Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. III (2004), pp. 1-

A Strongly Degenerate Quasilinear Equation: the Elliptic Case

I BOZZE

File: norm243.tex
22.maggio.2004

F. ANDREU – V. CASELLES – J. M. MAZÓN

Abstract. We prove existence and uniqueness of entropy solutions for the Neumann problem for the quasilinear elliptic equation $u - \operatorname{div} \mathbf{a}(u, Du) = v$, where $v \in L^\infty$, $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$, and f is a convex function of ξ with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions. In particular, this class includes the case where $f(z, \xi) = \varphi(z)\psi(\xi)$, $\varphi > 0$, ψ being a convex function with linear growth as $\|\xi\| \to \infty$. In the second part of this work, using Crandall-Ligget's iteration scheme, this result will permit us to prove existence and uniqueness of entropy solutions for the corresponding parabolic problem with bounded measurable initial data.

Mathematics Subject Classification (2000): MISSING.

1. – Introduction

Let Ω be a bounded set in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1. We are interested in the problem

$$\left\{ \begin{array}{ll} u - \operatorname{div}\ \mathbf{a}(u, Du) = v & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where $v \in L^\infty(\Omega)$, $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$, f being a function with linear growth as $\|\xi\| \to \infty$ and $\frac{\partial}{\partial \eta}$ is the Neumann boundary operator associated to $\mathbf{a}(u, Du)$, i.e.,

$$\frac{\partial u}{\partial \eta} := \mathbf{a}(u, Du) \cdot \nu,$$

with ν the unit outward normal on $\partial\Omega$.

Our purpose in this paper is to define a notion of entropy solution for (1.1), and prove existence and uniqueness results when the right hand side v is in $L^\infty(\Omega)$. Besides the fact that the elliptic problem is interesting by itself, this result permits us to associate to the expression $-div\, \mathbf{a}(u, Du)$ with Neumann

Pervenuto alla Redazione il ??.

boundary condition an accretive operator B in $L^1(\Omega)$ whose domain is contained in $L^\infty(\Omega)$ (which amounts to consider the right hand side v of (1.1) in $L^\infty(\Omega)$) and whose closure $\mathcal{B}$ is m-accretive in $L^1(\Omega)$, and, thus, generates a non-linear contraction semigroup $T(t)$ in $L^1(\Omega)$ [11], [18], [19]. However, even if $\mathcal{B}$ is not characterized in distributional terms, the knowledge of the operator B and the fact that if u is the entropy solution of (1.1) we have the estimate $\|u\|_\infty \le \|v\|_\infty$, permits to use Crandall-Ligget's iteration scheme and define the function

$$u(t) := T(t)u_0 = \lim_{n\to\infty} \Big(I + \frac{t}{n}B\Big)^{-n} u_0, \qquad u_0 \in L^\infty(\Omega).$$

which is a semigroup solution of the parabolic problem

$$\text{(1.2)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\ \mathbf{a}(u, Du) & \text{in } Q_T = (0,T)\times\Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } S_T = (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } x \in \Omega. \end{cases}$$

In a subsequent work [6] we shall define the notion of entropy solution for (1.2) and prove that the semigroup solution $u(t)$ is an entropy solution. Moreover, we shall also prove that entropy solutions of (1.2) are unique. As a technical tool both in this paper and in [6] we shall use the lower semi-continuity results proved in [21] for energy functionals whose density is a function $g(u, Du)$ convex in Du and with a linear growth rate in Du.

Particular instances of these PDE's have been studied in [12], [13], [14] and [20], when $N = 1$. Let us describe their results in some detail. In [12], [13], and [20] the authors considered the problem

$$\text{(1.3)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = (\varphi(u)\mathbf{b}(u_x))_x & \text{in } (0,T)\times\mathbb{R} \\ u(0,x) = u_0(x) & \text{in } x \in \mathbb{R} \end{cases}$$

corresponding to (1.2) when $N = 1$ and $\mathbf{a}(u, u_x) = \varphi(u)\mathbf{b}(u_x)$, where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is smooth and strictly positive, and $\mathbf{b} : \mathbb{R} \to \mathbb{R}$ is a smooth odd function such that $\mathbf{b}' > 0$ and $\lim_{s\to\infty} \mathbf{b}(s) = \mathbf{b}_\infty$. Such models appear as models for heat and mass transfer in turbulent fluids [8], or in the theory of phase transitions where the corresponding free energy functional has a linear growth rate with respect to the gradient [28]. As the authors observed, in general, there are no classical solutions of (1.2), they defined the notion of entropy solution and proved existence [12] and uniqueness [20] of entropy solutions of (1.3). Existence was proved for bounded strictly increasing initial conditions $u_0 : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{b}(u_0') \in C(\mathbb{R})$ (where $\mathbf{b}(u_0'(x_0)) = \mathbf{b}_\infty$ if u_0 is discontinuous at x_0), $\mathbf{b}(u_0'(x)) \to 0$ as $x \to \pm\infty$ [12]. The entropy condition was written in Oleinik's form and uniqueness was proved using Kruzhkov's method of doubling variables.

In [14], [15], the author considered the Neumann problem in an interval of $\mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t} = (\mathbf{a}(u, u_x))_x & \text{in } (0,T)\times(0,1) \\ u_x(t,0) = u_x(t,1) = 0 & \\ u(0,x) = u_0(x) & \text{in } x \in (0,1) \end{cases} \tag{1.4}$$

for functions $\mathbf{a}(u,v)$ of class $C^{1,\alpha}([0,\infty)\times\mathbb{R})$ such that $\frac{\partial}{\partial v}\mathbf{a}(u,v) < 0$ for any $(u,v) \in [0,\infty)\times\mathbb{R}$, $\mathbf{a}(u,0) = 0$ (and some other additional assumptions). After observing that there are no, in general, classical solutions of (1.2), the author associated an m-accretive operator to $-(\mathbf{a}(u,u_x))_x$ with Neumann boundary conditions, and proved the existence and uniqueness of a semigroup solution of (1.4). However, the accretive operator generating the semigroup was not characterized in distributional terms. An example of the equations considered in [14], [15] is the so called *plasma equation* (see [23])

$$\frac{\partial u}{\partial t} = \left(\frac{u^{5/2}u_x}{1+u|u_x|}\right)_x \quad \text{in} \quad (0,T)\times(0,1), \tag{1.5}$$

where the initial condition u_0 is assumed to be positive. In this case u represents the temperature of electrons and the form of the conductivity $\mathbf{a}(u,u_x) = \frac{u^{5/2}u_x}{1+u|u_x|}$ has the effect of limiting heat flux. Thus, existence and uniqueness results for higher dimensional problems were not considered. This will be the purpose of the present paper.

The case of equations of type

$$u - \operatorname{div}\ \mathbf{a}(x, Du) = v \quad \text{in} \quad \Omega, \tag{1.6}$$

where $v \in L^1(\Omega)$, or the corresponding parabolic problem

$$\frac{\partial u}{\partial t} = \operatorname{div}\ \mathbf{a}(x, Du) \quad \text{in} \quad (0,T)\times\Omega, \tag{1.7}$$

where $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, $f(x,\cdot)$ being a convex function of ξ with linear growth as $\|\xi\| \to \infty$ has been considered in [2], [3] and [4] (see also [5]), where existence and uniqueness results of entropy solutions were proved.

The present work can be considered as an extension of the previous works to the case where $\mathbf{a}$ depends on (u, Du) instead of (x, Du). We treat in this paper the elliptic case, the parabolic problem being considered in [6]. Entropy or renormalized solutions for elliptic or parabolic problems of types (1.1), (1.2), or (1.7), when $f(u,\xi)$ or $f(x,\xi)$ has a growth of order $p > 1$ as $\|\xi\| \to \infty$, were considered in [9], [16] and [17] (see also the references therein).

Finally, let us explain the plan of the paper. In Section 2 we recall some basic facts about functions of bounded variation, denoted by $BV(\Omega)$, Green's formula, and lower semi-continuity results for energy functionals defined in

$BV(\Omega)$. In Section 3 we introduce the main assumptions on the underlying operator, and define the notion of entropy solutions for (1.1). In Section 4 we prove an existence and uniqueness result for the entropy solutions of (1.1) when the right hand side v is in $L^\infty(\Omega)$. To prove existence we shall use the lower semi-continuity result for energy functionals whose density is a function $g(u, Du)$ convex in Du and with a linear growth rate in Du proved in [21], uniqueness will be proved by means of Kruzhkov's technique of doubling variables. Finally, in Section 5, we define an accretive operator associated to $-\operatorname{div}\mathbf{a}(u, Du)$ with Neumann boundary condition whose closure is m-accretive and generates a contraction semigroup in $L^1(\Omega)$, providing a solution of (1.2) in the semigroup sense. That semigroup solutions can be characterized in terms of entropy solutions will be the object of a subsequent paper [6].

Acknowledgements. The first and third authors have been supported by EC through the RTN Programme *Nonlinear Partial Differential Equations Describing Front propagation and other Singular Phenomena*, HPRN-CT-2002-00274, and by PNPGC project, reference BFM2002-01145. The second author acknowledges partial support by the Departament d'Universitats, Recerca i Societat de la Informació de la Generalitat de Catalunya and by PNPGC project, reference BFM2000-0962-C02-01.

2. – Preliminaries

We start with some notation. Here $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

Due to the linear growth condition on the Lagrangian, the natural energy space to study (1.1) is the space of functions of bounded variation. We recall briefly some facts about functions of bounded variation (for further information concerning functions of bounded variation we refer to [1], [24] or [29]).

A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \dots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i\varphi dx = -\int_\Omega \varphi d\mu_i \tag{2.1}$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \dots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$|Du|(\Omega) = \sup\left\{\int_\Omega u \operatorname{div}\varphi\, dx : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), |\varphi(x)| \le 1 \text{ for } x \in \Omega\right\}.$$

The space $BV(\Omega)$ is endowed with the norm

$$\| u \|_{BV} = \| u \|_{L^1(\Omega)} + |Du|(\Omega). \tag{2.2}$$

For $u \in BV(\Omega)$, the gradient Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, \mathcal{L}^N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. Let us denote by $D^s u = \overrightarrow{D^s u} |D^s u|$ the polar decomposition of $D^s u$, where $|D^s u|$ is the total variation measure of $D^s u$. We also split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. We denote by S_u the set of all $x \in \Omega$ such that x is not a Lebesgue point of u. We say that $x \in \Omega$ is an *approximate jump point of* u if there exist $u^+(x) \neq u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho^+(x, \nu_u(x)))} \int_{B_\rho^+(x, \nu_u(x))} |u(y) - u^+(x)| \, dy = 0$$

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{L}^N(B_\rho^-(x, \nu_u(x)))} \int_{B_\rho^-(x, \nu_u(x))} |u(y) - u^-(x)| \, dy = 0,$$

where

$$B_\rho^+(x, \nu_u(x)) = \{y \in B_\rho(x) \ : \ \langle y - x, \nu_u(x) \rangle > 0\}$$

and

$$B_\rho^-(x, \nu_u(x)) = \{y \in B_\rho(x) \ : \ \langle y - x, \nu_u(x) \rangle < 0\}.$$

We denote by J_u the set of approximate jump points of u. J_u is a Borel subset of S_u and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. We have

$$D^j u = D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \setminus S_u).$$

It is well known (see for instance [1]) that

$$D^j u = (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Moreover, if $x \in J_u$, then $\nu_u(x) = \frac{Du}{|Du|}(x)$, $\frac{Du}{|Du|}$ being the Radon-Nikodym derivative of Du with respect to its total variation $|Du|$.

Let Ω be a bounded open subset of $\mathbb{R}^N$. Given a Borel function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ such that

$$C\|\xi\| - D \leq g(x, z, \xi) \leq M(1 + \|\xi\|) \qquad \forall \ (x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{2.3}$$

for some constants $C > 0$, $M \geq 0$, we consider the energy functional

$$G(u) := \int_\Omega g(x, u(x), \nabla u(x)) \, dx$$

defined in the Sobolev space $W^{1,1}(\Omega)$. In order to get an integral representation of the relaxed energy associated with G, i.e.,

$$\mathcal{G}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \to \infty} G(u_n) \ : \ u_n \in W^{1,1}(\Omega), \ u_n \to u \ \in \ L^1(\Omega) \right\},$$

Dal Maso in [21] introduced the following functional for $u \in BV(\Omega)$:

$$\begin{aligned}\mathcal{R}_g(u) := &\int_\Omega g(x, u(x), \nabla u(x))\, dx + \int_\Omega g^0\left(x, u(x), \frac{Du}{|Du|}(x)\right) |D^c u| \\ &+ \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0(x, s, \nu_u(x))\, ds \right) d\mathcal{H}^{N-1}(x),\end{aligned} \tag{2.4}$$

where the recession function g^0 of g is defined as

$$g^0(x, z, \xi) = \lim_{t \to 0^+} t g\left(x, z, \frac{\xi}{t}\right). \tag{2.5}$$

It is clear that the function $g^0(x, z, \xi)$ is positively homogeneous of degree one in ξ, i.e.

$$g^0(x, z, s\xi) = s g^0(x, z, \xi) \qquad \text{for all } z, \xi \text{ and } s > 0.$$

Let us describe a different way of writing the functional $\mathcal{R}_g(u)$. Let us consider the function $\tilde{g} : \Omega \times \mathbb{R} \times \mathbb{R}^N \times]-\infty, 0] \to \mathbb{R}$ defined as

$$\tilde{g}(x, z, \xi, t) := \begin{cases} -g\left(x, z, -\dfrac{\xi}{t}\right) t & \text{if } t < 0 \\ g^0(x, z, \xi) & \text{if } t = 0. \end{cases} \tag{2.6}$$

As it is proved in [21], if g is a Borel function satisfying (2.3) and $g(x, z, \cdot)$ is convex in $\mathbb{R}^N$ for all $(x, z) \in \Omega \times \mathbb{R}$, then one has

$$\begin{aligned}\mathcal{R}_g(u) &= \int_{\Omega \times \mathbb{R}} \tilde{g}\left((x, s), \frac{d\alpha_u}{d|\alpha_u|}(x, s)\right)\, d|\alpha_u|(x, s) \\ &= \int_{\Omega \times \mathbb{R}} \tilde{g}\left((x, s), \nu[(x, s); N(u)]\right)\, d\mathcal{H}^N(x, s),\end{aligned} \tag{2.7}$$

where $\alpha_u = D\chi_{N(u)}$, with $N(u) := \{(x, s) \in \mathbb{R} \times \Omega : s < u_+(x)\}$ and $\nu[(x, s); N(u)]$ is the interior normal to $N(u)$ at (s, x) if it exists, and $\nu[(x, s); N(u)] = 0$ otherwise.

In [21] Dal Maso proved the following result:

THEOREM 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying* (2.3), *g^0 exists and such that $g(x, z, \cdot)$ is convex in $\mathbb{R}^N$. Then, $\mathcal{G}(u) = \mathcal{R}_g(u)$ for all $u \in BV(\Omega)$ and $\mathcal{R}_g(u)$ is lower semi-continuous respect to the $L^1(\Omega)$-convergence.*

We need to consider the following truncature functions. For $a < b$, let $T_{a,b}(r) := \max(\min(b, r), a)$. It is usual to denote $T_k = T_{-k,k}$.

PROPOSITION 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $g : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying (2.3), g^0 exists and such that $g(x, z, \cdot)$ is convex in $\mathbb{R}^N$. Let us define the functional*

$$R_g^{a,b}(u) = \int_\Omega g(x, u(x), \nabla T_{a,b}u(x))\,dx, \quad u \in W^{1,1}(\Omega).$$

For $u \in BV(\Omega)$, let

$$\begin{aligned}\mathcal{R}_g^{a,b}(u) := &\int_\Omega\int_a^b \tilde{g}\left((x,s), \nu[(x,s); N(u)]\right)\, d\mathcal{H}^{N-1}(x)\,ds\\ &+ \int_{[u\le a]} (g(x,u(x),0) - g(x,a,0))\,dx\\ &+ \int_{[u\ge b]} (g(x,u(x),0) - g(x,b,0))\,dx.\end{aligned}$$

Then $\mathcal{R}_g^{a,b}(u)$ is lower semi-continuous with respect to the $L^1(\Omega)$-convergence, and $\mathcal{R}_g^{a,b}$ coincides with the lower semi-continuous envelope of $R_g^{a,b}$.

PROOF. Observe that we have

$$\mathcal{R}_g(T_{a,b}(u)) = \int_\Omega\int_a^b \tilde{g}\left((x,s), \nu[(x,s); N(u)]\right)\, d\mathcal{H}^{N-1}(x)\,ds.$$

Hence, we may write

$$\begin{aligned}\mathcal{R}_g^{a,b}(u) = \mathcal{R}_g(T_{a,b}(u)) &+ \int_{[u\le a]} (g(x,u(x),0) - g(x,a,0))\,dx\\ &+ \int_{[u\ge b]} (g(x,u(x),0) - g(x,b,0))\,dx.\end{aligned}$$

Since the functional $\mathcal{R}_g$ is lower semi-continuous with respect to $L^1(\Omega)$ convergence (Theorem 2.1), we conclude that $\mathcal{R}_g^{a,b}$ is also lower semi-continuous. Moreover, if $u \in W^{1,1}(\Omega)$, we have

$$\begin{aligned}R_g^{a,b}(u) &= \int_{[a<u<b]} g(x,u(x),\nabla u(x))\,dx + \int_{[u\le a]} g(x,u(x),0)\,dx\\ &\quad + \int_{[u\ge b]} g(x,u(x),0)\,dx\\ &= \int_{[a<u<b]} g(x,u(x),\nabla u(x))\,dx + \int_{[u\le a]} g(x,a,0)\,dx\\ &\quad + \int_{[u\ge b]} g(x,b,0)\,dx + \int_{[u\le a]} (g(x,u(x),0) - g(x,a,0))\,dx\\ &\quad + \int_{[u\ge b]} (g(x,u(x),0) - g(x,b,0))\,dx\\ &= \mathcal{R}_g(T_{a,b}(u)) + \int_{[u\le a]} (g(x,u(x),0) - g(x,a,0))\,dx\\ &\quad + \int_{[u\ge b]} (g(x,u(x),0) - g(x,b,0))\,dx = \mathcal{R}_g^{a,b}(u).\end{aligned}$$

Therefore, $\mathcal{R}_g^{a,b}$ is a lower semi-continuous extension of $R_g^{a,b}$ to $BV(\Omega)$. □

Let $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying (2.3). Given $u \in BV(\Omega)$, let us define the measures

$$\mathcal{R}_g(u, \phi) := \int_{\mathbb{R}} \int_\Omega \phi(x) \tilde{g}\,(s, \nu[(s, x); N(u)])\; d\mathcal{H}^{N-1}(x)\, ds$$

and

$$\begin{aligned}\mathcal{R}_g^{a,b}(u, \phi) := &\int_a^b \int_\Omega \phi(x) \tilde{g}\,(s, \nu[(s, x); N(u)])\; d\mathcal{H}^{N-1}(x)\, ds \\ &+ \int_{[u \le a]} \phi(x)\,(g(u(x), 0) - g(a, 0))\; dx \\ &+ \int_{[u \ge b]} \phi(x)\,(g(u(x), 0) - g(b, 0))\; dx\end{aligned}$$

for any $\phi \in C(\overline{\Omega})$. For simplicity, we shall write

$$\mathcal{R}_g(u, \phi) = \int_\Omega \phi(x) g(u, Du)$$

and

$$\mathcal{R}_g^{a,b}(u, \phi) = \int_\Omega \phi(x) g(u, DT_{a,b}(u)).$$

The singular parts respect to the Lebesgue measure $\mathcal{L}^N$ of these measures will be denoted by

$$(\mathcal{R}_g)^s(u, \phi) = \int_\Omega \phi(x) g(u, Du)^s$$

and

$$(\mathcal{R}_g^{a,b})^s(u, \phi) = \int_\Omega \phi(x) g(u, DT_{a,b}(u))^s,$$

respectively.

We shall need several results from [7] (see also [25]) in order to give a sense to the integrals of bounded vector fields with divergence in L^p integrated with respect to the gradient of a BV function. Let $p \ge 1$ and $p' \ge 1$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Following [7], let

$$X_p(\Omega) = \{\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}(\mathbf{z}) \in L^p(\Omega)\}. \tag{2.8}$$

If $\mathbf{z} \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ we define the functional $(\mathbf{z}, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\mathbf{z}, Dw), \varphi \rangle := -\int_\Omega w\, \varphi \operatorname{div}(\mathbf{z})\, dx - \int_\Omega w\, \mathbf{z} \cdot \nabla \varphi\, dx. \tag{2.9}$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in Ω,

$$\int_\Omega (\mathbf{z}, Dw) = \int_\Omega \mathbf{z}\cdot\nabla w\, dx \qquad \forall\ w \in W^{1,1}(\Omega)\cap L^\infty(\Omega) \tag{2.10}$$

and

$$\left|\int_B (\mathbf{z}, Dw)\right| \le \int_B |(\mathbf{z}, Dw)| \le \|\mathbf{z}\|_\infty \int_B |Dw| \tag{2.11}$$

for any Borel set $B \subseteq \Omega$. Moreover, $(\mathbf{z}, Dw)$ is absolutely continuous with respect to $|Dw|$ with Radon-Nikodym derivative $\theta(\mathbf{z}, Dw, x)$ which is a $|Dw|$ measurable function from Ω to $\mathbb{R}$ such that

$$\int_B (\mathbf{z}, Dw) = \int_B \theta(\mathbf{z}, Dw, x)|Dw| \tag{2.12}$$

for any Borel set $B \subseteq \Omega$. We also have that

$$\|\theta(\mathbf{z}, Dw, .)\|_{L^\infty(\Omega,|Dw|)} \le \|\mathbf{z}\|_{L^\infty(\Omega,\mathbb{R}^N)}. \tag{2.13}$$

By writing

$$\mathbf{z}\cdot D^s u := (\mathbf{z}, Du) - (\mathbf{z}\cdot\nabla u)\ d\mathcal{L}^N,$$

we see that $\mathbf{z}\cdot D^s u$ is a bounded measure. Furthermore, in [25] it is proved that $\mathbf{z}\cdot D^s u$ is absolutely continuous with respect to $|D^s u|$ (and, thus, it is a singular measure with respect to $\mathcal{L}^N$), and

$$|\mathbf{z}\cdot D^s u| \le \|\mathbf{z}\|_\infty |D^s u|. \tag{2.14}$$

As a consequence of Theorem 2.4 of [7], we have:

$$\text{If}\quad \mathbf{z}\in X_p(\Omega)\cap C(\Omega,\mathbb{R}^N),\quad \text{then}\quad \mathbf{z}\cdot D^s u = (\mathbf{z}\cdot\overrightarrow{D^s u})\ d|D^s u|. \tag{2.15}$$

In [7], a weak trace on $\partial\Omega$ of the normal component of $\mathbf{z}\in X_p(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(\Omega)\to L^\infty(\partial\Omega)$ such that

$$\|\gamma(\mathbf{z})\|_\infty \le \|\mathbf{z}\|_\infty$$

$$\gamma(\mathbf{z})(x) = \mathbf{z}(x)\cdot\nu(x) \qquad \text{for all } x\in\partial\Omega \quad \text{if} \quad \mathbf{z}\in C^1(\overline{\Omega},\mathbb{R}^N).$$

We shall denote $\gamma(\mathbf{z})(x)$ by $[\mathbf{z},\nu](x)$. Moreover, the following *Green's formula*, relating the function $[\mathbf{z},\nu]$ and the measure $(\mathbf{z}, Dw)$, for $\mathbf{z}\in X_p(\Omega)$ and $w\in BV(\Omega)\cap L^{p'}(\Omega)$, is established:

$$\int_\Omega w\ \operatorname{div}(\mathbf{z})\ dx + \int_\Omega (\mathbf{z}, Dw) = \int_{\partial\Omega} [\mathbf{z},\nu] w\ d\mathcal{H}^{N-1}. \tag{2.16}$$

3. – Basic assumptions. The notion of entropy solution for the elliptic problem

This section deals with the elliptic problem

$$\begin{cases} v = -\operatorname{div}\ \mathbf{a}(u, Du) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here we assume that Ω is an open bounded set in $\mathbb{R}^N$, with boundary $\partial\Omega$ of class C^1, and the Lagrangian $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions, which we shall refer collectively as (H):

(H_1) f is continuous on $\mathbb{R} \times \mathbb{R}^N$ and is a convex differentiable function of ξ such that $\nabla_\xi f(z, \xi) \in C(\mathbb{R} \times \mathbb{R}^N)$. Further we require f to satisfy the linear growth condition

$$C_0\|\xi\| - D_0 \le f(z, \xi) \le M(\|\xi\| + 1). \tag{3.2}$$

for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $|z| \le R$ and some positive constants C_0, D_0, M depending on R. Moreover, we assume f^0 exists.

We consider the function $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$ associated to the Lagrangian f. By the convexity of f

$$\mathbf{a}(z, \xi) \cdot (\eta - \xi) \le f(z, \eta) - f(z, \xi), \tag{3.3}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(z, \eta) - \mathbf{a}(z, \xi)) \cdot (\eta - \xi) \ge 0. \tag{3.4}$$

Moreover, it is easy to see that

$$|\mathbf{a}(z, \xi)| \le M \qquad \forall\ (z, \xi) \in \mathbb{R} \times \mathbb{R}^N,\ |z| \le R. \tag{3.5}$$

We also assume that $\mathbf{a}(z, 0) = 0$ for all $z \in \mathbb{R}$. We consider the function $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(z, \xi) := \mathbf{a}(z, \xi) \cdot \xi.$$

By (3.4), we have

$$h(z, \xi) \ge 0 \qquad \forall \xi \in \mathbb{R}^N,\ z \in \mathbb{R}. \tag{3.6}$$

Moreover, from (3.3) and (3.2), it follows that

$$C_0\|\xi\| - D_1 \le h(z, \xi) \le M\|\xi\| \tag{3.7}$$

for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $|z| \le R$, where D_1 is a positive constant depending on R, C_0 and M being as above.

(H$_2$) We assume that $\frac{\partial \mathbf{a}}{\partial \xi_i}(z,\xi) \in C(\mathbb{R}\times\mathbb{R}^N)$ for any $i = 1,\dots,N$.

This assumption is not necessary for the case of separated variables described in Remark 3.1.

We assume that

(H$_3$) $h(z,\xi) = h(z,-\xi)$, for all $z \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ and h^0 exists.

Observe that we have

$$C_0\|\xi\| \le h^0(z,\xi) \le M\|\xi\| \quad \text{for any } (z,\xi) \in \mathbb{R}\times\mathbb{R}^N,\ |z| \le R.$$

(H$_4$) $f^0(z,\xi) = h^0(z,\xi)$, for all $\xi \in \mathbb{R}^N$ and all $z \in \mathbb{R}$.

(H$_5$) $\mathbf{a}(z,\xi)\cdot\eta \le h^0(z,\eta)$ for all $\xi,\eta \in \mathbb{R}^N$, and all $z \in \mathbb{R}$.

(H$_6$) We assume that $h^0(z,\xi)$ can be written in the form $h^0(z,\xi) = \varphi(z)\psi^0(\xi)$ with φ a C^1-function such that for any $R > 0$, we have $\varphi(z) > \alpha_R > 0$ for all $z \in \mathbb{R}$, $|z| \le R$, and ψ^0 being a convex function homogeneous of degree 1.

(H$_7$) For any $R > 0$, there is a constant $C > 0$ such that

$$|(\mathbf{a}(z,\xi) - \mathbf{a}(\hat z,\xi))\cdot(\xi - \hat\xi)| \le C|z-\hat z|\,\|\xi - \hat\xi\| \tag{3.8}$$

for any $(z,\xi),(\hat z,\hat\xi) \in \mathbb{R}\times\mathbb{R}^N$, $|z|,|\hat z| \le R$.

Observe that, by the monotonicity condition (3.4) and using (3.8), it follows that

$$(\mathbf{a}(z,\xi) - \mathbf{a}(\hat z,\hat\xi))\cdot(\xi - \hat\xi) \ge -C|z-\hat z|\,\|\xi - \hat\xi\| \tag{3.9}$$

for any $(z,\xi),(\hat z,\hat\xi) \in \mathbb{R}\times\mathbb{R}^N$, $|z|,|\hat z| \le R$.

Let us observe that under assumptions (H$_4$) and (H$_6$), applying the chain rule for BV-functions (see [1]), we have

$$\mathcal{R}_f(u) = \int_\Omega f(u,\nabla u)dx + \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|, \tag{3.10}$$

where $J_\varphi(r) = \int_0^r \varphi(s)\,ds$.

REMARK 3.1. An important particular case of Lagrangian f satisfying all assumptions (H) but (H$_2$), is the one given by $f(z,\xi) = \varphi(z)\psi(\xi)$ with φ a C^1-function such that for any $R > 0$, we have $\varphi(z) > \alpha_R > 0$ for all $z \in \mathbb{R}$, $|z| \le R$, and ψ a convex smooth function such that

$$C_0\|\xi\| - D_0 \le \psi(\xi) \le M(\|\xi\| + 1) \quad \forall \xi \in \mathbb{R}^N,$$

and there exists

$$\psi^0(\xi) = \lim_{t\to 0^+} t\psi\left(\frac{\xi}{t}\right).$$

In this case, if $\mathbf{b}(\xi) := \nabla\psi(\xi)$, we have $\mathbf{a}(z,\xi) = \varphi(z)\mathbf{b}(\xi)$, and $h(z,\xi) = \mathbf{a}(z,\xi)\cdot\xi = \varphi(z)\mathbf{b}(\xi)\cdot\xi$. Then, in order to have that (H) holds we need to assume that:

(i) $\mathbf{b}(-\xi)\cdot(-\xi) = \mathbf{b}(\xi)\cdot\xi \geq 0$ for all $\xi \in \mathbb{R}^N$ and there exists

$$\lim_{t\to 0^+} \mathbf{b}\left(\frac{\xi}{t}\right)\cdot\xi = \psi^0(\xi).$$

(ii) $\mathbf{b}(\xi)\cdot\eta \leq \psi^0(\xi)\cdot\eta$ for all $\xi,\eta \in \mathbb{R}^N$.

We note that in this case (H_2) is not necessary to obtain existence and uniqueness of solutions for problem (1.2). Let us prove that (H_7) holds. Indeed, by applying the mean value Theorem, we have

$$\begin{aligned}|(\mathbf{a}(z,\xi)-\mathbf{a}(\hat z,\xi))\cdot(\xi-\hat\xi)| &= |(\varphi(z)-\varphi(\hat z))||\mathbf{b}(\xi)\cdot(\xi-\hat\xi)|\\ &\leq \sup_{\tau\in[0,1]} |\varphi'(\tau z+(1-\tau)\hat z)|M\,|z-\hat z|\,\|\xi-\hat\xi\|\\ &\leq C|z-\hat z|\,\|\xi-\hat\xi\|,\end{aligned}$$

for some constant $C > 0$ depending on R and any $(z,\xi),(\hat z,\xi) \in \mathbb{R}\times\mathbb{R}^N$, $|z|, |\hat z| \leq R$.

REMARK 3.2. There are physical models for plasma fusion by inertial confinement in which the temperature evolution of the electrons satisfies an equation of type (1.2), where $\mathbf{a}(z,\xi) = \frac{|z|^{5/2}\xi}{1+|z||\xi|}$ which corresponds to $f(z,\xi) = |z|^{3/2}|\xi| - |z|^{1/2}\ln(1+|z||\xi|)$ [23], (see also [14] for a mathematical study in the one-dimensional case). It is easy to check that (H_1) (in particular (3.2) and (3.7)) holds for any $(z,\xi) \in \mathbb{R}\times\mathbb{R}^N$ with $z \in [a,R]$, $a > 0$, the constants in (3.2) and (3.7) depending on a, R. Note that (H_2) also holds. We also observe that $h^0(z,\xi) = |z|^{3/2}|\xi|$ and (H_3)-(H_6) hold. Finally, to check (H_7) we observe that

$$\frac{\partial\mathbf{a}}{\partial z}(z,\xi) = \frac{5}{2}\frac{z^{3/2}\xi}{1+z|\xi|} - \frac{z^{5/2}|\xi|\xi}{(1+z|\xi|)^2},$$

and therefore

$$\left|\frac{\partial\mathbf{a}}{\partial z}(z,\xi)\right| \leq \frac{7}{2}z^{1/2}$$

for any $z \in [a,R]$ and any $\xi \in \mathbb{R}^N$. It follows that

$$|\mathbf{a}(z,\xi)-\mathbf{a}(\hat z,\xi)| \leq \frac{7}{2}R^{1/2}|z-\hat z|$$

for any $z \in [a,R]$ and any $\xi \in \mathbb{R}^N$. Thus (H_7) also holds for the values of $z \in [a,R]$ and $\xi \in \mathbb{R}^N$. In this case, the results below will prove existence and uniqueness of entropy solutions of (1.1) for any initial condition $v \in L^\infty(\Omega)$ such that $v(x) \geq a > 0$ for some $a > 0$.

We need to consider the function space

$$TBV(\Omega) := \left\{ u \in L^1(\Omega) : T_k(u) \in BV(\Omega), \quad \forall\ k > 0 \right\},$$

and to give a sense to the Radon-Nikodym derivative ∇u of a function $u \in TBV(\Omega)$. Notice that the function space $TBV(\Omega)$ is closely related to the space $GBV(\Omega)$ of generalized functions of bounded variation introduced by E. Di Giorgi and L. Ambrosio ([22], see also [1]). In [3] we give the following results.

LEMMA 3.3. *For every $u \in TBV(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \mathcal{L}^N - \text{a.e.} \tag{3.11}$$

LEMMA 3.4. *If $u \in TBV(\Omega)$, then $p(u) \in BV(\Omega)$ for every Lipschitz continuous function $p : \mathbb{R} \to \mathbb{R}$ satisfying $p'(s) = 0$ for $|s|$ large enough. Moreover, $\nabla p(u) = p'(u)\nabla u$ $\mathcal{L}^N$-a.e.*

Thanks to Lemma 3.3 we define ∇u for a function $u \in TBV(\Omega)$ as the unique function v which satisfies (3.11). This notation will be used throughout in the sequel.

We introduce the following concept of solution for problem (3.1)

DEFINITION 3.5. Given $v \in L^\infty(\Omega)$, we say that u is an *entropy solution* of (3.1) if $u \in TBV(\Omega) \cap L^\infty(\Omega)$, $\mathbf{a}(u, \nabla u) \in X_1(\Omega)$ and satisfies:

$$v = -\text{div}\ \mathbf{a}(u, \nabla u) \qquad \text{in} \quad \mathcal{D}'(\Omega), \tag{3.12}$$

$$(\mathbf{a}(u, \nabla u), DT_{a,b}(u)) \geq h(u, DT_{a,b}(u)) \quad \text{as measures} \quad \forall\ a < b, \tag{3.13}$$

$$[\mathbf{a}(u, \nabla u), \nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e.} \quad \text{on } \partial\Omega. \tag{3.14}$$

Observe that (3.13) is equivalent to

$$\mathbf{a}(u, \nabla u) \cdot D^s T_{a,b}(u) \geq (\mathcal{R}_f^{a,b})^s(u) \quad \text{as measures} \quad \forall\ a < b. \tag{3.15}$$

We have the following characterization of entropy solutions.

PROPOSITION 3.6. *Let $v \in L^\infty(\Omega)$ and let $u \in TBV(\Omega) \cap L^\infty(\Omega)$ with $\mathbf{a}(u, \nabla u) \in X_1(\Omega)$, satisfying* (3.12) *and* (3.14). *Then u is an entropy solution of* (3.1) (*i.e., satisfies* (3.13)) *if and only if u satisfies*

$$\int_\Omega \phi h(u, DT_{a,b}(u)) + \int_\Omega T_{a,b}(u)\mathbf{a}(u, \nabla u) \cdot \nabla\phi\, dx \leq \int_\Omega \phi v T_{a,b}(u)\, dx \tag{3.16}$$

for any $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$.

PROOF. Assume that u is an entropy solution of (3.1). Multiplying (3.12) by $T_{a,b}(u)\phi$, integrating by parts and using (3.13) and (3.14), we obtain (3.16). Similarly, from (3.16), (3.13) and (3.14), we have

$$\begin{aligned}\int_\Omega \phi h(u, DT_{a,b}(u)) + \int_\Omega T_{a,b}(u)\mathbf{a}(u,\nabla u)\cdot\nabla\phi\,dx &\le \int_\Omega \phi v T_{a,b}(u)\,dx\\ &= -\int_\Omega \operatorname{div}\mathbf{a}(u,\nabla u)T_{a,b}(u)\phi\,dx = \int_\Omega (\mathbf{a}(u,\nabla u), D(T_{a,b}(u)\phi))\\ &= \int_\Omega \phi(\mathbf{a}(u,\nabla u), DT_{a,b}(u)) + \int_\Omega T_{a,b}(u)\mathbf{a}(u,\nabla u)\cdot\nabla\phi\,dx.\end{aligned}$$

Hence,

$$\int_\Omega \phi h(u, DT_{a,b}(u)) \le \int_\Omega \phi(\mathbf{a}(u,\nabla u), DT_{a,b}(u))$$

for all $\phi \in \mathcal{D}(\Omega)$, $\phi \ge 0$. This implies (3.13). □

4. – Existence and uniqueness of entropy solutions

This section is devoted to prove the following existence and uniqueness result.

THEOREM 4.1. *Assume that assumptions (H) hold. Then, for any $v \in L^\infty(\Omega)$ there exists a unique entropy solution $u \in BV(\Omega)\cap L^\infty(\Omega)$ of the problem*

$$\begin{cases} u - \operatorname{div}\mathbf{a}(u, Du) = v & \text{in } \Omega\\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

PROOF. *Step 1. Existence of entropy solutions.* Let $v \in L^\infty(\Omega)$. For every $n \in \mathbb{N}$, consider $\mathbf{a}_n(z,\xi) := \mathbf{a}(z,\xi) + \dfrac{1}{n}\xi$. Since $\mathbf{a}_n$ satisfies the classical Leray-Lions conditions [28], we know that for any $n \in \mathbb{N}$ there exists $u_n \in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that

$$\int_\Omega (w-u_n)(v-u_n)\,dx \le \int_\Omega \mathbf{a}_n(u_n,\nabla u_n)\cdot\nabla(w-u_n)\,dx \qquad \forall\, w \in W^{1,2}(\Omega). \tag{4.2}$$

Then, taking $w = u_n - (u_n - \|v\|_\infty)^+$ as test function in (4.2), we obtain

$$\int_\Omega (u_n - \|v\|_\infty)^+(u_n - v)\,dx \le 0.$$

Hence,

$$\int_{\{u_n > \|v\|_\infty\}} (u_n - \|v\|_\infty)^2\,dx \le \int_{\{u_n>\|v\|_\infty\}} (u_n - \|v\|_\infty)(v - \|v\|_\infty)\,dx \le 0.$$

Consequently, $u_n \le \|v\|_\infty$ a.e. in Ω. Similarly, taking $w = u_n - (u_n + \|v\|_\infty)^-$ as test function (where $r- := \min(r, 0)$), we get $-\|v\|_\infty \le u_n$ a.e. in Ω. Therefore,

$$\|u_n\|_\infty \le \|v\|_\infty \qquad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

Taking $w = 0$ in (4.2) and using (4.3) we get

$$\int_\Omega \mathbf{a}(u_n, \nabla u_n) \cdot \nabla u_n \, dx + \frac{1}{n} \int_\Omega |\nabla u_n|^2 \, dx \le \int_\Omega u_n v \, dx - \int_\Omega |u_n|^2 \, dx \le M_2.$$

Hence, by (3.7), we obtain

$$\int_\Omega |\nabla u_n| \, dx \le M_3 \quad \forall\ n \in \mathbb{N}. \tag{44}$$

and

$$\frac{1}{n} \int_\Omega |\nabla u_n|^2 \, dx \le M_4 \quad \forall\ n \in \mathbb{N} \tag{4.5}$$

Thus, $\{u_n : n \in \mathbb{N}\}$ is bounded in $W^{1,1}(\Omega)$ and, by extracting a subsequence if is necessary, we may assume that u_n converges in $L^1(\Omega)$ and almost everywhere to some $u \in L^1(\Omega) \cap BV(\Omega)$ as $n \to +\infty$. Now, by (4.3) and (4.4), we have that $u_n \to u$ in $L^2(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$.

Observe that by (3.5) and (4.5), $\{\mathbf{a}_n(u_n, \nabla u_n) : n \in \mathbb{N}\}$ is bounded in $L^2(\Omega, \mathbb{R}^N)$. Consequently we may assume that

$$\mathbf{a}_n(u_n, \nabla u_n) \rightharpoonup \mathbf{z} \quad \text{as } n \to \infty, \text{ weakly in } L^2(\Omega, \mathbb{R}^N). \tag{4.6}$$

Given $\phi \in C_0^\infty(\Omega)$, taking $w = u_n \pm \phi$ in (4.2) we obtain

$$\int_\Omega \phi(v - u_n) \, dx = \int_\Omega \mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla \phi \, dx.$$

Letting $n \to +\infty$, we obtain

$$\int_\Omega (v - u)\phi \, dx = \int_\Omega \mathbf{z} \cdot \nabla \phi \, dx,$$

that is,

$$v - u = -\mathrm{div}(\mathbf{z}), \qquad \text{in } \mathcal{D}'(\Omega) \tag{4.7}$$

and

$$\mathrm{div}\, \mathbf{a}_n(u_n, \nabla u_n) \rightharpoonup \mathrm{div}(\mathbf{z}) \qquad \text{weakly in } L^2(\Omega). \tag{4.8}$$

Since, by (4.5),

$$\frac{1}{n}|\nabla u_n| \to 0 \quad \text{in} \quad L^2(\Omega), \tag{4.9}$$

as a consequence of (4.6), it follows that

$$\mathbf{a}(u_n, \nabla u_n) \rightharpoonup \mathbf{z} \quad \text{as } n \to \infty, \text{ weakly in } L^2(\Omega, \mathbb{R}^N). \tag{4.10}$$

Moreover, by (3.5) we may assume that

$$\mathbf{a}(u_n, \nabla u_n) \rightharpoonup \mathbf{z} \quad \text{as } n \to \infty, \text{ weakly}^* \text{ in } L^\infty(\Omega, \mathbb{R}^N). \tag{4.11}$$

By (4.7), we have $\mathbf{z} \in X_2(\Omega)$. On the other hand, by (4.2), we have

$$[\mathbf{a}_n(u_n, \nabla u_n), \nu] = 0 \ \mathcal{H}^{N-1} - \text{a.e.} \quad \text{on} \quad \partial\Omega.$$

Then, by (4.6) and (4.8), we obtain that

$$[\mathbf{z}, \nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e.} \quad \text{on} \quad \partial\Omega. \tag{4.12}$$

Let us prove that

$$\mathbf{z}(x) = \mathbf{a}(u(x), \nabla u(x)) \qquad \text{a.e.} \quad x \in \Omega. \tag{4.13}$$

Let $0 \le \phi \in C_0^1(\Omega)$ and $g \in C^2(\Omega)$. By (3.4), we have

$$\int_\Omega \phi[\mathbf{a}(u_n, \nabla u_n) - \mathbf{a}(u_n, \nabla g)) \cdot \nabla(u_n - g)] \ dx \ge 0. \tag{4.14}$$

Now, since

$$\begin{aligned}
\int_\Omega \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla(u_n - g)\, dx &= \int_\Omega \phi \mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla(u_n - g)\, dx \\
&\quad - \frac{1}{n}\int_\Omega \phi \nabla u_n \cdot \nabla(u_n - g)\, dx \\
&\le -\int_\Omega \operatorname{div}(\mathbf{a}_n(u_n, \nabla u_n))\phi(u_n - g) \ dx \\
&\quad - \int_\Omega (u_n - g)\mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla\phi \ dx \\
&\quad + \frac{1}{n}\int_\Omega \phi \nabla u_n \cdot \nabla g\, dx,
\end{aligned}$$

we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \phi\mathbf{a}(u_n,\nabla u_n)\cdot\nabla(u_n-g)\,dx &\le -\int_\Omega \operatorname{div}(\mathbf{z})\phi(u-g)\;dx\\ &\quad -\int_\Omega (u-g)\mathbf{z}\cdot\nabla\phi\;dx\\ &= \int_\Omega \phi(\mathbf{z},D(u-g)).\end{aligned}$$

On the other hand, let us denote by

$$J_{\mathbf{a}_i}(x,r):=\int_0^r \mathbf{a}_i(s,\nabla g(x))\,ds,\quad \text{and}\quad J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x,r):=\int_0^r \frac{\partial}{\partial x_j}\mathbf{a}_i(s,\nabla g(x))\,ds,$$

$i,j\in\{1,\dots,N\}$, and observe that

$$\frac{\partial}{\partial x_j}J_{\mathbf{a}_i}(x,u_n(x))=\mathbf{a}_i(u_n(x),\nabla g(x))\frac{\partial u_n}{\partial x_j}(x)+J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x,u_n(x)).$$

We note that assumption (H_2) is used here, and, as we shall notice in Remark 4.3, (H_2) is not needed when $f(z,\xi)$ has the form described in Remark 3.1. Now, since

$$\frac{\partial}{\partial x_j}J_{\mathbf{a}_i}(x,u_n)\rightharpoonup\frac{\partial}{\partial x_j}J_{\mathbf{a}_i}(x,u)\qquad\text{weakly as measures,}$$

and $J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x,u_n(x))\to J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x,u(x))$ a.e., we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \phi\mathbf{a}(u_n,\nabla g)\cdot\nabla(u_n-g)dx &= \lim_{n\to\infty}\int_\Omega \phi\sum_{i=1}^N\left[\frac{\partial}{\partial x_i}J_{\mathbf{a}_i}(x,u_n(x))-J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x,u_n(x))\right]\\ &\quad -\lim_{n\to\infty}\int_\Omega \phi\,\mathbf{a}(u_n,\nabla g)\cdot\nabla g\,dx\\ &= \int_\Omega \phi\sum_{i=1}^N\left[\frac{\partial}{\partial x_i}J_{\mathbf{a}_i}(x,u)-J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x,u(x))\right]\\ &\quad -\int_\Omega \phi\,\mathbf{a}(u,\nabla g)\cdot\nabla g\,dx.\end{aligned}$$

Consequently, from (4.14), we obtain

$$\begin{aligned}&\int_\Omega \phi(\mathbf{z},D(u-g))\\ &\quad -\int_\Omega \phi\left(\sum_{i=1}^N\left[\frac{\partial}{\partial x_i}J_{\mathbf{a}_i}(x,u(x))-J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x,u(x))\right]-\mathbf{a}(u,\nabla g)\cdot\nabla g\right)\ge 0,\end{aligned}\tag{4.15}$$

for all $0 \le \phi \in C_0^1(\Omega)$. Thus the measure

$$(\mathbf{z}, D(u-g)) - \sum_{i=1}^{N} \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, u(x)) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, u(x)) \right] + \mathbf{a}(u, \nabla g) \cdot \nabla g \, \mathcal{L}^N \ge 0.$$

Then using chain's rule for BV functions ([1], Theorem 3.96) applied to $J_{\mathbf{a}_i}(u_1, u_2)$ with $u_1(x) = x$, $u_2(x) = u(x)$, $x \in \Omega$, we deduce that the absolutely continuous part of

$$\sum_{i=1}^{N} \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, u(x)) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, u(x)) \right]$$

is $\mathbf{a}(u, \nabla g) \cdot \nabla u \, \mathcal{L}^N$ and we obtain

$$\mathbf{z} \cdot \nabla(u-g) - \mathbf{a}(u, \nabla g) \cdot \nabla u + \mathbf{a}(u, \nabla g) \cdot \nabla g = (\mathbf{z} - \mathbf{a}(u, \nabla g)) \cdot \nabla(u-g) \ge 0 \quad \text{a.e..}$$

Since we may take a countable set of functions $g \in C^2(\Omega)$ dense in $C^1(\overline{\Omega})$ we have that the above inequality holds for all $x \in \tilde{\Omega}$, where $\tilde{\Omega} \subset \Omega$ is such that $\mathcal{L}^N(\Omega \setminus \tilde{\Omega}) = 0$, and all $g \in C^1(\overline{\Omega})$. Now, fixed $x \in \tilde{\Omega}$ and given $\xi \in \mathbb{R}^N$, there is $g \in C^1(\overline{\Omega})$ such that $\nabla g(x) = \xi$. Then

$$(\mathbf{z}(x) - \mathbf{a}(u(x), \xi)) \cdot (\nabla u(x) - \xi) \ge 0, \quad \forall \ \xi \in R^N.$$

These inequalities imply (4.13) by an application of Minty-Browder's method in $\mathbb{R}^N$.

From (4.13), (4.7) and (4.12), it follows that

$$v - u = -\operatorname{div} \mathbf{a}(u, \nabla u), \qquad \text{in} \quad \mathcal{D}'(\Omega) \tag{4.16}$$

and

$$[\mathbf{a}(u, \nabla u), \nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on} \quad \partial\Omega. \tag{4.17}$$

Therefore, to finish the existence part of the proof we only need to prove that

$$(\mathbf{a}(u, \nabla u), DT_{a,b}(u)) \ge h(u, DT_{a,b}(u)) \quad \text{as measures} \quad \forall \ a < b. \tag{4.18}$$

To do that, first let us prove

$$\limsup_n \int_\Omega \mathbf{a}(u_n, \nabla u_n) \cdot \nabla T_{a,b}(u_n) \phi(x) \, dx \le \int_\Omega \phi(\mathbf{a}(u, \nabla u), DT_{a,b}(u)) \tag{4.19}$$

for any $0 \le \phi \in \mathcal{D}(\Omega)$. By (4.2), we have

$$\int_\Omega w(v - u_n) \ dx = \int_\Omega \mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla w \ dx \qquad \forall \ w \in W^{1,2}(\Omega). \tag{4.20}$$

Then, taking $w = T_{a,b}(u_n)\phi$ as test function in (4.20), we obtain

$$\begin{aligned}\int_\Omega \phi \mathbf{a}(u_n, \nabla u_n)\cdot\nabla T_{a,b}(u_n)\,dx + \frac{1}{n}\int_\Omega \phi\nabla u_n\cdot\nabla T_{a,b}(u_n)\,dx \\ = \int_\Omega (v-u_n)T_{a,b}(u_n)\phi\,dx - \int_\Omega T_{a,b}(u_n)\mathbf{a}(u_n,\nabla u_n)\cdot\nabla\phi\,dx \\ - \frac{1}{n}\int_\Omega T_{a,b}(u_n)\nabla u_n\cdot\nabla\phi\,dx.\end{aligned}$$

Since the sign of the second term at the right hand side of the above inequality is positive, letting $n\to\infty$ we get

$$\begin{aligned}\limsup_n \int_\Omega \phi\mathbf{a}(u_n,\nabla u_n)\cdot\nabla T_{a,b}(u_n)\,dx &\le \int_\Omega \phi(v-u)T_{a,b}(u)\,dx \\ &\quad - \int_\Omega T_{a,b}(u)\mathbf{a}(u,\nabla u)\cdot\nabla\phi\,dx \\ &= -\int_\Omega \operatorname{div}\mathbf{a}(u,\nabla u)T_{a,b}(u)\phi\,dx \\ &\quad - \int_\Omega T_{a,b}(u)\mathbf{a}(u,\nabla u)\cdot\nabla\phi \\ &= \int_\Omega \phi(\mathbf{a}(u,\nabla u), DT_{a,b}(u)).\end{aligned}$$

Now, let us prove the following inequality for measures

$$f(u, DT_{a,b}(u)) \le (\mathbf{a}(u,\nabla u), DT_{a,b}(u)) + f(u,0). \tag{4.21}$$

Using the convexity of f, we have for any $w\in W^{1,1}(\Omega)$

$$\begin{aligned}\int_\Omega \phi f(u_n,\nabla T_{a,b}(u_n))\,dx &\le \int_\Omega \phi\mathbf{a}(u_n,\nabla T_{a,b}(u_n))\cdot\nabla T_{a,b}(u_n)\,dx \\ &\quad - \int_\Omega \phi\mathbf{a}(u_n,\nabla T_{a,b}(u_n))\cdot\nabla w\,dx \\ &\quad + \int_\Omega f(u_n,\nabla w)\phi\;dx.\end{aligned}$$

Choosing $w=0$, we obtain

$$\begin{aligned}\int_\Omega \phi f(u_n,\nabla T_{a,b}(u_n))\,dx &\le \int_\Omega \phi\mathbf{a}(u_n,\nabla T_{a,b}(u_n))\cdot\nabla T_{a,b}(u_n)\;dx \\ &\quad + \int_\Omega \phi f(u_n,0)\,dx.\end{aligned}$$

By Proposition 2.2 and (4.19), letting $n \to \infty$ we obtain

$$\begin{aligned}\mathcal{R}_f^{a,b}(u,\phi) &\le \liminf_n \int_\Omega f(u_n, \nabla T_{a,b}(u_n))\phi \ dx \\ &\le \liminf_n \int_\Omega \phi \mathbf{a}(u_n, \nabla T_{a,b}(u_n)) \cdot \nabla T_{a,b}(u_n) \ dx + \int_\Omega f(u,0)\phi \ dx \\ &\le \int_\Omega \phi(\mathbf{a}(u,\nabla u), DT_{a,b}(u)) + \int_\Omega \phi f(u,0)\, dx,\end{aligned}$$

and (4.21) holds.

Note that by the definition of $\mathcal{R}_f^{a,b}(u,\phi)$, using the chain rule for BV-functions, we deduce that

$$f(u, DT_{a,b}(u))^s = \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(T_{a,b}(u))|.$$

Then using (4.21), we have

$$\begin{aligned}(\mathbf{a}(u,\nabla u), DT_{a,b}(u)) &= (\mathbf{a}(u,\nabla u), DT_{a,b}(u))^{ac} + (\mathbf{a}(u,\nabla u), DT_{a,b}(u))^s \\ &\ge a(u,\nabla u)\cdot \nabla T_{a,b}(u) + \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(T_{a,b}(u))| \\ &= h(u, DT_{a,b}(u)).\end{aligned}$$

Step 2. Uniqueness of entropy solutions. Given $v, \overline{v} \in L^\infty(\Omega)$, let $u, \overline{u}$ be two bounded entropy solutions of the problems

$$\begin{cases} u - \operatorname{div}\ \mathbf{a}(u, Du) = v & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases} \tag{4.22}$$

and

$$\begin{cases} \overline{u} - \operatorname{div}\ \mathbf{a}(\overline{u}, D\overline{u}) = \overline{v} & \text{in } \Omega \\ \dfrac{\partial \overline{u}}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.23}$$

respectively.

Let ρ_n be a classical mollifiers in $\mathbb{R}^N$, and let us write $\xi_n(x,y) = \rho_n(x-y)$. If we denote $\mathbf{z}(y) = \mathbf{a}(u(x), \nabla u(x))$ and $\overline{\mathbf{z}}(x) = \mathbf{a}(\overline{u}(x), \nabla\overline{u}(y))$, we have

$$u - \operatorname{div}(\mathbf{z}) = v \quad \text{and} \quad \overline{u} - \operatorname{div}(\overline{\mathbf{z}}) = \overline{v} \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Then, multiplying by $T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)$ and integrating by parts , we obtain

$$\begin{aligned}&\int_\Omega u(y)T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)dy+\int_\Omega \xi_n(\mathbf{z},D_y(T_\varepsilon(u-\overline{u}(x))\\&+\int_\Omega T_\varepsilon(u(y)-\overline{u}(x))\mathbf{z}(y)\cdot\nabla_y\xi_n(x,y)\,dy\\&=\int_\Omega v(y)T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)\,dy\end{aligned}\tag{4.24}$$

and

$$\begin{aligned}&\int_\Omega \overline{u}(x)T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)dx+\int_\Omega \xi_n(\overline{\mathbf{z}},D_x(T_\varepsilon(u(y)-\overline{u}))\\&+\int_\Omega T_\varepsilon(u(y)-\overline{u}(x))\overline{z}(x)\cdot\nabla_x\xi_n(x,y)\,dx\\&=\int_\Omega \overline{v}(x)T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)\,dx.\end{aligned}\tag{4.25}$$

Integrating (4.24) in x and (4.25) in y, and taking differences we obtain

$$\begin{aligned}&\int_\Omega\int_\Omega (u(y)-\overline{u}(x))T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)\,dx\,dy\\&+\int_\Omega\left(\int_\Omega \xi_n(\mathbf{z},D_y(T_\varepsilon(u-\overline{u}(x)))\right)\,dx\\&+\int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\mathbf{z}(y)\cdot\nabla_y\xi_n(x,y)\,dy\,dx\\&-\int_\Omega\left(\int_\Omega \xi_n(\overline{\mathbf{z}},D_x(T_\varepsilon(u(y)-\overline{u})\right)\,dy\\&-\int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\overline{\mathbf{z}}(x)\cdot\nabla_x\xi_n(x,y)\,dx\,dy\\&=\int_\Omega\int_\Omega (v(y)-\overline{v}(x))T_\varepsilon(u(y)-\overline{u}(x))\xi_n(x,y)\,dx\,dy.\end{aligned}$$

Let I_1^n be the first term at the left hand side of the above identity, I_2^n the other part of the left hand side of the above identity and let I_3^n be the right hand side term. Note that, by applying Green's formula and using that $T_\varepsilon(-r)=-T_\varepsilon(r)$

and $\nabla_y \xi_n(x,y) + \nabla_x \xi_n(x,y) = 0$, we get

$$
\begin{aligned}
I_2^n &= \int_\Omega \left(\int_\Omega \xi_n(\mathbf{z}, D_y(T_\varepsilon(u-\overline{u}(x)))\right) dx - \int_\Omega \left(\int_\Omega \xi_n(\overline{\mathbf{z}}, D_x(T_\varepsilon(u(y)-\overline{u})))\right) dy \\
&\quad + \int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\mathbf{z}(y)\cdot\nabla_y\xi_n(x,y)\,dy\,dx \\
&\quad - \int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\overline{\mathbf{z}}(x)\cdot\nabla_x\xi_n(x,y)\,dx\,dy \\
&= \int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\left[(\mathbf{z}(y)-\overline{\mathbf{z}}(x))\cdot(\nabla_y\xi_n(x,y)+\nabla_x\xi_n(x,y))\right] dy\,dx \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n(\mathbf{z}, D_y(T_\varepsilon(u-\overline{u}(x)))\right) dx \\
&\quad + \int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\overline{\mathbf{z}}(x)\cdot\nabla_y\xi_n(x,y)\,dx\,dy \\
&\quad - \int_\Omega \left(\int_\Omega \xi_n(\overline{\mathbf{z}}, D_x(T_\varepsilon(u(y)-\overline{u})))\right) dy \\
&\quad - \int_{\Omega\times\Omega} T_\varepsilon(u(y)-\overline{u}(x))\mathbf{z}(y)\cdot\nabla_x\xi_n(x,y)\,dy\,dx \\
&= \int_\Omega \left(\int_\Omega \xi_n(\mathbf{z}, D_y(T_\varepsilon(u-\overline{u}(x)))\right) dx \\
&\quad - \int_\Omega \left(\int_\Omega \xi_n(\overline{\mathbf{z}}(x), D_y T_\varepsilon(u-\overline{u}(x))\right) dx \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n(\overline{\mathbf{z}}, D_x(T_\varepsilon(\overline{u}-u(y))))\right) dy \\
&\quad - \int_\Omega \left(\int_\Omega \xi_n(\mathbf{z}(y), D_x T_\varepsilon(\overline{u}-u(y)))\right) dy.
\end{aligned}
$$

Since $T_\varepsilon(\overline{u}(x)-u(y)) = T_{-\varepsilon+u(y),\varepsilon+u(y)}(\overline{u}(x))-u(y)$, if we take $\overline{u}_\varepsilon(x,y) := T_{-\varepsilon+u(y),\varepsilon+u(y)}(\overline{u}(x))$, we have

$$D_x(T_\varepsilon(\overline{u}-u(y)) = D_x\overline{u}_\varepsilon(\cdot,y).$$

Similarly,

$$D_y(T_\varepsilon(u-\overline{u}(x)) = D_y u_\varepsilon(\cdot,x),$$

being $u_\varepsilon(y,x) := T_{-\varepsilon+\overline{u}(x),\varepsilon+\overline{u}(x)}(u(y))$. Therefore, we can write

$$\begin{aligned}
I_2^n &= \int_\Omega \left(\int_\Omega \xi_n(x,y)\mathbf{z}(y)\cdot\nabla_y u_\varepsilon(y,x)\,dy \right) dx \\
&\quad - \int_\Omega \left(\int_\Omega \xi_n(x,y)\overline{\mathbf{z}}(x)\cdot\nabla_y u_\varepsilon(y,x)\,dy \right) dx \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n \mathbf{z}\cdot D_y^s u_\varepsilon(\cdot,x) \right) dx - \int_\Omega \left(\int_\Omega \xi_n \overline{\mathbf{z}}(x)\cdot D_y^s u_\varepsilon(\cdot,x) \right) dx \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n(x,y)\overline{\mathbf{z}}(x)\cdot\nabla_x \overline{u}_\varepsilon(x,y)\,dx \right) dy \\
&\quad - \int_\Omega \left(\int_\Omega \xi_n(x,y)\mathbf{z}(y)\cdot\nabla_x \overline{u}_\varepsilon(x,y)\,dx \right) dy \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n \overline{\mathbf{z}}\cdot D_x^s \overline{u}_\varepsilon(\cdot,y)\,dx \right) dy - \int_\Omega \left(\int_\Omega \xi_n \mathbf{z}(y)\cdot D_x^s \overline{u}_\varepsilon(\cdot,y)\,dx \right) dy \\
&= \int_{\Omega\times\Omega} \xi_n(x,y)(\mathbf{z}(y)-\overline{\mathbf{z}}(x))\cdot(\nabla_y u_\varepsilon(y,x) - \nabla_x \overline{u}_\varepsilon(x,y))\,dy\,dx \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n \overline{\mathbf{z}}\cdot D_x^s \overline{u}_\varepsilon(\cdot,y)\,dx \right) dy - \int_\Omega \left(\int_\Omega \xi_n \mathbf{z}(y)\cdot D_x^s \overline{u}_\varepsilon(\cdot,y)\,dx \right) dy \\
&\quad + \int_\Omega \left(\int_\Omega \xi_n \mathbf{z}\cdot D_y^s u_\varepsilon(\cdot,x) \right) dx - \int_\Omega \left(\int_\Omega \xi_n \overline{\mathbf{z}}(x)\cdot D_y^s u_\varepsilon(\cdot,x) \right) dx \\
&= I_{21}^n + I_{22}^n + I_{23}^n.
\end{aligned}$$

Let us compute I_{21}^n. By (3.9) it follows that

$$\begin{aligned}
I_{21}^n &= \int_{\Omega\times\Omega} \xi_n(x,y)(\mathbf{z}(y)-\overline{\mathbf{z}}(x))\cdot(\nabla_y u_\varepsilon(y,x) - \nabla_x \overline{u}_\varepsilon(x,y))\,dy\,dx \\
&= \int_{\Omega\times\Omega} \xi_n(x,y) T_\varepsilon'(u(y)-\overline{u}(x))(\mathbf{z}(y)-\overline{\mathbf{z}}(x))\cdot(\nabla_y u(y) - \nabla_x \overline{u}(x))\,dy\,dx \\
&\geq -C\int_{\Omega\times\Omega} \xi_n(x,y) T_\varepsilon'(u(y)-\overline{u}(x))|u(y)-\overline{u}(x)|\,\|\nabla_y u(y) - \nabla_x \overline{u}(x)\|\,dy\,dx \\
&\geq -C\varepsilon \int_\Omega \left(\int_{u(y)-\varepsilon}^{u(y)+\varepsilon} \xi_n(x,y)\|\nabla_y u(y) - \nabla_x \overline{u}(x)\|\,dx \right) dy.
\end{aligned}$$

Hence

$$\frac{1}{\varepsilon} I_{21}^n \geq o(\varepsilon) \qquad \forall\, n\in\mathbb{N}, \tag{4.26}$$

where $o(\varepsilon)$ is an expression that tends to 0 as $\varepsilon \to 0^+$.

Now we analyze I_{22}^n. Having in mind the condition (3.13) of the definition of entropy solution, we obtain that

$$\int_\Omega \left(\int_\Omega \xi_n \overline{\mathbf{z}}\cdot D_x^s \overline{u}_\varepsilon(\cdot,y)\,dx \right) dy \geq \int_\Omega \left(\int_\Omega \xi_n h(\overline{u}_\varepsilon(\cdot,y), D_x \overline{u}_\varepsilon(\cdot,y))^s \right) dy$$

On the other hand, by (H_5) and (H_6), we have

$$\mathbf{z}(y)\cdot D_x^s\overline{u}_\varepsilon(\cdot,y)=(\mathbf{z}(y)\cdot\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^s\overline{u}_\varepsilon|\le\varphi(u(y))\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^s\overline{u}_\varepsilon|.$$

Therefore we have

$$\begin{aligned}
I_{22}^n &\ge \int_\Omega\left(\int_\Omega \xi_n h(\overline{u}_\varepsilon, D_x\overline{u}_\varepsilon)^s\right)dy-\int_\Omega\left(\int_\Omega\xi_n\varphi(u(y))\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^s\overline{u}_\varepsilon|\right)dy\\
&=\int_\Omega\left(\int_\Omega\xi_n\varphi(\overline{u}_\varepsilon(x))\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^c\overline{u}_\varepsilon|\right)dy\\
&\quad-\int_\Omega\left(\int_\Omega\xi_n\varphi(u(y))\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^c\overline{u}_\varepsilon|\right)dy\\
&\quad+\int_\Omega\left(\int_{J_{\overline{u}_\varepsilon}}\xi_n\frac{1}{(\overline{u}_\varepsilon)^+(x)-(\overline{u}_\varepsilon)^-(x)}\left(\int_{(\overline{u}_\varepsilon)^-(x)}^{(\overline{u}_\varepsilon)^+(x)}\varphi(s)ds\right)\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^j\overline{u}_\varepsilon|\right)dy\\
&\quad-\int_\Omega\left(\int_\Omega\xi_n\varphi(u(y))\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^j\overline{u}_\varepsilon|\right)dy=J_1^n+J_2^n
\end{aligned}$$

where J_1^n denotes the first and second terms of the above expression, and J_2^n the third and fourth terms. Now, since φ is Lipschitz continuous, we have

$$\begin{aligned}
|J_1^n| &\le \int_\Omega\left(\int_\Omega\xi_n|\varphi(\overline{u}_\varepsilon(x))-\varphi(u(y))|\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^c\overline{u}_\varepsilon|\right)dy\\
&\le M_n\int_\Omega\left(\int_\Omega|\overline{u}_\varepsilon(x)-u(y)|T'_{-\varepsilon+u(y),\varepsilon+u(y)}(\overline{u}(x))|D_x^c\overline{u}|\right)dy\\
&= M_n\int_\Omega\left(\int_{\{x\in\Omega\ :\ -\varepsilon+u(y)<\overline{u}(x)<\varepsilon+u(y)\}}|\overline{u}_\varepsilon(x)-u(y)||D_x^c\overline{u}|\right)dy\\
&\le \varepsilon M_n\int_\Omega\left(\int_\Omega|D_x^c\overline{u}_\varepsilon|\right).
\end{aligned}$$

Using the coarea formula, we get

$$|J_1^n|\le\varepsilon M_n\int_\Omega\left(\int_{u(y)-\varepsilon}^{u(y)+\varepsilon}\mathrm{Per}(\{\overline{u}(x)\ge\lambda\})\,d\lambda\right)dy,$$

which yields

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon}J_1^n=0\qquad\forall\,n\in\mathbb{N}. \tag{4.27}$$

On the other hand, working in a similar way as before,

$$\begin{aligned}
|J_2^n| &\le \int_\Omega\left[\int_{J_{\overline{u}_\varepsilon}}\xi_n\frac{1}{(\overline{u}_\varepsilon)^+(x)-(\overline{u}_\varepsilon)^-(x)}\left(\int_{(\overline{u}_\varepsilon)^-(x)}^{(\overline{u}_\varepsilon)^+(x)}|\varphi(s)-\varphi(u(y))|ds\right)\psi^0(\overrightarrow{D^s\overline{u}_\varepsilon})|D_x^j\overline{u}_\varepsilon|\right]dy\\
&\le \varepsilon M_n\int_\Omega\left(\int_{u(y)-\varepsilon}^{u(y)+\varepsilon}\mathrm{Per}(\{\overline{u}(x)\ge\lambda\})\,d\lambda\right)dy,
\end{aligned}$$

and we obtain that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}J_2^n=0 \qquad \forall\, n\in\mathbb{N}. \tag{4.28}$$

Then, by (4.27) and (4.28), we get

$$\frac{1}{\varepsilon}I_{22}^n\geq o(\varepsilon) \qquad \forall\, n\in\mathbb{N}. \tag{4.29}$$

In a similar way, we also obtain that

$$\frac{1}{\varepsilon}I_{23}^n\geq o(\varepsilon) \qquad \forall\, n\in\mathbb{N}. \tag{4.30}$$

Now, from (4.26), (4.29) and (4.30), we deduce that

$$\frac{1}{\varepsilon}I_n^2\geq o(\varepsilon) \qquad \forall\, n\in\mathbb{N}.$$

Consequently, since

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}I_1^n=\int_\Omega |u(x)-\overline{u}(x)|\,dx,$$

and

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}I_3^n\leq\int_\Omega |v(x)-\overline{v}(x)|\,dx,$$

we deduce that

$$\int_\Omega |u(x)-\overline{u}(x)|\,dx\leq\int_\Omega |v(x)-\overline{v}(x)|\,dx,$$

and we conclude the proof of the Theorem. □

REMARK 4.2. When applying Kruzhkov's method, if instead of multiplying by $T_\varepsilon(u(y)-\overline{u}(x))$ we multiply by $T_\varepsilon(u(y)-\overline{u}(x))^+$, we obtain the estimate

$$\int_\Omega (u(x)-\overline{u}(x))^+\,dx\leq\int_\Omega (v(x)-\overline{v}(x))^+\,dx. \tag{4.31}$$

REMARK 4.3. We observe that (H_2) is not used when $f(z,\xi)=\varphi(z)\psi(\xi)$ as in Remark 3.1. Indeed, it suffices to prove the analogous of (4.15). Since $\mathbf{a}(z,\xi)=\varphi(z)\mathbf{b}(\xi)$, if we denote by $J_\varphi(r):=\int_0^r\varphi(s)\,ds$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \phi\,\mathbf{a}(u_n,\nabla g)\cdot\nabla(u_n-g)dx&=\lim_{n\to\infty}\int_\Omega \phi\,\varphi(u_n)\mathbf{b}(\nabla g)\cdot\nabla(u_n-g)\,dx\\&=\lim_{n\to\infty}\int_\Omega \phi\,\mathbf{b}(\nabla g)\cdot\big(\nabla J_\varphi(u_n)-\varphi(u_n)\nabla g\big)\,dx.\end{aligned}$$

Now, since $\nabla J_\varphi(u_n) \rightharpoonup DJ_\varphi(u)$ weakly as measures, we have

$$\lim_{n\to\infty} \int_\Omega \phi\, \mathbf{a}(u_n, \nabla g)\cdot\nabla(u_n - g)\,dx = \int_\Omega \phi[\mathbf{b}(\nabla g)\cdot DJ_\varphi(u) - \varphi(u)\mathbf{b}(\nabla g)\cdot\nabla g]$$

and (4.15) follows as in the proof above.

REMARK 4.4. If $\mathbf{a}(z,\xi) = \frac{|z|^{5/2}\xi}{1+|z||\xi|}$ [23], [14], then Theorem 4.1 holds for any $v \in L^\infty(\Omega)$ such that $v \geq a$, for some $a > 0$. Moreover, multiplying the equation

$$u_n - a - \operatorname{div}\mathbf{a}_n(u_n, \nabla u_n) = v - a$$

by $(u_n - a)^- = \min(u_n - a, 0)$ and integrating in Ω we obtain that

$$\int_\Omega ((u_n - a)^-)^2\,dx \leq \int_\Omega (v-a)(u_n-a)^-\,dx \leq 0$$

and we deduce that $u_n \geq a$. Hence $u \geq a$.

5. – Semigroup solution

In this section we shall associate an m-accretive operator in $L^1(\Omega)$ to the formal differential expression $-\operatorname{div}\,\mathbf{a}(u,\nabla u)$ together with the Neumann boundary conditions.

DEFINITION 5.1. $(u,v) \in B$ if and only if $u \in BV(\Omega)\cap L^\infty(\Omega)$, $v \in L^1(\Omega)$ and u is the entropy solution of problem (3.1), that is, $\mathbf{a}(u,\nabla u) \in X_1(\Omega)$ and satisfies:

$$v = -\operatorname{div}\,\mathbf{a}(u,\nabla u) \qquad \text{in} \quad \mathcal{D}'(\Omega) \tag{5.1}$$

$$\mathbf{a}(u,\nabla u)\cdot D^s T_{a,b}(u) \geq (\mathcal{R}_f^{a,b})^s(u) \quad \forall\, a < b, \tag{5.2}$$

$$[\mathbf{a}(u,\nabla u),\nu] = 0 \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial\Omega. \tag{5.3}$$

Let $(u,v) \in B$, and $w \in BV(\Omega)\cap L^\infty(\Omega)$. Multiplying (5.1) by $w-u$, using Green's formula (2.16) and having in mind that (5.2) is equivalent to (3.13) (using $a = -\|u\|_\infty$, $b = \|u\|_\infty$), we obtain

$$\begin{aligned}\int_\Omega (w-u)v\,dx &= -\int_\Omega (w-u)\operatorname{div}\,\mathbf{a}(u,\nabla u)\,dx \\ &= \int_\Omega (\mathbf{a}(u,\nabla u), Dw - Du) - \int_{\partial\Omega}[\mathbf{a}(u,\nabla u),\nu](w-u)\,d\mathcal{H}^{N-1} \\ &\leq \int_\Omega (\mathbf{a}(u,\nabla u), Dw) - \mathcal{R}_h(u).\end{aligned}$$

Therefore, if $(u, v) \in B$, we have that

$$\int_\Omega (w-u)v \ dx \le \int_\Omega (\mathbf{a}(u, \nabla u), Dw) - \mathcal{R}_h(u), \tag{5.4}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$.

PROPOSITION 5.2. *Assume we are under assumptions* (H). *Then B is accretive in $L^1(\Omega)$, $L^\infty(\Omega) \subset R(I+B)$ and $D(B)$ is dense in $L^1(\Omega)$.*

Again, we observe that assumption (H_2) is not required if $f(z,\xi) = \varphi(z)\psi(\xi)$, being φ a bounded smooth function such that $\varphi(z) \ge \alpha_R > 0$ for all $z \in \mathbb{R}$, $|z| \le R$ (see Remark 3.1).

PROOF. The accretivity of the operator B in $L^1(\Omega)$ and the range condition follows from Theorem 4.1.

To prove the density of $D(B)$ in $L^1(\Omega)$, we prove that $C_0^\infty(\Omega) \subseteq \overline{B}^{L^2(\Omega)}$. Let $v \in C_0^\infty(\Omega)$. By Theorem 4.1, $v \in R(I + \frac{1}{n}B)$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $u_n \in D(B)$, $\|u_n\|_\infty \le \|v\|_\infty$, such that $(u_n, n(v-u_n)) \in B$. Consequently, by (5.4), we get

$$\int_\Omega (w-u_n)n(v-u_n) \ dx \le \int_\Omega (\mathbf{a}(u_n, \nabla u_n), Dw) - \mathcal{R}_h(u_n),$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$. Taking $w = v$, we get

$$\begin{aligned}\int_\Omega (v-u_n)^2 \ dx &\le \frac{1}{n}\left(\int_\Omega \mathbf{a}(u_n, \nabla u_n)\cdot \nabla v \, dx - \mathcal{R}_h(u_n)\right)\\ &\le \frac{1}{n}\int_\Omega \mathbf{a}(u_n, \nabla u_n)\cdot \nabla v \, dx \le \frac{M}{n}\int_\Omega |\nabla v| \ dx.\end{aligned}$$

Letting $n \to \infty$, it follows that $u_n \to v$ in $L^2(\Omega)$. Therefore $v \in \overline{D(B)}^{L^2(\Omega)} \subset \overline{D(B)}^{L^1(\Omega)}$ and the proof is complete. □

From Proposition 5.2, if we denote by $\mathcal{B}$ the closure in $L^1(\Omega)$ of the operator B, it follows that $\mathcal{B}$ is m-accretive in $L^1(\Omega)$ and $\overline{D(\mathcal{B})}^{L^1(\Omega)} = L^1(\Omega)$. Therefore, according to the general theory of nonlinear semigroups (c.f., e.g., [11]), for any $u_0 \in L^1(\Omega)$ there exists a unique mild solution $u \in C([0,T]; L^1(\Omega))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}u(t) \ni 0, \qquad u(0) = u_0. \tag{5.5}$$

Moreover, $u(t) = T(t)u_0$ for all $t \ge 0$, being $(T(t))_{t\ge 0}$ the semigroup in $L^1(\Omega)$ generated by the Crandall-Liggett's exponential formula, i.e.,

$$T(t)u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}\mathcal{B}\right)^{-n} u_0.$$

LEMMA 5.3. *Given $\lambda > 0$, and $u \in L^q(\Omega)$,$1 \le q \le \infty$, if $v = (I + \lambda B)^{-1}u$, then*

$$\|v\|_q \le \|u\|_q. \tag{5.6}$$

As a consequence, for any $u_0 \in L^q(\Omega)$, $1 \le q \le \infty$, we have

$$T(t)u_0 \in L^q(\Omega) \quad \forall\, t \ge 0. \tag{5.7}$$

PROOF. Since $v = (I + \lambda B)^{-1}u$, we have $\left(v, \frac{1}{\lambda}(u - v)\right) \in B$. Then, $\mathbf{a}(v, \nabla v) \in X_1(\Omega)$ and the following holds:

$$\begin{aligned} \frac{1}{\lambda}(u - v) &= -\operatorname{div} \mathbf{a}(v, \nabla v) \quad \text{in} \quad \mathcal{D}'(\Omega), \\ \mathbf{a}(v, \nabla v) \cdot D^s v &\ge (\mathcal{R}_f)^s(v), \\ [\mathbf{a}(v, \nabla v), \nu] &= 0 \qquad \mathcal{H}^{N-1} - \text{a.e.} \quad \text{on } \partial\Omega. \end{aligned}$$

Let

$$\mathcal{P}_0 := \left\{ p \in C^\infty(\mathbb{R}) \ : \ 0 \le p' \le 1, \ \operatorname{supp}(p') \text{ compact}, \ 0 \notin \operatorname{supp}(p) \right\}.$$

For $p \in \mathcal{P}_0$, by Green's formula, we have

$$\begin{aligned} \int_\Omega p(v)(v - u)\, dx &= \lambda \int_\Omega p(v) \operatorname{div} \mathbf{a}(v, \nabla v)\, dx = -\lambda \int_\Omega (\mathbf{a}(v, \nabla v), Dp(v)) \\ &= -\lambda \int_\Omega p'(v)\mathbf{a}(v, \nabla v) \cdot \nabla v\, dx - \lambda \int_\Omega \mathbf{a}(v, \nabla v) \cdot D^s(p(v)). \end{aligned}$$

Now, by (3.6), it follows that

$$\int_\Omega p'(v)\mathbf{a}(v, \nabla v) \cdot \nabla v\, dx \ge 0.$$

On the other hand, by the chain rule for BV-functions (see [1]), we have $D^s(p(v)) = \overline{p}_v D^s v$ with $\overline{p}_v \ge 0$, $\overline{p}_v$ being the Vol'pert averaged superposition. Moreover, by [7],

$$\theta(\mathbf{a}(v, \nabla v), Dp(v), \cdot) = \theta(\mathbf{a}(v, \nabla v), Dv, \cdot) \quad |Dv| - \text{a.e.}$$

Then,

$$\begin{aligned} \mathbf{a}(v, \nabla v) \cdot D^s(p(v)) &= \theta(\mathbf{a}(v, \nabla v), Dp(v), \cdot)|D^s p(v)| \\ &= \overline{p}_v \theta(\mathbf{a}(v, \nabla v), Dv, \cdot)|D^s v| \\ &= \overline{p}_v \mathbf{a}(v, \nabla v) \cdot D^s v \ge \overline{p}_v (\mathcal{R}_f)^s(v) \ge 0. \end{aligned}$$

Therefore, we get

$$\int_\Omega p(v)(v - u)\, dx \le 0,$$

and consequently,

$$\int_\Omega p(v)v\, dx \le \int_\Omega p(v)u\, dx \qquad \forall\, p \in \mathcal{P}_0.$$

This implies (5.6), having in mind a result of [10]. Last assertion is a consequence of (5.6). □

In [6] we shall define the notion of entropy solution for (1.2), and we shall prove that entropy solutions are unique and coincide with semigroup solutions.

REFERENCES

[1] L. Ambrosio – N. Fusco – D. Pallara, "Functions of Bounded Variation and Free Discontinuity Problems", Oxford Mathematical Monographs, 2000.

[2] F. Andreu – V. Caselles – J.M. Mazón, *A parabolic quasilinear problem for linear growth functionals*, Rev. Mat. Iberoamericana **18** (2002), 135-185.

[3] F. Andreu – V. Caselles – J.M. Mazón, *Existence and uniqueness of solution for a parabolic quasilinear problem for linear growth functionals with L^1 data*, Math. Ann. **322** (2002), 139-206.

[4] F. Andreu – V. Caselles – J.M. Mazón, *The Cauchy Problem for Linear Growth Functionals with*, J. Evol. Equ. **3** (2003), 39-65.

[5] F. Andreu – V. Caselles – J.M. Mazón, "Parabolic Quasilinear Equations Minimizing Linear Growth Functionals", Progress in Mathematics, vol. 223, Birkhäuser-Verlag, 2004.

[6] F. Andreu – V. Caselles – J.M. Mazón, " A Strongly Degenerate Quasilinear Equation: the Parabolic Case", Preprint 2003.

[7] G. Anzellotti, *Pairings Between Measures and Bounded Functions and Compensated Compactness*, Ann. Mat. Pura Appl. (4) **135** (1983), 293-318.

[8] G.I. Barenblatt – M. Bertsch – R. Dal Passo – V.M. Prostokishin – M. Ughi, *A mathematical Model of Turbulent Heat and Mass Transfer in Stable Stratified Shear Flow*, J. Fluid Mech. **253** (1993), 341-358.

[9] Ph. Bénilan – L. Boccardo – T. Gallouet – R. Gariepy – M. Pierre – J.L Vazquez, *An L^1-Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), 241-273.

[10] Ph. Bénilan – M.G. Crandall, *Completely Accretive Operators*, In: "Semigroups Theory and Evolution Equations", Ph. Clement et al. (eds.), Marcel Dekker, 1991, pp. 41-76.

[11] Ph. Bénilan – M.G. Crandall – A. Pazy, "Evolution Equations Governed by Accretive Operators", Book in preparation.

[12] M. Bertsch – R. Dal Passo, *Hyperbolic Phenomena in a Strongly Degenerate Parabolic Equation*, Arch. Rational Mech. Anal. **117** (1992), 349-387.

[13] M. Bertsch – R. Dal Passo, *A Parabolic Equation with Mean-Curvature Type Operator*, In: "Progres Nonlinear Differential Equation Appl.", F. Browder (ed.), **7** Birkhäuser, 1992, 89-97.

[14] Ph. Blanc, *On the regularity of the solutions of some degenerate parabolic equations*, Comm. Partial Differential Equations **18** (1993), 821-846.

[15] Ph. Blanc, *Sur une classe d'equations paraboliques degeneréesa une dimension d'espace possedant des solutions discontinues*, Ph.D. Thesis, number 798, Ecole Polytechnique Federale de Lausanne, 1989.

[16] D. Blanchard – F. Murat, *Renormalized solutions on nonlinear parabolic problems with L^1 data: existence and uniqueness*, Proc. Royal Soc. Edinburgh Sect. A **127** (1997), 1137-1152.

[17] J. CARRILLO – P. WITTBOLD, *Uniqueness of Renormalized Solutions of Degenerate Elliptic-Parabolic problems*, Jour. Differential Equations **156** (1999), 93-121.

[18] M. G. CRANDALL, *Nonlinear Semigroups and Evolution Governed by Accretive Operators*, In: "Proceeding of Symposium in Pure Mat.", Part I, F. Browder (ed.) A.M.S., Providence 1986, 305-338.

[19] M. G. CRANDALL – T. M. LIGGETT, *Generation of Semigroups of Nonlinear Transformations on General Banach Spaces*, Amer. J. Math. **93** (1971), 265-298.

[20] R. DAL PASSO, *Uniqueness of the entropy solution of a strongly degenerate parabolic equation*, Comm. Partial Differential Equations **18** (1993), 265-279.

[21] G. DAL MASO, *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals*, Manuscripta Math. **30** (1980), 387-416.

[22] E. DE GIORGI – L. AMBROSIO, *Un nuovo tipo di funzionale del calcolo delle variazioni*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur., Rend. Lincei (9) Mat. Appl., 8 **82** (1988), 199-210.

[23] J. J. DUDERSTADT – G. A. MOSES, "Inertial Confinement Fusion", John Wiley and Sons, 1982.

[24] L. C. EVANS – R. F. GARIEPY, "Measure Theory and Fine Properties of Functions", Studies in Advanced Math., CRC Press, 1992.

[25] R. KOHN – R. TEMAM, *Dual space of stress and strains with application to Hencky plasticity*, Appl. Math. Optim. **10** (1983), 1-35.

[26] S. N. KRUZHKOV, *First order quasilinear equations in several independent variables*, Math. USSR-Sb. **10** (1970), 217-243.

[27] J. LERAY – J. L. LIONS, *Quelques resultats de Visik sur les problemes elliptiques semilineaires par les metode de Minty et Browder*, Bull. Soc. Math. France **93** (1965), 97-107.

[28] P. ROSENAU, *Free Energy Functionals at the High Gradient Limit*, Phys. Review A **41** (1990), 2227-2230.

[29] W.P. ZIEMER, "Weakly Differentiable Functions", GTM 120, Springer Verlag, 1989.

Universitat de Valencia
Dept. de Análisis Matemático

Universitat Pompeu-Fabra
Dept. de Tecnologia

Universitat de Valencia
Dept. de Análisis Matemático