Visualization in 3-Dimensional Geometry: In Search of a Framework

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Friend, when I look back now and ask myself, what, properly speaking, have I done for the education of humanity? - I find the following: I have established the highest basic principle of education by acknowledging sensory-perceptual observation (visualization) to be the absolute basis of all cognition.

Pestalozzi, cited in Antonovskii (1990, p. 5)

Abstract. *The usefulness of visualization and graphical representations in the teaching of mathematics is being recognized by most mathematics educators and teachers of mathematics, but much research is still necessary. In the first part of the paper a framework aimed to organize the field of visualization in mathematics is presented. Visualization in 3-dimensional geometry seems to be a neglected area, since only a few reports of research can be found in the literature. The second part of the paper is devoted to raising some questions related to the analysis of primary and secondary school students' behaviour when solving tasks in 3-dimensional geometry by using dynamic software. The analysis will focus on students' ways of using screen images, the mental images they create, and the processes and abilities of visualization they use to solve the tasks.*

Introduction.

Although Pestalozzi exaggerated in giving visualization the role of *the absolute basis* of cognition, it is true that visualization is one of its main basis. When entering the area of visualization, several terms appear immediately: Visual reasoning, imagination, spatial thinking, imagery, mental images, visual images, spatial images, and others. When looking at the electronic databases for papers related to the terms "visualization", "spatial ability", or "mental image", one can find that most of the papers one encounters are published in journals of psychology, and only a few of them in journals of mathematics education. Many publications can be found

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concerning developmental stages of individuals (from early childhood to adults), the relationship of visualization to drawing, writing or speech, construction and handling of 3-dimensional objects, and other issues related to psychology, mathematics, or mathematics education. But one can also find titles concerning engineering, art, medicine, economy, chemistry, car driving, and some other surprising specialities.

Some conclusions arise: 1) Psychologists were aware of the importance of visualization long time ago, and they have developed detailed theories to frame their work, and tools to observe and test individuals. 2) Visualization is important for many more activities than we could initially suspect, although each speciality is only interested in certain specific abilities and environments, those narrowly related to their research problems. 3) Persons coming from different activities may have developed different meanings for the same words. 4) The field of visualization is so wide and diverse that it is not reasonable to try to encompass it all.

There is no general agreement about the terminology to be used in this field: It may happen that an author uses, for instance, the term "visualization" and another author uses "spatial thinking", but we find that they are sharing the same meaning for different terms. On the other hand, a single term, like "visual image", may have different meanings if we take it from different authors. Such an apparent mess is merely a reflection of the diversity of areas where visualization is considered relevant and the variety of specialists who are interested in it. The first section of this paper is devoted to identifying the field of activity I am interested in, and define the meaning those confusing terms will have here. I present a complete theoretical framework integrating partial results from researchers like Bishop, Hoffer, Presmeg, or Yakimanskaya, who have characterized different components of visualization.

Pestalozzi's statement was made more than 150 years ago, but only recently the relevant position of visualization has been broadly acknowledged by mathematics educators. Since a few years ago, mathematics educators have been underlining the need to increase the use of visual elements as a part of the ordinary teaching of mathematics in the different educational levels, particularly in secondary schools and universities. As far as research is concerned, we can find many papers reporting the results of experimental studies where the use of visual representations of the concepts to be taught have, in most cases, helped or, in some cases, hindered students in the formation of such concepts. As for curriculum development, more teachers and textbook writers are paying more attention to the use of drawings, diagrams, pictures, etc. in the math classes.

Geometry can be considered as the origin of visualization in mathematics but, if we examine the papers or books published in the last years dealing with visualization in mathematics education, we find many of them focusing on the teaching or learning of calculus (i.e. advanced mathematical thinking), quite a lot on (pre-)algebra and number systems, some on plane geometry, and only a few focusing on space geometry. In some way, this is reasonable since visualization has always been recognized as a necessary component for the teaching and learning of geometry (maybe the only exception is the period of ‘modern mathematics’), and only recently has it gained the same recognition in other parts of mathematics. However, the technological revolution that has occurred in the last decade, with the popularization of computers and other multimedia tools, has provided teachers and researchers with new elements that may reshape the ways of teaching space geometry. These new possibilities have to be investigated and analyzed in depth, as a first step towards their implementation in the classrooms.

One of the new tools that can be used in the classrooms are computer programs giving a 3-dimensional representation of spatial objects and allowing users to transform those objects dynamically (transformations like rotations, translations, enlargements or sections by planes). In spite of the 3-dimensional aspect of the objects presented on the screen, they, like pictures, are plane representations of spatial objects, so some of the well-known difficulties students have when interpreting traditional plane representations of solids appear in computer environments too.

In the second part of the paper I raise some questions related to the analysis of primary and secondary school students’ behaviour when using 3-dimensional dynamic software, their ways of analyzing screen images, and their mental images when they are working in such environment. Such questions are discussed under the theoretical framework organized in the first part of the paper, and they are exemplified by excerpts from students who were observed by Adela Jaime and myself as part of an on-going research project that has been carried out since 1989 at the Departamento de Didáctica de la Matemática of the University of Valencia.

Setting the Borders for Visualization and Other Related Concepts.

In cognitive psychology, one meaning of “mental image”, supported by Denis, Kosslyn, Paivio, Shepard and others, is that of a quasi-picture created in the mind from memory, without any physical support. Kosslyn (1980) explains in detail his theory of mental images as having two major components: A *surface representation,* the quasi-pictorial entity present in the active memory, and a *deep representation,* the information stored in the long-term memory, from which the surface representation is derived. There are other cognitive psychologists, led by Pylyshyn, who argue against this concept of mental image owing to the many deficiencies they see in the picture-in-the-mind metaphor and the necessity they feel for a less vague definition. The work done by Minsky and Papert in the early 70s can be included in this paradigm. A third position is maintained by those who argue that the same representations are used in all kinds of cognitive processing, and mental images are just one case, so there is nothing special in mental images that makes them deserve a particular theory.

Researchers sharing the first or second positions are interested in the ways mental images are created and saved in a person’s mind. For this reason, many tests designed to assess students’ ability in the manipulation of mental images do not allow the use of paper and pencil or computers to answer the items (Zimmermann, Cunningham 1991, p. 3). A description of the main classical types of tests used by psychologists can be found in Denis (1989) and Clements (1981).

The meanings given by Kosslyn or Pylyshyn to the terms “visualization”, “mental images”, etc. are not shared by many educational psychologists nor by mathematicians, teachers of mathematics, or mathematics educators. These tend to give those terms a simpler more general meaning: A “mental image” is a mental representation of a mathematical concept or property containing information based on pictorial, graphical or diagrammatic elements. “Visualization”, or visual thinking, is the kind of reasoning based on the use of mental images.

One of the main reasons for such disagreement is that, in mathematics, the use of drawings, figures, diagrams, or computer representations is part of everyday activity in the classrooms. In opposition to the approach of cognitive psychologists, mathematics educators consider that mental images and external (i.e. non-mental) representations have to interact to achieve a better understanding and to solve problems. Visualization is the context where this interaction takes place (Zimmermann, Cunningham 1991, p. 3). Furthermore, many mental images used in mathematics do not have a pictorial base, since they are based on diagrams, other visual ways of representation of concepts, or even textual or symbolic information.

Within mathematics education we can find several interesting pieces of theoretical work about visualization. Although they have elements in common, they have not been stated as parts of a single theoretical body. One objective of the research we are carrying out is to define a framework integrating those pieces, and to provide experimental support for the resulting general theoretical organization.

For Yakimanskaya (1991) a “spatial image” is created from the sensory cognition of spatial relationships, and it may be expressed in a variety of verbal or graphical forms including diagrams, pictures, drawings, outlines, etc., so she stresses the above mentioned interaction between spatial images and external representations. Furthermore, spatial images must be dynamic, flexible, and operational. She describes “spatial thinking” as *a form of mental activity which makes it possible to create spatial images and manipulate them in the course of solving various practical and theoretical problems* (p. 21), including verbal and conceptual operations, and several perceptual events necessary to form mental images. Yakimanskaya, then, considers that *images are the basic operative units of spatial thinking* (p. 26), and geometric objects are the basic material used to create and manipulate spatial images.

In the 60s and early 70s, when the Russian research reported by Yakimanskaya was carried out, geometry, geography, art, and other areas with strong geometrical support were the areas to observe spatial thinking and where most research experiments took place. Nowadays the role of geometry and geometric objects continues to be central as a support for visualization in mathematics, but there are also other useful elements coming from different areas of mathematics like calculus, algebra or statistics that are used very often.

Clements (1982, p. 36) considers that the concept of “image” as a picture in the mind is valid in mathematics education. Lean, Clements (1981, p. 267-68), following the dominant psychological theories of the time, define: “Mental imagery” as *the occurrence of mental activity corresponding to the perception of an object, but when the object is not presented to the sense organ.* “Visual imagery” as *mental imagery which occurs as a picture in the mind’s eye.* “Spatial ability” as *the ability to formulate mental images and to manipulate these images in the mind.*

Presmeg (1986, p. 42) proposes to define a “visual image” as *a mental scheme depicting visual or spatial information,* with or without requiring the presence of an object or other external representation. As in Yakimanskaya’s definition, Presmeg wants to include within the concept of visual image all those images having a graphical support different from a picture in the mind. She acknowledges that this definition is broader than most previous definitions, in particular the one proposed by Lean, Clements (1981) and Clements (1982).

Along the same lines, Dreyfus (1995, p. 3) defines “visual imagery” as the use of *mental images with a strong visual component.*

These broad definitions of mental images allow the possibility of having several kinds of images. Presmeg (1986) reports the results of a piece of research seeking to establish different kinds of visual images. Those observed in her students are classified as (ibid., pp. 43-44):

- Concrete, pictorial images: The kind of ‘picture in the mind’ images referred to by other authors.
- Pattern images: Images representing abstract mathematical relationships in a visual way.
- Images of formulae: Some students can ‘see’ in their minds a formula as it appeared written on the blackboard or the textbook.
- Kinaesthetic images: Those images that are created, transformed, or communicated with the help of physical movements.
- Dynamic images: Those images with movement in the mind.

Looking at the problem of visualization from another point of view, Bishop (1983, p. 177) recognizes two abilities in visualization: The “visual processing” of information (abbreviated to VP), including the *translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and the transformation of one visual image into another*. The “interpretation of figural information” (abbreviated to IFI), involving *knowledge of the visual conventions and spatial ‘vocabulary’ used in geometric work, graphs, charts, and diagrams of all types … and … the ‘reading’ and interpreting of* visual images, either mental or physical, to get from them any relevant information that could help to solve a problem.

Bishop presents IFI and VP as abilities of persons, but, as defined, they fit better into the category of processes to be performed. The description of an ability should include information about the way it can be performed or the skills to be used. The description of a process should include information about the action to be done, but it is independent of the way of performing it in a specific case. For instance, the process of rotation of a mental image, which is a part of the IFI process, consists in converting the initial image into other one presenting the same object viewed while a rotation takes place or after it has been completed. The way such mental rotation is performed, i.e. the ability to be used, depends on the dimension of the rotation (in the plane or space), the position of the centre or axis of rotation relative to the figure (interior or exterior), the position of the axis of rotation relative to the subject (vertical, horizontal or orthogonal to the plane of vision), and the skills acquired by the subject. We can observe the presence of a variety of abilities by setting several problems to different students.

Yakimanskaya (1991, p. 101) describes two levels of activity in spatial thinking, the creation of mental images and their manipulation or use, as two closely interrelated processes (ibid., p. 102). The likeness of these processes to VP and IFI is evident.

Kosslyn (1980) identifies four processes applicable to visualization and mental images: Generating a mental image from some given information; Inspecting a mental image to observe its position or the presence of parts or elements; Transforming a mental image by rotating, translating, scaling, or decomposing it; Using a mental image to answer questions. The transformations investigated by Kosslyn are only part of those made with mental images in mathematics. For instance, it is quite usual to deform figures to solve problems of geometry. As an example, when students are learning relationships among quadrilaterals, they can imagine a rectangle shrinking to become a square and then again a rectangle.

Although Kosslyn's concept of mental image is different from that of the mathematics educators mentioned except Clements, the first process he defines is equivalent to the VP process, and the three others are parts of the more general IFI process. We can see, then, that Kosslyn, Yakimanskaya, and Bishop refer to the same processes of visualization, with the only difference being that Kosslyn's model is more detailed than the other two.

The list of abilities necessary to process mental images may be very long if we are interested in a general description of the field from the psychological point of view. For instance, McGee (1979), summarizing results from previous research on spatial abilities, describes ten different abilities, distributed into two classes:

Abilities of spatial visualization: 1) Ability to imagine the rotation of a depicted object, the (un)folding of a solid, and the relative changes of position of objects in space. 2) Ability to visualize a configuration in which there is movement among its parts. 3) Ability to comprehend imaginary movements in three dimensions, and to manipulate objects in the imagination. 4) Ability to manipulate or transform the image of a spatial pattern into other arrangement.

Abilities of spatial orientation: 1) Ability to determine relationships between different spatial objects. 2) Ability to recognize the identity of an object when it is seen from different angles, or when the object is moved. 3) Ability to consider spatial relations where the body orientation of the observer is essential. 4) Ability to perceive spatial patterns and to compare them with each other. 5) Ability to remain unconfused by the varying orientations in which a spatial object may be presented. 6) Ability to perceive spatial patterns or to maintain orientation with respect to objects in space.

Hoffer (1977) identifies several physio-psychological abilities relevant to the learning of mathematics: Eye-motor coordination; Figure-ground perception; Perceptual constancy; Perception of positions in space; Perception of spatial relationships; Visual discrimination; Visual memory (to remember mental images or objects no longer seen).

Some abilities in the previous lists overlap, part of them are general abilities useful in many ordinary life or professional activities, and others may be seen as specific to mathematized contexts. If we limit ourselves to the environment of geometry, only a part of the abilities are pertinent.

A complete framework characterizing the activity of visualization in mathematics can be defined by unifying the terminology used by several of the above mentioned authors, and integrating the concepts defined by them into a single network. In the following I will restrict myself to the context of mathematics, so the definitions given below do not try to go further or to be applicable outside the teaching and learning of mathematics. With respect to the vocabulary, the terms mental image, spatial image and visual image defined by Yakimanskaya, Dreyfus and Presmeg can be considered as basically equivalent, and the terms visualization, visual imagery, and spatial thinking can also be considered as equivalents.

I therefore consider “visualization” in mathematics as the kind of *reasoning activity based on the use of visual or spatial elements, either mental or physical,* performed to solve problems or prove properties. Visualization is integrated by four main elements: Mental images, external representations, processes of visualization, and abilities of visualization.

A “mental image” is *any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements.* Like Yakimanskaya (1991), I consider mental images as the basic element in visualization. The types of mental images identified by Presmeg (1986) can probably be completed if research is made to identify them in other specific areas of mathematics, like probability, functional analysis, or analytic geometry. Usually only a few types of mental images are necessary to solve a certain kind of task. For instance, only concrete, kinaesthetic, and dynamic images were used by our students to solve the tasks we proposed them.

An “external representation” pertinent to visualization is *any kind of verbal or graphical representation of concepts or properties including pictures, drawings,* *diagrams, etc. that helps to create or transform mental images and to do visual reasoning.* A research question can be raised at this point: Do external representations have an absolute or individual character? That is, does the property of being visual belong to an external representation or does it depend on the way a person uses the representation? This question has to do with the distinction between geometric (visualizers), analytic (non-visualizers), and harmonic (mixed) types of mathematical reasoning made by Krutetskii (1976).

A "process" of visualization is *a mental or physical action where mental images are involved.* There are two processes performed in visualization: "Visual interpretation of information" to create mental images, and "interpretation of mental images" to generate information. The first process corresponds to Bishop's VP. The second process corresponds to Bishop's IFI, and it is made up of three sub-processes, as identified by Kosslyn: Observation and analysis of mental images, transformation of mental images into other mental images, and transformation of mental images into other kinds of information.

Individuals should acquire and improve a set of "abilities" of visualization to perform the necessary processes with specific mental images for a given problem. Depending on the characteristics of the mathematics problem to be solved and the images created, students should be able to choose among several visual abilities. These abilities may have quite different foundations, the main ones being:

- "Figure-ground perception": The ability to identify a specific figure by isolating it out of a complex background.
- "Perceptual constancy": The ability to recognize that some properties of an object (real or in a mental image) are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations.
- "Mental rotation": The ability to produce dynamic mental images and to visualize a configuration in movement.
- "Perception of spatial positions": The ability to relate an object, picture, or mental image to oneself.
- "Perception of spatial relationships": The ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself.
- "Visual discrimination": The ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them.

To conclude, the diagram in Figure 1 summarizes the steps to be followed when using visualization to solve a task: The statement of the task is interpreted by the students as an external representation suitable to generate a mental image. This first image initiates a process of visual reasoning where, depending on the task and students' abilities, they use some of their visual abilities to perform different processes, and other mental images and/or external representations may be generated before the students arrive at the answer.

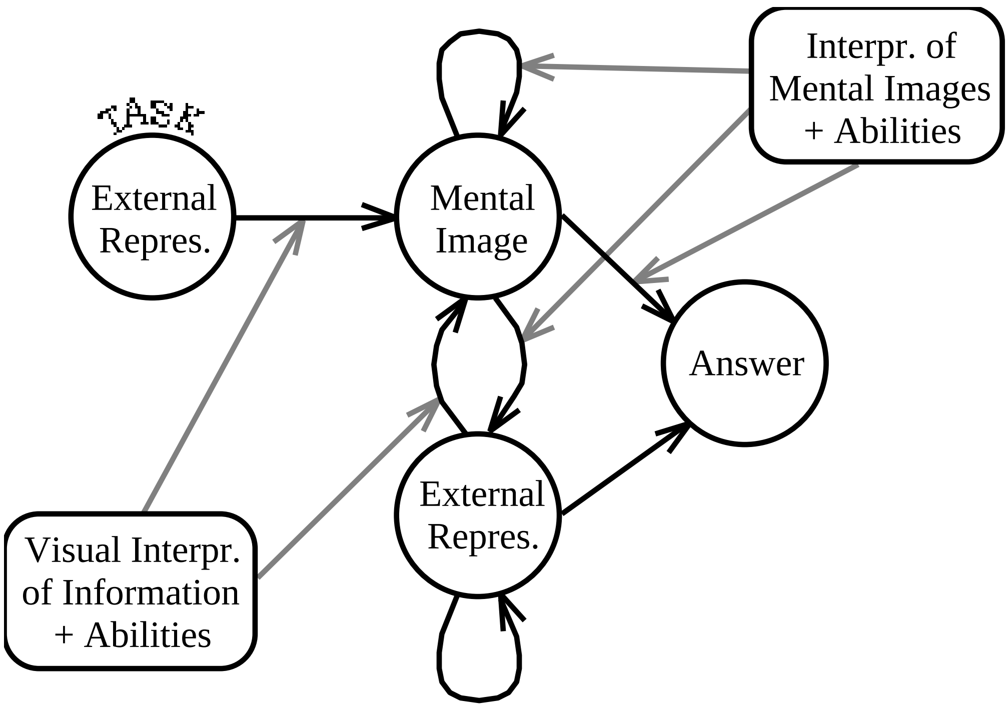

Figure 1. Main visualization elements integrating the solution of a mathematics task.

The Role of Visualization in 3-Dimensional Geometry. A Case.

I believe there is a general agreement that visualization is a basic component in learning and teaching 3-dimensional geometry. However, there is a very limited research activity in this specific area. On this research activity, most publications deal with students' difficulties when they have to move between 3-dimensional objects and some of their usual 2-dimensional representations, and only a few have approached the problems of the students using some specific kind of representation.

The only way textbooks have to present 3-dimensional geometry to students is by means of plane representations, usually perspective, parallel, or orthogonal projections. In many cases the teachers mitigate the limitation of textbooks by using wood or cardboard models. Until a few years ago, these were the only two possibilities available for most teachers all over the world, but now they can have access to a third way of representation: Computers with special software allowing students to see a solid represented in several possible ways on the screen and to transform it. Some important advantages students can gain from the use of this kind of software are:

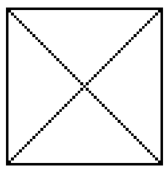

Figure 2.

Students will see polyhedra and other solids in many more different positions on the screen than in the textbooks. As a consequence, they will gain a rich experience that will allow them to form richer mental images than from textbooks. In particular, students will greatly improve their ability to create dynamic mental images. For instance, a student who has only seen the pictures in the textbooks will hardly recognize the drawing in Figure 2 as a representation of a pyramid or an octahedron. However, when rotating these solids on a computer, this special position appears as part of a continuum of images, and it gets meaning in this set of linked mental images.

As pointed out by Yakimanskaya (1991, p. 103), *the creation of images is possible because of the accumulation of representations that serve as the starting point and as necessary conditions for the realization of thought. The richer and more diverse the store of spatial representations, the more highly perfected the methods of creating representations and the easier it is to use images*. Computers can play a very relevant part in helping students to acquire and develop abilities of visualization in the context of space geometry.

When a person handles a real 3-dimensional solid and rotates it, the rotations made with the hands are so fast, unconscious, and accurate, even in the case of young primary school students, that one can hardly reflect on such actions; However, a software package limiting the directions of rotation forces the students to devise strategies of movement and to anticipate the result of a given turn.

Many advantages derived from the use of computers to teach geometry have been reported by research, and also several problems have already been highlighted in relation to the use of computers. In space geometry, students tend to base their arguments and conclusions on the appearance of the solid on the screen (Dreyfus, Hadas 1991, p. 87); For instance, they may accept a right angle as acute because it looks acute on the screen. Then, in the same way that students have to learn to interpret plane drawings correctly, they have to learn to interpret computer drawings correctly, and to use the tools provided by the software efficiently. On the other hand, when selecting a piece of software and a type of activity to be solved with that software, several variables should be taken into consideration (Gutiérrez, 1992 a; Gutiérrez, 1996): The type of representation of solids; The way the software allows a representation to be transformed, in other words, how user-friendly the software is; The range of students' abilities required by the software and the activities.

Working in that direction, we have made extensive experiments with students from a wide range of primary and secondary grades, aged from 7 to 17 years old. We have selected several computer programs that represent polyhedra in perspective and that allow the users to rotate them around the three standard coordinate axes (vertical, horizontal, and orthogonal to the screen), and we have asked the students to solve several types of activities. In particular, we asked them to rotate solids on the computer screen from an initial position to a target position drawn on paper (a hard copy of the computer screen). A more detailed description of this environment can be found in Gutiérrez, Jaime (1993). One of the aims of this line of research is to analyze the variables mentioned in the previous paragraph. Another aim, relevant to this paper, is to analyze the ways students solve the different activities, paying attention to the kinds of mental images and abilities of visualization they have used.

When trying to know the mental images created by students and the abilities put to work, one has to be aware that a mental image can be perceived only by means of some form of external support, verbal, graphical, gestural, etc. (Sutherland 1991, p.

71). From a methodological point of view, researchers should not ask the subjects to describe their mental images while they are solving a task, since the subjects may not be aware of their own images (as is the case of young primary school students) or, if they were aware of them, the dialogue could certainly distort the subjects' process of reasoning. Then, the best possibility for researchers seems to be to interpret the actions produced as a consequence of the subjects' activity with those mental images (ibid., p. 71), although, as Dreyfus (1991, p. 6) points, such an interpretation is influenced by the researcher's theory, and there may be some elements connecting the mental image and the researcher that influence and mediate the interpretation.

Figure 3 shows the three kinds of solid used in our research: A cube with pictures on the faces, several polyhedra with shaded faces, and the same polyhedra with transparent faces.

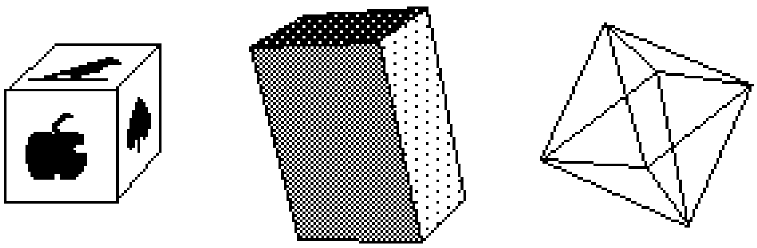

Figure 3.

The figurative cube was used in a HyperCard stack designed to help the younger primary school students to enter into the manipulation of 3-dimensional objects on the computer. This program only allows rotations of 90°, that are performed after clicking on one of a set of buttons shown on the screen (Figure 4).

One of the tasks presented to the students was to rotate a given figurative cube from its current position to match exactly a picture shown on a sheet of paper. The students were also asked to make the movement with the minimum number of rotations. Even this simple and easy task is rich enough to show different students' strategies, and the use of mental images and abilities of visualization with different grades of expertise.

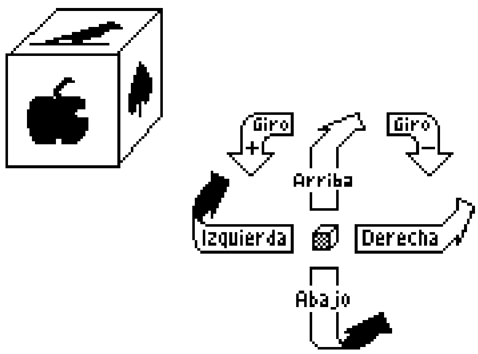

Figure 4.

After spending one hour solving several activities of this kind on the second day of the experiment, a second grader (7-8 years old) was asked to move the cube from its current position (schematized in Figure 5.1) to the target position (Figure 5.3). The first student's action was to move the cursor to the button ⇓.

1 Researcher: *Wait a moment before clicking. You are going to move it down. What is going to happen with the spade?*

2 Student: *It will go here* [pointing to the hidden bottom face of the cube on the screen].

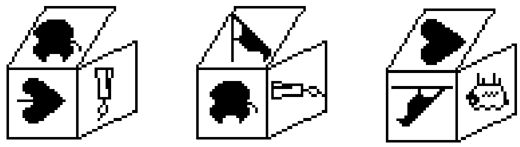

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Figure 5.

3 R.: *You will not see it, will you?*
4 S.: *No.*
5 R.: *What about the apple?*
6 S.: [It goes] *here* [pointing to the front face of the cube].
7 R.: *And, what about the candle?*
8 S.: … [he hesitates, pointing to the left and right faces of the cube] *here* [pointing to the top face of the cube].
9 R.: *Let's see what happens.*
10 S.: [he clicks on ⇓ (Figure 5.2)] … *No. It comes this way. And the bird.*
11 R.: [pointing to the cube on the screen] *The spade has come down, where you said, the apple has come here, but this one* [the candle] *has stayed on the same face but it has …* [S interrupts R]
12 S.: *Now the bird is here* [pointing to the top face] … [clicks on ⇓] … *Now the turn minus* [clicks on (Figure 5.3)] … *I got it.*

The student continued solving another task like the previous one. Now he has to rotate the cube from the position in Figure 6.1 to that in Figure 6.4.

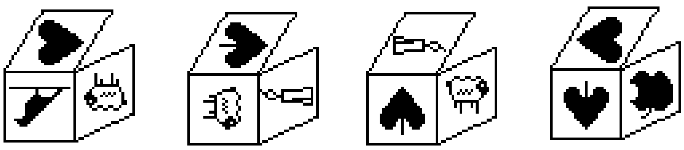

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Figure 6.

13 S. [talking to himself while working alone]: [clicks on ⇓, ⇐ (Figure 6.2)] *This one?* [clicks on ⇓] … *Yes, but now I am going to make the turn plus* [clicks on (Figure 6.3)] *I got it … Oh, no. The apple isn't there.*
14 R.: *What has happened?*
15 S.: *The apple is there or there or there* [pointing to each of the hidden faces].
16 R.: *Where is the apple? Look at the* [target cube on the] *sheet of paper, where do you say the apple is?*
17 S.: *There on the back* [pointing to the hidden back face].
18 R.: *On the back? Why?*
19 S.: … *No, here down here* [pointing to the bottom face] *because before … No. It is here because I have made lots of turns and it has gone there.*
20 R.: *But, look, the apple is next to the spade, isn't it?*
21 S.: *Yes.*

22 R.: *Then, may it be on the opposite face* [to the spade] *?*

23 S.: *… Well … This* [the spade] *has to be upside down* [clicks on ⥥] *… Another one* [clicks on ⥥] *… Yes, I got it.*

24 R.: *Where was the apple?*

25 S.: *Here* [pointing to the back face].

26 R.: *But you have rotated it* [the cube] *this way* [going ⥥ with the hand].

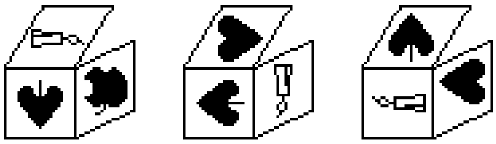
27 S.: *Then it was here* [pointing to the bottom face].

In the excerpt of the first task, we see that the student is able to create and use dynamic mental images, since he can anticipate the position of a figure after a turn (paragraphs 2, 6, and 8), although he cannot create a dynamic mental image of the whole cube, but only of one face, and he has difficulty in answering to the question in 7. When he is asked for the positions of the different faces after the turn ⇓, he creates a new mental image for each face. As a consequence, the student does not recognize his mistake when he thinks of the candle. many primary school students tend to pay attention only to the front face of the cube. This is the origin of the misconception of assuming that, after a rotation moving the front face’s figure to another face, every figure is also moved to another face. From another viewpoint, this mistake reflects a lack of the abilities of perceptual constancy or perception of spatial relationships.

The last part of the excerpt (10 to 12) shows the student’s resolution strategy for these problems: He looks for the figure on the front face of the target cube, making turns at random if it is hidden. When this figure appears on the screen, he moves it to the front face, and then he makes turns to move it to its target position. This student has difficulty in focusing on another figure to take decisions about the rotations to be made. This behaviour is typical of students in lower primary grades; Students in middle primary grades can improve this strategy and pay attention to any face with the help of the teacher, and students in the upper primary grades can improve it by themselves. However, to have this strategy does not mean to be able to apply it efficiently. I showed in Gutiérrez (1992 b) the case of a sixth primary school student who stated the strategy correctly while solving several tasks like the previous one. Although these tasks can be solved by sequences of four or fewer rotations, this student needed as many as 9, 12, or even 21 rotations to move the cubes to the target positions. The origin of her difficulties was the lack of certain abilities of visualization, like the ability of mental rotation and those of perception of spatial relationships or positions.

The excerpt of the second task confirms the conclusions drawn from the previous one. The student made the first series of turns (13) looking for the spade and trying to move it to its target position, although in this case the most efficient way of solving the problem would be to pay attention to the top face (the heart). The lack of the abilities of perceptual constancy or perception of spatial relationships is still more apparent in the second task (16 to 19). Although this child is able to rotate the mental image of a cube’s face, he can only do rotations by a single turn. Now, as several rotations were made one after the other, the student cannot reproduce the movement of the faces (19).

Finally, I will show an eighth grade girl solving a similar task. This kind of activity is very easy for students in grade 8 or older, so now the researchers changed their approach and asked the students to predict the position of the cube after one or more rotations. The cube had to be moved from the position in Figure 7.1 to that in Figure 7.3.



- 1 - - 2 - - 3 -

Figure 7.

1 S.: *The candle has to be on the top the other way around.*
2 R.: *What should you do?*
3 S.: *A rotation like this* [making the rotation ⬐ with the hand], *and then the candle would be like this, wouldn't it?* [she shows the position with the hand] *Then to the right.*
4 R.: *Where are you going to move the candle first? How?*
5 S.: *Here* [pointing to the hidden left face].
6 R.: *On the hidden face?*
7 S.: *Yes, I think … I can't see the heart … I can't see the heart. Maybe it's here* [pointing to the hidden left face] *or down here* [pointing to the hidden bottom face].
8 R.: *Where do you think the heart is?*
9 S.: *Down here.*
10 R.: *Down here? How do you know?*
11 S.: *I think so, because if it is next to this one* [the spade], *it may be either here* [pointing to the left face] *or down here …* [she rotates the sheet of paper with the target cube] … *It is here* [pointing to the left face] … *Then, if it is here, we will have to go …* [she rotates the sheet] … *we will have to turn it like this. Then, if we rotate it this way* [going ⬎ with the hand], *it would go here, wouldn't it? The heart would come here* [pointing to the top face].
12 R.: *So, you want to put the candle here* [pointing to the right face] *to make the heart appear, don't you?*
13 S.: *Yes* [she clicks on ⬎ (Figure 7.2)] … *Now it* [the candle] *is like that* [pointing to the right face on the target cube on the sheet of paper and the screen].
14 R.: *What do you have to do now?*
15 S.: *Another turn* [going ⬎ with the hand] *to put the candle down here, and then another turn up* [going ⇑ with the hand], *don't you?* [she clicks on ⬎ and ⇑] *OK.*

The most noticeable fact from this excerpt is the extensive use the student makes of her hands to show the rotations, i.e. of kinaesthetic images. Another form of this type of mental image appears in paragraph 11 when she rotates the sheet of paper to see the position of some figures after a turn. In some moments (11 and 15) the student shows her ability to foresee the result of a series of rotations, thanks to the use of the ability of perception of spatial positions. In paragraphs 8 to 11 we can also observe how the student uses the ability of perception of spatial relationships by looking at the target cube, although with only partial mastery since, at first (9), she is able to determine that the heart is situated on a face next to the spade but she does not pay attention to their relative positions to determine the correct face of the heart.

To Conclude.

Five years ago T. Dreyfus' plenary address in PME-15 was a call for paying more attention to visualization and visual reasoning in the teaching of mathematics, since he *attempted to show that visual reasoning in mathematics is important in its own right and that therefore we need to develop and give full status to purely visual mathematics activities* (Dreyfus 1991, p. 46).

Since then, there have been several attempts in this direction, mainly designs of teaching units, but for the moment there is still a need for a theory (Dreyfus 1995, p. 16) emerging from mathematics education explaining how mental images of mathematical concepts are formed, how students can gain mastery in creating and using mental images, what role mental images play in the understanding of mathematical concepts and in problem solving, when visualization is more (or less) useful to students than analytical methods, how mental images can be transmitted, etc.

In this paper I have outlined a model characterizing the field of visualization in mathematics and defining its four main elements, mental images, external representations, processes, and abilities of visualization. This model is an attempt to integrate and complete several elements previously defined by Presmeg, Bishop, Clements, and others, that partially explained teachers' and students' activity when they use visualization as a component of their teaching, learning or reasoning in mathematics. On the other hand, the Van Hiele levels of reasoning give us the possibility to complete the model of visualization and to incorporate the assessment of activities of visualization into the context of assessment in mathematics (Gutiérrez 1992 b).

I have also shown the application of the model to analyze some cases of students solving tasks with the help of software allowing them to manipulate 3-dimensional objects. When completed this research project, we will probably give better answers to some of the many questions about the use of visualization and mental images in 3-dimensional geometry. Research and experimentation both in 3-dimensional geometry and other parts of mathematics is needed to complete the framework.

But there is still a long way to go.

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