This material corresponds to the full version of the appendix in the paper

"The Manufacturer's Choice of Distribution Policy under Successive Duopoly"

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Appendix A: The Third Stage Equilibrium Outcomes.

a) **The** $(1,0)$, $(2,0)$, $(0,1)$ **and** $(0,2)$ **distribution systems.**

In any of these cases there is only one brand and one retailer present in the market. The distribution system $(i,0)$ means that M_1 is distributing its brand through R_i, $i=1,2$, while M_2 is out of the market; similarly for $(0,i)$. We analyze for example the subgame $(1,0)$. R_1 chooses q_{11} to maximize,

$$R_1(q_{11}) = (a_1 - bq_{11} - w_1)q_{11}$$

the equilibrium output, price and retailer's profits are

$$q_1(1,0) \equiv q_{11}(1,0) = \tfrac{a_1-w_1}{2b} \quad p_1(1,0) = \tfrac{a_1+w_1}{2} \quad R_1(1,0) = \tfrac{(a_1-w_1)^2}{4b}$$

while for R_2 it is obvious that it distributes zero and gets zero profits. The restriction on the parameter space to get interior (nonnegative) output equilibria for subgames $(1,0)$, $(2,0)$, $(0,1)$ and $(0,2)$ is that $(a_i - w_i) > 0$ for $i = 1,\ 2$.

b) **The** $(1,1)$ **and** $(2,2)$ **distribution systems.**

Under these distribution systems, both manufacturers have chosen the same retailer which is a multi-product monopolist retailer. Take for example the case $(1,1)$, R_1 chooses q_{11} q_{21} to maximize

$$R_1(q_{11}, q_{21}) = (a_1 - bq_{11} - dq_{21} - w_i)q_{11} + (a_2 - bq_{21} - dq_{11} - w_2)q_{21}$$

which results in the following equilibrium outcomes

$$q_1(1,1) = q_{11}(1,1) + q_{21}(1,1) =$$
$$= \begin{cases} 0 + \frac{(a_2 - w_2)}{2b} & \text{for } 0 < \alpha_R < \frac{d}{b} \\ \frac{b(a_1 - w_1) - d(a_2 - w_2)}{2(b^2 - d^2)} + \frac{b(a_2 - w_2) - d(a_1 - w_1)}{2(b^2 - d^2)} & \text{for } \frac{d}{b} \leq \alpha_R \leq \frac{b}{d} \\ \frac{(a_1 - w_1)}{2b} + 0 & \text{for } \frac{b}{d} < \alpha_R \end{cases}$$

where $\alpha_R = (a_1 - w_1)/(a_2 - w_2)$ denotes the relative per unit profitability of brands for retailers..Retailers' prices and profits are,

$$p_1(1,1) = \tfrac{a_1 + w_1}{2};\ p_2(1,1) = \tfrac{a_2 + w_2}{2} \text{ for } \alpha_R > 0$$

$$R_1(1,1) = \begin{cases} \frac{(a_2 - w_2)^2}{4b} & \text{for } 0 < \alpha_R < \frac{d}{b} \\ \frac{b(a_1 - w_1)^2 + b(a_2 - w_2)^2 - 2d(a_1 - w_1)(a_2 - w_2)}{4(b^2 - d^2)} & \text{for } \frac{d}{b} \leq \alpha_R \leq \frac{b}{d} \\ \frac{(a_1 - w_1)^2}{4b} & \text{for } \frac{b}{d} < \alpha_R \end{cases}$$

In this case, to get interior equilibrium outputs, we need to restrict α_R to the next interval

$$\frac{d}{b} \leq \alpha_R \leq \frac{b}{d}$$

Note that whenever $0 < \alpha_R < \frac{d}{b}$, the retailer does not sell brand one and sells the amount of brand 2 equal to that in under distribution system $(0, 1)$. By the same token, for $\alpha_R > \frac{b}{d}$ we have $q_{21}(1, 1) = 0$ and the retailer sells an amount of brand 1 $q_{11}(1, 1) = q_{11}(1, 0)$.

c) **The** $(12, 0)$ **and** $(0, 12)$ **distribution systems.**

In both cases we have a homogenous duopoly in the downstream market. One of the manufacturers employs both retailers to distribute its brand while the other manufacturer's brand is not sold in the market. Take as an example the case of $(12, 0)$. Retailers one and two maximize profits choosing q_{11}, and q_{12}, respectively.

$$\begin{aligned} R_1(q_{11}, q_{12}) &= (a_1 - b(q_{11} + q_{12}) - w_1)q_{11} \\ R_2(q_{11}, q_{12}) &= (a_1 - b(q_{12} + q_{11}) - w_1)q_{12} \end{aligned}$$

The equilibrium quantities are obtained by solving the two-equation system of first order conditions for q_{11} and q_{12}. These are:

$$\begin{aligned}
q_1(12,0) &\equiv q_{11}(12,0) = \frac{a_1 - w_1}{3b} \\
q_2(12,0) &\equiv q_{12}(12,0) = \frac{a_1 - w_1}{3b} \\
p_1(12,0) &= \frac{a_1 + 2w_1}{3} \\
R_1(12,0) &= R_2(12,0) = \frac{(a_1 - w_1)^2}{9b}
\end{aligned}$$

with the same restriction as in the case presented in the first place.

d) **The** $(1,2)$ **and** $(2,1)$ **distribution systems.**

Here, each manufacturer uses one retailer and a different one from the retailer employed by the other manufacturer. Therefore, there is a differentiated duopoly in the downstream market. Consider the case $(1,2)$. Each retailer maximizes profits by choosing quantities q_{11} and q_{22}.

$$\begin{aligned}
R_1(q_{11}, q_{22}) &= (a_1 - bq_{11} - dq_{22} - w_1)q_{11} \\
R_2(q_{11}, q_{22}) &= (a_2 - bq_{22} - dq_{11} - w_2)q_{22}
\end{aligned}$$

We obtain the following equilibrium outcomes

$$q_1(1,2) \equiv q_{11}(1,2) = \begin{cases} 0 & \text{for } 0 < \alpha_R < \frac{d}{2b} \\ \frac{2b(a_1-w_1)-d(a_2-w_2)}{4b^2-d^2} & \text{for } \frac{d}{2b} \le \alpha_R \le \frac{2b}{d} \\ \frac{(a_1-w_1)}{2b} & \text{for } \frac{2b}{d} < \alpha_R \end{cases}$$

$$q_2(1,2) \equiv q_{22}(1,2) = \begin{cases} \frac{(a_2-w_2)}{2b} & \text{for } 0 < \alpha_R < \frac{d}{2b} \\ \frac{2b(a_2-w_2)-d(a_1-w_1)}{4b^2-d^2} & \text{for } \frac{d}{2b} \le \alpha_R \le \frac{2b}{d} \\ 0 & \text{for } \frac{2b}{d} < \alpha_R \end{cases}$$

Retailers' prices and profits are,

$$p_1(1,2) = \begin{cases} \emptyset & \text{for } 0 < \alpha_R < \frac{d}{2b} \\ \frac{2b^2a_1+(2b^2-d^2)w_1-bd(a_2-w_2)}{4b^2-d^2} & \text{for } \frac{d}{2b} \le \alpha_R \le \frac{2b}{d} \\ \frac{a_1+w_1}{2} & \text{for } \frac{2b}{d} < \alpha_R \end{cases}$$

$$p_2(1,2) = \begin{cases} \frac{a_2+w_2}{2} & \text{for } 0 < \alpha_R < \frac{d}{2b} \\ \frac{2b^2a_2+(2b^2-d^2)w_2-bd(a_1-w_1)}{4b^2-d^2} & \text{for } \frac{d}{2b} \le \alpha_R \le \frac{2b}{d} \\ \emptyset & \text{for } \frac{2b}{d} < \alpha_R \end{cases}$$

$$R_1(1,2) = \begin{cases} 0 & \text{for } 0 < \alpha_R < \frac{d}{2b} \\ \frac{b[2b(a_1-w_1)-d(a_2-w_2)]^2}{(4b^2-d^2)^2} & \text{for } \frac{d}{2b} \leq \alpha_R \leq \frac{2b}{d} \\ \frac{(a_1-w_1)^2}{4b} & \text{for } \frac{2b}{d} < \alpha_R \end{cases}$$

$$R_2(1,2) = \begin{cases} \frac{(a_2-w_2)^2}{4b} & \text{for } 0 < \alpha_R < \frac{d}{2b} \\ \frac{b[2b(a_2-w_2)-d(a_1-w_1)]^2}{(4b^2-d^2)^2} & \text{for } \frac{d}{2b} \leq \alpha_R \leq \frac{2b}{d} \\ 0 & \text{for } \frac{2b}{d} < \alpha_R \end{cases}$$

where the restriction in order to get interior equilibrium outputs becomes:

$$\frac{d}{2b} < \alpha_R < \frac{2b}{d}$$

e) **The** $(12,1)$, $(12,2)$ **and** $(1,12)$, $(2,12)$ **distribution systems.**

These are asymmetric cases where one of the manufacturers employs both retailers, while the other employs only one. Then, we have a multi-product retailer facing a single-product one. Take as an example the distribution system $(12,1)$. Each retailer maximizes,

$$\begin{aligned} \max_{q_{11},q_{21}} R_1(q_{11},q_{12},q_{21}) &= (a_1 - b(q_{11}+q_{12}) - dq_{21} - w_1)q_{11} \\ &\quad +(a_2 - bq_{21} - d(q_{11}+q_{12}) - w_2)q_{21} \end{aligned}$$

$$\max_{q_{12}} R_2(q_{11}, q_{12}, q_{21}) \quad = \quad (a_1 - b(q_{11} + q_{12}) - dq_{21} - w_1)q_{12}$$

The equilibrium outputs are the solution to the three-equation system of first order conditions for q_{11}, q_{12}, and q_{21}. We obtain

$q_1(12,1) \equiv q_{11}(12,1) + q_{21}(12,1) =$

$$\begin{cases} 0 + \frac{2b(a_2-w_2)-d(a_1-w_1)}{4b^2-d^2} & \text{for } 0 < \alpha_R < \frac{3bd}{2b^2+d^2} \\ \frac{(2b^2+d^2)(a_1-w_1)-3bd(a_2-w_2)}{6b(b^2-d^2)} + \frac{b(a_2-w_2)-d(a_1-w_1)}{2(b^2-d^2)} & \text{for } \frac{3bd}{2b^2+d^2} \leq \alpha_R \leq \frac{b}{d} \\ \frac{(a_1-w_1)}{3b} + 0 & \text{for } \frac{b}{d} < \alpha_R \end{cases}$$

$$q_2(12,1) \equiv q_{12}(12,1) \begin{cases} \frac{2b(a_1-w_1)-d(a_2-w_2)}{4b^2-d^2} & \text{for } 0 < \alpha_R < \frac{3bd}{2b^2+d^2} \\ \frac{(a_1-w_1)}{3b} & \text{for } \frac{3bd}{2b^2+d^2} \leq \alpha_R \end{cases}$$

Retailers' prices and profits are,

$$p_1(12,1) \quad = \quad \begin{cases} \frac{2b^2a_1+(2b^2-d^2)w_1-bd(a_2-w_2)}{4b^2-d^2} & \text{for } 0 < \alpha_R < \frac{3bd}{2b^2+d^2} \\ \frac{a_1+2w_1}{3} & \text{for } \frac{3bd}{2b^2+d^2} \leq \alpha_R \end{cases}$$

$$p_2(12,1) \quad = \quad \begin{cases} \frac{2b^2a_2+(2b^2-d^2)w_2-bd(a_1-w_1)}{4b^2-d^2} & \text{for } 0 < \alpha_R < \frac{3bd}{2b^2+d^2} \\ \frac{3b(a_2+w_2)-d(a_1-w_1)}{6b} & \text{for } \frac{3bd}{2b^2+d^2} \leq \alpha_R \leq \frac{b}{d} \\ \emptyset & \text{for } \frac{b}{d} < \alpha_R \end{cases}$$

$$R_1(12,1) = \begin{cases} \frac{b[2b(a_2-w_2)-d(a_1-w_1)]^2}{(4b^2-d^2)^2} & \text{for } 0 < \alpha_R < \frac{3bd}{2b^2+d^2} \\ \frac{(4b^2+5d^2)(a_1-w_1)^2+9b^2(a_2-w_2)^2-18bd(a_1-w_1)(a_2-w_2)}{36b(b^2-d^2)} & \text{for } \frac{3bd}{2b^2+d^2} \leq \alpha_R \leq \frac{b}{d} \\ \frac{(a_1-w_1)^2}{9b} & \text{for } \frac{b}{d} < \alpha_R \end{cases}$$

$$R_2(12,1) = \begin{cases} \frac{b[2b(a_1-w_1)-d(a_2-w_2)]^2}{(4b^2-d^2)^2} & \text{for } 0 < \alpha_R < \frac{3bd}{2b^2+d^2} \\ \frac{(a_1-w_1)^2}{9b} & \text{for } \frac{3bd}{2b^2+d^2} \leq \alpha_R \end{cases}$$

where it is clear from above that the restriction to ensure that $q_{11}(12,1) = q_{12}(1,12)$ are nonnegative is,

$$\frac{3bd}{2b^2+d^2} < \alpha_R$$

while, by a similar analysis as above, the restriction applying for $q_{21}(12,2) = q_{22}(2,12)$ is,

$$\alpha_R < \frac{2b^2+d^2}{3bd}$$

Appendix B: Proof of Proposition 1.

We compute the Nash equilibrium in pure strategies in the payoff matrix given by Table 3. The strategy of the proof consists of constructing the best-response function for each manufacturer when the rival hires either one retailer or two retailers. These best responses are presented in four lemmatas, the combination of which yields the equilibria reported in the proposition.

Five different payoffs for each player need to be considered. For the sake of the proof we will make use the following:

i) $\alpha_M = \frac{a_1 - c}{a_2 - c}$belongs to the interval $[1, \frac{2b^2 - d^2}{db}]$ and,

ii) $\frac{d}{b} \in (0, 1)$ since $b > d > 0$.

1) In order to construct the best response function for M_1, suppose that:

A) M_2 chooses $s_2 = 1$ (respectively, $s_2 = 2$).

The best response for M_1 follows from ranking the next expressions:

$$M_1(1,1) = M_1(2,2) = \frac{b((2b^2 - d^2)(a_1 - c) - bd(a_2 - c))^2}{2(b^2 - d^2)(4b^2 - d^2)^2}$$

$$M_1(2,1) = M_1(1,2) = \frac{2b((8b^2 - d^2)(a_1 - c) - 2bd(a_2 - c))^2}{(4b^2 - d^2)(16b^2 - d^2)^2}$$

$$M_1(12,1) = M_1(12,2) = \frac{(4b^2 - d^2)((8b^2 - 5d^2)(a_1 - c) - 3bd(a_2 - c))^2}{6b(b^2 - d^2)(16b^2 - 7d^2)^2}$$

Note that the strategy $s_1 = 0$ is always dominated. We start by comparing

$M_1(2,1)$ and $M_1(1,1)$ (which is equivalent to comparing $M_1(1,2)$ and $M_1(2,2)$).

The difference $M_1(2,1) - M_1(1,1)$ defines a concave quadratic function in C. Since the roots of the quadratic function set the range of C for which the function is either positive or negative and since it is assumed that the range of C is bounded, the strategy of the proof amounts to ranking the roots and the boundaries of $[1, \frac{2b^2-d^2}{db}]$. Let r_A be the upper root of that function (the lower root, whenever it exists, is the same expression as the upper one up to a negative sign before the term with the square root),

$$r_A = \frac{b[256b^6 + 32b^4d^2 - 70b^2d^4 + 7d^6 + 2(16b^2 - d^2)(4b^2 + d^2)\sqrt{(4b^2 - d^2)(b^2 - d^2)}]}{d(384b^6 - 204b^4d^2 + 48b^2d^4 - 3d^6)}$$

It is easily proven that r_A is greater than one since $b > d$ and it is smaller than $\frac{2b^2-d^2}{db}$ if $\frac{d}{b} < 0.708$. Likewise, the lower root is smaller than one since $b > d$. Therefore, it follows that

$$\begin{aligned} M_1(2,1) &\geq M_1(1,1) \text{ if } \begin{cases} \text{either } \frac{d}{b} \in (0, 0.708] & \text{and } C \in [1, r_A] \\ \text{or } \frac{d}{b} \in (0.708, 1) & \text{and } C \in [1, \frac{2b^2-d^2}{db}] \end{cases} \\ & \text{while,} \\ M_1(1,1) &\geq M_1(2,1) \text{ if } \frac{d}{b} \in (0, 0.708] \text{ and } C \in [r_A, \frac{2b^2 - d^2}{db}]. \end{aligned}$$

By the same token, the difference $M_1(12,1) - M_1(1,1)$ defines a convex quadratic function in C. Since the upper root of that equation,

$r_B = \frac{bd[3d^4(14b^2-5d^2)+2b(16b^2-7d^2)(4b^2-d^2)\sqrt{3(4b^2-d^2)}]}{1024b^8-1408b^6d^2+756b^4d^4-208b^2d^6+25d^8}$, is always smaller than one, we conclude that:

$$M_1(12,1) > M_1(1,1) \text{ if } \frac{d}{b} \in (0,1) \text{ and } C \in [1, \frac{2b^2-d^2}{db}].$$

Finally, the difference $M_1(12,1) - M_1(2,1)$ defines a convex quadratic function in C. It is easily proven that the lower root is always smaller than one. The upper one, denoted by r_C,

$$\begin{aligned} r_C \;=\; & \frac{3bd(16384b^{10} - 8192b^8d^2 - 4288b^6d^4 + 2360b^4d^6 - 184b^2d^8 - 5d^{10})}{65536b^{12} - 73728b^{10}d^2 + 36096b^8d^4 - 12992b^6d^6 + 3780b^4d^8 - 492b^2d^{10} + 25d^{12}} \\ & + \frac{2b^2d(4b^2-d^2)(16b^2-d^2)(16b^2-7d^2)(8b^2+7d^2)\sqrt{3(b^2-d^2)}}{65536b^{12} - 73728b^{10}d^2 + 36096b^8d^4 - 12992b^6d^6 + 3780b^4d^8 - 492b^2d^{10} + 25d^{12}} \end{aligned}$$

is increasing with the ratio $\frac{d}{b}$, and satisfies the following:

$$\begin{array}{ll} 0 < r_C \leq 1 & \text{if } \frac{d}{b} \in (0, 0.682] \\ 1 < r_C \leq \frac{2b^2-d^2}{bd} & \text{if } \frac{d}{b} \in (0.682, 0.907] \\ \frac{2b^2-d^2}{bd} < r_C & \text{if } \frac{d}{b} \in (0.907, 1) \end{array}$$

Therefore, it is true that:

$$M_1(12,1) \geq M_1(2,1) \text{ if } \begin{cases} \text{either } \frac{d}{b} \in (0, 0.682] & \text{and } C \in [1, \frac{2b^2-d^2}{db}] \\ \text{or } \frac{d}{b} \in (0.682, 0.907] & \text{and } C \in [r_C, \frac{2b^2-d^2}{db}] \end{cases}$$

while,

$$M_1(2,1) \geq M_1(12,1) \text{ if } \begin{cases} \text{either } \frac{d}{b} \in (0.682, 0.907] & \text{and } C \in [1, r_C] \\ \text{or } \frac{d}{b} \in (0.907, 1) & \text{and } C \in [1, \frac{2b^2-d^2}{db}] \end{cases}$$

The above discussion in summarized in the following lemma:

Lemma 1 *The best response to* $s_2 = 1$ *(respectively, to* $s_2 = 2$*) is,*

a) $s_1 = 12$ *if*

either $\frac{d}{b} \in (0, 0.682]$ *and* $C \in [1, \frac{2b^2-d^2}{db}]$,

or $\frac{d}{b} \in (0.682, 0.907]$ *and* $C \in [r_C, \frac{2b^2-d^2}{db}]$.

b) $s_1 = 2$ *(respectively,* $s_1 = 1$*) if*

either $\frac{d}{b} \in (0.682, 0.907]$ *and* $C \in [1, r_C]$,

or $\frac{d}{b} \in (0.907, 1)$ *and* $C \in [1, \frac{2b^2-d^2}{db}]$.

B) M_2 **chooses** $s_2 = 12$.

The best response for M_1 follows from comparing:

$$M_1(1,12) = M_1(2,12) = \frac{(b(8b^2-5d^2)(a_1-c)-d(4b^2-d^2)(a_2-c))^2}{2b(b^2-d^2)(16b^2-7d^2)^2}$$

$$M_1(12,12) = \frac{2b((2b^2-d^2)(a_1-c)-bd(a_2-c))^2}{3(b^2-d^2)(4b^2-d^2)^2}$$

The difference $M_1(12,12) - M_1(1,12)$ (or equivalently $M_1(12,12) - M_1(2,12)$) defines a convex quadratic function in C. Let r_D denote the upper root of that equation,

$$r_D = \frac{d[512b^8 - 704b^6d^2 + 280b^4d^4 + 8b^2d^6 - 15d^8 + 2d^2(64b^6 - 108b^4d^2 + 51b^2d^4 - 7d^6)\sqrt{3}]}{b(1024b^8 - 2304b^6d^2 + 2080b^4d^4 - 840b^2d^6 + 121d^8)}$$

This root is smaller or equal than one if $\frac{d}{b} \in (0,\, 0.909]$ while it belongs to the interval $[1, \frac{2b^2-d^2}{db}]$ when $\frac{d}{b} \in (0.909,\, 1)$. Then, the next lemma is stated:

Lemma 2 *The best response to* $s_2 = 12$ *is*

a) $s_1 = 12$ *if*

either $\frac{d}{b} \in (0, 0.909]$ *and* $C \in [1, \frac{2b^2-d^2}{db}]$,

or $\frac{d}{b} \in (0.909, 1]$ *and* $C \in [r_D, \frac{2b^2-d^2}{db}]$.

b) $s_1 = 1$ *and* $s_1 = 2$ *if* $\frac{d}{b} \in (0.909, 1]$ *and* $C \in [1, r_D]$.

2)The next step is to obtain the best response function for M_2, suppose that:

A) M_1 chooses either $s_1 = 1$ (respectively $s_1 = 2$).

To compute the best response for M_2 the relevant payoffs are:

$$M_2(1,1) = M_2(2,2) = \frac{b((2b^2-d^2)(a_2-c)-bd(a_1-c))^2}{2(b^2-d^2)(4b^2-d^2)^2}$$

$$M_2(1,2) = M_2(2,1) = \frac{2b((8b^2-d^2)(a_2-c)-2bd(a_1-c))^2}{(4b^2-d^2)(16b^2-d^2)^2}$$

$$M_2(1,12) = M_2(2,12) = \frac{(4b^2-d^2)((8b^2-5d^2)(a_2-c)-3bd(a_1-c))^2}{6b(b^2-d^2)(16b^2-7d^2)^2}$$

The difference $M_2(1,2) - M_2(1,1)$ (equivalently $M_2(2,1) - M_2(2,2)$) defines a concave quadratic function in C. It is easily proven that the upper root of that equation,

$r_a = \frac{256b^6+32b^4d^2-70b^2d^2+7d^6+2(16b^2-d^2)(4b^2+d^2)\sqrt{(4b^2-d^2)(b^2-d^2)]}}{3bd(64b^4+16b^2d^2-5d^4)}$ is greater than $\frac{2b^2-d^2}{db}$ and that the lower root is smaller than one since $b > d$. It follows that

$$M_2(1,2) = M_2(2,1) > M_2(1,1) = M_2(2,2) \text{ if } \frac{d}{b} \in (0,1) \text{ and } C \in [1, \frac{2b^2-d^2}{db}].$$

Similarly, the difference $M_2(1,12) - M_2(1,1)$ (equivalently $M_2(2,12) - M_2(2,2)$) defines a concave quadratic function in C. Since the upper root of that equation, $r_b = \frac{-3d^4(14b^2-5d^2)+2b(16b^2-7d^2)(4b^2-d^2)\sqrt{3(4b^2-d^2)}}{bd(192b^4-48b^2d^2-9d^4)}$, is always greater than $\frac{2b^2-d^2}{db}$ and the lower one is smaller than one, we conclude that:

$$M_2(1,12) = M_2(2,12) > M_2(1,1) = M_2(2,2) \text{ if } \frac{d}{b} \in (0,1) \text{ and } C \in [1, \frac{2b^2-d^2}{db}]$$

Finally, the difference $M_2(1,12) - M_1(1,2)$ (equivalently $M_2(2,12) - M_2(2,1)$) defines a convex quadratic function in C. It is easily proven that the upper root is always greater than $\frac{2b^2-d^2}{db}$. The lower one, denoted by r_c,

$$r_c = \frac{3(16384b^{10} - 8192b^8d^2 - 4288b^6d^4 + 2360b^4d^6 - 184b^2d^8 - 5d^{10})}{3bd(8192b^8 - 2784b^4d^4 + 664b^2d^6 + 3d^8)} - \frac{2b(4b^2-d^2)(16b^2-d^2)(16b^2-7d^2)(8b^2+7d^2)\sqrt{3(b^2-d^2)}}{3bd(8192b^8 - 2784b^4d^4 + 664b^2d^6 + 3d^8)}$$

is decreasing with the ratio $\frac{d}{b}$, and satisfies the following:

$$\begin{array}{ll} 1 < r_c < \frac{2b^2-d^2}{bd} & \text{if } \frac{d}{b} \in (0, 0.682] \\ 0 < r_c < 1 & \text{if } \frac{d}{b} \in (0.682, 1] \end{array}$$

Therefore, it follows that:

$$M_2(1,2) \geq M_2(1,12) \text{ if } \begin{cases} \text{either } \frac{d}{b} \in (0, 0.682] & \text{and } C \in [r_c, \frac{2b^2-d^2}{db}] \\ \text{or } \frac{d}{b} \in (0.682, 1] & \text{and } C \in [1, \frac{2b^2-d^2}{db}] \end{cases}$$

while,

$$M_2(1,12) \geq M_2(1,2) \text{ if } \frac{d}{b} \in (0, 0.682] \text{ and } C \in [1, r_c]$$

The following lemma summarizes the above analysis:

Lemma 3 *The best response to $s_1 = 1$ (respectively, to $s_1 = 2$) is,*

a) $s_2 = 12$ if $\frac{d}{b} \in (0, 0.682]$ and $C \in [1, r_c]$

b) $s_2 = 2$ (respectively, $s_2 = 1$) if

either $\frac{d}{b} \in (0, 0.682]$ and $C \in [r_c, \frac{2b^2-d^2}{db}]$,

or $\frac{d}{b} \in (0.682, 1]$ and $C \in [1, \frac{2b^2-d^2}{db}]$.

B) M_1 **chooses** $s_1 = 12$.

The best response for M_2 follows from comparing of the next expressions:

$$M_2(12,1) = M_2(12,2) = \frac{(4b^2-d^2)((8b^2-5d^2)(a_2-c)-3bd(a_1-c))^2}{6b(b^2-d^2)(16b^2-7d^2)^2}$$

$$M_2(12,12) = \frac{2b((2b^2-d^2)(a_2-c)-bd(a_1-c))^2}{3(b^2-d^2)(4b^2-d^2)^2}$$

The difference $M_2(12,12) - M_2(12,1)$ (equivalently $M_2(12,12) - M_2(12,2)$) defines a convex quadratic function in C. Let r_d be the lower root of that equation,

$$r_d = \frac{b(512b^8 - 704b^6d^2 + 280b^4d^4 + 8b^2d^6 - 15d^8) - 2d(64b^6 - 108b^4d^2 + 51b^2d^4 - 7d^6)\sqrt{3}}{d(256b^8 - 128b^6d^2 - 92b^4d^4 + 48b^2d^6 - 3d^8)}$$

This root is smaller than one if $\frac{d}{b} \in (0.909,\ 1)$ while it belongs to the interval $(1, \frac{2b^2-d^2}{bd})$ when $\frac{d}{b} \in (0,\ 0.909)$. The upper root is always greater than $\frac{2b^2-d^2}{bd}$. Then, the next lemma states M_2's best response to $s_1 = 12$.

Lemma 4 *The best response to* $s_1 = 12$ *is*

a) $s_2 = 1$ *and* $s_2 = 2$ *if*

either $\frac{d}{b} \in (0, 0.909]$ *and* $C \in [r_d, \frac{2b^2-d^2}{db}]$,

or $\frac{d}{b} \in (0.909, 1]$ *and* $C \in [1, \frac{2b^2-d^2}{db}]$.

b) $s_2 = 12$ *if* $\frac{d}{b} \in (0, 0.909]$ *and* $C \in [1, r_d]$.

Taking into account the above lemmatas the range of $\frac{d}{b}$ is partitioned into four intervals: i) $\frac{d}{b} \in (0, 0.682]$; ii) $\frac{d}{b} \in (0.682, 0.907]$; iii) $\frac{d}{b} \in (0.907,\ 0.909]$ and iv)

$\frac{d}{b} \in (0.909,\, 1)$. Now, we establish the Nash equilibrium for each of these intervals as a function of C.

1)Let us assume that $\frac{d}{b} \in (0, 0.682]$.

By Lemmatas 1 and 2, the best response of M_1 to $s_2 = 1$, $s_2 = 2$ and $s_2 = 12$ is $s_1 = 12$ for all $C \in [\,1, \frac{2b^2-d^2}{bd}]$. That is, $s_1 = 12$ is a dominant strategy.

By Lemma 3 the best response of M_2 to $s_1 = 1$ (respectively, to $s_1 = 2$) is either $s_2 = 12$ if $C \in [1, r_c]$ or $s_2 = 2$ (respectively, $s_2 = 1$) if $C \in [r_c, \frac{2b^2-d^2}{bd}]$. By Lemma 4 the best response of M_2 to $s_1 = 12$ is either $s_2 = 12$ if $C \in [1, r_d]$ or $s_2 = 1$ and $s_2 = 2$ if $C \in [r_d, \frac{2b^2-d^2}{bd}]$. Furthermore, it can be proven that $r_c < r_d$ for $\frac{d}{b} \in (0, 0.682]$. Therefore we conclude that:

1.1) for $\frac{d}{b} \in (0, 0.682]$ and $C \in [1, r_c]$ the Nash Equilibrium is the pair of strategies $(12, 12)$.

1.2) for $\frac{d}{b} \in (0, 0.682]$ and $C \in [r_c, r_d]$ the Nash Equilibrium is also $(12, 12)$.

1.3) for $\frac{d}{b} \in (0, 0.682]$ and $C \in [r_d, \frac{2b^2-d^2}{bd}]$ the Nash Equilibria are both $(12, 1)$ and $(12, 2)$.

2)Let us assume that $\frac{d}{b} \in (0.682, 0907]$.

Both r_d and r_C appear as relevant in this interval. Remind that r_d is decreasing with $\frac{d}{b}$ in the interval $(0.682, 0.907]$ and it equals one for $\frac{d}{b} = 0.909$, while r_C is increasing with $\frac{d}{b}$ in the same interval, being equal to one for $\frac{d}{b} = 0.682$ and equal to $\frac{2b^2-d^2}{bd}(= 1.298)$ for $\frac{d}{b} = 0.907$. Then it is easy to find that r_d and r_C cross each other at $\frac{d}{b} = 0.805$, and therefore, we analyze the cases $\frac{d}{b} \in (0.682, 0.805]$ and $\frac{d}{b} \in (0.805, 0907]$ separately.

2.1) For $\frac{d}{b} \in (0.682, 0.805]$ it is the case that $r_C < r_d$ and we conclude that:

2.1.a) for $\frac{d}{b} \in (0.682, 0.805]$ and $C \in [1, r_C]$ the Nash Equilibria are $(12, 12)$, $(1, 2)$ and $(2, 1)$.

2.1.b) for $\frac{d}{b} \in (0.682, 0.805]$ and $C \in [r_C, r_d]$ the Nash Equilibrium is $(12, 12)$.

2.1.c) for $\frac{d}{b} \in (0.682, 0.805]$ and $C \in [r_d, \frac{2b^2-d^2}{bd}]$ the Nash Equilibria are both $(12, 1)$ and $(12, 2)$.

2.2) For $\frac{d}{b} \in (0.805, 0.907]$ it is the case that $r_d < r_C$. Then, we conclude that:

2.2.a) for $\frac{d}{b} \in (0.805, 0.907]$ and $C \in [1, r_d]$ the Nash Equilibria are $(12, 12)$,

$(1,2)$ and $(2,1)$.

2.2.b) for $\frac{d}{b} \in (0.805, 0.907]$ and $C \in [r_d, r_C]$ the Nash Equilibria are $(1,2)$ and $(2,1)$.

ii.2.c) for $\frac{d}{b} \in (0.805, 0.907]$ and $C \in [r_C, \frac{2b^2-d^2}{bd}]$ the Nash Equilibria are both $(12,1)$ and $(12,2)$.

3) Let us assume that $\frac{d}{b} \in (0.907, 0.909]$.

Two different conclusions are reached depending on the size of C.

3.1) for $\frac{d}{b} \in (0.907, 0.909]$ and $C \in [1, r_d]$ the Nash Equilibria are $(12,12)$, $(1,2)$ and $(2,1)$.

3.2)for $\frac{d}{b} \in (0.907, 0.909]$ and $C \in [r_d, \frac{2b^2-d^2}{bd}]$ the Nash Equilibria are $(1,2)$ and $(2,1)$.

4) Let us assume that $\frac{d}{b} \in (0.909, 1)$.

Following the lemmatas above we conclude that for $\frac{d}{b} \in (0.909, 1)$ and $C \in [1, \frac{2b^2-d^2}{bd}]$ the Nash Equilibria are $(1,2)$ and $(2,1)$.

The different Nash equilibria presented above are shown in Proposition 1 noting the difference between a unique equilibrium system at equilibrium and a multiplicity of them. Q.E.D.