{|\Omega_1||\Omega_2|}{\operatorname{Per}(\Omega_1)|\Omega_2| - |\Omega_1|(\operatorname{Per}(B_{R_3}) - \operatorname{Per}(B_{R_2}))}\,, \quad \Omega_3 = B_{R_3} \setminus \overline{B_{R_1}}\,,$$

$$b_1 = b + \left(\frac{\operatorname{Per}(\Omega_3)}{|\Omega_3|} - \frac{\operatorname{Per}(\Omega_1)}{|\Omega_1|}\right)T_1\,,$$

$$S_1 = (b_1 - a)\frac{|B_{R_1}||\Omega_3|}{\mathrm{Per}(B_{R_1})|\Omega_3| + \mathrm{Per}(\Omega_3)|B_{R_1}|}\,,$$

$$a_1 = a + S_1\left(\frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|} + \frac{\mathrm{Per}(B_{R_3})}{|B_{R_3}|}\right);$$

if we assume that $T_1 \le S_1$.

(iii) *If* $a = c < b$, $u(t)$ *is given by*

$$u(t) = \begin{cases} \left(a + \frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|}t\right)\chi_{B_{R_1}} + \left(b - \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|}t\right)\chi_{\Omega_1}\,, & \\ \quad + \left(c - \frac{\mathrm{Per}(B_{R_3}) - \mathrm{Per}(B_{R_2})}{|\Omega_2|}t\right)\chi_{\Omega_2}\,, & 0 \le t \le T_1\,, \\ \left(a + \frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|}t\right)\chi_{B_{R_1}} + \left(b - \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|}t\right)\chi_{\Omega_1}\,, & T_1 \le t \le S_1\,, \\ \left(a_1 - \frac{\mathrm{Per}(B_{R_2})}{|B_{R_2}|}t\right)\chi_{B_{R_3}}\,, & t \ge S_1\,, \end{cases}$$

where

$$T_1 = \frac{a|\Omega_2|}{\mathrm{Per}(B_{R_3}) - \mathrm{Per}(B_{R_2})}\,,$$

$$S_1 = (b - a)\frac{|\Omega_1||B_{R_1}|}{\mathrm{Per}(B_{R_1})|\Omega_1| + \mathrm{Per}(\Omega_1)|B_{R_1}|}\,,$$

$$a_1 = a + S_1\left(\frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|} + \frac{\mathrm{Per}(B_{R_2})}{|B_{R_2}|}\right);$$

if we assume that $T_1 \le S_1$.

Proof. (i) We look for a solution of the form $u(t) = a(t)\chi_{B_{R_1}} + b(t)\chi_{\Omega_1} + c(t)\chi_{\Omega_2}$, with $b(t) \le a(t) \le c(t)$ on some time interval $(0, T_1)$. Then, we shall look for some $z(t) \in X_1(\mathbb{R}^N)$, with $\|z(t)\|_\infty \le 1$, such that

$$u'(t) = \mathrm{div}(z(t)) \text{ in } \mathcal{D}'(\mathbb{R}^N)\,, \tag{4.2}$$

$$\int_{\mathbb{R}^N}(z(t), Du(t)) = \int_{\mathbb{R}^N}|Du(t)|(\mathbb{R}^N)\,. \tag{4.3}$$

Now, by the coarea formula, if $E_s = \{x \in \mathbb{R}^N : u(t)(x) > s\}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}|Du(t)|(\mathbb{R}^N) \\ &\quad = \int_0^\infty \int_{\mathbb{R}^N}|D\chi_{E_s}|ds = \int_0^{b(t)}\int_{\mathbb{R}^N}|D\chi_{B_{R_3}}|ds \\ &\qquad + \int_{b(t)}^{a(t)}\int_{\mathbb{R}^N}|D\chi_{B_{R_1}\cup\Omega_2}|ds + \int_{a(t)}^{c(t)}\int_{\mathbb{R}^N}|D\chi_{\Omega_2}|ds \end{aligned}$$

$$= b(t)\mathrm{Per}(B_{R_3}) + (a(t) - b(t))(\mathrm{Per}(B_{R_1}) + \mathrm{Per}(\Omega_2)) + (c(t) - a(t))\mathrm{Per}(\Omega_2)$$
$$= b(t)\mathrm{Per}(B_{R_3}) + a(t)\mathrm{Per}(B_{R_1}) - b(t)(\mathrm{Per}(B_{R_1}) + \mathrm{Per}(\Omega_2)) + c(t)\mathrm{Per}(\Omega_2)$$
$$= a(t)\mathrm{Per}(B_{R_1}) - b(t)\mathrm{Per}(\Omega_1) + c(t)\mathrm{Per}(\Omega_2)\,.$$

1 On the other hand,

$$u'(t) = a'(t)\chi_{B_{R_1}} + b'(t)\chi_{\Omega_1} + c'(t)\chi_{\Omega_2}\,.$$

Then, by Green formula, in order to have (4.2) and (4.3), we need

$$\begin{aligned}
&a(t)\mathrm{Per}(B_{R_1}) - b(t)\mathrm{Per}(\Omega_1) + c(t)\mathrm{Per}(\Omega_2)\\
&\quad = \int_{\mathbb{R}^N} |Du(t)|(\mathbb{R}^N)\\
&\quad = \int_{\mathbb{R}^N} (z(t), Du(t)) = -\int_{\mathbb{R}^N} \mathrm{div}(z(t))u(t) = -\int_{\mathbb{R}^N} u'(t)u(t)\\
&\quad = -a'(t)a(t)|B_{R_1}| - b'(t)b(t)|\Omega_1| - c'(t)c(t)|\Omega_2|\,.
\end{aligned}$$

Therefore, we must have

$$\begin{aligned}
&a'(t) = -\frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|}\,, \quad a(0) = a \Rightarrow a(t) = \left(a - \frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|}t\right)^+\,,\\
&b'(t) = \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|}\,, \quad b(0) = 0 \Rightarrow b(t) = \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|}t\,,\\
&c'(t) = -\frac{\mathrm{Per}(\Omega_2)}{|\Omega_2|}\,, \quad c(0) = c \Rightarrow c(t) = \left(c - \frac{\mathrm{Per}(\Omega_2)}{|\Omega_2|}t\right)^+\,.
\end{aligned}$$

If we consider the vector field $z(t)$ defined by

$$z(t)(x) := \begin{cases}
-\dfrac{x}{R_1}\,, & \text{if } x \in B_{R_1}\\[2mm]
\left[(R_2^{N-1} + R_1^{N-1}) - (R_2 + R_1)\dfrac{R_2^{N-1}R_1^{N-1}}{\|x\|^N}\right]\dfrac{x}{R_2^N - R_1^N}\,, & \text{if } x \in \Omega_1\\[2mm]
\left[(R_3 + R_2)\dfrac{R_2^{N-1}R_3^{N-1}}{\|x\|^N} - (R_3^{N-1} + R_2^{N-1})\right]\dfrac{x}{R_3^N - R_2^N}\,, & \text{if } x \in \Omega_2\\[2mm]
-\dfrac{xR_3}{\|x\|^N}\,, & \text{if } x \in \mathbb{R}^N \setminus \overline{B_{R_3}}\,,
\end{cases}$$

we have

$$\mathrm{div}(z(t)) = -\frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|} \text{ in } B_{R_1}\,, \quad z(t)(x)_{|\partial B_{R_1}} = -\frac{x}{\|x\|}\,,$$

$$\mathrm{div}(z(t)) = \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|} \text{ in } \Omega_1\,, \quad z(t)(x)_{|\partial B_{R_1}} = -\frac{x}{\|x\|}\,, \quad z(t)(x)_{|\partial B_{R_2}} = \frac{x}{\|x\|}\,,$$

$$\mathrm{div}(z(t)) = -\frac{\mathrm{Per}(\Omega_2)}{|\Omega_2|} \text{ in } \Omega_2\,, \quad z(t)(x)_{|\partial B_{R_2}} = \frac{x}{\|x\|}\,, \quad z(t)(x)_{|\partial B_{R_3}} = -\frac{x}{\|x\|}\,,$$

$$\mathrm{div}(z(t)) = 0 \text{ in } \mathbb{R}^N \setminus \overline{B_{R_3}}\,, \quad z(t)(x)_{|\partial B_{R_3}} = -\frac{x}{\|x\|}\,.$$

Hence, $u(t) = \left(a - \frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|}t\right)\chi_{B_{R_1}} + \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|}t\chi_{\Omega_1} + \left(c - \frac{\mathrm{Per}(\Omega_2)}{|\Omega_2|}t\right)\chi_{\Omega_2}$ is a strong solution of (1.1) in $[0, T_1]$, with T_1 such that

$$a - \frac{\mathrm{Per}(B_{R_1})}{|B_{R_1}|}T_1 = \frac{\mathrm{Per}(\Omega_1)}{|\Omega_1|}T_1\,,$$

i.e.,

$$T_1 = \frac{a|B_{R_1}||\Omega_1|}{\mathrm{Per}(B_{R_1})|\Omega_1| + \mathrm{Per}(\Omega_1)|B_{R_1}|}\,.$$

Now,

$$u(T_1) = \frac{a|B_{R_1}|\mathrm{Per}(\Omega_1)}{\mathrm{Per}(B_{R_1})|\Omega_1| + \mathrm{Per}(\Omega_1)|B_{R_1}|}\chi_{B_{R_2}} + \left(c - \frac{\mathrm{Per}(\Omega_2)}{|\Omega_2|}T_1\right)\chi_{\Omega_2}\,.$$

Then, for $t > T_1$ the solution $u(t)$ is equal to the solution starting from $u(T_1)$ (at time T_1) as it is described in (ii) of Lemma 4.1.

The proofs of cases (ii) and (iii) are similar and we shall omit the details. □

Theorem 4.3. *Let μ be the measure $\mu = \chi_{B_R(0)} + \alpha\mathcal{H}^{N-1} \llcorner \Gamma_r$, $\alpha > 0$, and let $u(t)$ be the limit solution of* (1.1) *constructed using the approximations* (3.1) *for the initial datum μ. Then, we have*

(i) *If $R < r$, $u(t)$ is given by*

$$u(t) = \begin{cases} \left(1 - \dfrac{\mathrm{Per}(B_R(0))}{|B_R(0)|}t\right)\chi_{B_R(0)} + \dfrac{\mathrm{Per}(\Omega_{r,R})}{|\Omega_{r,R}|}t\chi_{\Omega_{r,R}} & \\ \quad + \left(1 - \dfrac{2}{\alpha}t\right)\alpha\mathcal{H}^{N-1} \llcorner \Gamma_r\,, & \leq t \leq T\,, \\ \left(\beta + \dfrac{\mathrm{Per}(B_r(0))}{|B_r(0)|}t\right)\chi_{B_r(0)} & \\ \quad + \left(1 - \dfrac{2}{\alpha}t\right)\alpha\mathcal{H}^{N-1} \llcorner \Gamma_r\,, & T \leq t \leq \dfrac{\alpha}{2}\,, \\ \left(\beta + \dfrac{\alpha}{2}\dfrac{\mathrm{Per}(B_r(0))}{|B_r)0)|} - \dfrac{\mathrm{Per}(B_r(0))}{|B_r(0)|}t\right)^+ \chi_{B_r(0)}\,, & t \geq \dfrac{\alpha}{2}\,, \end{cases}$$

where

$$T = \frac{|B_R||\Omega_{r,R}|}{\mathrm{Per}(B_R)|\Omega_{r,R}| + \mathrm{Per}(\Omega_{r,R})|B_R|}\,, \beta = T\left(\frac{\mathrm{Per}(\Omega_{r,R})}{|\Omega_{r,R}|} - \frac{\mathrm{Per}(B_r)}{|B_r|}\right);$$

if we assume that $T \leq \frac{\alpha}{2}$.

(ii) *If* $R = r$, $u(t)$ *is given by*

$$u(t) = \begin{cases} \left(1 + \dfrac{\mathrm{Per}(B_r)}{|B_r|} t\right) \chi_{B_r} + \left(1 - \dfrac{2}{\alpha} t\right) \alpha \mathcal{H}^{N-1} \llcorner \Gamma_r \,, & 0 \leq t \leq \dfrac{\alpha}{2} \,, \\ \left(1 + \alpha \dfrac{\mathrm{Per}(B_r)}{|B_r|} - \dfrac{\mathrm{Per}(B_r)}{|B_r|} t\right)^{+} \chi_{B_r} \,, & t \geq \dfrac{\alpha}{2} \,. \end{cases}$$

(iii) *If* $R > r$, $u(t)$ *is given by*

$$u(t) = \begin{cases} \left(1 + \dfrac{\mathrm{Per}(B_r)}{|B_r|} t\right) \chi_{B_r} + \left(1 - \dfrac{2}{\alpha} t\right) \alpha \mathcal{H}^{N-1} \llcorner \Gamma_r & \\ \quad + \left(1 - \dfrac{\mathrm{Per}(B_R) - \mathrm{Per}(B_r)}{|\Omega_{R,r}|} t\right) \chi_{\Omega_{R,r}} \,, & 0 \leq t \leq T \,, \\ \left(1 + \dfrac{\mathrm{Per}(B_r)}{|B_r|} t\right) \chi_{B_r} + \left(1 - \dfrac{2}{\alpha} t\right) \alpha \mathcal{H}^{N-1} \llcorner \Gamma_r \,, & T \leq t \leq \dfrac{\alpha}{2} \,, \\ \left(1 + \dfrac{\alpha}{2} \left(\dfrac{\mathrm{Per}(B_r)}{|B_r|} + \dfrac{\mathrm{Per}(B_r)}{|B_r|}\right) - \dfrac{\mathrm{Per}(B_r)}{|B_r|} t\right)^{+} \chi_{B_R} \,, & t \geq \dfrac{\alpha}{2} \,, \end{cases}$$

where

$$T = \frac{|\Omega_{R,r}|}{\mathrm{Per}(B_R) - \mathrm{Per}(B_r)} \,,$$

if we assume that $T \leq \frac{\alpha}{2}$.

Proof. (i) For every $n \in \mathbb{N}$, let $u_{0,n}(\mu) = \chi_{B_R(0)} + \frac{\alpha \mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} \chi_{I_n(\Gamma_r)}$. If n is large enough, we have

$$u_{0,n}(\mu) = \begin{cases} 1 \,, & \text{in } B_R(0) \\ \dfrac{\alpha \mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} \,, & \text{in } I_n(\Gamma_r) \,. \end{cases}$$

Then, applying Lemma 4.2 (i), with $R_1 = R$, $R_2 = r - \frac{1}{n}$, $R_3 = r + \frac{1}{n}$, $a = 1$ and $c = \frac{\alpha \mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|}$, we have that if $u_n(t)$ is the unique strong solution of (1.1) for the initial datum $u_{0,n}(\mu)$, then

$$u_n(t) = \begin{cases} \left(1 - \frac{\mathrm{Per}(B_R)}{|B_R|}t\right)\chi_{B_R} + \frac{\mathrm{Per}(\Omega_{r-\frac{1}{n},R})}{|\Omega_{r-\frac{1}{n},R}|}t\chi_{\Omega_{r-\frac{1}{n},R}}\,, & \\ \quad + \left(\frac{\alpha\mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} - \frac{\mathrm{Per}(I_n(\Gamma_r))}{|I_n(\Gamma_r)|}t\right)\chi_{I_n(\Gamma_r)}\,, & 0 \le t \le T_n\,, \\ \left(a_n + \frac{\mathrm{Per}(B_{r-\frac{1}{n}})}{|B_{r-\frac{1}{n}}|}t\right)\chi_{B_{r-\frac{1}{n}}} & \\ \quad + \left(\frac{\alpha\mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} - \frac{\mathrm{Per}(I_n(\Gamma_r))}{|I_n(\Gamma_r)|}t\right)\chi_{I_n(\Gamma_r)}\,, & T_n \le t \le S_n\,, \\ \left(c_n - \frac{\mathrm{Per}(B_{r+\frac{1}{n}})}{|B_{r+\frac{1}{n}}|}t\right)\chi_{B_{r+\frac{1}{n}}}\,, & t \ge S_n\,, \end{cases}$$

where

$$T_n = \frac{|B_R||\Omega_{r-\frac{1}{n},R}|}{\mathrm{Per}(B_R)|\Omega_{r-\frac{1}{n},R}| + \mathrm{Per}(\Omega_{r-\frac{1}{n},R})|B_R|}\,,$$

$$a_n = T_n\left(\frac{\mathrm{Per}(\Omega_{r-\frac{1}{n},R})}{|\Omega_{r-\frac{1}{n},R}|} - \frac{\mathrm{Per}(B_{r-\frac{1}{n}})}{|B_{r-\frac{1}{n}}|}\right)\,,$$

$$S_n = \left(\frac{\alpha\mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} - a_n\right)\frac{|I_n(\Gamma_r)||B_{r-\frac{1}{n}}|}{\mathrm{Per}(B_{r-\frac{1}{n}})|I_n(\Gamma_r)| + \mathrm{Per}(I_n(\Gamma_r))|B_{r-\frac{1}{n}}|}\,,$$

$$c_n = \frac{\alpha\mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} + S_n\left(\frac{\mathrm{Per}(B_{r+\frac{1}{n}})}{|B_{r+\frac{1}{n}}|} - \frac{\mathrm{Per}(I_n(\Gamma_r))}{|I_n(\Gamma_r)|}\right)\,.$$

Now

$$T := \lim_{n\to\infty} T_n = \frac{|B_R||\Omega_{r,R}|}{\mathrm{Per}(B_R)|\Omega_{r,R}| + \mathrm{Per}(\Omega_{r,R})|B_R|}\,,$$

$$\beta := \lim_{n\to\infty} a_n = T\left(\frac{\mathrm{Per}(\Omega_{r,R})}{|\Omega_{r,R}|} - \frac{\mathrm{Per}(B_r)}{|B_r|}\right)\,,$$

$$\lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left(\frac{\alpha\mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} - a_n\right)\frac{|I_n(\Gamma_r)||B_{r-\frac{1}{n}}|}{\mathrm{Per}(B_{r-\frac{1}{n}})|I_n(\Gamma_r)| + \mathrm{Per}(I_n(\Gamma_r))|B_{r-\frac{1}{n}}|} = \frac{\alpha}{2}\,,$$

$$\lim_{n\to\infty} c_n$$

$$= \frac{\alpha\mathcal{H}^{N-1}(\Gamma_r)}{|I_n(\Gamma_r)|} + S_n\left(\frac{\mathrm{Per}(B_{r+\frac{1}{n}})}{|B_{r+\frac{1}{n}}|} - \frac{\mathrm{Per}(I_n(\Gamma_r))}{|I_n(\Gamma_r)|}\right) = \beta + \frac{\alpha}{2}\frac{\mathrm{Per}(B_r)}{|B_r|}\,.$$

The proofs of cases (ii) and (iii) are similar and we shall omit the details. □

Remark 4.4. (i) Limit solutions corresponding to initial conditions given by a measure are, in general, measures. There is no regularizing effect.

(ii) Not all measures are treated in the same way. The solutions $u(t)$ corresponding to measures μ for which the dimension of the support of its singular part μ_s is estrictly less than $N-1$ satisfy that the singular part of $u(t)$ does not move (examples (i) and (ii) of Theorem 4.3). If the support of μ_s has dimension $N-1$, the singular part of $u(t)$ evolves (example (iii) of Theorem 4.3). We shall prove in Sec. 5 that this corresponds to the behaviour of limit solutions.

(iii) It is interesting to compare example (i) with what happens with the p-Laplacian operator

$$\Delta_p(u) = \operatorname{div}(|Du|^{p-2}Du)\,, \quad p > 1\,.$$

Di Benedetto and Herrero [15] introduce the concept of local weak solution for the Cauchy problem

$$u_t = \Delta_p u \text{ in } Q_T =]0,T[\times\mathbb{R}^N\,, \tag{4.4}$$

and prove that a nonnegative local weak solution of (4.4) in Q_T admits a unique initial trace u_0 which is a σ-finite Borel measure. Moreover, for $p > \frac{2N}{N+1}$, they prove the solvability of the Cauchy problem (4.4) when the initial datum is a σ-Borel positive measure in $\mathbb{R}^N$. Now, if $p > 2$, Kamin and Vázquez [20] (see also [21]) prove that the Barenblatt selfsimilar solution

$$w(t,x) = t^{-k}(C - q\|\xi\|^{p/(p-1)})_+^{(p-1)/(p-2)}\,,$$

where

$$\xi = xt^{-k/N}\,, \quad k = \left(p - 2 + \frac{p}{N}\right)^{-1}\,, \quad q = \frac{p-2}{p}\left(\frac{k}{N}\right)^{t/(p-1)}\,,$$

is the unique nonnegative weak solution of the Cauchy problem (4.4) satisfying

$$u(0,x) = 0 \quad \text{for } x \neq 0\,, \quad \lim_{t\to 0}\int_{B_r(0)} u(t,x)dx = M\,.$$

Therefore, in this case, δ_0 evolves. Nevertheless, in the limit case $p = 1$, δ_0 does not evolve. To our knowledge it is not known if δ_0 does not evolve in the case $1 < p \leq \frac{2N}{N+1}$.

5. Characterization of Limit Solutions

Let $0 \leq \mu \in \mathcal{M}(\mathbb{R}^N)$ and $0 \leq k \leq N$. Recall that the upper and lower k-dimensional densities of μ at x are respectively defined by

$$\Theta_k^*(\mu,x) := \limsup_{\rho\to 0^+}\frac{\mu(B_\rho(x))}{\omega_k\rho^k}\,, \quad \Theta_{*k}(\mu,x) := \liminf_{\rho\to 0^+}\frac{\mu(B_\rho(x))}{\omega_k\rho^k}\,.$$

If $\Theta_k^*(\mu,x) = \Theta_{*k}(\mu,x)$ their common value is denoted by $\Theta_k(\mu,x)$.

Theorem 5.1. *Let $0 \leq \mu \in \mathcal{M}_b(\mathbb{R}^N)$ be such that $\mu_{ac} \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Suppose that $\mu_s = \alpha\mathcal{H}^k \llcorner S$, where S is a compact k-manifold in $\mathbb{R}^N$ of class $W^{3,\infty}$. Let $u_n(t)$ be the strong solution of problem (1.1) with initial datum $u_{0,n}(\mu)$ and let $u(t)$ be the corresponding limit solution. Then, for any $T > 0$ there is a constant $K_T = K(N, T, \|\mu_{ac}\|)$, independent of n, such that*

$$u_n(t,x) \leq K_T \ \forall\, x \in \mathbb{R}^N \setminus I_n(S)\,, \quad \forall\, t \in (0,T)\,. \tag{5.1}$$

Moreover, up to extraction of a subsequence if necessary, we have

$$u_n(t) \to u(t)_{ac}\,, \quad \mathcal{L}^N\text{-a.e.} \quad \text{for all } t > 0\,, \tag{5.2}$$

$$u_n(t)\chi_{\mathbb{R}^N \setminus I_n(S)} \to u(t)_{ac} \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N)\,, \quad \forall\, p \in [1,\infty)\,, \tag{5.3}$$

$$u_n(t)\chi_{I_n(S)} \rightharpoonup u(t)_s \text{ weakly}^* \text{ as measures}\,, \tag{5.4}$$

$$\operatorname{supp}(u(t)_s) = S \begin{cases} \text{for all } t \geq 0\,, & \text{if } k < N-1\,, \\[2ex] \text{for } 0 \leq t < \dfrac{\alpha}{2N}\,, & \text{if } k = N-1\,, \end{cases} \tag{5.5}$$

and

$$u(t)_s \leq \mu_s \text{ for all } t \geq 0\,. \tag{5.6}$$

Moreover,

$$\text{if } k < N-1\,, \quad \text{we have that } u(t)_s \geq \mu_s\,, \quad \text{for all } t \geq 0\,. \tag{5.7}$$

Therefore,

$$\text{if } k < N-1\,, \quad \text{we have that } u(t)_s = \mu_s\,, \quad \text{for all } t \geq 0\,. \tag{5.8}$$

Proof. Let us prove (5.1). Since S has bounded curvatures, there exists $r > 0$ such that, for every $x \in \mathbb{R}^N \setminus I_n(S)$ one can find $y_x \in \mathbb{R}^N$ and $r_x \geq r$ such that $x \in B_{r_x}(y_x)$ and $B_{r_x}(y_x) \cap I_n(S) = \emptyset$. Then, given $x \in \mathbb{R}^N \setminus I_n(S)$, and

$$v^x_{0,n} := \| u_{0,n}(\mu) \|_\infty \chi_{\mathbb{R}^N \setminus \overline{B_{r_x}(y_x)}} + \|\mu_{ac}\|_\infty \chi_{B_{r_x}(y_x)}\,,$$

we have that $0 \leq u_{0,n}(\mu) \leq v^x_{0,n}$. Note that the solution $v^x_n(t)$ of (1.1) with initial datum $v^x_{0,n}$ is

$$v^x_n(t) = \| u_{0,n}(\mu) \|_\infty \chi_{\mathbb{R}^N \setminus \overline{B_{r_x}(y_x)}} + \inf\left\{ \|\mu_{ac}\|_\infty + \frac{Nt}{r_x}, \| u_{0,n}(\mu) \|_\infty \right\} \chi_{B_{r_x}(y_x)}$$

Using the comparison principle for solutions in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ [8], we obtain that $u_n(t) \leq v^x_n(t)$, Therefore, we have

$$0 \leq u_n(t) \leq \|\mu_{ac}\|_\infty + \frac{Nt}{r_x} \leq \|\mu_{ac}\|_\infty + \frac{Nt}{r} \text{ in } B_{r_x}(y_x)\,, \tag{5.9}$$

for any $t \geq 0$, $x \in \mathbb{R}^N \setminus I_n(S)$ and (5.1) follows.

Let $v_n(t) := u_n(t)\chi_{\mathbb{R}^N \setminus I_n(S)}$. Using (3.6), (3.7), (3.11), and the compact embedding of $BV([\tau,T] \times B_R(0))$ in $L^1([\tau,T] \times B_R(0))$ for any $\tau > 0$ and $R > 0$, by extracting a subsequence, if necessary, we may assume that

$$v_n \to v \text{ a.e. in }]0,T[\times\mathbb{R}^N , \tag{5.10}$$

and

$$v_n \to v \text{ in } L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^N) \quad \text{and} \quad v_n(t) \to v(t) \text{ in } \; L^1_{\mathrm{loc}}(\mathbb{R}^N) \;\forall\, t \in [0,T] . \tag{5.11}$$

Thus, using (5.1) we also have

$$v_n(t) \to v(t) \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \;\forall\, t \in [0,T] \quad \text{and in } L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^N) , \tag{5.12}$$

for all $p \in [1,\infty)$. Now, by estimates (3.6) and (3.7), as in Sec. 3, we may assume that $v_n(t)$ converges in $C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))$ to some measure $\tilde{v}(t)$. According to (3.6) and (5.11), we have that $\tilde{v}(t) = v(t)$ for all $t \in [0,T]$. On the other hand, we may also assume that

$$u_n(t)\chi_{I_n(S)} \rightharpoonup w(t) \text{ weakly}^* \text{ as measures} .$$

Hence, $w(t)$ is singular respect to the Lebesgue measure $\mathcal{L}^N$. Since $u_n(t) = v_n(t) + u_n(t)\chi_{I_n(S)}$, we have $u(t) = v(t) + w(t)$, for all $t \in [0,T]$, with $v(t)$ absolutely continuos respect to $\mathcal{L}^N$ and $w(t)$ singular respect to $\mathcal{L}^N$. It follows that $v(t) = u(t)_{ac}$ and $w(t) = u(t)_s$, and we conclude the proof of (5.2) and (5.3).

From (5.3) is is easy to deduce that $\mathrm{supp}(u(t)_s) \subset S$ for all $t > 0$. Let us prove the opposite inclusion. Given $p \in S$, we have

$$u_{0,n}(\mu) \geq w^p_{0,n} = \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\chi_{B_{\frac{1}{n}}(p)} .$$

Using the comparison principle and having in mind Lemma 4.1, we have

$$u_n(t) \geq \left(\frac{\alpha\mathcal{H}^k(S)}{\|I_n(S)\|} - nNt\right)^+ \chi_{B_{\frac{1}{n}}(p)} .$$

Since the above inequality is true for all $p \in S$, we deduce that

$$u_n(t) \geq \left(\frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|} - nNt\right)^+ \chi_{I_n(S)} . \tag{5.13}$$

As a consequence, for all $x \in S$ and all $m \geq n$ we have

$$\begin{aligned} &\frac{1}{|B_{\frac{1}{n}}(x)|}\int_{B_{\frac{1}{n}}(x)} u_m(t) \\ &\qquad \leq \left(\frac{|I_m(S) \cap B_{\frac{1}{n}}(x)|}{|B_{\frac{1}{n}}(x)|}\right)\left(\frac{\alpha\mathcal{H}^k(S)}{|I_m(S)|}\right)\left(1 - \frac{mN|I_m(S)|t}{\alpha\mathcal{H}^k(S)}\right)^+ . \end{aligned} \tag{5.14}$$

Now, by [2, Theorem 2.104], we have

$$\lim_{m\to\infty} \frac{mN|I_m(S)|t}{\alpha\mathcal{H}^k(S)} = \begin{cases} 0 , & \text{if } k < N-1 , \\ \dfrac{Nt\omega_1}{\alpha} , & \text{if } k = N-1 . \end{cases}$$

Then, taking limits in (5.14), and applying Lemma 3.1, we obtain

$$\limsup_{m\to\infty} \frac{1}{|B_{\frac{1}{n}}(x)|} \int_{B_{\frac{1}{n}}(x)} u_m(t) \geq \begin{cases} \dfrac{\mu_s(B_{\frac{1}{n}}(x))}{|B_{\frac{1}{n}}(x)|}\,, & \text{if } k < N-1\,, \\ \dfrac{\mu_s(B_{\frac{1}{n}}(x))}{|B_{\frac{1}{n}}(x)|} \left(1 - \dfrac{Nt\omega_1}{\alpha}\right)^+\,, & \text{if } k = N-1\,. \end{cases}$$

Hence, since $u_m(t) \rightharpoonup u(t)$ weakly* as measures, we have

$$\frac{u(t)(\overline{B_{\frac{1}{n}}(x)})}{|B_{\frac{1}{n}}(x)|} \geq \begin{cases} \dfrac{\mu_s(B_{\frac{1}{n}}(x))}{|B_{\frac{1}{n}}(x)|}\,, & \text{if } k < N-1\,, \\ \dfrac{\mu_s(B_{\frac{1}{n}}(x))}{|B_{\frac{1}{n}}(x)|} \left(1 - \dfrac{Nt\omega_1}{\alpha}\right)^+\,, & \text{if } k = N-1\,. \end{cases}$$

Since $x \in \operatorname{supp}(\mu_s)$, from the above inequalities we deduce that

$$\limsup_{n\to\infty} \frac{u(t)(\overline{B_{\frac{1}{n}}(x)})}{|B_{\frac{1}{n}}(x)|} = \begin{cases} 0\,, & \text{for } t \geq \dfrac{\alpha}{2N},\ \text{if } k = N-1\,, \\ +\infty\,, & \text{otherwise}\,, \end{cases}$$

which implies, using Besicovitch derivation Theorem (see [2]), that $x \in \operatorname{supp}(u(t)_s)$ for all $t \geq 0$ if $k < N-1$, and for $0 \leq t < \frac{\alpha}{2N}$ in case $k = N-1$. This concludes the proof of (5.5).

Let us prove (5.6). Let $v_{0,n} = \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}$. Observe that $v_{0,n} \geq u_{0,n}(\mu)$ for n large enough. Hence, using the comparison principle, we have

$$u_n(t) \leq \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|} \text{ for all } t \geq 0\,.$$

Then, if $x \in S$, using (5.1), we have

$$\begin{aligned} \Theta_k^*(u(t),x) &= \limsup_{\rho\to 0^+} \frac{u(t)(B_\rho(x))}{\omega_k\rho^k} \leq \limsup_{\rho\to 0^+} \limsup_{n\to\infty} \frac{\int_{B_\rho(x)} u_n(t)}{\omega_k\rho^k} \\ &= \limsup_{\rho\to 0^+} \limsup_{n\to\infty} \frac{\int_{B_\rho(x)\cap I_n(S)} u_n(t) + \int_{B_\rho(x)\setminus I_n(S)} u_n(t)}{\omega_k\rho^k} \\ &\leq \limsup_{\rho\to 0^+} \limsup_{n\to\infty} \frac{\frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|} \int_{B_\rho(x)} \chi_{I_n(S)} + K_t|B_\rho(x)|}{\omega_k\rho^k} \\ &\leq \limsup_{\rho\to 0^+} \frac{\limsup_n \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|} \int_{B_\rho(x)} \chi_{I_n(S)}}{\omega_k\rho^k} \\ &\leq \limsup_{\rho\to 0^+} \frac{\mu_s(\overline{B_\rho(x)})}{\omega_k\rho^k} = \Theta_k^*(\mu_s,x)\,. \end{aligned}$$

Using [2, Theorem 2.83], it follows that

$$\Theta_k^*(u(t)_s, x) \leq \Theta_k^*(\mu_s, x) = \alpha \quad \mathcal{H}^k \text{ a.e. in } S\,.$$

Then, by (5.5), it follows that $u(t)_s \leq \mu_s$.

Finally, let us prove (5.7). Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. Then, using (5.4), (5.13) and Lemma 3.1, we have

$$\begin{aligned}\langle u(t)_s, \varphi\rangle &= \lim_n \langle u_n(t)\chi_{I_n(S)}, \varphi\rangle \geq \lim_n \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\int_{I_n(S)}\left(1 - \frac{nN|I_n(S)|t}{\alpha\mathcal{H}^k(S)}\right)^+ \varphi(x)dx \\ &= \lim_n \frac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\int_{I_n(S)} \varphi(x)dx = \langle \mu_s, \varphi\rangle\,.\end{aligned}$$

Note that in the above derivation we have used that

$$\left|\frac{nNt}{\alpha\mathcal{H}^k(S)}\int_{I_n(S)}\varphi(x)dx\right| \leq \frac{\|\varphi\|_\infty\, nNt|I_n(S)|}{\alpha\mathcal{H}^k(S)} \leq Cn^{k+1-N} \to 0 \quad \text{as } n \to \infty\,,$$

since $k < N-1$. We conclude that $u(t)_s \geq \mu_s$. □

5.1. *Singular part of μ of dimension $N-1$*

We assume that $\mu = \mu_{ac} + \alpha\mathcal{H}^{N-1} \llcorner S$, with $\alpha \geq 0$, $\mu_{ac} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and S is a compact $(N-1)$-manifold of class $W^{3,\infty}$. We want to describe the precise behaviour of $u(t)$ and, in particular, compute $u(t)_s$. For that we need precise estimates for the evolution of $u_n(t)\chi_{I_n(S)}$.

Our first purpose will be to prove the following result.

Theorem 5.2. *In the time interval* $[0, \frac{\alpha}{2}]$ *we have*

$$u(t)_s = (\alpha - 2t)^+\mathcal{H}^{N-1} \llcorner S = \left(1 - \frac{2}{\alpha}t\right)^+ \mu_s\,. \tag{5.15}$$

Let $\mathbb{R}^N \setminus S = C_1 \cup C_2$, where C_1 is the open bounded component of $\mathbb{R}^N \setminus S$. Let $\Omega_n^1 := (\mathbb{R}^N \setminus I_n(S)) \cap C_1$ and $\Omega_n^2 = (\mathbb{R}^N \setminus I_n(S)) \cap C_2$. Let ν_n^1, ν_n^2, ν_n denote the outer unit normals to $\partial\Omega_n^1$, $\partial\Omega_n^2$ and $\partial I_n(S)$, respectively.

Lemma 5.3. *Let $0 \leq T < \frac{\alpha}{2N}$. For n large enough and almost all $t \in [0, T]$, we have that*

$$\begin{aligned}&\int_{\Omega_n^i}(z_n(t), Du_n(t)) = \int_{\Omega_n^i}|Du_n(t)|\,, \quad i = 1, 2\,, \\ &\int_{I_n(S)}(z_n(t), Du_n(t)) = \int_{I_n(S)}|Du_n(t)|\,,\end{aligned} \tag{5.16}$$

and

$$[z_n(t), \nu_n^1] = [z_n(t), \nu_n^2] = 1 \;\; \mathcal{H}^{N-1}\text{-a.e.} \tag{5.17}$$

Proof. Since u_n are strong solutions of (1.1), by Theorem 2.7, we know that $(z_n(t), Du_n(t)) = |Du_n(t)|$ as measures in $\mathbb{R}^N$ for almost all $t \geq 0$. This implies (5.16).

Note that estimates (5.9), (5.13) prove that for n large enough and for any $t \in [0,T]$ there is a jump discontinuity in $u_n(t)$ in $\partial I_n(S)$. This implies (5.17). Indeed, let u_{n1}^- and u_{n1}^+ be the traces of u_n in $\partial\Omega_n^1$ taken from inside and from outside the domain, respectively, and let u_{n2}^- and u_{n2}^+ be the traces of u_n in $\partial\Omega_n^2$ taken from outside and from inside the domain, respectively. We have

$$\begin{aligned}
\int_{\mathbb{R}^N} &|Du_n(t)| \\
&= \int_{\mathbb{R}^N} (z_n(t), Du_n(t)) = -\int_{\mathbb{R}^N} \mathrm{div}(z_n(t))u_n(t) \\
&= -\int_{\Omega_n^1} \mathrm{div}(z_n(t))u_n(t) - \int_{\Omega_n^2} \mathrm{div}(z_n)(t)u_n(t) - \int_{I_n(S)} \mathrm{div}(z_n)(t)u_n(t) \\
&= \int_{\Omega_n^1} (z_n(t), Du_n(t)) + \int_{\Omega_n^2} (z_n(t), Du_n(t)) + \int_{I_n(S)} (z_n(t), Du_n(t)) \\
&\quad - \int_{\partial\Omega_n^1} [z_n(t), \nu_n^1]u_n(t) - \int_{\partial\Omega_n^2} [z_n(t), \nu_n^2]u_n(t) - \int_{\partial I_n(S)} [z_n(t), \nu_n]u_n(t) \\
&= \int_{\mathbb{R}^N \setminus \partial I_n(S)} |Du_n(t)| + \int_{\partial\Omega_n^1} [z_n(t), \nu_n^1(t)](u_{n1}^+(t) - u_{n1}^-(t)) \\
&\quad + \int_{\partial\Omega_n^2} [z_n(t), \nu_n^2](u_{n2}^+(t) - u_{n2}^-(t))\,.
\end{aligned}$$

Since for any $t \in [0,T]$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N} &|Du_n(t)| \\
&= \int_{\mathbb{R}^N \setminus \partial I_n(S)} |Du_n(t)| + \int_{\partial\Omega_n^1} |u_{n1}^+(t) - u_{n1}^-(t)| + \int_{\partial\Omega_n^2} |u_{n2}^+(t) - u_{n2}^-(t)|
\end{aligned}$$

and by (5.9), (5.13), we know that for n large enough and any $t \in [0,T]$, $|u_{ni}^+ - u_{ni}^-| > 0 \mathcal{H}^{N-1}$ a.e. in $\partial\Omega_n^i$, comparing the previous two formulas we deduce (5.17). $\square$

To obtain a more precise estimate, we observe that

$$\frac{\mathrm{Per}(I_n(S))}{|I_n(S)|} = n + o_n\,,$$

with $\frac{o_n}{n} = 0$ as $n \to \infty$. We denote by d the signed distance function $d(x) := \mathrm{dist}(x, C_1) - \mathrm{dist}(x, C_2)$. It is well known that if S is of class C^p, then there exists $n_0 \in \mathbb{N}$ such that $d \in C^p(I_n(S))$ for all $n \geq n_0$ (see [18]).

Lemma 5.4. *Let* $0 < T < \frac{\alpha}{2}$.

(i) *Let* $\alpha_n = n + |o_n| + \sqrt{n}$, $w_n(t) = (\alpha\frac{\mathcal{H}^{N-1}(S)}{|I_n(S)|} - \alpha_n t)^+\chi_{I_n(S)}$, *with* $\eta_n(x) = -nd(x)\nabla d(x)$, $t \in [0,T]$. *Then, for* n *large enough we have*

$$(w_n)_t \leq \operatorname{div}(\eta_n) \text{ on } I_n(S)\,,$$

and $\eta_n \cdot \nu_n = -1$ *where* ν_n *is the outer unit normal to* $I_n(S)$.

(ii) *Let* $\beta_n = n - |o_n| - \sqrt{n}$, $W_n(t) = (b + \alpha\frac{\mathcal{H}^{N-1}(S)}{|I_n(S)|} - \beta_n t)^+\chi_{I_n(S)}$, $t \in [0,T]$, *with* $b \geq \mu_{ac}(x)$ *for almost all* $x \in I_n(S)$ *and* $\eta_n(x) = -nd(x)\nabla d(x)$. *Then, for* n *large enough we have*

$$(W_n)_t \geq \operatorname{div}(\eta_n) \text{ on } I_n(S)\,,$$

and $\eta_n \cdot \nu_n = -1$ *where* ν_n *is the outer unit normal to* $I_n(S)$.

Proof. We only prove (i) since the proof of (ii) is similar. Observe that by our choice of T, for n large enough we have that $(\alpha\frac{\mathcal{H}^{N-1}(S)}{|I_n(S)|} - \alpha_n T) > 0$. Observe that $(w_n)_t = -\alpha_n$ and

$$\operatorname{div}(\eta_n)(x) = -n(\nabla d(x) \cdot \nabla d(x)) - nd(x)\Delta d(x) = -n - nd(x)\Delta d(x)\,.$$

Then $(w_n)_t \leq \operatorname{div}(\eta_n)$ if and only if

$$-\alpha_n \leq -n - nd(x)\Delta d(x)\,,$$

on $I_n(S)$, i.e., if and only if

$$1 + d(x)\Delta d(x) \leq \alpha_n \frac{1}{n} = 1 + (|o_n| + \sqrt{n})\frac{1}{n}\,,$$

on $I_n(S)$, i.e., if and only if

$$d(x)\Delta d(x) \leq (|o_n| + \sqrt{n})\frac{1}{n}\,.$$

Now,

$$\Delta d(x) = \sum_{i=1}^{N-1} \frac{-k_i}{1 - k_i d(x)}\,,$$

where k_i are the principal curvatures of S at $y(x) \in S$, such that $d(x) = \|x - y(x)\|$. Hence, having in mind that S has bounded curvatures, for n large enough we have

$$\begin{aligned} d(x)\Delta d(x) &\leq d(x)\sum_{i=1}^{N-1} \frac{|k_i|}{1 - k_i d(x)} \leq d(x)\sum_{i=1}^{N-1} |k_i|(1 + 2|k_i|d(x)) \\ &\leq C(d(x) + d(x)^2)\,, \quad \forall\, x \in I_n(S)\,, \end{aligned}$$

where C is a constant bounding $\sum_{i=1}^{N-1} |k_i|$ and $\sum_{i=1}^{N-1} |k_i|^2$. Then by choosing n large enough we have that

$$d(x)\Delta d(x) \leq (|o_n| + \sqrt{n})\frac{1}{n}\,, \quad \forall\, x \in I_n(S)\,.$$

The condition $\eta_n \cdot \nu_n = -1$ follows immediately from the definition of η_n. □

Lemma 5.5. *For any* $t \in [0, \frac{\alpha}{2}[$ *we have*

$$w_n(t) \leq u_n(t) \leq W_n(t) \text{ on } I_n(S)\,. \tag{5.18}$$

Proof. First we consider a time interval $[0, \tau]$ with $\tau < \frac{\alpha}{2N}$. Now, by Lemma 5.3, we have that $u_n(t)\chi_{\mathrm{int}(I_n(S))}$, for n large enough, is the strong solution of the problem

$$\begin{cases} w_t = \mathrm{div}\left(\dfrac{Dw}{|Dw|}\right), & \text{in } (0,\tau) \times \mathrm{int}(I_n(S))\,, \\ [z, \nu_n] = -1\,, & \text{on } (0,\tau) \times \partial I_n(S)\,, \\ w(0) = \mu_{ac} + \dfrac{\alpha \mathcal{H}^k(S)}{|I_n(S)|}\,, & \text{in int } (I_n(S))\,. \end{cases} \tag{5.19}$$

Then, by the above lemma we have

$$(u_n)_t - (W_n)_t \leq \mathrm{div}(z_n(t)) - \mathrm{div}(\eta_n) \text{ on } I_n(S)\,.$$

Hence, applying Green's formula we get

$$\begin{aligned} &\int_{I_n(S)} ((u_n)_t - (W_n)_t)(u_n(t) - W_n(t))^+ \\ &\quad \leq \int_{I_n(S)} (\mathrm{div}(z_n(t)) - \mathrm{div}(\eta_n))(u_n(t) - W_n(t))^+ \\ &\quad = -\int_{I_n(S)} (z_n(t), D(u_n(t) - W_n(t))^+) + \int_{\partial I_n(S)} [z_n(t), \nu_n](u_n(t) - W_n(t))^+) \\ &\qquad + \int_{I_n(S)} (\eta_n, D(u_n(t) - W_n(t))^+) - \int_{\partial I_n(S)} [\eta_n, \nu_n](u_n(t) - W_n(t))^+) \\ &\quad = -\int_{I_n(S)} (z_n(t), D(u_n(t) - W_n(t))^+) + \int_{I_n(S)} (\eta_n, D(u_n(t) - W_n(t))^+)\,. \end{aligned}$$

Now, by the chain rule in BV (see [2]), there exists $0 \leq \xi(t)$, such that

$$D(u_n(t) - W_n(t))^+ = \xi(t) D(u_n(t) - W_n(t)) = \xi(t) D u_n(t)\,.$$

Then, since

$$\int_{I_n(S)} (z_n(t), D(u_n(t) - W_n(t))^+) = \int_{I_n(S)} |D(u_n(t) - W_n(t))^+|\,,$$

and

$$\int_{I_n(S)} (\eta_n, D(u_n(t) - W_n(t))^+) \leq \int_{I_n(S)} |D(u_n(t) - W_n(t))^+|\,,$$

it follows that

$$\frac{d}{dt}\int_{I_n(S)} \frac{1}{2}(u_n(t) - W_n(t))^+)^2$$

$$= \int_{I_n(S)} ((u_n)_t - (W_n)_t)(u_n(t) - W_n(t))^+ \le 0\,.$$

Thus, since $u_n(0) \le W_n(0)$, we obtain that $u_n(t) \le W_n(t)$ on $I_n(S)$. In a similar way, we obtain that $u_n(t) \ge w_n(t)$ on $I_n(S)$. Therefore, we conclude that (5.18) holds for any $t \in [0, \tau]$. Since this holds for any $\tau < \frac{\alpha}{2N}$, having in mind (3.7), we have that (5.18) holds in $[0, \frac{\alpha}{2N}]$. Observe that, for any $t > 0$

$$u_n(t) \le \| \mu_{ac} \|_\infty + \frac{Nt}{r} \text{ on } \mathbb{R}^N \setminus I_n(S)\,.$$

On the other hand we have

$$u_n\left(\frac{\alpha}{2N}\right) \ge w_n\left(\frac{\alpha}{2N}\right) \text{ on } I_n(S)\,.$$

Thus, using the above estimate and working as in (5.13) which is obtained by comparison with balls, for n large enough, we have still a jump in the solution $u_n(t)$ during the time interval $[\frac{\alpha}{2N}, 2\frac{\alpha}{2N} - \frac{\alpha}{2N^2})$. This means that Lemma 5.3 still holds in this time interval. Thus we may proceed as above in the proof to conclude that (5.18) holds in $[0, 2\frac{\alpha}{2N} - \frac{\alpha}{2N^2}]$. Thus, since in the k-iteration the time interval obtained is $[0, \frac{\alpha}{2}(1-(\frac{N-1}{N})^{k+1})]$, iteratively we prove that, for n large enough, (5.18) holds in $[0, \frac{\alpha}{2}[$. □

Remark 5.6. The estimates of Lemma 5.5 permit us to prove that $[z_n(t), \nu_n^i] = 1$, $i = 1, 2$, for any $t \in [0, T]$ (where $T < \frac{\alpha}{2}$) and n large enough.

Proof of Theorem 5.2. We observe that

$$w_n(t) \rightharpoonup (\alpha - 2t)^+ \mathcal{H}^{N-1} \llcorner S\,,$$

and also

$$W_n(t) \rightharpoonup (\alpha - 2t)^+ \mathcal{H}^{N-1} \llcorner S\,,$$

weakly* as measures. This and Lemma 5.5 imply that (5.15) holds in $[0, \frac{\alpha}{2})$. Now, using (3.8) we have

$$\int_{\mathbb{R}^N} \left| u_s\left(\frac{\alpha}{2}\right) - u_s\left(\frac{\alpha}{2} - h\right)\right| \le \int_{\mathbb{R}^N} \left| u\left(\frac{\alpha}{2}\right) - u\left(\frac{\alpha}{2} - h\right)\right|$$

$$\le \frac{4}{\alpha - 2h} h |\mu|(\mathbb{R}^N)\,,$$

and letting $h \to 0+$ we deduce that $u_s(\frac{\alpha}{2}) = 0$. Thus, (5.15) holds in $[0, \frac{\alpha}{2}]$. □

By (5.1), for each $T > 0$ there exists a constant $K_T = \|\mu_{ac}\|_\infty + \frac{NT}{r} > 0$ such that

$$0 \leq u_n(t,x) \leq K_T \ \forall\, x \in \mathbb{R}^N \setminus I_n(S)\,, \quad 0 \leq t \leq T\,. \tag{5.20}$$

hence $\operatorname{sign}(K_T + 1 - u_n(t,x)) = 1$ for all $x \in \partial\Omega_n^i$, $i = 1, 2$. Let $C_T = K_T + 1$. Using Remark 5.6 we have that for n large enough

$$[z_n(t), \nu_n^i] = \operatorname{sign}(C_T - u_n(t)) \tag{5.21}$$

when $t \in [0,T]$ and $T < \frac{\alpha}{2}$. Thus, according to Theorem 2.2, for n large enough $u_n(t)|_{\Omega_n^i}$ is the strong solution of the Dirichlet problem

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T) \times \Omega_n^i\,, \\ v = C_T\,, & \text{on } \partial\Omega_n^i \times (0,T)\,, \\ v(0) = \mu_{ac}\,, & \text{in } \Omega_n^i\,. \end{cases} \tag{5.22}$$

Now, for any $T < \frac{\alpha}{2}$ and n large enough we have that

$$C_T + 1 \leq w_n(T)\,.$$

Hence $\operatorname{sign}(C_T - u_n(t,x)) = -1$ for all $x \in \partial I_n(S)$. Again, using Remark 5.6 we have that $[z_n(t), \nu_n] = -1$ on $\partial I_n(S)$ when $t \in [0,T]$ and $T < \frac{\alpha}{2}$. According to Theorem 2.2, $u_n(t)|_{I_n(S)}$ is the strong solution of the Dirichlet problem

$$\begin{cases} w_t = \operatorname{div}\left(\dfrac{Dw}{|Dw|}\right), & \text{in } (0,T) \times \operatorname{int}(I_n(S))\,, \\ w = C_T\,, & \text{on } \partial I_n(S) \times (0,T)\,, \\ w(0) = \mu_{ac} + \dfrac{\alpha\mathcal{H}^k(S)}{|I_n(S)|}\,, & \text{in int } (I_n(S))\,. \end{cases} \tag{5.23}$$

We summarize the above discussion in the following Lemma.

Lemma 5.7. *Let $T < \frac{\alpha}{2}$. For n large enough, we have that $u_n(t)|_{\Omega_n^i}$, $i = 1, 2$, is the strong solution of problem* (5.22) *in* $[0,T]$; *and $u_n(t)|_{\operatorname{int}(I_n(S))}$ is the strong solution of problem* (5.23) *in* $[0,T]$.

Lemma 5.8. *The sequence u_n is bounded in $C([0,T], L^2(\Omega_n^i))$. More precisely, for any $\delta > 0$ we have*

$$\int_{\Omega_n^i} |(u_n)_t|^2 dx \leq C(\delta) \ \forall\, t \in [\delta, T]\,, \quad i = 1, 2\,. \tag{5.24}$$

Moreover, we have that

$$u_n(t)|_{\Omega_n^i} \to u(t)|_{C_i} \text{ in } L^2(C_i)\,, \quad i = 1, 2\,. \tag{5.25}$$

Proof. The assertion (5.24) is a consequence of Lemma 5.7, Theorem 2.2 and Proposition 2.4. Since C_1 is bounded, the convergence of $u_n(t)|_{\Omega_n^1} \to u(t)|_{C_1}$ in

$L^2(C_1)$ is a consequence of Theorem 5.1. To prove the corresponding asertion in C_2, let us prove the equiintegrability of u_n^2 at infinity. For that, let $M > 0$ be such that C_1 is contained in $B(0, M/4)$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be such that $\varphi = 0$ on $B(0, M/2)$, $\varphi = 1$ outside $B(0, M)$ and it increases linearly from 0 to 1 in $B(0, M) \setminus B(0, M/2)$. Since $u_n(t)|_{\Omega_n^2}$ is the strong solution of (5.22) ($i = 2$) by Lemma 5.7, multiplying the equation in (5.22) ($i = 2$) by $u_n\varphi^2$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_n^2} u_n(t)^2\varphi^2 + \int_{\Omega_n^2} |Du_n(t)|\varphi^2 = -\int_{\Omega_n^2} u_n(t) z_n(t) \cdot \nabla\varphi^2 \,.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_n^2} u_n(t)^2\varphi^2 \le 2\int_{\Omega_n^2} u_n(t)\varphi|\nabla\varphi| \le 2 \parallel u_n(t)\varphi \parallel_{p'} \parallel \nabla\varphi \parallel_p \,,$$

where $p > N$ and p' is its conjugate exponent. Since $|\nabla\varphi| \le \frac{2}{M}$, we have

$$\parallel \nabla\varphi \parallel_p \le \frac{2}{M}(CM^N)^{1/p} \le \frac{C}{M^{1-N/p}} \,.$$

Since $\parallel u_n(t) \parallel_{p'}$ is bounded independently of n by the complete accretivity of the operator underlying (5.22) ($i = 2$) we have that $\parallel u_n(t)\varphi \parallel_{p'}$ is bounded independently of n and M, and we may write

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_n^2} u_n(t)^2\varphi^2 \le \frac{C}{M^{1-N/p}} \,.$$

Thus, integrating in $[0, t]$, given $\epsilon > 0$ we find M large enough so that

$$\int_{\Omega_n^2} u_n(t)^2\varphi^2 \le \int_{\Omega_n^2} \mu_{ac}^2\varphi^2 + \epsilon \le 2\epsilon \,,$$

for all n. Now, using this and (5.3) we conclude that $u_n(t)|_{\Omega_n^2} \to u(t)|_{C_2}$ in $L^2(C_2)$. $\square$

Consider the following Dirichlet problems, $i = 1, 2$,

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T) \times C_i \,, \\ v = C_T \,, & \text{on } (0,T) \times S \,, \\ v(0) = \mu_{ac} \,, & \text{in } C_i \,. \end{cases} \tag{5.26}$$

Theorem 5.9. *$u(t)_{ac}|_{C_i}$ is the strong solution of problem* (5.26) *in* $[0, \frac{\alpha}{2})$, $i = 1$, 2.

Proof. Let $T < \frac{\alpha}{2}$. We shall prove in detail only the case $v(t) := u(t)_{ac}|_{C_1}$, the other case being similar. We divide the proof in three steps.

Step 1. By Lemma 5.8 we know that

$$v_n(t) := u_n(t)|_{\Omega_n^1} \to v(t) \text{ in } L^2(C_1) \,. \tag{5.27}$$

and

$$v_{nt} \to v_t \text{ weakly in } L^2_{\mathrm{loc}}((0,T), L^2(C_1)) . \tag{5.28}$$

Since $\|z_n\|_\infty \le 1$ for all $n \in \mathbb{N}$, we may assume that

$$z_n \rightharpoonup z \in L^\infty(]0,T[\times C_1, \mathbb{R}^N) \text{ weakly}^* . \tag{5.29}$$

Passing to the limit we deduce that

$$v_t = \mathrm{div}_x(z) \text{ in } \mathcal{D}'(]0,T[\times C_1) . \tag{5.30}$$

On the other hand, if we take $\eta(t,x) = \phi(t)\psi(x)$ with $\phi \in \mathcal{D}(]0,T[)$ and $\psi \in \mathcal{D}(C_1)$, the same calculation as above shows that

$$v_t(t) = \mathrm{div}_x(z(t)) \text{ in } \mathcal{D}'(C_1) \text{ a.e. } t \in [0,T] . \tag{5.31}$$

Step 2. Consider the functions $\tilde{v}_n(t)$ defined by

$$\tilde{v}_n(t)(x) := \begin{cases} v_n(t)(x) , & \text{if } x \in \Omega_n^1 , \\ C_T , & \text{if } x \in I_n(S) \cap C_1 . \end{cases}$$

Let $\Phi : L^2(C_1) \to]-\infty, +\infty]$ the functional defined by

$$\Phi(w) := \begin{cases} \displaystyle\int_{C_1} |Dw| + \int_{\partial C_1} |C_T - w| d\mathcal{H}^{N-1} , & \text{if } w \in L^2(C_1) \cap BV(C_1) , \\ +\infty , & \text{if } w \notin BV(C_1) . \end{cases}$$

Since the functional Φ is lower semicontinuous [4] and we have (5.27), we may write

$$\begin{aligned} &\int_{C_1} |Dv(t)| + \int_{\partial C_1} |C_T - v(t)| d\mathcal{H}^{N-1} \\ &\quad = \Phi(v(t)) \le \liminf_{n\to\infty} \Phi(\tilde{v}_n(t)) \\ &\quad = \liminf_{n\to\infty} \int_{C_1} |D\tilde{v}_n(t)| = \liminf_{n\to\infty} \left(\int_{\Omega_n^1} |Dv_n(t)| + \int_{\partial\Omega_n^1} (C_T - v_n^1(t)) d\mathcal{H}^{N-1} \right) \\ &\quad = \liminf_{n\to\infty} \left(- \int_{\Omega_n^1} (v_n)_t v_n(t) + \int_{\partial\Omega_n^1} C_T d\mathcal{H}^{N-1} \right) \\ &\quad = \liminf_{n\to\infty} \left(-\frac{d}{dt} \int_{\Omega_n^1} \frac{1}{2} |v_n(t)|^2 + \int_{\partial\Omega_n^1} C_T d\mathcal{H}^{N-1} \right) . \end{aligned}$$

Hence, using Fatou's Lemma, we have

$$\begin{aligned}
&\int_0^T \int_{C_1} |Dv(t)| + \int_0^T \int_{\partial C_1} |C_T - v(t)| d\mathcal{H}^{N-1} \\
&\quad \le \liminf_{n\to\infty} \left(\int_{\Omega_n^1} \left(\frac{1}{2}|v_n(0)|^2 - \frac{1}{2}|v_n(T)|^2 \right) + \int_0^T \int_{\partial\Omega_n^1} C_T d\mathcal{H}^{N-1} \right) \\
&\quad = \int_{C_1} \left(\frac{1}{2} v(0)^2 - \frac{1}{2} v(T)^2 \right) + \int_0^T \int_{\partial C_1} C_T d\mathcal{H}^{N-1}\,.
\end{aligned}$$

Therefore, $v(t) \in BV(C_1)$ for almost all $t \in [0, T]$.

Let ν^1 be the outer unit normal to ∂C_1. Then, since $\|[z_n(t), \nu^1]\|_\infty \le \|z_n(t)\|_\infty \le 1$, up to extraction of a subsequence, if necessary, we may assume that

$$[z_n(\cdot), \nu^1] \rightharpoonup \rho \ \sigma[L^\infty((0,T)\times\partial C_1), L^1((0,T)\times\partial C_1)]\,.$$

Now, working as in the proof of Step 4 of [4, Theorem 1], we get

$$\rho(t) = [z(t), \nu^1]\mathcal{H}^{N-1}\text{-a.e. on } \partial C_1, \text{a.e. } t \in [0,T]\,.$$

Let us prove that $\rho(t) = 1$. For that, let $w(t) := \eta(t)\chi_{C_1}$ where $\eta(t) \in \mathcal{D}(0,T)$. Using Lemma 5.8, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega_n^1} v_n'(t,x) w(t,x) dx dt \\
&\quad \to \int_0^T \int_{C_1} v_t(t,x) w(t,x) dx dt = \int_0^T \int_{C_1} v_t(t,x)\eta(t) dx dt\,.
\end{aligned}$$

Now

$$\begin{aligned}
\int_0^T \int_{\Omega_n^1} v_n' w dx dt &= \int_0^T \eta(t) \int_{\Omega_n^1} \operatorname{div}(z_n) dx dt \\
&= \int_0^T \eta(t) \int_{\partial\Omega_n^1} [z_n(t), \nu_n^1] d\mathcal{H}^{N-1} dt \\
&= \mathcal{H}^{N-1}(\partial\Omega_n^1) \int_0^T \eta(t) dt\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^T \int_{C_1} v_t \eta dx dt &= \int_0^T \eta(t) \int_{C_1} \operatorname{div}(z) dx dt \\
&= \int_0^T \eta(t) \int_{\partial C_1} [z(t), \nu^1] d\mathcal{H}^{N-1} dt\,.
\end{aligned}$$

Thus, we have

$$\int_0^T \eta(t) \int_{\partial C_1} [z(t), \nu^1] d\mathcal{H}^{N-1} dt = \mathcal{H}^{N-1}(C_1) \int_0^T \eta(t) dt\,.$$

It follows that $[z(t), \nu^1] = 1\mathcal{H}^{N-1}$-a.e. on ∂C_1 and a.e. $t \in [0, T]$.

Step 3. Finally, we are going to prove that v verifies the inequalities (3.3). Let $w \in C^1(\overline{C_1})$ and $\eta \in \mathcal{D}(0,T)$. Then, working as in Step 2 and using (5.21) we have

$$\begin{aligned}
&\int_0^T \int_{C_1} (u(t)-w)u_t(t)\eta(t)dxdt + \int_0^T \int_{C_1} |Du(t)|\eta(t)dt \\
&\qquad + \int_0^T \int_{\partial C_1} |C_T - u(t)| d\mathcal{H}^{N-1} dt \\
&\quad \leq \liminf_n \int_0^T \int_{\Omega_n^1} (u_n(t)-w)u_{nt}(t)\eta(t)dxdt \\
&\qquad + \int_0^T \int_{\Omega_n^1} |Du_n(t)|\eta(t)dt + \int_0^T \int_{\partial\Omega_n^1} |C_T - u_n(t)| d\mathcal{H}^{N-1} dt \\
&\quad = \liminf_n \int_0^T \int_{\Omega_n^1} (u_n(t)-w)\operatorname{div}(z_n(t))\eta(t)dxdt + \int_0^T \int_{\Omega_n^1} |Du_n(t)|\eta(t)dt \\
&\qquad + \int_0^T \int_{\partial\Omega_n^1} |C_T - u_n(t)| d\mathcal{H}^{N-1} dt \\
&\quad \leq \liminf_n \Bigg(-\int_0^T \int_{\Omega_n^1} z_n(t)\cdot D(u_n(t)-w)\eta(t)dxdt \\
&\qquad + \int_0^T \int_{\partial\Omega_n^1} [z_n(t),\nu_n^1](u_n(t)-w)\eta d\mathcal{H}^{N-1}dt + \int_0^T \int_{\Omega_n^1} |Du_n(t)|\eta(t)dt \\
&\qquad + \int_0^T \int_{\partial\Omega_n^1} |C_T - u_n(t)| d\mathcal{H}^{N-1} dt \Bigg) \\
&\quad = \liminf_n \int_0^T \int_{\Omega_n^1} z_n(t)\cdot Dw\eta(t)dxdt \\
&\qquad + \int_0^T \int_{\partial\Omega_n^1} |C_T - w|\eta d\mathcal{H}^{N-1} dt \\
&\quad = \int_0^T \int_{C_1} (z(t), Dw)\eta(t)dxdt \\
&\qquad + \int_0^T \int_{\partial C_1} |C_T - w|\eta d\mathcal{H}^{N-1} dt\,.
\end{aligned}$$

Observe that in the last limit we have used the fact that

$$\mathcal{H}^{N-1} \llcorner \partial\Omega_n^1 \to \mathcal{H}^{N-1} \llcorner \partial C_1 \text{ in the distributional sense},$$

which is true because ∂C_1 has bounded curvatures. Now, approximating a function $w \in L^2(C_1)\cap W^{1,1}(C_1)$ by functions in $C^1(\overline{C_1})$ we obtain that the above inequality also holds for all $w \in L^2(C_1) \cap W^{1,1}(C_1)$. This implies that the inequalities (3.3) hold for all $w \in L^2(C_1) \cap W^{1,1}(C_1)$ and a.e. in $[0,T]$. Finally, approximating functions in $L^2(C_1) \cap BV(C_1)$ by functions in $L^2(C_1) \cap W^{1,1}(C_1)$ we obtain that the inequalities (3.3) hold for all $w \in L^2(C_1) \cap BV(C_1)$ and a.e. in $[0,T]$. □

From Theorem 5.9, we have the following characterization of limit solutions.

Theorem 5.10. *Assume that* $\mu = \mu_{ac} + \alpha \mathcal{H}^{N-1} \llcorner S$, *with* $\alpha \geq 0$, $\mu_{ac} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *and* S *is a compact* $(N-1)$*-manifold in* $\mathbb{R}^N$ *of class* $W^{3,\infty}$. *If* $u(t)$ *is the limit solution of problem* (1.1) *corresponding to the initial condition* μ, *then in the time interval* $[0, \frac{\alpha}{2}]$ *we have that* $u(t)_{ac}|_{C_i}$ *is the strong solution of problem* (5.26), $i = 1, 2$, *and we have*

$$u(t)_s = \left(1 - \frac{2}{\alpha} t\right)^+ \mu_s \,. \tag{5.32}$$

For $t \geq \frac{\alpha}{2}$, $u(t)_s = 0$ *and* $u(t) = u(t)_{ac}$ *is the entropy* (*or equivalently, strong*) *solution of* (1.1) *in* $[\frac{\alpha}{2}, \infty) \times \mathbb{R}^N$ *with initial condition* $u(\frac{\alpha}{2})$.

Proof. The behaviour of $u(t)$ in $[0, \frac{\alpha}{2})$ was described in Theorems 5.2 and 5.9. According to (5.4) and (5.15), for $t = \frac{\alpha}{2}$ we deduce

$$u_n\left(\frac{\alpha}{2}\right) \chi_{I_n(S)} \to 0 \text{ in } L^1(\mathbb{R}^N) \,.$$

Now, by (5.1), there is a positive constant C such that

$$u_n\left(\frac{\alpha}{2}\right) = u_n\left(\frac{\alpha}{2}\right) \chi_{\mathbb{R}^N \setminus I_n(S)} + u_n\left(\frac{\alpha}{2}\right) \chi_{I_n(S)} \leq C + u_n\left(\frac{\alpha}{2}\right) \chi_{I_n(S)} \,.$$

Hence,

$$\left(u_n\left(\frac{\alpha}{2}\right) - C\right)^+ \leq u_n\left(\frac{\alpha}{2}\right) \chi_{I_n(S)} \,.$$

Now, by estimate (3.4) we have

$$u_n(t) << u_n\left(\frac{\alpha}{2}\right) \text{ for any } t \geq \frac{\alpha}{2} \,, \tag{5.33}$$

consequently

$$\int_{\mathbb{R}^N} (u_n(t) - C)^+ \leq \int_{\mathbb{R}^N} \left(u_n\left(\frac{\alpha}{2}\right) - C\right)^+ \text{ for } t \geq \frac{\alpha}{2} \,,$$

and we have that

$$\int_{\mathbb{R}^N} (u_n(t) - C)^+ \to 0 \text{ for } t \geq \frac{\alpha}{2} \,.$$

Thus, having in mind that $|I_n(S)| \to 0$ and $u_n(t) \leq (u_n(t) - C)^+ + C$, we deduce that

$$u_n(t) \chi_{I_n(S)} \to 0 \text{ in } L^1(\mathbb{R}^N) \text{ for } t \geq \frac{\alpha}{2} \,.$$

Since $u_n(t)\chi_{I_n(S)} \rightharpoonup u(t)_s$ weakly* as measures we conclude that $u(t)_s = 0$ for $t \geq \frac{\alpha}{2}$.

Hence

$$u_n(t) = u_n(t)\chi_{\mathbb{R}^N \setminus I_n(S)} + u_n(t)\chi_{I_n(S)} \to u(t)_{ac} \text{ in } L^1_{\text{loc}}(\mathbb{R}^N)\,.$$

By the equintegrability in time given by estimate (3.7), the above convergence can be taken locally uniformly in $(0,T)$. Since $u_n(t)$ is an entropy solution of (1.1) and converges in $L^1_{\text{loc}}((\frac{\alpha}{2},\infty)\times\mathbb{R}^N)$ to $u(t) = u(t)_{ac}$, then $u(t)$ is an entropy solution of (1.1) in $[\frac{\alpha}{2},\infty)\times\mathbb{R}^N$ [8]. Since $u(t) \in L^2(\mathbb{R}^N)$ and entropy solutions coincide with strong solutions, we have that $u(t)$ is also a strong solution of (1.1) for $t \geq \frac{\alpha}{2}$ [8]. □

We could write an entropy condition for the solutions described in Theorem 5.10, similar to the one considered in Sec. 5.1, but not being satisfactory for a flexible treatment of uniqueness in the general case, we shall not pursue this here.

Remark 5.11. As it was observed to us by the referee, Theorem 5.10 can be extended to more irregular $(N-1)$-manifolds. Indeed, it can be extended to the case where S is a closed and Lipschitz $(N-1)$-manifold which can be approximated by closed $(N-1)$-manifolds S_n of class $W^{3,\infty}$ in the sense that

$$\text{(i)} \quad \|\mathcal{H}^{N-1} \llcorner S - \mathcal{H}^{N-1} \llcorner S_n\|_{\mathcal{M}_b(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty\,,$$

and (ii) if Q_n denotes the set inside S_n, and Q denotes the set inside S, then $Q \cap Q_n$ is an increasing sequence whose union is Q. To justify this assertion, let $u_n(t)$ be the strong solution of (1.1) such that $u_n(0) = \mu_{ac} + \alpha\mathcal{H}^{N-1} \llcorner S_n$ given by Theorem 5.10. Since $u_n(t)_s = (1-\frac{2}{\alpha}t)^+\mathcal{H}^{N-1}\llcorner S_n$, we have that $u_n(t)_s \to= (1-\frac{2}{\alpha}t)^+\mathcal{H}^{N-1}\llcorner S$ in the norm of measures. On the other hand, we know that $u_n(t)_{ac}|_{Q_n}$ is the strong solution of the problem

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T)\times Q_n\,, \\ [z,\nu] = 1\,, & \text{on } (0,T)\times\partial Q_n\,, \\ v(0) = \mu_{ac}\,, & \text{in } Q_n\,. \end{cases} \tag{5.34}$$

Then, after some standard calculations and using that

$$\partial(Q\cap Q_n) \subseteq [S\cap S_n \cap S_m] \cup [S\Delta S_n] \cup [S_n \Delta S_m]\,,$$

we prove that

$$\begin{aligned} \frac{d}{dt}\int_{Q\cap Q_n} |u_n(t) - u_m(t)| &\leq \int_{\partial(Q\cap Q_n)} |[z_n - z_m, \nu]| \\ &\leq 2(\mathcal{H}^{N-1}(S\Delta S_n) + \mathcal{H}^{N-1}(S_n\Delta S_m))\,, \end{aligned}$$

for every $n \geq 1$, and every $m \geq n$. In particular, since $u_m(0) = u_n(0)$, we have

$$\int_{Q\cap Q_n} |u_n(t) - u_m(t)| \leq 2T(\mathcal{H}^{N-1}(S\Delta S_n) + \mathcal{H}^{N-1}(S_n\Delta S_m))\,,$$

for any $t \in [0,T]$, every $n \geq 1$, and every $m \geq n$. Thus

$$\int_{Q\cap Q_p} |u_n(t) - u_m(t)| \leq 2T(\mathcal{H}^{N-1}(S\Delta S_n) + \mathcal{H}^{N-1}(S_n \Delta S_m))\,,$$

for any $t \in [0,T]$, and for all $m \geq n \geq p$. We deduce that $\{u_n\}$ is a Cauchy sequence in $C([0,T], L^1(Q\cap Q_p))$ for any $p \geq 1$. Since $u_n(t)$ is bounded in $L^\infty([0,T], L^1(Q_n))$, there is a function $u \in L^\infty([0,T], L^1(Q))$ such that, passing to a subsequence, if necessary, u_n converges to $u(t)$ in $C([0,T], L^1(Q\cap Q_p))$ for any $p \geq 1$. Moreover, we may assume that $z_n \to z$ weakly* in $L^\infty(]0,T[\times Q)$, and we obtain that

$$u_t = \mathrm{div}(z) \text{ in } \mathcal{D}'(]0,T[\times Q\,.$$

To prove that $[z(t),\nu] = 1\mathcal{H}^{N-1}$ a.e. in S, let $\varphi(t,x) = \phi(x)\eta(t)$ where $\phi \in C_0^\infty(Q)$ and $\eta \in C_0^\infty(]0,T[)$. Since for p large enough, and $n \geq p$, we have

$$\int_0^T \int_{Q\cap Q_p} u_n \varphi' = \int_0^T \int_{Q\cap Q_p} z_n \cdot \nabla\varphi + \int_0^T \int_{\partial(Q\cap Q_p)} [z_n(t),\nu]\varphi\,,$$

letting $n \to \infty$ and $p \to \infty$ in this order, we get

$$\int_0^T \int_Q u\varphi' = \int_0^T \int_Q z \cdot \nabla\varphi + \int_0^T \int_S \varphi\,.$$

Consequently, $[z(t),\nu] = 1\mathcal{H}^{N-1}$ a.e. in S and for almost all $t \in]0,T[$. Now, working in a similar way as in [4], it can be proved that $u(t)|_Q$ is an entropy solution of problem

$$\begin{cases} v_t = \mathrm{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T)\times Q\,, \\ [z,\nu] = 1\,, & \text{on } (0,T)\times \partial Q\,, \\ v(0) = \mu_{ac}\,, & \text{in } Q\,. \end{cases} \tag{5.35}$$

In a similar way we can prove that there is a subsequence of $u_n|_{\mathbb{R}^N\setminus Q_n}$ converging to an entropy solution $u(t)|_{\mathbb{R}^N\setminus Q}$ of

$$\begin{cases} v_t = \mathrm{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T)\times (\mathbb{R}^N\setminus Q)\,, \\ [z,\nu] = 1\,, & \text{on } (0,T)\times \partial(\mathbb{R}^N\setminus Q)\,, \\ v(0) = \mu_{ac} \text{ in } \mathbb{R}^N\setminus Q\,. \end{cases} \tag{5.36}$$

We conclude with this our sketch of the proof. A complete discussion of this problem will be detailed elsewhere.

5.2. *Singular part of μ of dimension $k < N-1$*

In the case $k < N-1$, we assume S to be a compact k-manifold of class $W^{3,\infty}$. We have $C_1 = \emptyset$. Thus, $\Omega_n^1 = \emptyset$ and $\Omega_n^2 = \mathbb{R}^N \setminus I_n(S)$. Let us rename $\tilde{\Omega}_n := \mathbb{R}^N \setminus I_n(S)$. Let $T > 0$. Since $|I_n(S)|$ behaves as $\frac{1}{n^{N-k}}$ as $n \to \infty$, we note that estimates (5.9), (5.13) prove that, for n large enough, there is a jump discontinuity in $u_n(t)$ in $\partial I_n(S)$ for any $t \in [0,T]$. As in Lemma 5.3 we have

Lemma 5.12. *Let $T > 0$. For n large enough we have that*

$$[z_n(t), \nu_n^2] = -[z_n(t), \nu_n] = 1 \quad \mathcal{H}^{N-1}\text{-a.e. and a.e. in } [0,T]\,. \tag{5.37}$$

With the same notation for C_T as in the previous section we have the following result.

Lemma 5.13. *Let $T > 0$. For n large enough, $u_n(t)|_{\tilde{\Omega}_n}$ is the strong solution of problem*

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T) \times \tilde{\Omega}_n\,, \\ v = C_T\,, & \text{on } (0,T) \times \partial\tilde{\Omega}_n\,, \\ v(0) = \mu_{ac}\,, & \text{in } \tilde{\Omega}_n\,. \end{cases} \tag{5.38}$$

and $u_n(t)|_{\mathrm{int}(I_n(S))}$ is the strong solution of problem (5.23).

Consider the Dirichlet problem

$$\begin{cases} v_t = \operatorname{div}\left(\dfrac{Dv}{|Dv|}\right), & \text{in } (0,T) \times \mathbb{R}^N\,, \\ v(0) = \mu_{ac}\,, & \text{in } \mathbb{R}^N\,. \end{cases} \tag{5.39}$$

Working as in the proof of Theorem 5.9, we obtain the following result.

Theorem 5.14. *$u(t)_{ac}$ is the strong solution of problem* (5.39).

By (5.8) we also have that $u(t)_s = \mu_s$ for all $t \geq 0$. Then, by Theorem 5.14, we have the following characterization of the limit solutions

Theorem 5.15. *Assume that $k < N-1$. Let $\mu = \mu_{ac} + \alpha\mathcal{H}^k \llcorner S$, with $\alpha \geq 0$, $\mu_{ac} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and S is a compact k-manifold of class $W^{3,\infty}$. If $u(t)$ is the limit solution of problem* (1.1) *corresponding to the initial condition $u(0) = \mu$, then $u(t)_{ac}$ is the strong solution of problem* (5.39) *and we have $u(t)_s = \mu_s$ for any $t \geq 0$. In the particular case that $\mu_{ac} = 0$, we have $u(t) = \mu$ for all $t \geq 0$.*

Remark 5.16. In a similar way as we noted in Remark 5.11, Theorem 5.15 can be extended to more irregular k-manifolds ($k < N-1$). Indeed, it can be extended to the case where S is a closed and Lipschitz k-manifold which can be approximated by closed k-manifolds S_n of class $W^{3,\infty}$ in the sense that

$$\|\mathcal{H}^{N-1} \llcorner S - \mathcal{H}^{N-1} \llcorner S_n\|_{\mathcal{M}_b(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty\,.$$

To justify this assertion, let $u_n(t)$ be the solution of (1.1) such that $u_n(0) = \mu_{ac} + \alpha\mathcal{H}^k \llcorner S_n$ given by Theorem 5.15. Since $u_n(t)_s = \alpha\mathcal{H}^k \llcorner S_n$, we have that $u_n(t)_s \to u(t)_s := \alpha\mathcal{H}^k \llcorner S$ in the norm of measures. Since $u_n(t)_{ac}$ is the strong solution of

$u_t = \mathrm{div}(\frac{Du}{|Du|})$ in $\mathbb{R}^N$ with initial datum $u_n(0)_{ac} = \mu_{ac}$, we conclude that $u(t) = u(t)_{ac} + \alpha\mathcal{H}^k \llcorner S$, where $u(t)_{ac}$ is the strong solution of $u_t = \mathrm{div}(\frac{Du}{|Du|})$ in $\mathbb{R}^N$ with initial datum $u(0)_{ac} = \mu_{ac}$. Thus, Theorem 5.15 also holds in this case.

5.2.1. *The entropy condition when $k < N-1$*

Since $u(t)_{ac}$ is the strong solution of the Cauchy problem (5.39), there exists $z \in L^\infty(]0,T[\times\mathbb{R}^N)$ with $\|z\|_\infty \le 1$ such that

$$(u_{ac})_t = \mathrm{div}(z) \text{ in } \mathcal{D}'(]0,T[\times\mathbb{R}^N)\,. \tag{5.40}$$

Moreover $u(t)_{ac}$ satifies the entropy condition [8]

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^N} j_p(u(t)_{ac} - l)\eta_t + \int_0^T\int_{\mathbb{R}^N} \eta(t)|D(p(u(t)_{ac} - l) \\ &+ \int_0^T\int_{\mathbb{R}^N} z(t)\cdot\nabla\eta(t)p(u(t)_{ac} - l) \le 0\,, \end{aligned} \tag{5.41}$$

for all $l \in \mathbb{R}$ and $0 \le \eta(t,x) = \phi(t)\psi(x)$, with $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$ and for all $p \in \mathcal{T}$, being $j_p(r) = \int_0^r p(s)ds$.

Moreover, since $u(t)_s = \mu_s$ for all $t \ge 0$, given $\nu = \beta\mathcal{H}^k \llcorner S$, $\beta \in \mathbb{R}$, we have

$$-\int_0^T\int_{\mathbb{R}^N} j_p(u(t)_s - \nu)\eta_t = 0\,.$$

Hence, from (5.41) we obtain the following entropy condition for the limit solution $u(t)$:

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^N} j_p(u(t) - l - d\nu)\eta_t + \int_0^T\int_{\mathbb{R}^N} \eta(t)|D(p(u(t)_{ac} - l)| \\ &+ \int_0^T\int_{\mathbb{R}^N} z(t)\cdot\nabla\eta(t)p(u(t)_{ac} - l) \le 0\,, \end{aligned} \tag{5.42}$$

for all $l \in \mathbb{R}$, $\nu = \beta\mathcal{H}^k \llcorner S$, $\beta \in \mathbb{R}$, and $0 \le \eta(t,x) = \phi(t)\psi(x)$, with $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$ and for all $p \in \mathcal{T}$, being $j_p(r) = \int_0^r p(s)ds$.

Let us prove in which sense limit solutions are characterized by the entropy condition (5.42). Indeed, let $v \in C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))$ be such that $v(0) = \mu$, $v(t)_s = f(t)\mathcal{H}^k \llcorner S$, $0 < a \le f(t) \le A$, and satisfies $v_t = \mathrm{div}(\xi)$ in $\mathcal{D}'(]0,T[\times\mathbb{R}^N)$ and (5.42). Then, if we take in (5.42) $p = T_k^+$ and $\beta > A$, we get

$$\begin{aligned} &-\int_0^T\int_{\mathbb{R}^N} j_k^+(v(t)_{ac} - l)\eta_t + \int_0^T\int_{\mathbb{R}^N} \eta(t)|D(T_k^+(v(t)_{ac} - l)| \\ &+ \int_0^T\int_{\mathbb{R}^N} \xi(t)\cdot\nabla\eta(t)T_k^+(v(t)_{ac} - l) \le 0\,. \end{aligned} \tag{5.43}$$

Similarly, taking $p = T_k^-$ and $\beta < a$, we get

$$-\int_0^T \int_{\mathbb{R}^N} j_k^-(v(t)_{ac} - l)\eta_t + \int_0^T \int_{\mathbb{R}^N} \eta(t)|D(T_k^-(v(t)_{ac} - l)| + \int_0^T \int_{\mathbb{R}^N} \xi(t) \cdot \nabla\eta(t) T_k^-(v(t)_{ac} - l) \leq 0\,. \tag{5.44}$$

Now, from (5.43) and (5.44), using the doubling variables method of Kruzhkov (see [8]), it follows that $v(t)_{ac} = u(t)_{ac}$. On the other hand, since $v(t)_{ac} \in L^\infty(\mathbb{R}^N)$, taking $p = T_k^+$ in (5.42) and l large enough, we obtain that

$$-\int_0^T \int_{\mathbb{R}^N} j_k^+(v(t)_s - \nu)\eta_t \leq 0\,,$$

for $\nu = \beta\mathcal{H}^k \llcorner S$ and $0 \leq \eta(t,x) = \phi(t)\psi(x)$, with $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$. Then,

$$k\int_0^T \int_{\mathbb{R}^N} \frac{d}{dt}(v(t)_s - \nu)^+ \eta \leq 0\,.$$

Now, taking $\nu = v(0)_s$, it follows that

$$v(T)_s \leq v(0)_s = \mu_s\,.$$

Similarlly, working with T_k^-, we get $v(T)_s \geq v(0)_s = \mu_s$. Consequently, we obtain that

$$v(t)_s = \mu_s\,, \quad \forall\, t \geq 0\,.$$

6. Solutions Obtained by Approximating the Singular Part of $\boldsymbol{\mu}$ by Convolution in the Case $\boldsymbol{k < N - 1}$

Let $u_0 = \mu_{ac} + \mu_s$ with $\mu_s = a\mathcal{H}^k \llcorner S$ with $k < N - 1$, $a > 0$, and S being a compact k-manifold of class $W^{3,\infty}$. We assume that $\mu_{ac} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Let $\rho \in C_0^\infty(\mathbb{R}^N)$ be a radial decreasing function such that $\rho \geq 0$, whose support coincides with $B(0,1)$, and $\int_{\mathbb{R}^N} \rho(x)dx = 1$. Let $\rho_n(x) = n^N\rho(nx)$. Let us prove that if we approximate u_0 by $u_{0n} = \mu_{ac} + \rho_n * \mu_s$ and $u_n(t)$ denotes the solution of (1.1) with initial condition $u(0) = u_{0n}$, then $u_n(t)$ converges to the limit solution of (1.1) with initial condition $u(0) = u_0$, and consequently, $u(t)_s = \mu_s$ for all $t \geq 0$.

We fix $T > 0$. Let $\nu = \mathcal{H}^k \llcorner S$. In a first step, we shall need the following condition on ρ

$(H)_\rho : \rho \in C^\infty(B(0,1))$ is a radial decreasing function with $\rho \geq 0$, $\rho(x) = 0$ outside $B(0,1)$, $\int_{\mathbb{R}^N} \rho(x)dx = 1$, and if we write $\rho = \rho(\|x\|)$ the behavior of $\rho(1-r)$ near $r = 0$ is as γr^β for some $\gamma > 0$, $0 < \beta < \infty$.

Lemma 6.1. *Let $0 < \epsilon < \mathcal{H}^k(S)$, $\alpha_j \geq 0$, $\alpha_0 = 0$, $\alpha_j < \alpha_{j+1}$ be such that $\frac{\alpha_j}{\alpha_{j+1}} \geq (1-\epsilon)$ for all $j \geq 1$ and $\int_{[0 \leq \rho_n * \nu \leq \alpha_1]} (\rho_n * \nu)(x)dx \leq \epsilon$. Then*

$$\sum_{j=0}^{\infty} \alpha_j |[\alpha_j \leq \rho_n * \nu < \alpha_{j+1}]| \geq (1-\epsilon)(\mathcal{H}^k(S) - \epsilon)\,. \tag{6.1}$$

As a consequence, if $\nu_n(x) = \sum_{j=0}^{\infty} \alpha_j \chi_{[\alpha_j \leq \rho_n * \nu < \alpha_{j+1}]}$, we have

$$\int_{\mathbb{R}^N} (\rho_n * \nu - \nu_n)(x)dx \leq \epsilon(1 + (\mathcal{H}^k(S) - \epsilon))\,. \tag{6.2}$$

Proof.

$$\begin{aligned}
&\sum_{j=0}^{\infty} \alpha_j |[\alpha_j \leq \rho_n * \nu < \alpha_{j+1}]| \\
&\qquad \geq (1-\epsilon) \sum_{j=1}^{\infty} \alpha_{j+1} |[\alpha_j \leq \rho_n * \nu < \alpha_{j+1}]| \\
&\qquad \geq (1-\epsilon) \sum_{j=1}^{\infty} \int_{[\alpha_j \leq \rho_n * \nu < \alpha_{j+1}]} (\rho_n * \nu)(x)dx \\
&\qquad = (1-\epsilon) \int_{\mathbb{R}^N} (\rho_n * \nu)(x)dx - (1-\epsilon) \int_{[0 \leq \rho_n * \nu \leq \alpha_1]} (\rho_n(x) * \nu)(x)dx \\
&\qquad \geq (1-\epsilon)\mathcal{H}^k(S) - \epsilon(1-\epsilon) = (1-\epsilon)(\mathcal{H}^k(S) - \epsilon)\,.
\end{aligned}$$

The inequality (6.2) follows from (6.1) and the observation that $\nu_n \leq \rho_n * \nu$. □

Notice that we always have

$$\begin{aligned}
\rho_n * \nu(x) &\leq n^N \int_{\mathbb{R}^N} \chi_{B(0,\frac{1}{n})}(x-y)d\nu(y) \\
&\leq n^N \sup_{x \in \mathbb{R}^N} \nu\left(B\left(x, \frac{1}{n}\right)\right) \leq C n^{N-k}\,.
\end{aligned}$$

Lemma 6.2. *Assume that ρ satisfies $(H)_\rho$. Let $0 < q < 1$, $\alpha > 0$. Then*

$$\int_{[0 \leq \rho_n * \nu \leq \alpha n^{N-k-q}]} (\rho_n * \nu)(x)dx \to 0 \text{ as } n \to \infty\,. \tag{6.3}$$

Proof. Let $p > 0$ such that $p(\beta + \frac{k}{2}) < q$. Let $x \in \mathbb{R}^N$, $d(x,S) < \frac{1}{n} - \frac{1}{n^{1+p}}$. There is $Y \in S$ such that $\|x - Y\| < \frac{1}{n} - \frac{1}{n^{1+p}}$. Moreover, we may assume that $Y - x$ is orthogonal to S. Let us consider the k-plane H_k tangent to S at the point Y. Let H be the $N-1$ plane containing H_k and orthogonal to $Y - x$. By taking n large enough we may assume that, locally around Y, $S \cap B(x, \frac{1}{n})$ is the graph of a function $(X_{k+1}, \ldots, X_N) = g_Y(X_1, \ldots, X_k)$, $(X_1, \ldots, X_k) \in H_k \cap B(x, \frac{1}{n})$. Observe that, in the $(X_1, \ldots, X_N)$ coordinate system, the point Y has coordinates $(0, \ldots, 0)$ and $g_Y(0) = 0$, $Dg_Y(0) = 0$. The intersection of H and $B(x, \frac{1}{n})$ is a $N-1$ ball B_Y of radius, at least, $\frac{1}{n^{(p+2)/2}}$. Indeed, we take the $N-1$ plane H' parallel to H and tangent to $B(x, \frac{1}{n})$. We consider the surface of $B(x, \frac{1}{n})$ as a graph over H'. We compare the surface of $B(x, \frac{1}{n})$ with the paraboloid $x_N = k(x_1^2 + \cdots + x_{N-1}^2)$. By rotation invariance in the $N-1$ first coordinates we may reduce the situation to the comparison of the circle $y = \frac{1}{n} - \sqrt{(\frac{1}{n})^2 - x^2}$ and the parabola $y = \frac{kx^2}{2}$. Observe that if $k = n$, the parabola $y = kx^2$ is above the circle $y = \frac{1}{n} - \sqrt{(\frac{1}{n})^2 - x^2}$. Thus, when $y = \frac{1}{n^{p+1}}$ the x coordinate of the circle is, at least, $\frac{1}{n^{(p+2)/2}}$, which is the corresponding abscissa of the parabola. We conclude that the radius of B_Y is, at least, $\frac{1}{n^{(p+2)/2}}$. Now, observe that, since g_Y is smooth and the curvatures of S are bounded, there is a constant $C > 0$ such that, if $(X_1, \ldots, X_k) \in H_k \cap B(Y, \frac{1}{2n^{(p+2)/2}})$, then $\|(X_{k+1}, \ldots, X_N)\| \leq \frac{C}{n^{p+2}}$. Thus the distance of the graph of g_Y over $H_k \cap B(Y, \frac{1}{2n^{(p+2)/2}})$ (call it S_1) to $\partial B(x, \frac{1}{n})$ is greater or equal than the distance of the point of coordinates $(\frac{1}{n} - \frac{1}{n^{p+1}} + \frac{C}{n^{p+2}}, \frac{1}{2n^{(p+2)/2}})$ to the boundary of the ball $B(0, \frac{1}{n})$ in $\mathbb{R}^2$. This distance is greater or equal than $\frac{1}{4n^{p+1}}$. Thus, if $y \in S_1$, then the distance from ny to the boundary of $B(nx, 1)$ is greater or equal than $\frac{1}{4n^p}$. By our choice of ρ, we have that $\rho(n(x-y)) \geq \gamma \frac{1}{4n^{p\beta}}$ for all $y \in S_1$. Now,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \rho_n(x-y)\,d\nu(y) \\
&\quad = n^N \int_{\mathbb{R}^N} \rho(n(x-y))\,d\nu(y) \\
&\quad \geq n^N \int_{S_1} \rho(n(x-y))\,d\nu(y) \geq n^N \gamma \frac{1}{4n^{p\beta}} \nu(S_1) \geq \gamma' \frac{n^N}{n^{p\beta} n^{k(p+2)/2}}\,.
\end{aligned}$$

1 Thus, if $x \in \mathbb{R}^N$ is such that $d(x, S) < \frac{1}{n} - \frac{1}{n^{p+1}}$ and $\rho_n * \nu(x) < \alpha n^{N-k-q}$, then

$$\gamma' \frac{n^N}{n^{p\beta} n^{k(p+2)/2}} \leq \alpha n^{N-k-q}\,,$$

which implies that $q \leq p(\beta + \frac{k}{2})$, a contradiction. Thus, if $\rho_n * \nu(x) < \alpha n^{N-k-q}$, then $d(x, S) \geq \frac{1}{n} - \frac{1}{n^{p+1}}$. Hence

$$\begin{aligned}
\int_{[0 \leq \rho_n * \nu \leq \alpha n^{N-k-q}]} \rho_n * \nu\,dx &\leq \alpha n^{N-k-q} |[0 < \rho_n * \nu \leq \alpha n^{N-k-q}]| \\
&\leq \alpha n^{N-k-q} \left| \left\{ x \in \mathbb{R}^N : \frac{1}{n} - \frac{1}{n^{p+1}} \leq d(x, S) < \frac{1}{n} \right\} \right| \\
&\leq \alpha C n^{N-k-q} \frac{1}{n^{N-k+p}} = C\alpha \frac{1}{n^{p+q}} \to 0 \text{ as } n \to \infty\,,
\end{aligned}$$

3 where $C > 0$ is a constant. □

Lemma 6.3. *Assume that ρ satisfies $(H)_\rho$. Then given $\epsilon = \frac{1}{h} > 0$, there are constants $\alpha_j^{h,n} \geq 0$ such that $\alpha_0^{h,n} = 0$, $\alpha_j^{h,n} < \alpha_{j+1}^{h,n}$, $\frac{\alpha_j^{h,n}}{\alpha_{j+1}^{h,n}} \geq (1 - \frac{1}{h})$ for all j, h, $n \geq 1$ and $n_h \geq 1$ such that*

$$\int_{[0\leq \rho_n \leq \alpha_1^n]} (\rho_n * \nu)(x)dx \leq \epsilon \text{ for } n \geq n_h\,.$$

Therefore, if we denote $A_j^{h,n} = [\alpha_j^{h,n} \leq \rho_n * \nu(x) < \alpha_{j+1}^{h,n}]$, $j \geq 0$, and we define $\nu_{h,n}(x) = \sum_{j=1}^{\infty} \alpha_j^{h,n} \chi_{A_j^{h,n}}$, we have

$$\int_{\mathbb{R}^N} (\rho_n * \nu - \nu_{h,n})dx \leq \frac{1}{h}\left(1 + \mathcal{H}^k(S) - \frac{1}{h}\right) \text{ for all } n \geq n_h\,. \tag{6.4}$$

Proof. Let us choose p, $q > 0$ such that $p(\beta + \frac{k}{2}) < q \leq 1$. Given $\epsilon = \frac{1}{h}$, there is m_h such that $\frac{r}{r+1} \geq 1 - \frac{1}{h}$ for all $r \geq m_h$. We define $\alpha_0^{h,n} = 0$, $\alpha_1^{h,n} = m_h n^{N-k-q}$, $\alpha_j^{h,n} = (m_h + j - 1)n^{N-k-q}$. Then, by Lemma 6.2, we have

$$\int_{[0\leq \rho_n \leq \alpha_1^{h,n}]} (\rho_n * \nu)(x)dx \leq \frac{1}{h}\,,$$

for n large enough, say for $n \geq n_h$. By our choice of m_h, we have that $\frac{\alpha_j^{h,n}}{\alpha_{j+1}^{h,n}} \geq (1 - \frac{1}{h})$ for all $j \geq 1$. By Lemma 6.1, we have that (6.4) holds for all $n \geq n_h$. $\square$

In the rest of the section and until we consider the general case we assume that ρ satisfies condition $(H)_\rho$. Note that (6.4) holds for $n = n_h$. Thus, there is a sequence n_i such that $\rho_{n_i} * \nu - \nu_{i,n_i} \to 0$ in $L^1(\mathbb{R}^N)$. Thus, for simplicity of notation, we shall denote ν_n instead of ν_{i,n_i}, α_j^n instead of α_j^{i,n_i}, and A_j^n instead of A_j^{i,n_i}. With this notation and for further reference, we have

$$\int_{\mathbb{R}^N} (\rho_n * \nu - \nu_n)dx \to 0 \text{ as } n \to \infty\,. \tag{6.5}$$

The sets A_j^n are not far from being level sets of the distance function $d(x, S)$. To prove that we need the following Lemma.

Lemma 6.4. *Let $\lambda > 0$. Let S be a compact k-manifold of class $W^{2,\infty}$. Let ρ be the radial convolution kernel introduced above with the assumption that $\rho(1 - r)$ behaves as γr^2 near $r = 0$ for some $\gamma > 0$. Let*

$$\Phi(t, \vec{e}, H) = \int_{(t\vec{e}+H)\cap B(0,1)} \rho(u)d\mathcal{H}^k(u)\,, \tag{6.6}$$

for $t \in [0, 1]$, $\vec{e}$ a unit vector in $\mathbb{R}^N$, H a k-hyperplane orthogonal to $\vec{e}$. Then Φ depends only on t and, if x is such that $\rho_n * \nu(x) = \lambda n^{N-k-q}$, then we have

$$d(x, S) = \frac{1}{n}\Phi^{-1}\left(\frac{\lambda}{n^q} + n^k F(x)\right) \tag{6.7}$$

where

$$F(x) = O\left(\frac{1}{n^{k+1}}\right), \quad \nabla F(x) = O\left(\frac{1}{n^k}\right). \tag{6.8}$$

Proof. To avoid any confusion in the formalism and help the intuition let us observe that the result is obviously true if $k = 0$ and S is reduced to a finite number of points. Thus, we may assume that $k \geq 1$. Let x be such that

$$\int_S \rho_n(x-y)\,d\mathcal{H}^k(y) = \lambda n^{N-k-q}.$$

Then

$$\int_S \rho(n(x-y))\,d\mathcal{H}^k(y) = \frac{\lambda}{n^{k+q}}. \tag{6.9}$$

Let $x_0 \in S$ be such that $\|x - x_0\| = d(x, S)$ and such that $x - x_0$ is orthogonal to S. Let H_k be the tangent plane to S at x_0 passing by 0. We assume n to be large enough so that $S \cap B(x, \frac{1}{n})$ can be parameterized by a function $\psi : H_k \cap B(0, \frac{1}{n}) \to \mathbb{R}^{N-k}$ where $\psi \in W^{2,\infty}$. Thus we may write $S \cap B(x, \frac{1}{n})$ as the set of points $y = x_0 + (z, \psi(z))$ where $z \in Q := [(x_0 + H_k) \cap B(x, \frac{1}{n})] - x_0$. Moreover, we assume that $|\psi(z)| \leq C\|z\|^2$. Thus, using that ρ has compact support, we may write (6.9) as

$$\int_Q \rho(n(x-x_0) - n(z, \psi(z)))J_k(\psi)(z)\,dz = \frac{\lambda}{n^{k+q}}. \tag{6.10}$$

where $J_k(\psi)$ denotes the "corresponding" Jacobian [17]. Then, we have

$$\begin{aligned}
&\int_Q \rho(n(x-x_0) - n(z,0))\,dz \\
&\quad = \frac{\lambda}{n^{k+q}} + \int_Q [\rho(n(x-x_0) - n(z,0)) - \rho(n(x-x_0) - n(z,\psi(z)))J_k(\psi)(z)]\,dz \\
&\quad = \frac{\lambda}{n^{k+q}} + F(x)
\end{aligned}$$

where

$$\begin{aligned}
F(x) &= \int_Q [\rho(n(x-x_0) - n(z,0)) - \rho(n(x-x_0) - n(z,\psi(z)))J_k(\psi)(z)]\,dz \\
&= \int_Q [\rho(n(x-x_0) - n(z,0)) - \rho(n(x-x_0) - n(z,\psi(z)))]J_k(\psi)(z)\,dz \\
&\quad + \int_Q \rho(n(x-x_0) - n(z,0))[1 - J_k(\psi)(z)]\,dz \\
&=: \tau_1 + \tau_2
\end{aligned}$$

Let us estimate both terms τ_1 and τ_2,

$$|\tau_1| \le Cn \int_Q |\psi(z)| J_k(\psi)(z)dz \le Cn \int_Q \|z\|^2 J_k(\psi)(z)dz$$
$$\le \frac{C}{n}\int_Q J_k(\psi)(z)dz \le \frac{C}{n^{k+1}}$$

(the constant C denotes a different constant on each line). Now, using that $|1 - J_k(\psi)(z)| \le C\|z\|$, we have

$$|\tau_2| \le C\int_Q \rho(n(x-x_0) - n(z,0))|z|dz \le \frac{C}{n}\int_Q dz = \frac{C}{n^{k+1}}\,.$$

Similarly, since ρ is of class C^2, we bound

$$|\nabla\tau_1| \le \frac{C}{n^k}\,,$$

and

$$|\nabla\tau_2| \le \frac{C}{n^k}\,.$$

Summarizing, we have

$$\int_Q \rho(n(x-x_0) - n(z,0))dz = \frac{\lambda}{n^{k+q}} + F(x)\,, \tag{6.11}$$

with

$$F(x) = O\left(\frac{1}{n^{k+1}}\right), \quad \nabla F(x) = O\left(\frac{1}{n^k}\right). \tag{6.12}$$

Let $\vec{e}$ be the unit vector in the direction of $x - x_0$, so that $x - x_0 = \vec{e}d(x,S)$ and let $u = n(x-x_0) - n(z,0)$. Observe that $(z,0) \in Q$ if and only if $u \in (n\vec{e}d(x,S) + H_k) \cap B(0,1)$. Then we have

$$\int_Q \rho(n(x-x_0) - n(z,0))dz$$
$$= \frac{1}{n^k}\int_{(n\vec{e}d(x,S)+H_k)\cap B(0,1)} \rho(u)du = \frac{\lambda}{n^{k+q}} + F(x)\,. \tag{6.13}$$

Let Φ be the function defined in (6.6). Let us prove that Φ only depends on t. Let R be a rotation in $\mathbb{R}^N$ such that $R^t\vec{e} = \vec{e}$, i.e., such that $R(H)$ is orthogonal to $\vec{e}$. Then

$$\Phi(t, \vec{e}, R(H)) = \Phi(t, \vec{e}, H)\,.$$

Thus $\Phi(t, \vec{e}, H)$ is independent of H. Let us write it as $\Phi(t, \vec{e})$. Now, let R be any rotation in $\mathbb{R}^N$. Then, using any k-hyperplane H orthogonal to $R\vec{e}$, we have

$$\Phi(t, R\vec{e}) = \int_{(tR\vec{e}+H)\cap B(0,1)} \rho(u)du = \int_{(t\vec{e}+R^tH)\cap B(0,1)} \rho(u)du = \Phi(t, \vec{e})\,,$$

since $R^t(H)$ is orthogonal to $\vec{e}$. Thus, $\Phi = \Phi(t)$ only depends on t. Then Φ is continuous, and strictly decreasing. We may write (6.13) as

$$\Phi(nd(x,S)) = \Phi(nd(x,S), \vec{e}, H_k) = \frac{\lambda}{n^q} + n^k F(x)\,, \tag{6.14}$$

hence

$$d(x,S) = \frac{1}{n}\Phi^{-1}\left(\frac{\lambda}{n^q} + n^k F(x)\right). \tag{6.15}$$

□

Remark 6.5. By the properties of Φ we may write $\Phi(t) = \Phi_k(t)$,

$$\Phi_k(t) = \int_{(te_{k+1}+H'_k)\cap B_{k+1}(0,1)} \rho(u)du\,,$$

where $e_{k+1} = (0,\ldots,1) \in \mathbb{R}^{k+1}$, $H'_k = \{x : x_k = 0\}$, $B_{k+1}(0,1)$ the unit ball in $\mathbb{R}^{k+1}$.

Let us write $\alpha_j^n = \lambda_j^n n^{N-k-q}$. By Lemma 6.4, we may write

$$A_j^n = \left[\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q} + n^k F(x)\right) < d(x,S) \le \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q} + n^k F(x)\right)\right].$$

Let

$$B_j^n = \left[\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q}\right) < d(x,S) \le \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right)\right]$$

and let us define

$$\nu'_n = \sum_{j=1}^{\infty} \alpha_j^n \chi_{B_j^n}\,. \tag{6.16}$$

Lemma 6.6. *We have*

$$\int_{\mathbb{R}^N} |\nu_n - \nu'_n| dx \to 0 \text{ as } n \to \infty\,. \tag{6.17}$$

Proof. Let us prove that

$$A_j^n \subseteq \bigcup_{|j-i|\le 1} B_i^n\,, \tag{6.18}$$

and

$$B_j^n \subseteq \bigcup_{|j-i|\le 1} A_i^n\,. \tag{6.19}$$

With this, since $\alpha_{j+1}^n - \alpha_j^n = \alpha_j^n - \alpha_{j-1}^n = n^{N-k-q}$, we may write

$$\int_{\mathbb{R}^N} |\nu_n - \nu'_n| dx \le \sum_{j=1}^{Cn^q} n^{N-k-q} |A_j^n \Delta B_j^n|\,. \tag{6.20}$$

First, observe that, since (6.18) implies that $A_i^n \cap B_j^n = \emptyset$ if $|i-j| \geq 2$, (6.19) is a consequence of (6.18). Now, let us take n large enough so that

$$|n^k F(x)| \leq \frac{C}{n} \leq \frac{1}{n^q}\,.$$

Then, if $x \in A_j^k$, using the fact that Φ^{-1} is a decreasing function, we have

$$d(x,S) \leq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q} + n^k F(x)\right) \leq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q} - \frac{1}{n^q}\right) = \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j-1}^n}{n^q}\right),$$

and

$$\begin{aligned} d(x,S) &> \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q} + n^k F(x)\right) \\ &\geq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q} + \frac{1}{n}\right) = \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+2}^n}{n^q}\right). \end{aligned}$$

Both inequalities prove the inclusion (6.18).

Let

$$\begin{aligned} \epsilon(j,n,x) &= \sup_{i=j,j+1}\left|\Phi^{-1}\left(\frac{\lambda_i^n}{n^q} + n^k F(x)\right) - \Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right)\right|, \\ \epsilon_n^1 &= \sup_{j,x}|\epsilon(j,n,x)|, \\ \epsilon_n^2 &= \Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q}\right), \end{aligned}$$

and

$$\epsilon_n = \sup(\epsilon_n^1, \epsilon_n^2)\,.$$

Since Φ^{-1} is continuous and $n^k F(x) = O(\frac{1}{n})$ we have that $\epsilon_n \to 0$ as $n \to \infty$. Let us prove that

$$A_j^n \Delta B_j^n \subseteq \left[\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n}{n} \leq d(x,S) \leq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{\epsilon_n}{n}\right]. \tag{6.21}$$

Indeed, if $x \in B_j^n \setminus A_j^n$, then either (i) $d(x,S) > \frac{1}{n}\Phi^{-1}(\frac{\lambda_j^n}{n^q} + n^k F(x))$ or (ii) $d(x,S) < \frac{1}{n}\Phi^{-1}(\frac{\lambda_{j+1}^n}{n^q} + n^k F(x))$.

In case (i),

$$\begin{aligned} d(x,S) &> \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q} + n^k F(x)\right) - \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) \\ &\geq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon(j,n,x)}{n} \geq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n}{n}. \end{aligned}$$

On the other hand, since $x \in B_j^n$, we have

$$d(x,S) \leq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) \leq \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{\epsilon_n}{n}\,.$$

In case (ii),

$$d(x,S) < \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q} + n^k F(x)\right) \le \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q} - \frac{1}{n}\right)$$
$$= \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) \le \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{\epsilon_n}{n}\,.$$

On the other hand, since $x \in B_j^n$, we have

$$d(x,S) > \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_{j+1}^n}{n^q}\right) \ge \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n^2}{n} \ge \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n}{n}\,.$$

In a similar way we prove that $A_j^n \setminus B_j^n$ is contained in the right-hand side of (6.21). Since, by [2, Theorem 2.104],

$$\lim_{n\to\infty} \frac{|[d(x,S) < \frac{1}{n}\Phi^{-1}(\frac{\lambda_j^n}{n^q}) + \frac{\epsilon_n}{n}]|}{\omega_{N-k}\left(\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{\epsilon_n}{n}\right)^{N-k}} = \mathcal{H}^k(S)\,,$$

and

$$\lim_{n\to\infty} \frac{|[d(x,S) < \frac{1}{n}\Phi^{-1}(\frac{\lambda_j^n}{n^q}) - \frac{\epsilon_n}{n}]|}{\omega_{N-k}\left(\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n}{n}\right)^{N-k}} = \mathcal{H}^k(S)\,,$$

given $\delta > 0$, for n large enough, we have

$$\begin{aligned}
&|A_j^n \Delta B_j^n| \\
&\quad \le \left|\left[\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n}{n} \le d(x,S) \le \frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{\epsilon_n}{n}\right]\right| \le \omega_{N-k}\mathcal{H}^k(S) \\
&\qquad \cdot \left[(1+\delta)\left(\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \frac{\epsilon_n}{n}\right)^{N-k} - (1-\delta)\left(\frac{1}{n}\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \frac{\epsilon_n}{n}\right)^{N-k}\right] \\
&\quad = \mathcal{H}^k(S)\frac{\omega_{N-k}}{n^{N-k}}\left[\left(\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \epsilon_n\right)^{N-k} - \left(\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \epsilon_n\right)^{N-k}\right] \\
&\qquad + \delta\mathcal{H}^k(S)\frac{\omega_{N-k}}{n^{N-k}}\left[\left(\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) + \epsilon_n\right)^{N-k} + \left(\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right) - \epsilon_n\right)^{N-k}\right] \\
&\quad \le C\mathcal{H}^k(S)\frac{\omega_{N-k}}{n^{N-k}}\left(\Phi^{-1}\left(\frac{\lambda_j^n}{n^q}\right)\right)^{N-k-1}\epsilon_n + C\delta\mathcal{H}^k(S)\frac{\omega_{N-k}}{n^{N-k}} \\
&\quad \le C\mathcal{H}^k(S)\frac{\omega_{N-k}}{n^{N-k}}(\epsilon_n + \delta)\,.
\end{aligned}$$

Introducing the above estimate in (6.20), we obtain

$$\int_{\mathbb{R}^N} |\nu_n - \nu_n'|dx \le C\mathcal{H}^k(S)\sum_{j=1}^{Cn^q} n^{N-k-q}\frac{\omega_{N-k}}{n^{N-k}}(\epsilon_n + \delta) \le C(\epsilon_n + \delta)\,.$$

Letting $n \to \infty$, we obtain

$$\limsup_n \int_{\mathbb{R}^N} |\nu_n - \nu_n'| dx \le \delta\,.$$

Since this is true for all $\delta > 0$, we obtain (6.17). $\square$

Let $C_1^n = [d(x,S) \le \frac{1}{n}\Phi^{-1}(\frac{\lambda_1^n}{n^q})]$. Let us note that since $\frac{\lambda_1^n}{n^q} \to 0$ as $n \to \infty$ and $\Phi^{-1}(0+) = 1$, we have that $\Phi^{-1}(\frac{\lambda_1^n}{n^q})$ may be taken arbitrarily near to 1. According to (6.17), if $u_n(t)$ and $v_n(t)$ denote the solutions of (1.1) corresponding to the initial conditions $u_n(0) = \mu_{ac} + \rho_n \star (a\nu)$ and $v_n(0) = v_{0n} = \mu_{ac} + a\nu_n'$, then

$$\int_{\mathbb{R}^N} |u_n(t) - v_n(t)| dx \to 0 \text{ as } n \to \infty\,.$$

In particular, both $u_n(t)$ and $v_n(t)$ converge to the same solution $u(t)$ of (1.1) corresponding to the initial condition $u(0)$. Since the value α_1^n grows as n^{N-k-q} as $n \to \infty$ and $k \le N-2$, we have that α_1^n grows at least as n^{2-q} as $n \to \infty$. Thus the comparison given by estimates (5.9), (5.13) prove that for all $T > 0$ and n large enough, the solution $v_n(t)$ has a jump discontinuity at ∂C_1^n, and, therefore, if $\xi_n(t,x)$ denotes the vector field associated to $v_n(t)$, i.e., the vector field satisfying

$$(v_n)_t = \operatorname{div}(\xi_n) \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^N)\,,$$

and

$$\int_{\mathbb{R}^N} (\xi_n(t), Dv_n(t)) = \int_{\mathbb{R}^N} |Dv_n(t)|\,,$$

then we have that $[\xi_n(t), \nu^{C_1^n}] = -1$ a.e. t, and $\mathcal{H}^{N-1}$ in ∂C_1^n. Moreover we also obtain, like in Theorem 5.1, that

$$v_n(t)\chi_{C_1^n} \rightharpoonup u(t)_s \text{ weakly}^* \text{ as measure}\,. \tag{6.22}$$

Let us prove that $u(t)_s = \mu_s = a\mathcal{H}^k \llcorner S$. Given $\eta(t,x) := \phi(t)\psi(x)$, with $\phi \in \mathcal{D}(]0,T[)$ and $\psi \in \mathcal{D}(\mathbb{R}^N)$, since $v_n(t)|_{C_1^n}$ is the strong solution of problem

$$\begin{cases} w_t = \operatorname{div}\left(\dfrac{Dw}{|Dw|}\right), & \text{in } (0,T) \times \operatorname{int}(C_1^n)\,, \\ w = C_T\,, & \text{on } \partial C_1^n \times (0,T)\,, \\ w(0) = \mu_{ac} + a\nu_n'\,, & \text{in int } (C_1^n)\,. \end{cases} \tag{6.23}$$

We have

$$\begin{aligned} -\int_0^T \phi'(t) \int_{C_1^n} v_n(t)\psi &= -\int_0^T \int_{C_1^n} v_n(t)\eta_t = \int_0^T \int_{C_1^n} v_n'(t)\eta = \int_0^T \int_{C_1^n} \operatorname{div}(\xi_n(t))\eta \\ &= \int_0^T \phi(t) \left(-\int_{C_1^n} (\xi_n(t), \nabla\psi) + \int_{\partial C_1^n} [\xi_n(t), \nu_n]\psi d\mathcal{H}^{N-1} \right). \end{aligned}$$

Now, by (6.22) we have

$$\int_{C_1^n} v_n(t)\psi \to \langle u(t)_s, \psi\rangle\,.$$

Moreover,

$$\int_{C_1^n} (\xi_n(t), \nabla\psi) \to 0$$

and

$$\int_{\partial C_1^n} [\xi_n(t), \nu_n]\psi d\mathcal{H}^{N-1} = -\int_{\partial C_1^n} \psi d\mathcal{H}^{N-1} \to 0\,.$$

Therefore, taking limits as $n \to \infty$, we obtain that

$$\int_0^T \phi'(t)\langle u(t)_s, \psi\rangle dt = 0\,, \quad \forall\ \phi \in \mathcal{D}(]0, T[)\,.$$

Hence

$$\frac{d}{dt}\langle u(t)_s, \psi\rangle = 0 \text{ in } \mathcal{D}'(]0, T[)\,, \quad \forall\ \psi \in \mathcal{D}(\mathbb{R}^N)\,.$$

Thus, we have proved that

$$u(t)_s = \mu_s\,, \quad \forall\ t \geq 0,\ \text{if}\ k < N-1\,.$$

In conclusion, under the assumption that ρ satisfies $(H)_\rho$ with $\beta = 2$ we have proved the following result.

Theorem 6.7. *Assume that $k < N-1$. Let $\mu = \mu_{ac} + a\mathcal{H}^k \llcorner S$, with $a \geq 0$, $\mu_{ac} \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, S a compact k-manifold of class $W^{3,\infty}$. Then, if $u(t)$ is the limit solution of problem* (1.1) *corresponding to the initial condition $\mu = \mu_{ac}+\mu_s$ obtained as a limit of the solutions $u_n(t)$ of* (1.1) *corresponding to $u_n(0) = \mu_{ac} + \rho_n * \mu_s$, we have $u(t)_s = \mu_s$ and $u(t)_{ac}$ is the strong solution of problem* (5.39)*. In particular $u(t)$ is the limit solution of problem* (1.1) *corresponding to the initial condition μ.*

Let us complete the proof in case that $\rho \in C_0^\infty(\mathbb{R}^N)$ be a radial decreasing function such that $\rho \geq 0$, whose support coincides with $B(0,1)$, and $\int_{\mathbb{R}^N} \rho(x)dx = 1$. Let $u_{n0} = \mu_{ac} + \rho_n * \mu_s$ and let $u_n(t)$ be the solutions of (1.1) with $u_n(0) = u_{n0}$. We know that $u_n(t) \to U(t)$ weakly* as measures for some function $U \in C_w([0,T], \mathcal{M}_b(\mathbb{R}^N))$. Given $\epsilon > 0$, let $\tilde{\rho}$ be a kernel satisfying $(H)_\rho$ with $\beta = 2$ for which we already know that Theorem 6.7 holds. Let $\tilde{\nu}_n'$ be the measure constructed with the kernel $\tilde{\rho}$ which satisfies Lemma 6.6. Let $\tilde{v}_n(t)$ be the solution of (1.1) corresponding to the initial condition $\tilde{v}_n(0) = \mu_{ac} + a\tilde{\nu}_n'$. By Theorem 6.7 we know that $\tilde{v}_n(t)$ converges to $u(t)$ where $u(t)_s = \mu_s$ for all $t \geq 0$ and $u(t)_{ac}$ is the strong solution of (5.39) corresponding to the initial condition $u(0)_{ac}(0) = \mu_{ac}$. Since

$$\|\rho_n * \mu_s - \tilde{\rho}_n * \mu_s\|_1 \leq \|\rho_n - \tilde{\rho}_n\|_1\|\mu_s\|_1 \leq \epsilon\|\mu_s\|_1\,,$$

we have that

$$\|u_n(0) - \tilde{v}_n(0)\|_1 \leq \epsilon \|\mu_s\|_1 \,,$$

and, therefore,

$$\|u_n(t) - \tilde{v}_n(t)\|_1 \leq \epsilon \|\mu_s\|_1 \quad \forall\, t \geq 0 \,.$$

Letting $n \to \infty$ we obtain

$$\|U(t) - u(t)\|_1 \leq \epsilon \|\mu_s\|_1 \quad \forall\, t \geq 0 \,.$$

Since this is true for all $\epsilon > 0$, we conclude that $U(t) = u(t)$ for all $t \geq 0$. This concludes the proof of Theorem 6.7.

7. Guy David Measure Initial Conditions

Lemma 7.1. *Let $u_0 \in L^1(\mathbb{R}^N)$ be and $u(t)$ the unique strong solution of* (1.1) *with initial datum u_0. Then, for every set $E \subset \mathbb{R}^N$ of finite perimeter, we have*

$$\int_E u_t(t) dx \leq \mathrm{Per}(E) \qquad \text{a.e. } t > 0 \,. \tag{7.1}$$

$$\int_E u(t) dx \leq \int_E u_0 dx + t\,\mathrm{Per}(E) \quad \text{a.e. } t > 0 \,. \tag{7.2}$$

Proof. Taking $w = T_k(u(t)) - \chi_E$ as test function in the definition of strong solution, we have

$$\int_{\mathbb{R}^N} \chi_E u_t(t) \leq -\int_{\mathbb{R}^N} (z(t), D\chi_E) \leq \mathrm{Per}(E) \,.$$

Then, integrating in time, we get

$$\int_{\mathbb{R}^N} \chi_E (u(t) - u_0) dx \leq t\,\mathrm{Per}(E) \,. \qquad \square$$

Proposition 7.2. *Let μ be a Guy David measure and $u(t)$ the limit solution of* (1.1) *corresponding to the initial condition μ. Then, for any $t > 0$, $u(t)$ is also a Guy David measure.*

Proof. Since $u_{0,n}(\mu) \rightharpoonup \mu$ and $u_n(t) \rightharpoonup u(t)$ locally weakly* as measures, by Lemma 7.1, for any $x \in \mathbb{R}^N$ and $r > 0$, we have

$$\begin{aligned} u(t)(B_r(y)) &\leq \liminf_{n\to\infty} \int_{B_r(y)} u_n(t) dx \leq \liminf_{n\to\infty} \int_{B_r(y)} u_{0,n} dx + t\,\mathrm{Per}(B_r(y)) \\ &\leq \limsup_{n\to\infty} \int_{\overline{B_r(y)}} u_{0,n} dx + t\,\mathrm{Per}(B_r(y)) \leq \mu(\overline{B_r(y)}) + t\,\mathrm{Per}(B_r(y)) \,. \end{aligned}$$

Now, using Theorem 2.1, we deduce that $u(t)$ is a Guy David measure. $\square$

8. Distributional Solutions of (1.1) and the Equation $-\mathrm{div}(z) = \mu$

In [8], extending a result of [22], the following result is established.

Lemma 8.1. *Let $f \in L^2(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$. The following statements are equivalent*:

(i) *The function $u \equiv 0$ is the solution of*

$$\min_{w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)} D(w)\,, \quad D(w) := \int_{\mathbb{R}^N} |Dw| + \frac{1}{2}\int_{\mathbb{R}^N} (w-f)^2 dx\,. \tag{8.1}$$

(ii) *There exists $z \in X_2(\mathbb{R}^N)$ with $\|z\|_\infty \le 1$ satisfying*

$$-\mathrm{div}\,(z) = f \text{ in } \mathcal{D}'(\mathbb{R}^N)\,.$$

(iii)

$$\|f\|_* := \sup\left\{\left|\int_{\mathbb{R}^N} f(x)w(x)dx\right| : w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \int_{\mathbb{R}^N} |Dw| \le 1\right\} \le 1\,.$$

Now we are going to study the equation $-\mathrm{div}(z) = \mu$, where $\mu \in \mathcal{M}(\mathbb{R}^N) \cap BV(\mathbb{R}^N)^*$. We denote

$$Z(\mathbb{R}^N) := \{z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) : \mathrm{div}(z) \in BV(\mathbb{R}^N)^*\}\,.$$

Given $z \in Z(\mathbb{R}^N)$ and $u \in BV(\mathbb{R}^N)$, we can define the distribution (z, Du) in $\mathbb{R}^N$, by

$$\langle (z, Du), \varphi\rangle := -\langle \mathrm{div}(z), \varphi u\rangle_{BV^*, BV} - \int_{\mathbb{R}^N} z \cdot \nabla\varphi u dx\,, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^N)\,.$$

Definition 8.2. Given $\mu \in \mathcal{M}(\mathbb{R}^N) \cap BV(\mathbb{R}^N)^*$, we say that $z \in Z(\mathbb{R}^N)$, with $\|z\|_\infty \le 1$, is a solution of

$$-\mathrm{div}(z) = \mu \text{ in } BV(\mathbb{R}^N)^*\,,$$

if

$$-\mathrm{div}(z) = \mu \text{ in } \mathcal{D}'(\mathbb{R}^N)\,,$$

and (z, Du) is a Radon measure satisfying

$$\int_{\mathbb{R}^N} |(z, Du)| \le \int_{\mathbb{R}^N} |Du|\,, \quad \int_{\mathbb{R}^N} (z, Du) = \langle \mu, u\rangle_{BV^*, BV}\ \forall\, u \in BV(\mathbb{R}^N)\,.$$

Theorem 8.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^N) \cap BV(\mathbb{R}^N)^*$. There is a solution $z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $\|z\|_\infty \le 1$ of*

$$-\mathrm{div}(z) = \mu \text{ in } BV(\mathbb{R}^N)^*\,, \tag{8.2}$$

if and only if

$$\|\mu\|_{BV(\mathbb{R}^N)^*} = \sup\left\{\langle \mu, v\rangle_{BV^*, BV}| : v \in BV(\mathbb{R}^N)\,, \int_{\mathbb{R}^N} |Dv| \le 1\right\} \le 1\,.$$

Proof. If $z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $\|z\|_\infty \le 1$ is a solution of (8.2), then for any $v \in BV(\mathbb{R}^N)$ we have

$$|\langle \mu, v\rangle_{BV^*,BV}| = \left| \int_{\mathbb{R}^N} (z, Dv) \right| \le \int_{\mathbb{R}^N} |Dv| \,.$$

Thus $\|\mu\|_{BV(\mathbb{R}^N)^*} \le 1$.

Assume that $\mu \in \mathcal{M}(\mathbb{R}^N) \cap BV(\mathbb{R}^N)^*$ is such that $\|\mu\|_{BV(\mathbb{R}^N)^*} \le 1$. Let $\rho \in C_0^\infty(\mathbb{R}^N)$ with $\rho \ge 0$, $\int_{\mathbb{R}^N} \rho(x)dx = 1$ and $\rho_n(x) = \frac{1}{n^N}\rho(\frac{x}{n})$. Let $\mu_n = \rho_n * \mu$. Then $\mu_n \in C^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \cap L^N(\mathbb{R}^N) \cap BV(\mathbb{R}^N)^*$ and

$$\|\mu_n\|_{BV(\mathbb{R}^N)^*} \le \|\mu\|_{BV(\mathbb{R}^N)^*} \le 1 \,.$$

Thus, by Lemma 8.1, there is a vector field $z_n \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $\|z_n\|_\infty \le 1$ such that

$$-\mathrm{div}(z_n) = \mu_n \text{ in } \mathcal{D}'(\mathbb{R}^N)\,. \tag{8.3}$$

We may assume that $z_n \to z$ weakly* in $L^\infty(\mathbb{R}^N)$ and $-\mathrm{div}(z_n) \to \xi$ weakly* in $BV(\mathbb{R}^N)^*$ with $\|z\|_\infty \le 1$ and $\|\xi\|_{BV(\mathbb{R}^N)^*} \le 1$. Thus we may pass to the limit in (8.3) and obtain that $\xi = -\mathrm{div}(z)$ in $\mathcal{D}'(\mathbb{R}^N)$. Thus, we have $z \in Z(\mathbb{R}^N)$. Let us see that (z, Du) is a Radon measure in $\mathbb{R}^N$ for all $u \in BV(\mathbb{R}^N)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$, then by the integration by parts formula (2.6), we have

$$\begin{aligned}\langle (z, Du), \varphi\rangle &= \langle \xi + \mathrm{div}(z_n), u\varphi\rangle_{BV^*,BV} - \int_{\mathbb{R}^N} \mathrm{div}(z_n)u\varphi dx - \int_{\mathbb{R}^N} z\cdot\nabla\varphi u dx \\ &= \langle \xi + \mathrm{div}(z_n), u\varphi\rangle_{BV^*,BV} + \int_{\mathbb{R}^N} (z_n - z)\cdot\nabla\varphi u dx + \int_{\mathbb{R}^N} \varphi(z_n, Du)\,.\end{aligned}$$

Then, taking limits in n, we get

$$|\langle (z, Du), \varphi\rangle| \le \|\varphi\|_\infty \int_{\mathbb{R}^N} |Du| \,,$$

consequently, (z, Du) is a Radon measure in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} |(z, Du)| \le \int_{\mathbb{R}^N} |Du| \,.$$

Moreover,

$$\langle \xi, u\rangle_{BV^*,BV} = \int_{\mathbb{R}^N} (z, Du)\,, \quad \forall\, u \in BV(\mathbb{R}^N)\,. \tag{8.4}$$

Indeed, let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi \ge 0$, $\varphi(x) = 1$ for $x \in B(0,1)$, $\mathrm{supp}(\varphi) \subseteq B(0,2)$, and $\varphi_n(x) = \varphi(\frac{x}{n})$. Since $u\varphi_n \to u$ in $BV(\mathbb{R}^N)$ as $n \to \infty$ and

$$-\int_{\mathbb{R}^N} z\cdot\nabla\varphi_n u dx \le \frac{\|\nabla\varphi\|_\infty}{n} \int_{n \le \|x\| \le 2n} |u| \to 0\,,$$

as $n \to \infty$, we have

$$\langle \xi, u\rangle_{BV^*,BV} = \int_{\mathbb{R}^N} (z, Du)\,,$$

for all $u \in BV(\mathbb{R}^N)$.

Finally, let us prove that $\xi = \mu$. For that, by an approximation procedure, we only need to prove that $\langle \xi, u\rangle_{BV^*,BV} = \langle \mu, u\rangle_{BV^*,BV}$ for any $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with compact support. We know that $\rho_n * u(x) \to u^*(x)\mathcal{H}^{N-1}$-a.e. in $\mathbb{R}^N$ [2], hence, also μ-a.e., since μ vanishes on $\mathcal{H}^{N-1}$ null sets [26, Theorem 5.12.4]. Then

$$\begin{aligned}
\langle \mu, u\rangle_{BV^*,BV} &= \int_{\mathbb{R}^N} u^* d\mu = \lim_n \int_{\mathbb{R}^N} \rho_n * u(x) d\mu(x) \\
&= \lim_n \int_{\mathbb{R}^N} u(x)\rho_n * \mu(x) dx = \lim_n \langle u, \mu_n\rangle_{BV^*,BV} \\
&= \lim \langle u, -\mathrm{div}(z_n)\rangle_{BV^*,BV} = \langle u, \xi\rangle_{BV^*,BV}\,.
\end{aligned}$$

$\square$

If $u \in BV(\mathbb{R}^N)$, we have that the equality (2.1) holds modulo an $\mathcal{H}^{N-1}$ null set. Then for any rectifiable set Γ we have

$$\begin{aligned}
\langle \mathcal{H}^{N-1} \llcorner \Gamma, u\rangle_{BV^*,BV} &= \int_\Gamma u^*(x) d\mathcal{H}^{N-1}(x) \\
&= \int_\Gamma \int_0^\infty (\chi_{[u>t]})^*(x) dt d\mathcal{H}^{N-1}(x) \\
&= \int_0^\infty \int_\Gamma (\chi_{[u>t]})^*(x) d\mathcal{H}^{N-1}(x) dt \\
&= \int_0^\infty \langle \mathcal{H}^{N-1} \llcorner \Gamma, \chi_{[u>t]}\rangle_{BV^*,BV} dt\,.
\end{aligned}$$

Let us consider first the simpler case of the Hausdorff measure restricted to a rectifiable Jordan curve.

Proposition 8.4. *Let Γ be a rectifiable Jordan curve in $\mathbb{R}^2$. Then,*

$$\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} \le 1 \text{ if and only if } \Gamma \text{ is a convex curve}\,.$$

Proof. Assume that Γ is a convex curve. Let $u \in BV(\mathbb{R}^2)$, $u \ge 0$. Then, by the coarea formula, we have

$$\begin{aligned}
\langle \mathcal{H}^1 \llcorner \Gamma, u\rangle_{BV^*,BV} &= \int_0^\infty \langle \mathcal{H}^1 \llcorner \Gamma, \chi_{[u>t]}\rangle_{BV^*,BV} dt \\
&\le \int_0^\infty \mathrm{Per}([u>t]) dt = \int_{\mathbb{R}^2} |Du|\,,
\end{aligned}$$

in other words, $\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} \le 1$.

Now, assume that $\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} \le 1$. Suppose that Γ is not convex. Let $V = co(\Gamma)$ (where $co(\Gamma)$ denotes the convex enveloppe of Γ). Then $\mathrm{Per}(V) < \mathcal{H}^1(\Gamma)$. Choose $\epsilon > 0$ small enough so that, if $U = V + B(0, \epsilon)$, then $\mathrm{Per}(U) < H^1(\Gamma)$. Then we have

$$\langle \mathcal{H}^1 \llcorner \Gamma, \chi_U \rangle_{BV^*,BV} = \mathcal{H}^1(\Gamma) > \operatorname{Per}(U) = \int_{\mathbb{R}^2} |D\chi_U| \,,$$

hence, $\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} > 1$, a contradiction. □

We need to recall the following definition given in [1]. Let $D \subset \mathbb{R}^N$ be a set of finite perimeter, D is said to be *decomposable* if there exists a partition (A, B) of D such that $\operatorname{Per}(D) = \operatorname{Per}(A) + \operatorname{Per}(B)$ and both $|A|$ and $|B|$ are estrictly positive. D is said to be *indecomposable* if it is not decomposable.

Theorem 8.5. *Let Γ_i, $i = 1, \ldots, m$ be disjoint rectifiable Jordan curves in $\mathbb{R}^2$. Then, if we take $\Gamma := \bigcup_{i=1}^m \Gamma_i$ we have $\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} \le 1$ if and only if the following two conditions hold*:

(i) Γ_i *is convex for all* $i = 1, \ldots, m$,

(ii) *let C_i the bounded open set with boundary Γ_i and let $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ be a k-tuple of indices with $0 \le k \le m$; if we denote by $E_{i_1,\ldots,i_k}$ a solution of the variational problem*

$$\min \left\{ \operatorname{Per}(E) : E \text{ of finite perimeter, } \bigcup_{j=1}^{k} C_{i_j} \subseteq E \subseteq \mathbb{R}^2 \setminus \bigcup_{j \notin \{i_1,\ldots,i_k\}} C_j \right\} ,$$

we have

$$\operatorname{Per}(E_{i_1,\ldots,i_k}) \ge \sum_{j=1}^{k} \operatorname{Per}(C_{i_j}) = \mathcal{H}^1 \left(\bigcup_{j=1}^{k} \Gamma_{i_j} \right) . \tag{8.5}$$

Proof. We recall that by the coarea formula, for any rectifiable set Γ and $u \in BV(\mathbb{R}^N)$ we have

$$\langle \mathcal{H}^{N-1} \llcorner \Gamma, u \rangle_{BV^*,BV} = \int_0^\infty \langle \mathcal{H}^{N-1} \llcorner \Gamma, \chi_{[u>t]} \rangle_{BV^*,BV} \, dt \,.$$

Assume now that $\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} \le 1$. Suppose first that exists $i \in \{1, \ldots, m\}$ such that Γ_i is not convex. Let $V = co(\Gamma_i)$. Then $\operatorname{Per}(V) < \mathcal{H}^1(\Gamma_i)$. Choose $\varepsilon > 0$ small enough such that, if $U = V + B(0, \varepsilon)$, $\operatorname{Per}(U) < \mathcal{H}^1(\Gamma_i)$. Then we have

$$\langle \mathcal{H}^1 \llcorner \Gamma, \chi_U \rangle_{BV^*,BV} \ge \mathcal{H}^1(\Gamma_i) > \operatorname{Per}(U) = \int_{\mathbb{R}^2} |D\chi_U| \,,$$

hence, $\|\mathcal{H}^1 \llcorner \Gamma\|_{BV^*} > 1$, a contradiction.

Suppose now that condition (8.5) does not hold. Then we have for suitable $\{i_1 \ldots, i_k\}$ that there exists $E_{i_1,\ldots,i_k}$ such that

$$\operatorname{Per}(E_{i_1,\ldots,i_k}) < \mathcal{H}^1 \left(\bigcup_{j=1}^{k} \Gamma_{i_j} \right) .$$

Choose $\varepsilon > 0$ small enough such that, if $U = E_{i_1,\ldots,i_k} + B(0,\varepsilon)$, $\mathrm{Per}(U) < \mathcal{H}^1(\bigcup_{j=1}^k \Gamma_{i_j})$. Then we obtain

$$\langle \mathcal{H}^1 \llcorner \Gamma, \chi_U \rangle_{BV^*,BV} \geq \mathcal{H}^1 \left(\bigcup_{j=1}^k \Gamma_i \right) > \mathrm{Per}(U) = \int_{\mathbb{R}^2} |D\chi_U| \,.$$

To prove the other implication we recall that if we take $\Omega := \bigcup_{i=1}^m C_i$, as it is proved in [8], from (8.5) we have,

$$\mathcal{H}^1 \left(D \cap \left(\bigcup_{j=1}^k \Gamma_{i_j} \right) \right) \leq \mathrm{Per}(D, \mathbb{R}^2 \setminus \bar{\Omega}) \leq \mathrm{Per}(D)\,, \tag{8.6}$$

for any bounded indecomposable set of finite perimeter D [1], where $\{i_1,\ldots,i_k\} \subset \{1,\ldots,m\}$ is the set of indexes such that $D \cup \bigcup_{j=1}^k C_{i_j}$ is connected. Let $u \in BV(\mathbb{R}^N)$. Since $[u > t]$ has finite perimeter, there exists a countable family $\{D_p\}$ of indecomposable sets such that $\mathcal{H}^1([u > t]) = \sum_p \mathrm{Per}(D_p)$ (see [1]). Then

$$\begin{aligned} \langle \mathcal{H}^1 \llcorner \Gamma, u \rangle_{BV^*,BV} &= \int_0^\infty \langle \mathcal{H}^1 \llcorner \Gamma, \chi_{[u>t]} \rangle_{BV^*,BV}\, dt \\ &= \int_0^\infty \mathcal{H}^1([u > t] \cap \Gamma) dt = \int_0^\infty \sum_p \mathcal{H}^1(D_p \cap \Gamma) dt \,. \end{aligned}$$

Now, if $D_p \cap \Gamma = D_p \cap \bigcup_{j=1}^{k(p)} \Gamma_{i(p)_j}$, using (8.6), we finally obtain

$$\begin{aligned} \langle \mathcal{H}^1 \llcorner \Gamma, u \rangle_{BV^*,BV} &= \int_0^\infty \sum_p \mathcal{H}^1 \left(D_p \cap \bigcup_{j=1}^{k(p)} \Gamma_{i(p)_j} \right) dt \\ &\leq \int_0^\infty \sum_p P(D_p) dt = \int_0^\infty \mathrm{Per}([u > t]) dt = \int_{\mathbb{R}^2} |Du| \,. \end{aligned}$$

□

Definition 8.6. Let $\mu \in \mathcal{M}_b(\mathbb{R}^N)$. We say that $u(t)$ is a *distributional solution* of (1.1) in $[0,T] \times \mathbb{R}^N$ corresponding to the initial condition $u(0) = \mu$ if $u \in C((0,T], \mathcal{M}_b(\mathbb{R}^N))$, $u(t) \rightharpoonup u(0)$ weakly* as measures and there exists $z \in L^\infty((0,T) \times \mathbb{R}^N)$ with $\|z\|_\infty \leq 1$ such that

$$(z(t), DT_k(u_{ac})) = |DT_k(u_{ac}(t))| \quad \text{a.e. } t \in (0,T)\,, \quad \forall\, k > 0\,, \tag{8.7}$$

and

$$u_t = \mathrm{div}(z) \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^N)\,. \tag{8.8}$$

Let $\mu \in \mathcal{M}_b(\mathbb{R}^N)$ be a singular measure. Then $u(t) = \mu$ is a distributional solution of (1.1). Indeed, it suffices to take $z = 0$. Note that $u(t)$ satisfies the family of inequalities (5.42), hence is an entropy solution of (1.1) in this sense. Thus, if $\mu = \alpha \mathcal{H}^{N-1} \llcorner S$ where S is a compact $N-1$-manifold of class $W^{3,\infty}$, and $\alpha > 0$, we cannot take condition (5.42) as the notion of entropy solution for (1.1) but a different family of inequalities which one could obtain as in Sec. 5.1. Assume now that $\|\mu\|_{BV(\mathbb{R}^N)^*} \leq 1$. Then $u(t) = (1-t)\mu$ is also a distributional solution of (1.1). In this case, we take z as a solution of $-\mathrm{div}(z) = \mu$ with $\|z\|_\infty \leq 1$.

Remark 8.6. Let Γ_i, $i = 1, \ldots, m$, be convex curves and let C_i denote the bounded open set with boundary Γ_i. Let $\Gamma = \bigcup_{i=1}^m \Gamma_i$. We assume that Γ_i are of class $C^{1,1}$ satisfying

$$\operatorname*{ess\,sup}_{p\in\Gamma_i} k_{\Gamma_i}(p) \leq \frac{\mathrm{Per}(C_i)}{|C_i|}\,,$$

where k_{Γ_i} denotes the curvature of Γ_i, and the assumption (ii) of Theorem 8.5. Let us see that the limit solution of problem (1.1) corresponding to the initial datum $u_0 = \alpha\mathcal{H}^1 \llcorner \Gamma$, $\alpha > 0$ is given by

$$u(t) = \begin{cases} (\alpha - 2t)\mathcal{H}^1 \llcorner \Gamma + \displaystyle\sum_{i=1}^m \frac{\mathrm{Per}(C_i)}{|C_i|} t\chi_{C_i}\,, & t \in \left[0, \dfrac{\alpha}{2}\right]\,, \\ \displaystyle\sum_{i=1}^m \frac{\mathrm{Per}(C_i)}{|C_i|}(\alpha - t)^+\chi_{C_i}\,, & t \geq \dfrac{\alpha}{2}\,. \end{cases} \tag{8.9}$$

Indeed, by results in [8] we know that there is a vector field $\xi \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\|\xi\|_\infty \leq 1$ such that

$$-\mathrm{div}(\xi) = \sum_{i=1}^m \frac{\mathrm{Per}(C_i)}{|C_i|}\chi_{C_i} \tag{8.10}$$

and

$$[\xi, \nu^i] = -1 \quad i = 1, \ldots, m\,,$$

where ν^i is the outer unit normal to C_i, $i = 1, \ldots, m$. Now, for $t \in [0, \frac{\alpha}{2}]$, we define the vector field

$$\xi'(t,x) = \begin{cases} -\xi(x)\,, & x \in C_i\,, \quad i = 1, \ldots, m\,, \\ \xi(x)\,, & x \in \mathbb{R}^2 \setminus \cup_{i=1}^m \bar{C}_i\,. \end{cases}$$

Then, $u_t = \mathrm{div}\,\xi'$ in $\mathcal{D}'((0, \frac{\alpha}{2}) \times C_i)$ and we have that $u(t)|_{C_i}$ is the strong solution of (5.26) in $(0, \frac{\alpha}{2}) \times C_i$. In the same way we prove that $u(t)|_{\mathbb{R}^2 \setminus \cup_{i=1}^m \bar{C}_i}$ is the strong solution of (5.26) in $(0, \frac{\alpha}{2}) \times \mathbb{R}^2 \setminus \cup_{i=1}^m C_i$. Thus, by Theorem 5.10, $u(t)$ coincides in $[0, \frac{\alpha}{2}]$ with the limit solution of (1.1) corresponding to the initial condition u_0. For times $t \geq \frac{\alpha}{2}$, the limit solution $u(t)$ is described by the strong solution of (1.1) corresponding to the initial data $u(\frac{\alpha}{2}) = \frac{\alpha}{2}\sum_{i=1}^m \frac{\mathrm{Per}(C_i)}{|C_i|}\chi_{C_i}$ and is given by (see [8])

$$u\left(\frac{\alpha}{2} + t, x\right) = \sum_{i=1}^m \frac{\mathrm{Per}(C_i)}{|C_i|}\left(\frac{\alpha}{2} - t\right)^+ \chi_{C_i}(x) \quad t \geq 0, x \in \mathbb{R}^N\,.$$

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Appendix

In this Appendix, for the sake of completeness, we outline the proof of (3.2) using the ideas of Minkowski's content [2].

First of all, we note that, if f is a continuous function and S is a countably $\mathcal{H}^k$-rectifiable set, we can write its integral respect to the Hausdorff measure as

$$\int_S f(z)d\mathcal{H}^k(z) = \sup\left\{\sum_i \inf_{t\in K_i} f(t)\mathcal{H}^k(K_i); K_i \subset S \text{ compact, pairwise disjoint}\right\}. \quad \text{(A.1)}$$

We denote by G_k the set of orthogonal projections onto k-dimensional subspaces of $\mathbb{R}^N$. A slight modification of the proof of [2, Proposition 2.66] give us the following result.

Proposition A.1. *For any countably $\mathcal{H}^k$-rectifiable set E,*

$$\int_E f(x)d\mathcal{H}^k = \sup\left\{\sum_i \inf_{t\in K_i} f(t)L^k(\pi_i(K_i)) : \pi_i \in G_k, K_i \subset E \text{ compact, pairwise disjoint}\right\}. \quad \text{(A.2)}$$

With the same technique used to prove [2, Proposition 2.101 and Lemma 2.102], we can stablish the following two results.

Proposition A.2 (Lower bound). *For any countably $\mathcal{H}^k$-rectifiable closed set S, if f is a positive continuous function, the following inequality holds*

$$\liminf_{\rho\to 0^+} \frac{\int_{I_\rho(S)} f(z)d\mathcal{L}^N(z)}{w_{N-k}\rho^{N-k}} \geq \int_S f(z)d\mathcal{H}^k(z),$$

where $I_\rho(S) := \{x \in \mathbb{R}^N : \text{dist}(x,S) \leq \rho\}$.

Lemma A.3. *Let $S \subset \mathbb{R}^n$ be a countably $\mathcal{H}^k$-rectifiable set and $\tau > 1$. Then, for any $\beta > 0$, $\mathcal{H}^k$-almost all of S can be covered by a sequence (S_i) of pairwise disjoint compact sets with diameter less or equal than β satisfying*

$$\limsup_{\rho \to o^+} \frac{\int_{I_\rho(S_i)} d\mathcal{L}^N(z)}{w_{N-k}\rho^{N-k}} \leq \tau \mathcal{H}^k(S_i) < \infty \,. \tag{A.3}$$

In the next Theorem we need to assume that S satisfies the following density lower bound

$$\nu(B_\rho(x)) \geq \gamma \rho^k \,, \quad \forall\, x \in S \,, \quad \rho \in (0,1) \,, \tag{A.4}$$

for a suitable measure ν absolutely continuous with respect to $\mathcal{H}^k$.

Theorem A.4. *Let $S \subset \mathbb{R}^N$ be a countably rectifiable compact set and assume that (A.4) holds for some $\gamma > 0$ and some Radon measure ν in $\mathbb{R}^N$ absolutely continuous with respect to $\mathcal{H}^k$. Then, we have*

$$\int_S f(z) d\mathcal{H}^k(z) = \lim_{\rho \to 0^+} \frac{\int_{I_\rho(S)} f(z) d\mathcal{L}^N(z)}{w_{N-k}\rho^{N-k}} \,, \tag{A.5}$$

for all continuous function f.

Proof. First note that it is enough to prove (A.5) assuming f is positive. By Proposition A.2, we only have to prove one inequality, and we may also assume that $\int_S f(z) d\mathcal{H}^k(z) < \infty$.

Given $\eta > 0$, as f is continuous, we can find $\beta > 0$ such that if K is a subset of $\mathbb{R}^N$ whose diameter is less than β, we have that the oscillation of f in K, $osc_k(f)$, is less or equal than η.

On the other hand, given $\epsilon > 0$, by Lemma A.3, we can find compact pairwise disjoint sets S_i with diameter less than β such that

$$\mathcal{H}^k\left(S \setminus \bigcup_i S_i\right) = 0 \,,$$

and

$$\limsup_{\rho \to o^+} \frac{\int_{I_\rho(S_i)} d\mathcal{L}^N(z)}{w_{N-k}\rho^{N-k}} \leq (1+\epsilon)\mathcal{H}^k(S_i) \,.$$

Moreover, we have that there exists n such that $\nu(S) < \epsilon + \sum_{i=1}^n \nu(S_i)$.

Let $E = S \setminus \bigcup_{i=1}^n S_i$ and, for λ, ρ fixed, define

$$\tilde{S}_\rho := \left\{ x \in S : \text{dist}\left(x, \bigcup_{i=1}^n S_i\right) \geq \lambda\rho \right\}.$$

If we apply now Besicovitch's covering theorem (see [2]), we can find a cover of $\tilde{S}_\rho$ by balls $\{B_{\lambda\rho}(x_j)\}_{j\in J}$ centered at points of $\tilde{S}_\rho$ with overlapping controlled by ξ. By the definition of $\tilde{S}_\rho$ and the lower estimate bound (A.4), we can control the cardinality of such J since

$$\sum_{j\in J} \gamma(\lambda\rho)^k \leq \sum_{j\in J} \nu(B_{\lambda\rho}(x_j)) \leq \xi\nu\left(I_{\lambda\rho}(S) \setminus \bigcup_{i=1}^n S_i\right) \leq \xi\epsilon\,,$$

for ρ sufficiently small. Thus, we obtain that the cardinal of J is less than $\frac{(\xi\epsilon)}{(\gamma\lambda^k\rho^k)}$. As a consequence,

$$\mathcal{L}^N(I_{(1+\lambda)\rho}(\tilde{S}_\rho)) \leq \sum_{j\in J} \mathcal{L}^N(B_{(1+2\lambda)\rho}(x_j)) \leq \frac{\omega_N(1+2\lambda)^N\xi\epsilon}{\gamma\lambda^k}\rho^{N-K}.$$

We notice now that we have the following inclusions

$$I_\rho(S) \subset I_\rho(E) \cup \bigcup_{i=1}^n I_\rho(S_i) \subset I_{(1+\lambda)\rho}(\tilde{S}\rho) \cup \bigcup_{i=1}^n I_\rho(S_i)\,.$$

Therefore, having in mind Lemma A.3 and (A.1), we have

$$\begin{aligned}
&\limsup_{\rho\to 0^+} \frac{\int_{I_\rho(S)} f(z)d\mathcal{L}^N}{w_{N-k}\rho^{N-k}} \\
&\quad \leq \limsup_{\rho\to 0^+} \frac{\int_{I_{(1+\lambda)\rho}(\tilde{S}\rho)} f(z)d\mathcal{L}^N}{w_{N-k}\rho^{N-k}} \\
&\qquad + \limsup_{\rho\to 0^+} \frac{\int_{\cup_{i=1}^n I_\rho(S_i)} f(z)d\mathcal{L}^N}{w_{N-k}\rho^{N-k}} \leq \|f\|_\infty \limsup_{\rho\to 0^+} \frac{|I_{(1+\lambda)\rho}(\tilde{S}_\rho)|}{\omega_{N-k}\rho^{N-k}} \\
&\qquad + \sum_{i=1}^n \limsup_{\rho\to 0} \frac{\int_{I_\rho(S_i)} f(z)d\mathcal{L}^N}{\omega_{N-k}\rho^{N-k}} \leq \|f\|_\infty \frac{\omega_N(1+2\lambda)^N\xi\epsilon}{\omega_{N-k}\gamma\lambda^k} \\
&\qquad + \sum_{i=1}^n \limsup_{\rho\to 0^+} \sup_{z\in I_\rho(S_i)} f(z)\frac{|I_\rho(S_i)|}{\omega_{N-k}\rho^{N-k}} \\
&\quad \leq C\epsilon + (1+\epsilon)\sum_{i=1}^n \left(\limsup_{\rho\to 0^+} \inf_{z\in I_\rho(S_i)} f(z) + \eta\right)\mathcal{H}^k(S_i) \\
&\quad \leq C\epsilon + (1+\epsilon)\left(\int_S f d\mathcal{H}^k + \eta\mathcal{H}^k(S)\right).
\end{aligned}$$

We conclude by taking limits as ϵ, $\eta \to 0^+$. □

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