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Solitons and solitary vortices in continuous and discrete 2D models

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The objective of the talk is to present an overview of fundamental dynamical models for the pattern formation in nonlinear two-dimensional (**2D**) media. The overview will present both the basic models and their basic solutions, which describe solitary patterns (***2D solitons***) in them. Physical realizations of the solutions will be considered too.

Both **conservative** and **dissipative** models will be reviewed. Different parts of the talk are defined according to basic mechanisms which provide for the **stability** of the **2D** solitons, as the stability is the main issue in this field:

(A) Conservative systems with trapping potentials;
(B) Conservative systems with the **cubic-quintic** (**CQ**) nonlinearity;
(C) Dissipative models based on **2D** complex Ginzburg-Landau (**CGL**) equations;
(D): Stable 2D composite solitons in **spin-orbit-coupled** self-attractive **Bose-Einstein condensates** in free space.

Part A: Conservative systems with trapping potentials

(1) *Introduction*. The simplest model which may give rise to solitons and solitary vortices: the **2D** nonlinear Schrödinger equation (**NLSE**), alias the Gross-Pitaevskii equation (**GPE**):

$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-|u|^2 u+W(x,y)u$$

(in optical models, t is replaced by the propagation distance, z).

In the presence of the **axially symmetric** trapping potential, $W=W(r)$, where (r,θ) are the polar coordinates in the (x,y) plane, solutions with integer **vorticity** S are looked for as

$$u=\exp\left(-i\mu t+iS\theta\right)U(r),$$

with $\mu<0$ and real function $U(r)$ obeying the stationary equation:

$$\mu U=-\frac{1}{2}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{S^2}{r^2}U\right)-U^3+W(r)U.$$

Localized solutions, with asymptotic forms

$U(r) \sim r^{S}$ at $r \to 0$, and $U(r) \sim \exp(-(-2\mu)^{1/2}r)$,

exist even in the absence of the external potential, $W = 0$, but they all are ***completely unstable***. The **fundamental solitons** [with $\boldsymbol{S = 0}$, alias ***Townes' solitons***, R.Y. Chiao, E. Garmire & C. H. Townes, Phys . Rev. Lett . **13**, 479 (1964)] are unstable against the ***collapse***, and vortical solitons (with $\boldsymbol{S \geq 1}$) are vulnerable to a still stronger instability against ***azimuthal perturbations*** which break their axial symmetry and split the vortices into a few separating segments (fundamental solitons, which will later be destroyed by the collapse).

2. *Stabilization of single-component vortices*

A fundamental issue: how can solitons and vortices be **stabilized** in physically relevant settings?

The simplest possibility is to use an axially symmetric ***trapping potential***. Typically, it is taken as the ***harmonic-oscillator potential***, $\boldsymbol{W(r)=(\Omega^2/2)r^2}$. Here, basic results will be presented as per the following paper:

PHYSICAL REVIEW A 73, 043615 (2006)

Vortex stability in nearly-two-dimensional Bose-Einstein condensates with attraction

Dumitru Mihalache,[1,2] Dumitru Mazilu,[1,2] Boris A. Malomed,[3] and Falk Lederer[1]

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[2]*National Institute of Physics and Nuclear Engineering, Department of Theoretical Physics, Institute of Atomic Physics, P.O. Box MG-6, Bucharest, Romania*

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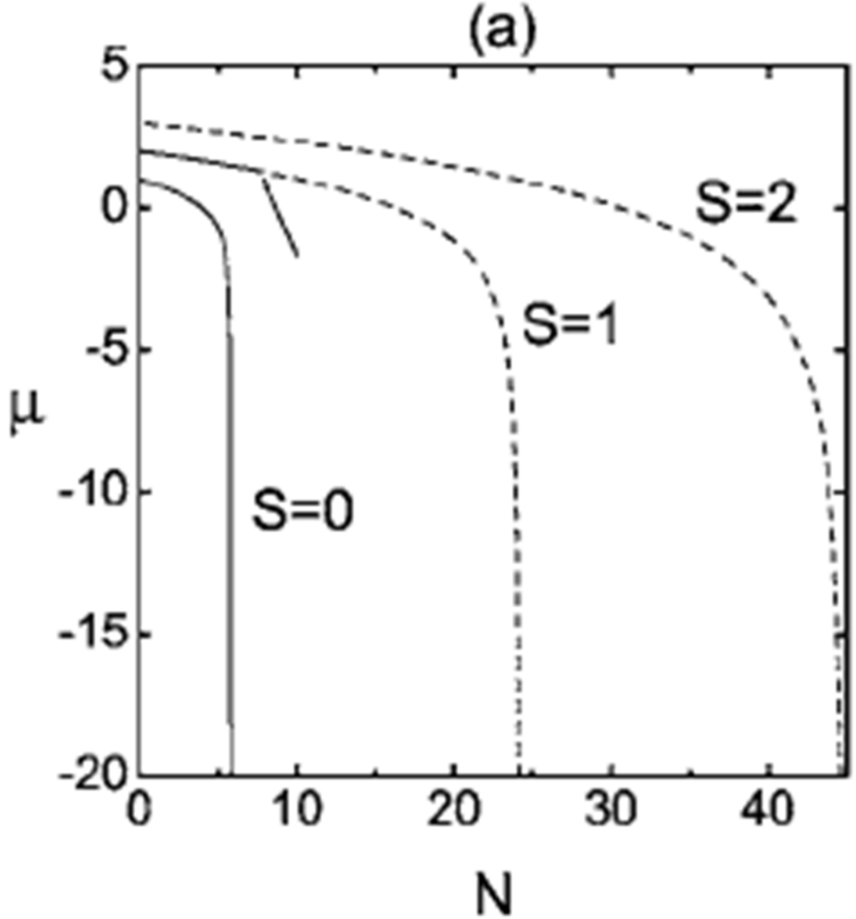
The basic equation with the ***harmonic-oscillator trapping potential*** is written as

$$i\frac{\partial \psi}{\partial t} = \left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2}\Omega_r^2(x^2 + y^2) + g|\psi|^2 \right]\psi,$$

with $g < 0$ (which corresponds to the self-attraction). The norm of solution (~ number of atoms in the BEC, or the total power in the optical model) is defined as

$$N = 2\pi \int_0^\infty |\psi(r)|^2 r^2 dr.$$

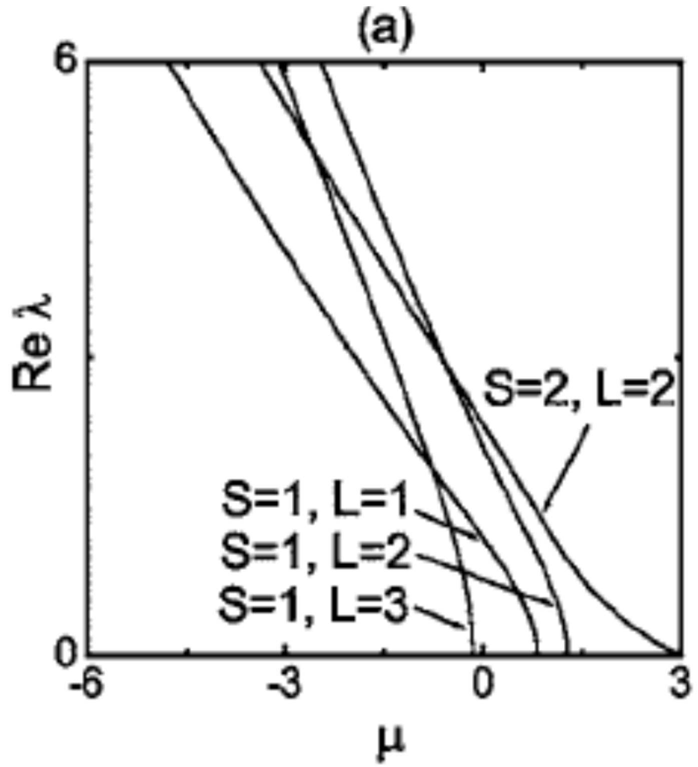
The main result of the analysis: families of fundamental solitons and vortices are represented by ***N*(*μ*)** curves, in which ***stable subfamilies*** are shown by **continuous curves**. For ***S* = 1**, the ***stability region*** (its edge is indicated by the arrow) amounts to **≈1/3** of the respective ***existence region***:



For the analysis of the stability of these solutions against small perturbations, perturbed solutions were looked for as

$$\psi(x,y,t) = [R(r) + u(r)\exp(\lambda t + iL\theta) + v^*(r)\exp(\lambda^* t - iL\theta)]\exp(iS\theta - i\mu t),$$

(in the application to BEC, the respective linearization is called the ***Bogoliubov-de Gennes equations***). As a result, the instability growth rates are calculated in the following form, featuring a ***stability window*** for ***S = 1***, but not for ***S = 2***:



An example of direct simulations of the perturbed evolution of a ***stable vortex*** (intensity and phase fields are shown):

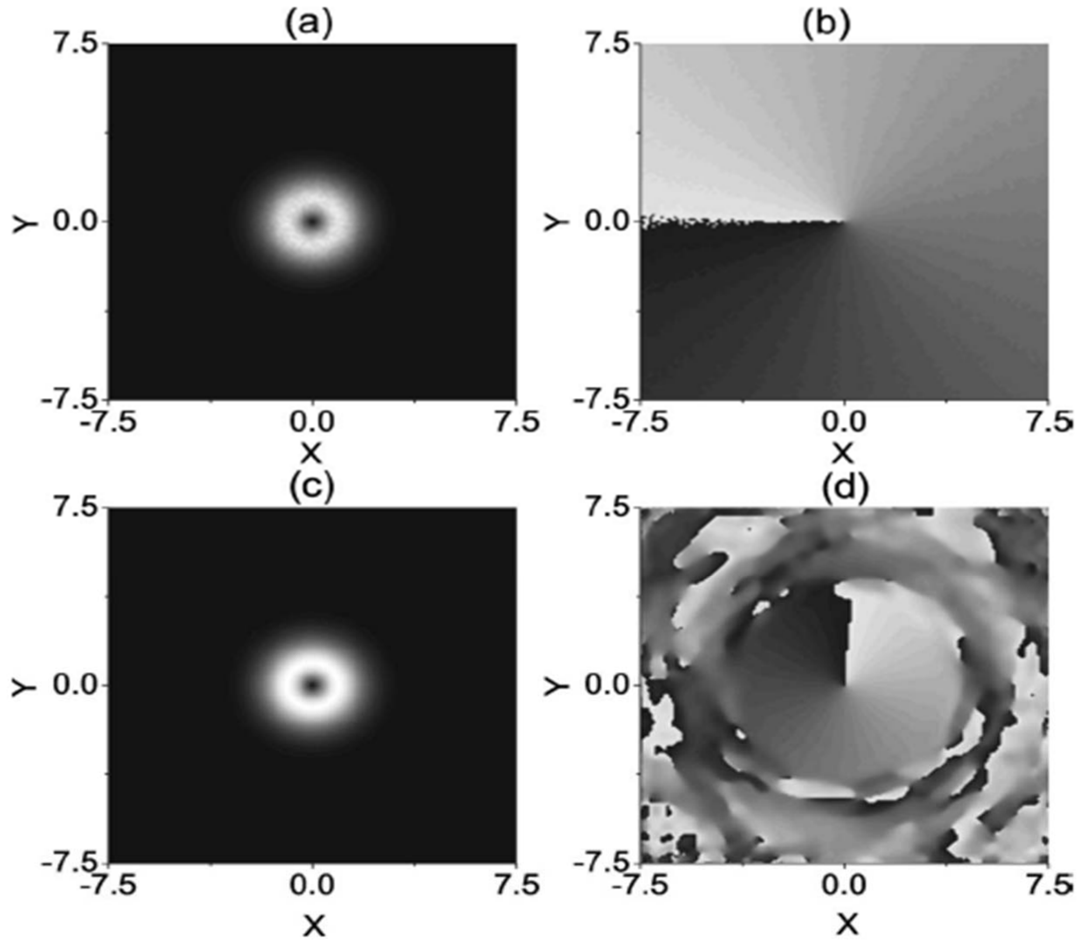
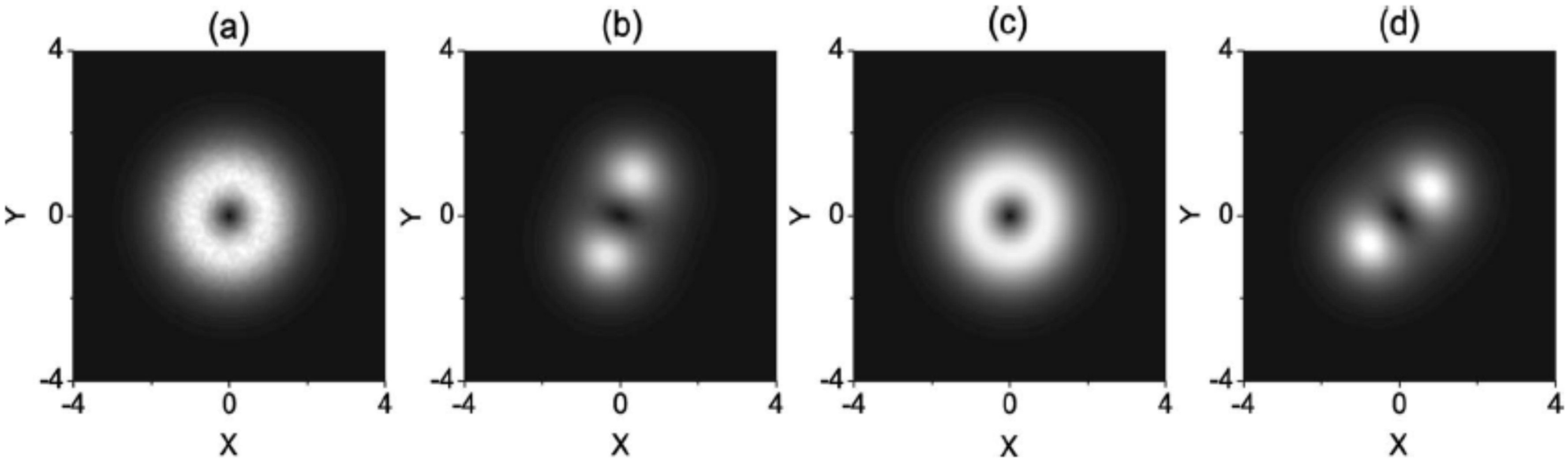


FIG. 3. Grey-scale plots illustrating recovery of the perturbed stable vortex with $S=1$ for $\mu=1.4$. (a),(b) Intensity and phase distributions in the initial configuration including random noise. (c),(d) The same in the self-cleaned vortex soliton at $t=120$. The norms of

In the interval of the norm between **1/3** and **0.43** of the ***existence region***, the vortex with ***S* = 1** is ***semi-unstable***, periodically splitting into two fragments and ***recombining***:



3. *Another general problem*: a possibility of stabilizing two-component states with ***hidden vorticity*** in a system of two coupled **2D NLSE**s with the axisymmetric trapping potential. The presentation is based on the following paper:

PHYSICAL REVIEW A **82**, 053610 (2010)

Hidden vorticity in binary Bose-Einstein condensates

Marijana Brtka,[1] Arnaldo Gammal,[2] and Boris A. Malomed[3]

[1]*Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, 09210-170 Santo André, São Paulo (SP), Brazil*

[2]*Instituto de Física, Universidade de São Paulo, 05508-090 São Paulo, São Paulo, Brazil*

[3]*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*

The system of the coupled equations is set as:

$$i\frac{\partial \psi_1}{\partial t} = \left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(x^2 + y^2) - (|\psi_1|^2 + \beta|\psi_2|^2)\right]\psi_1,$$

$$i\frac{\partial \psi_2}{\partial t} = \left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(x^2 + y^2) - (|\psi_2|^2 + \beta|\psi_1|^2)\right]\psi_2.$$

The ***hidden-vorticity solution*** is looked for as the one with ***opposite values*** of the angular momentum in the two components, so that the ***total angular momentum is zero***:

$$\psi_{1,2}(r,\theta,t) = R(r)\exp\left(iS_{1,2}\theta - i\mu t\right),$$

$$S_1 = +1,\ S_2 = -1,$$

with the **common radial amplitude** of both modes, $R(r)$, satisftying

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{d^2R}{dr^2} + \left(2\mu - \frac{S^2}{r^2} - r^2\right)R + 2\left(1+\beta\right)R^3 = 0.$$

The stability analysis was based on the use of the ***Bogoliubov – de Gennes*** equations for perturbed solutions, taken as

$$\psi_{1,2}(r,\theta,t)=\left[R(r)+u_{1,2}(r)e^{-i\omega t-iL\theta}+v_{1,2}^{*}(r)e^{i\omega^{*}t+iL\theta}\right]$$

$$\times\exp\left(-i\mu t+iS_{1,2}\theta\right),$$

where L is integer, and ω is the stability eigenvalue, the **instability** occurring if $\mathrm{Im}\{\omega\}\neq 0.$

The main result of the analysis – the ***stability diagram*** for the **hidden-vorticity** (**HV**) modes. Note that both $\boldsymbol{\beta > 0}$ and $\boldsymbol{\beta < 0}$ are included.

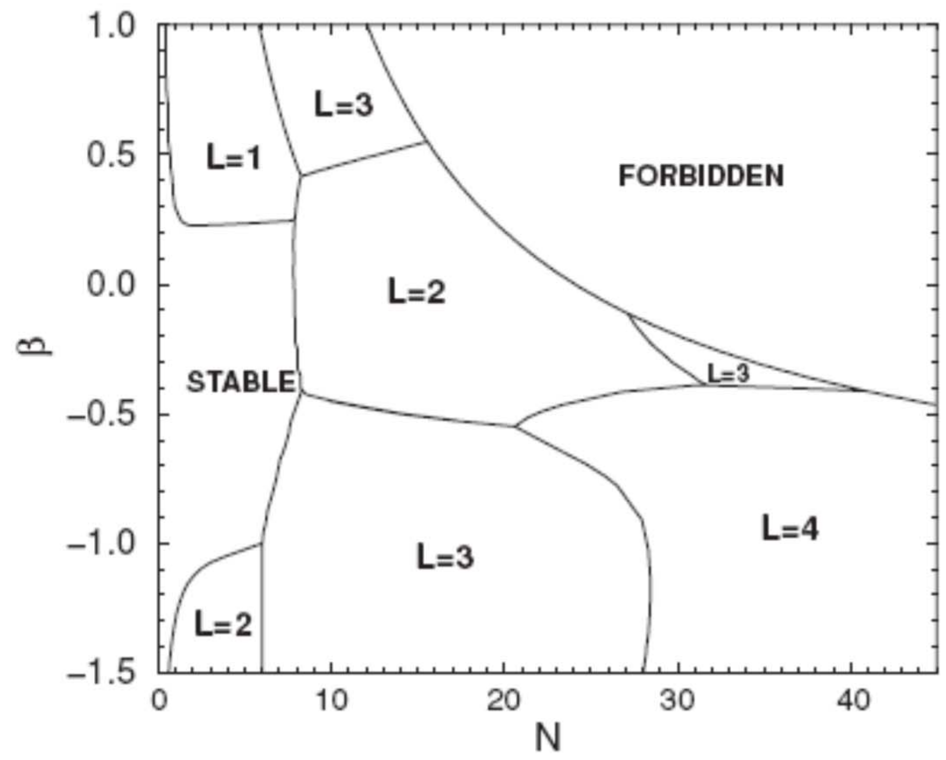
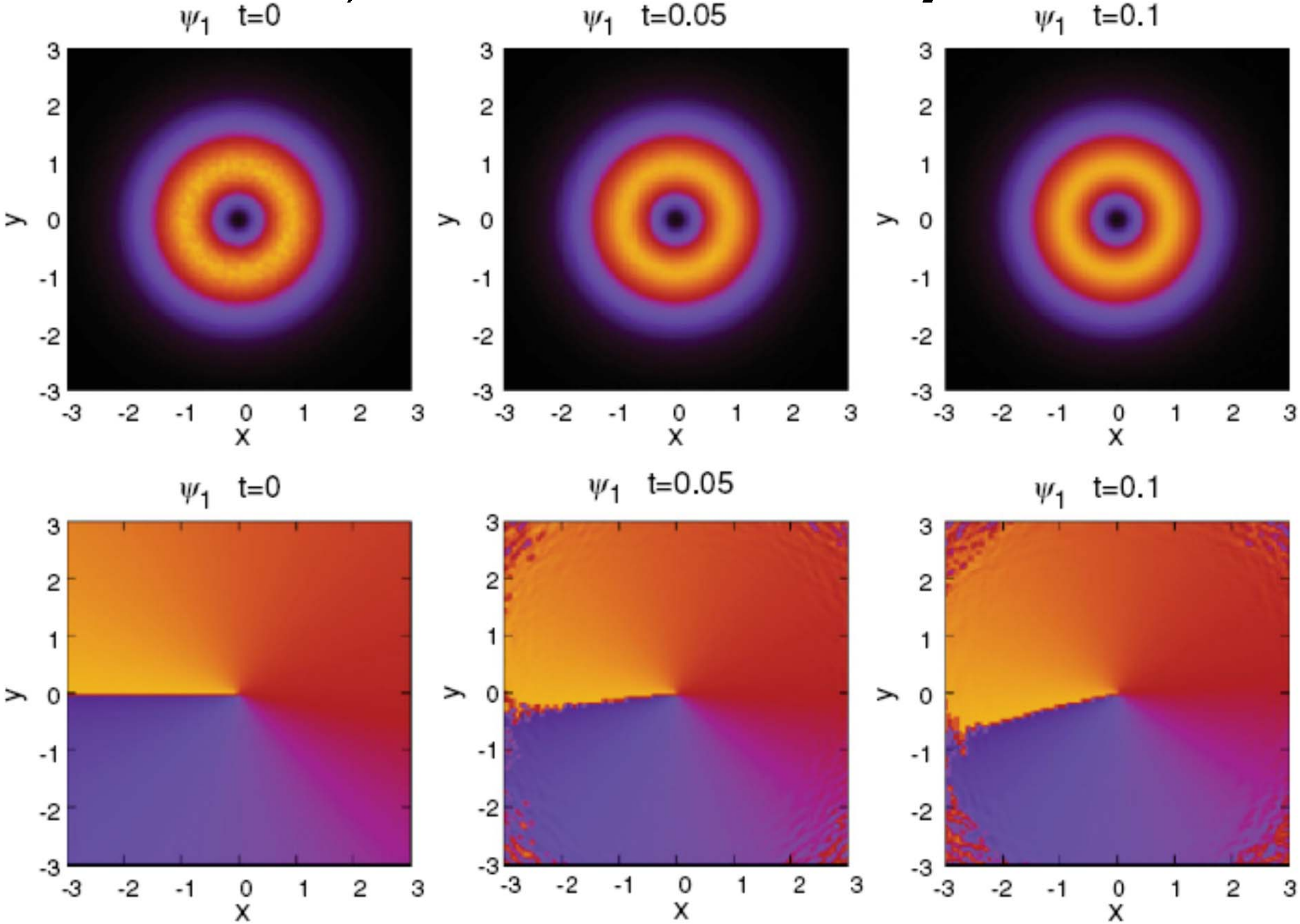
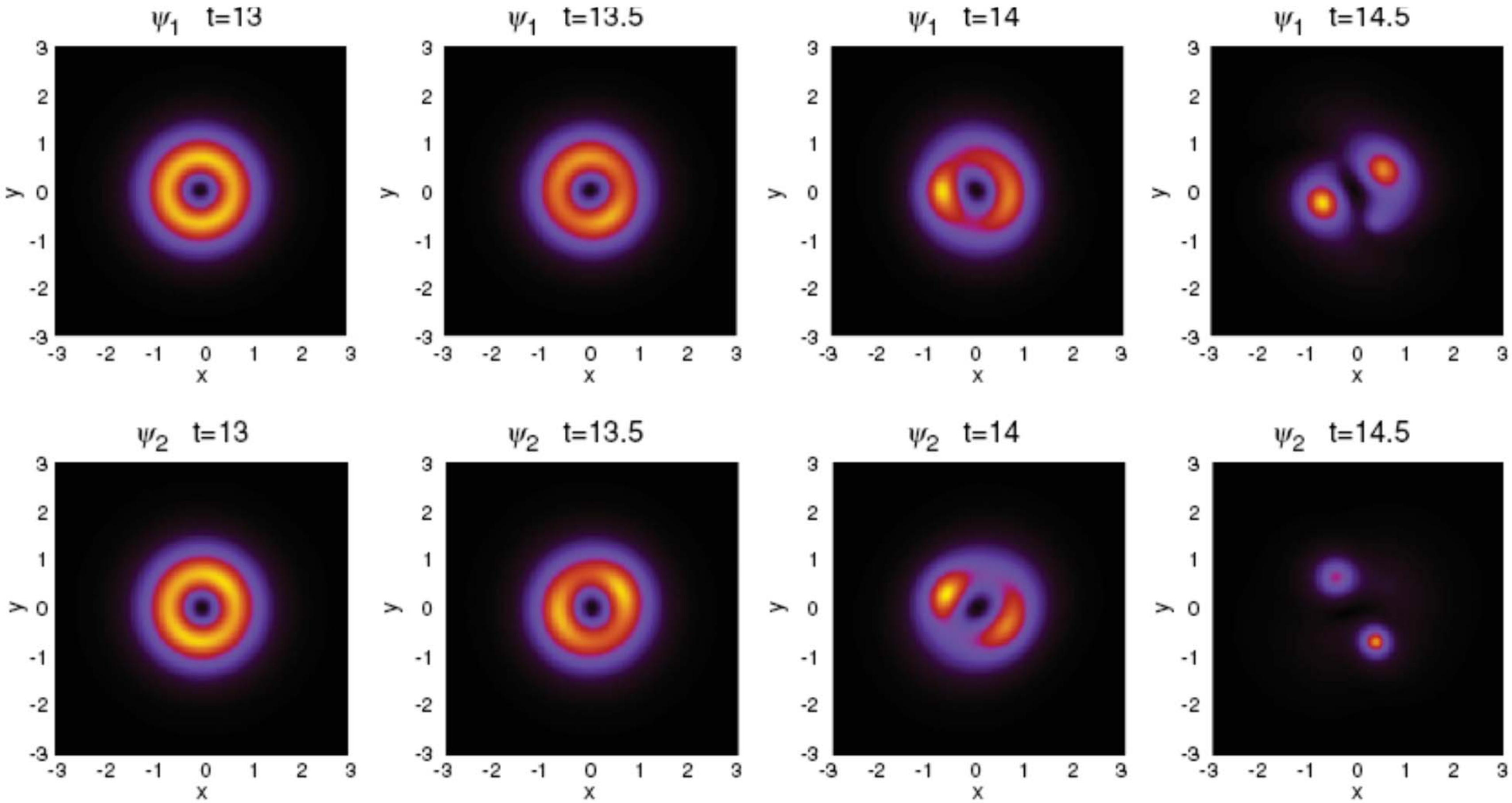


FIG. 3. The stability diagram for symmetric HV modes, in the plane of the norm (of one component) and interaction coefficient. Instability areas are labeled by the azimuthal index of the dominating perturbation eigenmode.

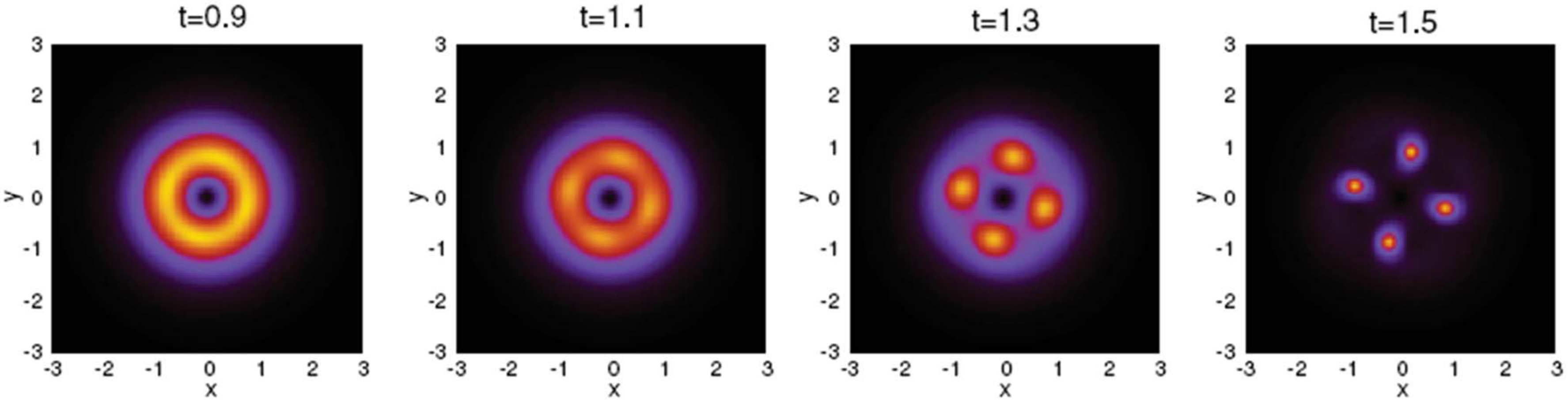
Verification of the predicted stability by direct simulations, for ***N* = 14.10** and ***β* =+0.2**:



Verification of the predicted instability (splitting into **2 segments** which collapse later) for ***N* = 13.49** and ***β* = +0.5**:



Verification of the predicted instability (splitting into **4 segments** which collapse later) for ***N* = 26.98** and ***β* = +0.5**:



4. *The use of **periodic potentials** for the stabilization of 2D solitons and vortices: an overview*

Periodic potentials, that may be induced by ***material*** or ***photonic (virtual) lattices*** in optical media, or by ***optical lattices*** (**OLs**) in **BECs**, can ***create*** and/or ***stabilize*** various types of localized modes (***solitons***) which ***do not exist*** at all, or ***are definitely unstable***, in the respective ***uniform*** media.

A natural link between the description of ***continuous media*** with periodic potentials and ***discrete lattices*** is established by the consideration of the ***limit case*** of a **very strong** periodic (cellular) potential, which, in the ***tightly-binding approximation***, effectively splits the wave field into an ***array of weakly coupled droplets/filaments***.

This mechanism underlies the concept of ***discrete nonlinear optics***, which has grown into a huge research area, see a **review**: "***Discrete solitons in optics***", by *F. Lederer, G.I. Stegeman, D.N. Christodoulides, G. Assanto, M. Segev, and Y. Silberberg*: Phys. Rep. **463**, 1 (2008).

This approach leads to the ***discrete nonlinear Schrödinger equation*** (**DNLSE**). In the **1D** setting, this equation is

$$\mathbf{i}\frac{d\psi_n}{dt} = -\frac{1}{2}\left(\psi_{n+1} + \psi_{n-1} - 2\psi_n\right) + g\,|\psi_n|^2\,\psi_n.$$

It can be derived from the continual ***Gross-Pitaevskii equation*** (**GPE**), which describes **BEC** trapped in a **deep optical lattice**:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + g\,|\psi|^2\,\psi - \varepsilon\cos\left(\frac{2\pi x}{L}\right)\psi.$$

5. ***Vortex solitons in continuous periodic media***

The fundamental model of the **2D** continuous medium equipped with the ***periodic (lattice) potential***, in optics and **BEC** alike, is based on the following continuous **NLSE/GPE** with the self-focusing cubic nonlinearity:

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+|u|^2 u-\varepsilon[\cos(2x)+\cos(2y)]u=0$$

(the period of the potential is scaled here to be π).

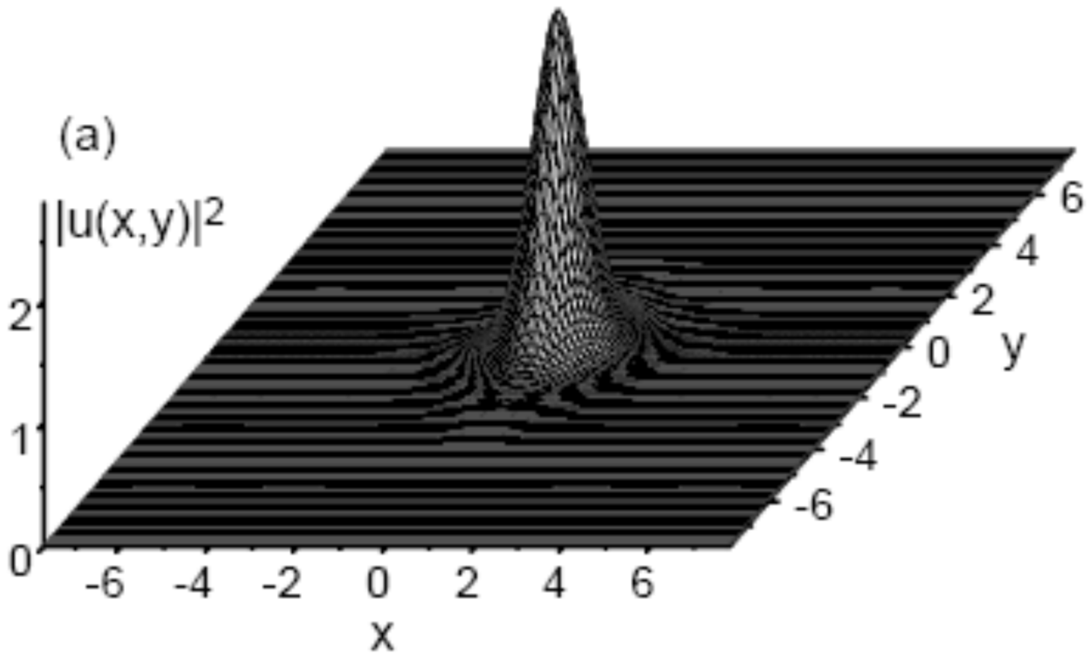
In optics, the evolutional variable is z instead of t.

The possibility of the ***stabilization*** of both ***fundamental solitons*** and ***solitary vortices*** by means of ***lattice potentials*** was first demonstrated, independently, in the following works:

B.B. Baizakov, B.A. Malomed, and M. Salerno, *Multidimensional solitons in periodic potentials*, Europhys. Lett. **63**, 642 (2003);

J. Yang and Z.H. Musslimani, *Fundamental and vortex solitons in a two-dimensional optical lattice*, Opt. Lett. **28**, 2094 (2003).

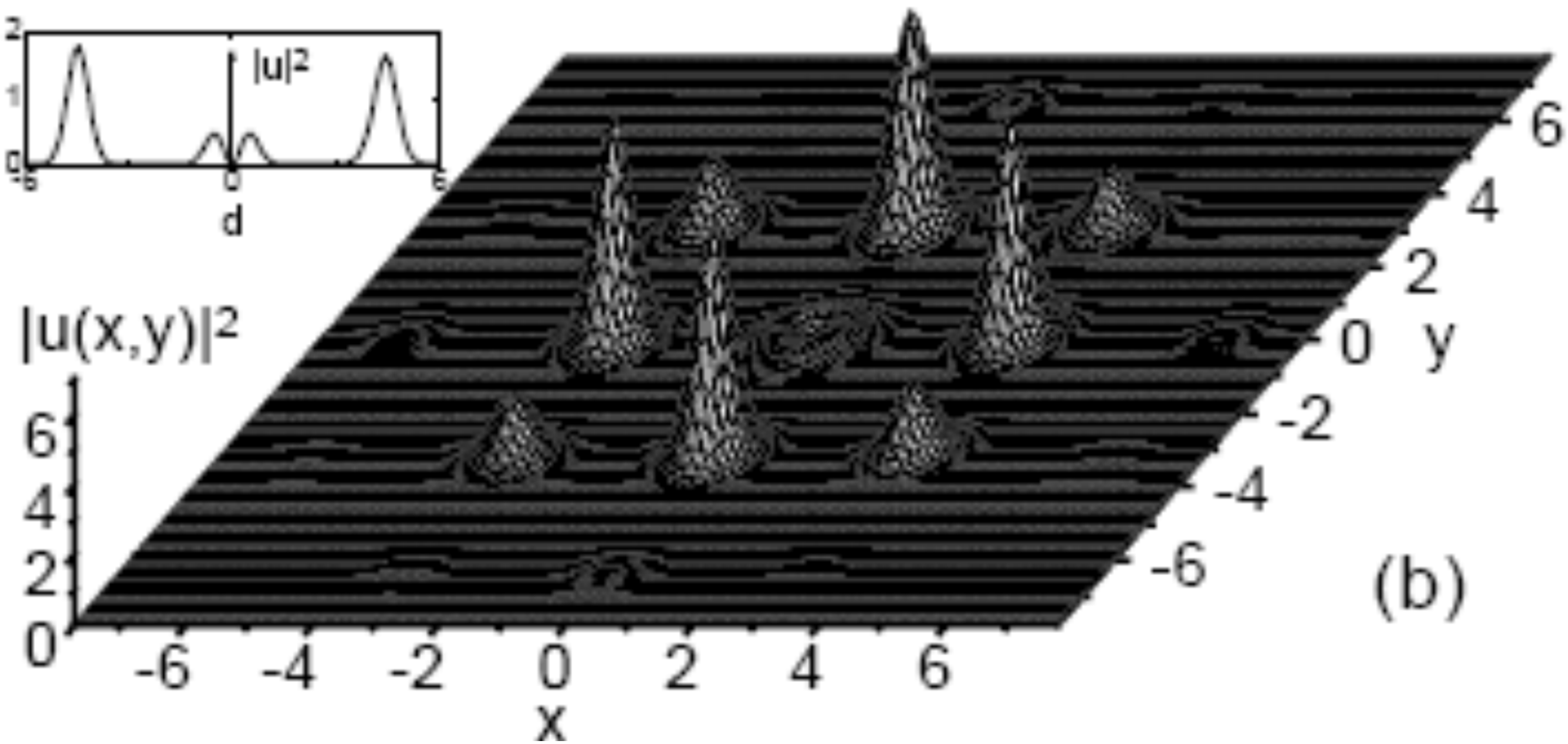
Stable fundamental solitons found in these works are ***single-peak nearly isotropic*** objects, slightly deformed by the underlying **square-lattice potential** (this example is shown for the lattice's strength $\boldsymbol{\varepsilon}$ **= 0.92**, and the integral norm $\boldsymbol{N}$ **= 2**$\boldsymbol{\pi}$):



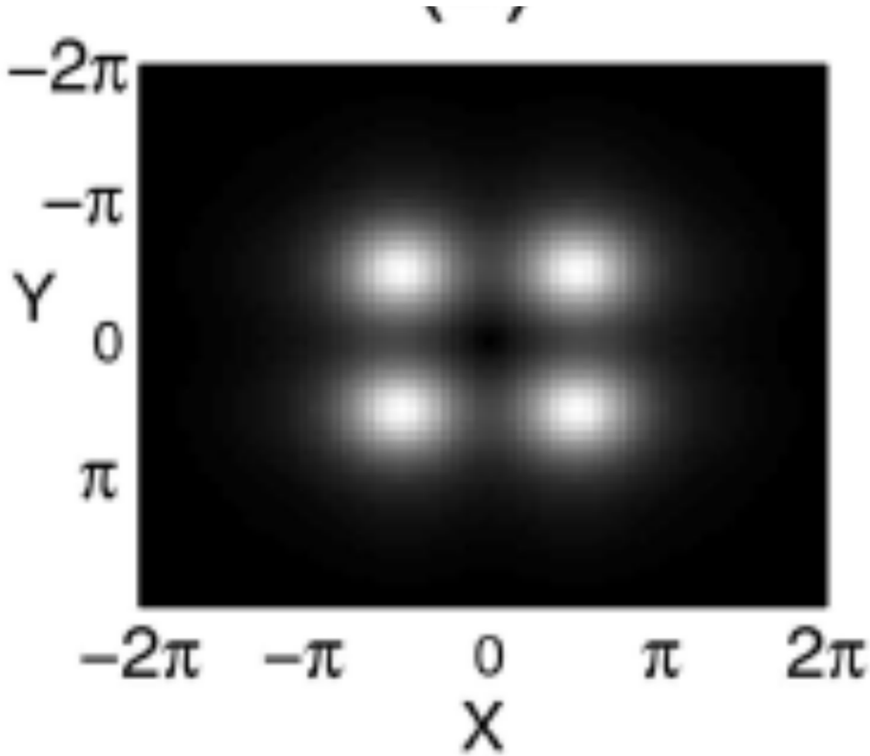
Stable objects identified as ***vortex solitons*** were actually built as ***4-peak complexes***. The respective vorticity, ***S***, is represented by ***phase shifts*** between the wave functions at adjacent peaks: $\boldsymbol{\Delta\Phi = \pi/2}$, hence the ***total phase circulation*** around the ***pivot*** of the **4-peak complex** is $\boldsymbol{2\pi}$, which corresponds to $\boldsymbol{S = 1}$**.**

Two different types of **stable** **4-peak** ***vortical modes*** (with $\boldsymbol{S = 1}$) can be thus built: ***on-site-centered vortices***, and ***off-site-centered vortices.***

A typical example of the ***stable on-site centered vortex*** (with norm $\boldsymbol{N = 2\pi}$ and $\boldsymbol{\varepsilon = 10}$). Note the presence of the ***empty site*** at the center:



A typical example of the ***inter-site-centered square-shaped vortex*** (a ***densely packed*** pattern, ***without*** an empty site in the middle), for **ε = 0.5**, is shown by means of a **contour plot** for the **2D** intensity distribution:



6. *Discrete vortex solitons*

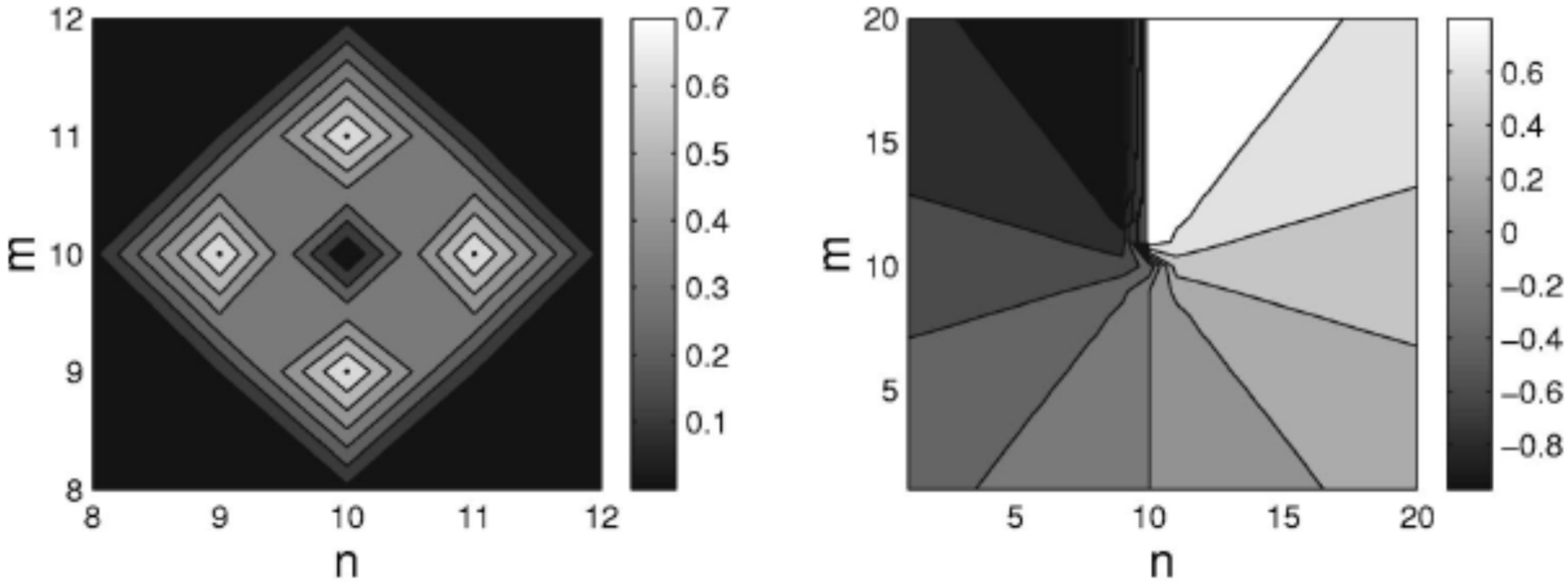
The discovery of the ***stable vortex complexes***, supported by the lattice potentials acting in the continuous media, suggests a possibility of the existence of ***stable vortex solitons*** in ***discrete lattices***, obtained, as said above, from the continuum models as the ***limit case*** corresponding to a ***very strong lattice***. Such a discrete lattice is described by the **2D DNLSE** (recall the ***evolution variable*** is ***z*** in optics):

$$i\frac{\partial u_{m,n}}{\partial t} = -C\left(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n}\right)$$

$$-|u_{m,n}|^2 u_{m,n}.$$

Stable solutions to the **2D DNLSE**, which explicitly represent ***discrete vortex solitons***, were reported by *B.A. Malomed and P.G. Kevrekidis*, Phys. Rev. E **64**, 026601 (2001).

An example of a ***stable vortex*** at ***C* = 0.05** (the intensity and phase distributions are displayed):



Generally, the stationary solutions are sought for as

$$u_{m,n}(t) = \exp(i\Lambda t) U_{m,n}.$$

The solutions may be normalized by fixing $\boldsymbol{\Lambda = 0.32}$ (for instance) and varying ***C***. The basic result is that the vortices are **stable** at

$C < C_{\mathrm{cr}}^{(S=1)} \approx \mathbf{0.13}$. For comparison, the fundamental discrete solitons, with $\mathbf{S = 0}$, are **stable** in an essentially **broader** interval, $C < C_{\mathrm{cr}}^{(S=0)} \approx \mathbf{0.29}$.

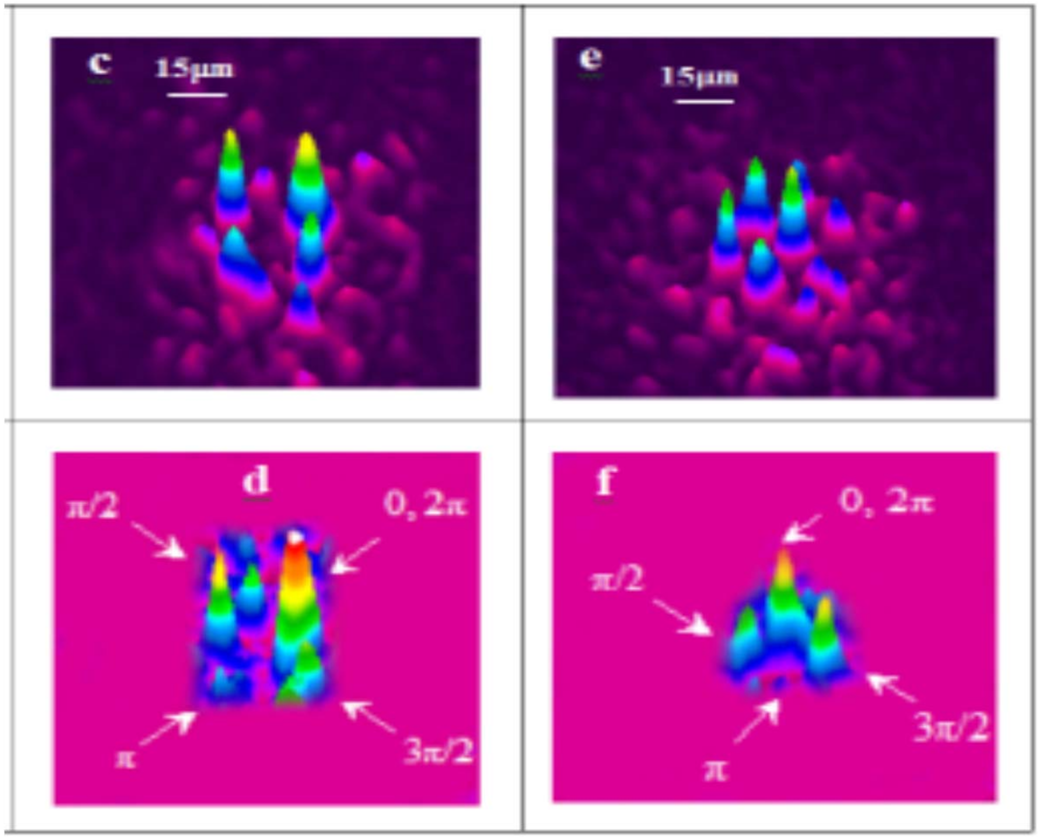
7. *Experimental results for discrete vortices*

Quasi-discrete **vortex solitons** were created in ***photorefractive*** materials with a ***photonic lattice*** induced by the illumination of the sample in directions orthogonal to that of the probe beam by counter-propagating pairs of beams, launched in the ***ordinary polarization*** (while the probe beam carries the ***extraordinary polarization***):

D.N. Neshev, T.J. Alexander, E.A. Ostrovskaya, Y. S. Kivshar, H. Martin, I. Makasyuk, and Z. Chen, Phys. Rev. Lett. 92, 123903 (2004).

J. W. Fleischer, G. Bartal, O. Cohen, O. Manela, M. Segev, J. Hudock, and D.N. Christodoulides, Phys. Rev. Lett. **92**, 123904 (2004).

Characteristic examples of the experimentally observed ***stable*** localized vortex structures:

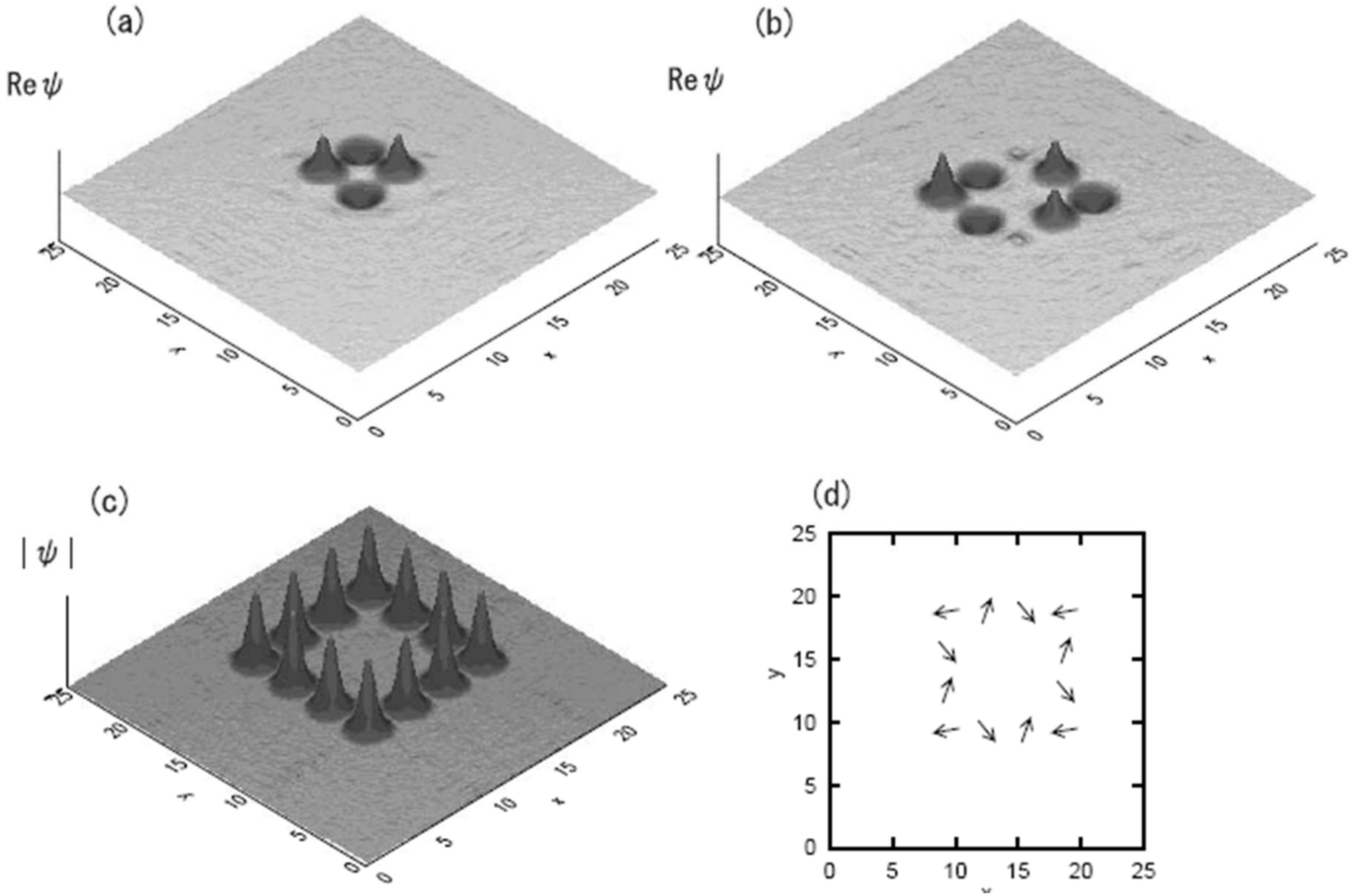


8. *Stable higher-order* (S > 1) *vortices*

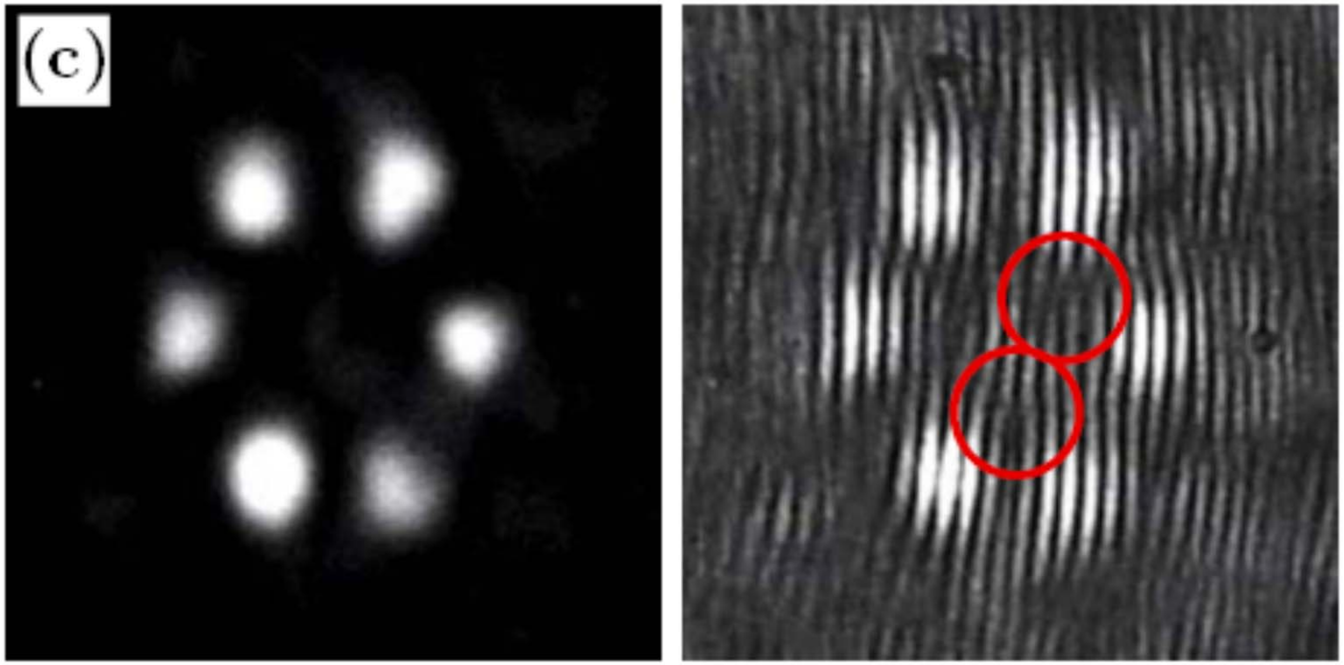
In the **2D continuous models** including the usual ***periodic potential***, and the ***cubic self-focusing nonlinearity***, ***stable multi-peak vortex complexes***, carrying the vorticity up to ***S*** **= 6**, were reported in: *H. Sakaguchi and B.A. Malomed*, Europhys. Lett. **72**, 698 (2005).

Examples of ***stable*** ***higher-order vortex solitons***:

(a) ***S* = 2**; (b) ***S* = 3**; (c,d) ***S* = 4**:



The ***stability*** of the ***vortex soliton*** with ***S* = 2** (and ***instability*** of the one with ***S* = 1**) in a ***hexagonal*** photonic lattice was ***experimentally demonstrated*** by *B. Terhalle, T. Richter, K. J. H. Law, D. Göries, P. Rose, T.J. Alexander, P.G. Kevrekidis, A.S. Desyatnikov, W. Krolikowski, F. Kaiser, C. Denz, and Y.S. Kivshar*, Phys. Rev. A **79**, 043821 (2009):



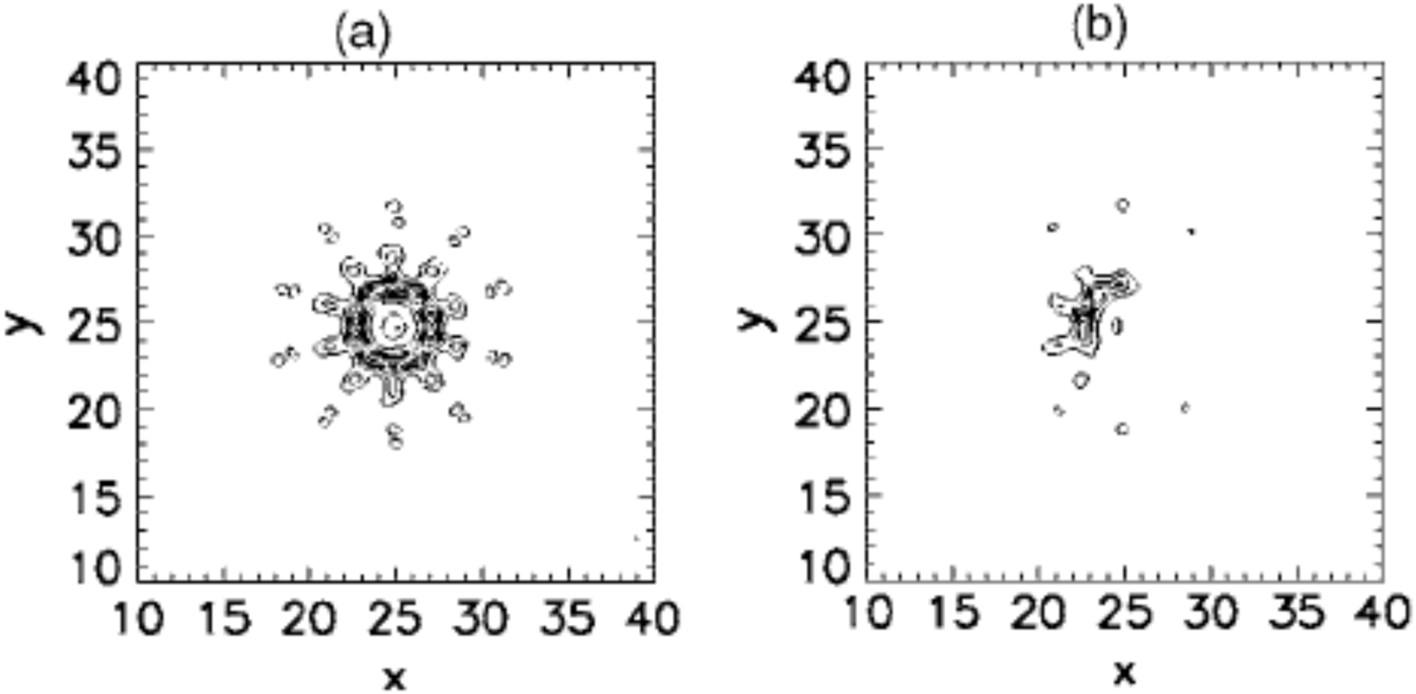
9. *Vortex solitons in quasi-periodic 2D lattices*

For the ***self-defocusing cubic*** nonlinearity, ***stable gap-mode vortex solitons*** were reported by *H. Sakaguchi and B.A. Malomed*, Phys. Rev. E **74**, 026601 (2006), in the **2D GPE/NLSE** with the potential in the form of the ***five-fold*** ***Penrose tiling***:

$$i\frac{\partial\phi}{\partial t}=-\frac{1}{2}\nabla^2\phi+|\phi|^2\phi-\varepsilon\sum_{n=1}^{5}\cos\left(\mathbf{k}^{(n)}\cdot\mathbf{r}\right)\phi,$$

where five vectors $\mathbf{k}^{(n)}$ form a **star**, with angles $\mathbf{2\pi/5}$ between adjacent vectors.

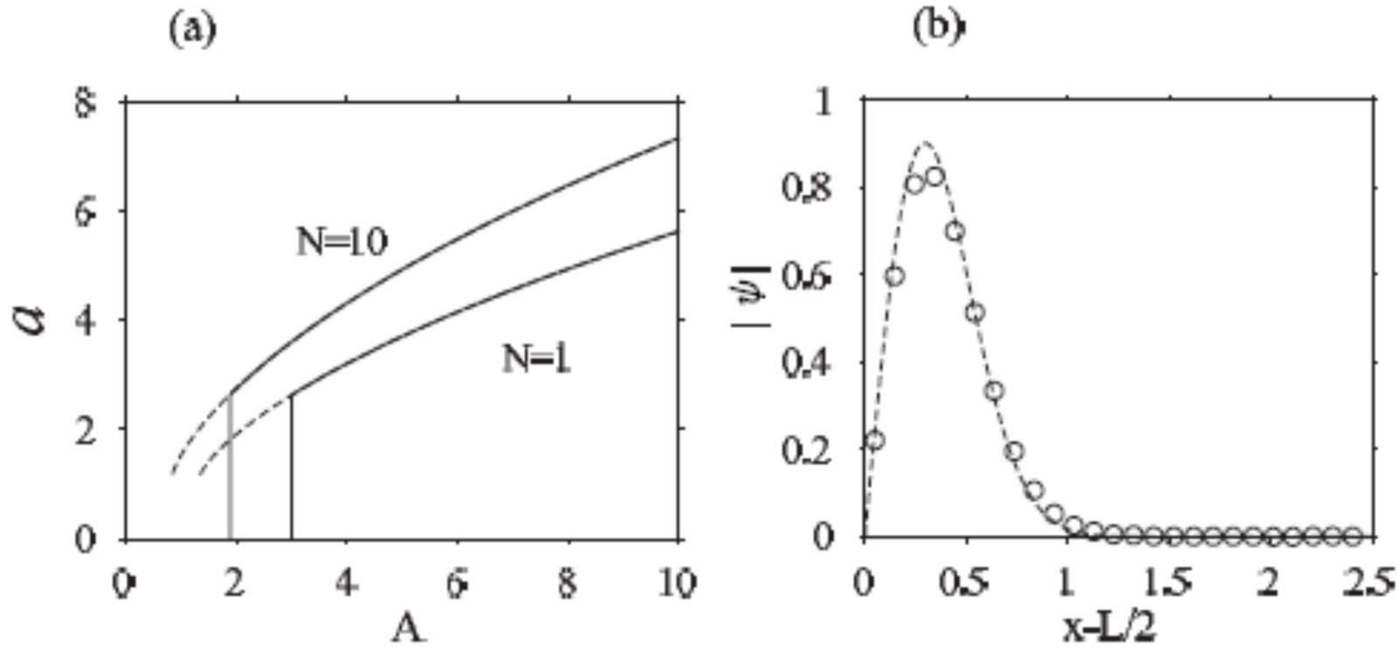
An example of a ***stable vortex gap soliton*** with ***S* = 1** [(**a**) and (**b**) display contour plots of **|*Φ*(*x*,*y*)|** and **Re(*Φ*(*x*,*y*))]**:



10. ***Crater-shaped vortices in continuous models***.

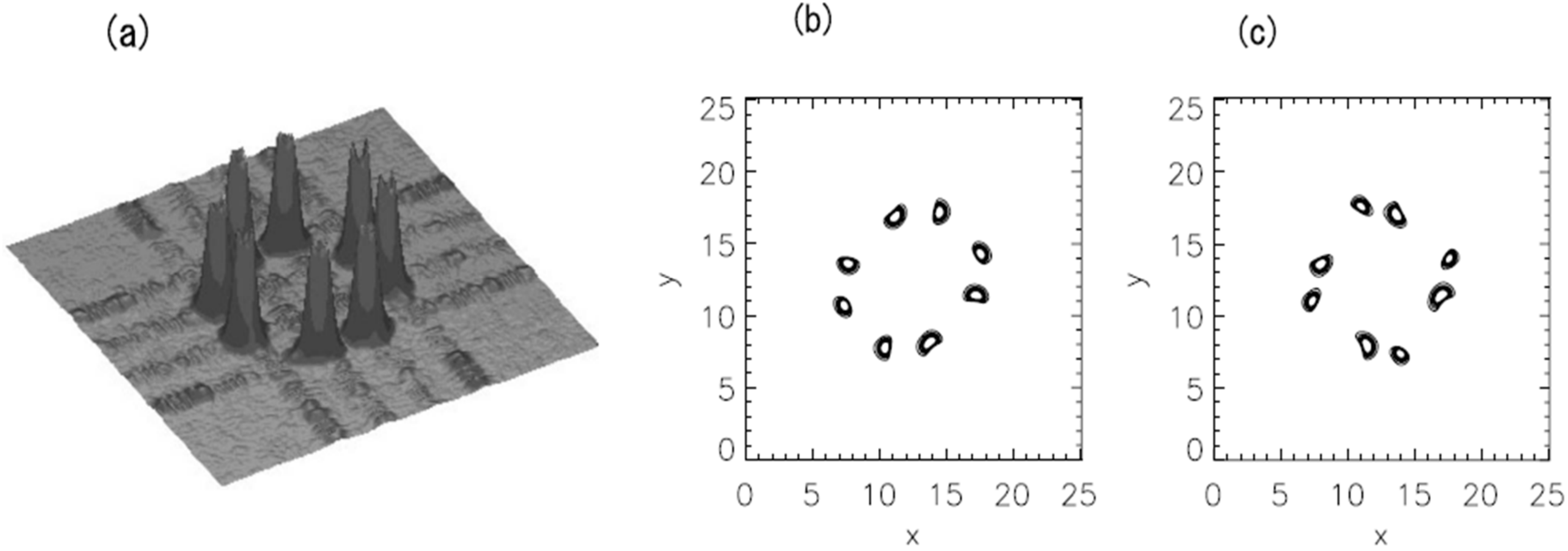
"Crater" is a vortex soliton squeezed into a ***single cell*** of the underlying **lattice potential**. In many cases, unlike the ***multi-peak vortex complexes***, such ***compact vortices*** are ***completely unstable***.

Nevertheless, in the usual **2D** model with the ***cosinusoidal cellular potential*** and ***self-focusing cubic nonlinearity***, a ***stability region*** for the "***craters***" was found, provided that the potential is ***deep*** (***strong***) ***enough*** [*H. Sakaguchi and B.A. Malomed*, Phys. Rev. A **79**, 043606 (2009)]. In this picture, ***A*** is the ***depth*** of the potential:



Moreover, taking the compact ***crater-shaped vortices*** as ***building blocks***, one can arrange them into a ***ring***, onto which a ***global vorticity***, ***S***, may be imprinted. This yields patterns ("***supervortices***") with ***two different vorticities***, namely, ***individual vorticity*** ***s* = 1** of each individual "***crater***", and ***global vorticity*** ***S***. Therefore, the ***supervortices*** with ***S* = +1** and ***S* = -1** are ***not*** equivalent, if ***s* = +1** is fixed:
H. Sakaguchi and B.A. Malomed, Europhys. Lett. **72**, 698 (2005); Phys. Rev. A **79**, 043606 (2009).

An example of two ***stable non-equivalent supervortices*** with global vorticities ***S* = +1** (**a,b**) and ***S* = -1** (**c**) in the **2D GPE/NLSE** equation:



Part B: Conservative systems with the cubic-quintic nonlinearity

Besides the use of trapping potentials, the stabilization of **2D** fundamental and vortical solitons can be provided, in the uniform space, by a combination of ***self-focusing cubic*** and ***self-defocusing quintic*** nonlinear terms.

The model equation is written in the normalized form, for the spatial-domain propagation of a stationary optical beam in a bulk medium:

$$i\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+|u|^2 u-|u|^4 u=0.$$

Stationary solutions for *vortex solitons* with topological charge m are looked for as

$$u(z,x,y)=R(r)\exp(ikz+im\theta),$$

with $R(r)$ satisfying the radial equation:

$$-kR+\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{m^2}{r^2}R\right)+R^3-R^5=0.$$

At $\boldsymbol{r} \to \infty$, the radial equation becomes asymptotically **one-dimensional**, which gives rise to the well-known ***exact soliton solution*** (*Kh.I. Pushkarov, D.I. Pushkarov, and I.V. Tomov*, Opt. Quant. Electr. **11**, 471 (1979)):

$$U(x) = 2\sqrt{\frac{k}{1+\sqrt{1-16k/3}\cosh\left(2\sqrt{k}r\right)}} \ ,$$

which exists at $\mathbf{k < 3/16}$.

Broad **2D** solitons are asymptotically equivalent, at $r \to \infty$, to the **1D** solitons in the radial direction, therefore **2D** solitons of any type may exist solely at

$\mathbf{k < 3/16 = 0.1875 \equiv k_{max}}$.

Numerous theoretical and ***experimental*** works demonstrate that the **cubic-quintic** nonlinearity occurs in real optical media, such as chalcogenide glasses, some organic materials, and suspensions of metallic nanoparticles:

G.S. Agarwal, S. Dutta, Gupta, Phys. Rev. A 38 (1988) 5678;
K. Dolgaleva, R.W. Boyd, J.E. Sipe, Gupta, Phys. Rev. A 76 (2007) 063806.
F. Smektala, C. Quemard, V. Couderc, A. Barthélémy, J. Non-Cryst. Solids 274 (2000) 232;
K. Ogusu, J. Yamasaki, S. Maeda, M. Kitao, M. Minakata, Opt. Lett. 29 (2004) 265;
C. Zhan, D. Zhang, D. Zhu, D. Wang, Y. Li, D. Li, Z. Lu, L. Zhao, Y. Nie, J. Opt. Soc. Amer. B 19 (2002) 369;
G. Boudebs, S. Cherukulappurath, H. Leblond, J. Troles, F. Smektala, F. Sanchez, Opt. Commun. 219 (2003) 427;
R.A. Ganeev, M. Baba, M. Morita, A.I. Ryasnyansky, M. Suzuki, M. Turu, H. Kuroda, J. Opt. A: Pure Appl. Opt. 6 (2004) 282;
E.L. Falcão-Filho, C.B. de Araújo, J.J. Rodrigues Jr., J. Opt. Soc. Amer. B 24 (2007) 2948.

Experimentally, the stability of **(2+1)D** ***fundamental*** (***m*** **= 0**) solitons in an optical ***cubic-quintic*** medium was demonstrated in

PRL **110,** 013901 (2013) PHYSICAL REVIEW LETTERS week ending 4 JANUARY 2013

Robust Two-Dimensional Spatial Solitons in Liquid Carbon Disulfide

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(Received 29 March 2012; published 2 January 2013)

The excitation of near-infrared (2 + 1)D solitons in liquid carbon disulfide is demonstrated due to the simultaneous contribution of the third- and fifth-order susceptibilities. Solitons propagating free from diffraction for more than 10 Rayleigh lengths although damped, were observed to support the proposed soliton behavior. Numerical calculations using a nonlinear Schrödinger-type equation were also performed.

The stability of ***fundamental solitons*** (***m* = 0**) in the framework of this equation is obvious. A nontrivial problem is the ***stability of vortex solitons*** against splitting by azimuthal perturbations. For the first time, this possibility was reported,

on the basis of direct simulations. in:

J. Opt. Soc. Am. B/Vol. 14, No. 8/August 1997 M. Quiroga-Teixeiro and H. Michinel

Stable azimuthal stationary state in quintic nonlinear optical media

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Institute for Electromagnetic Field Theory, Chalmers University of Technology, S-412 96, Göteborg, Sweden

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Accurate results for this problem have been reported in the following paper:

J. Nonlinear Sci. Vol. 12: pp. 347–394 (2002)
DOI: 10.1007/s00332-002-0475-3

Journal of Nonlinear Science

Spectrally Stable Encapsulated Vortices for Nonlinear Schrödinger Equations

R. L. Pego[1] and H. A. Warchall[2,3]

The vortex solitons with topological charge ***m*** are stable if their radius is large enough, in intervals $\boldsymbol{k_{cr} < k < k_{max} \equiv 0.1875}$. Values of $\boldsymbol{k_{cr} \equiv \omega_{cr}}$ are collected in the table (the relative width of the stability region is $\boldsymbol{(k_{max} - k_{cr})/ k_{max} = 0.21}$ for ***m* = 1**, and only **0.04** for ***m* = 5**):

m	ω_{cr}
1	0.1487
2	0.1619
3	0.1700
4	0.1769
5	0.1806

Profiles of the vortices with different topological charges ***m*** at the respective ***stability boundaries***:

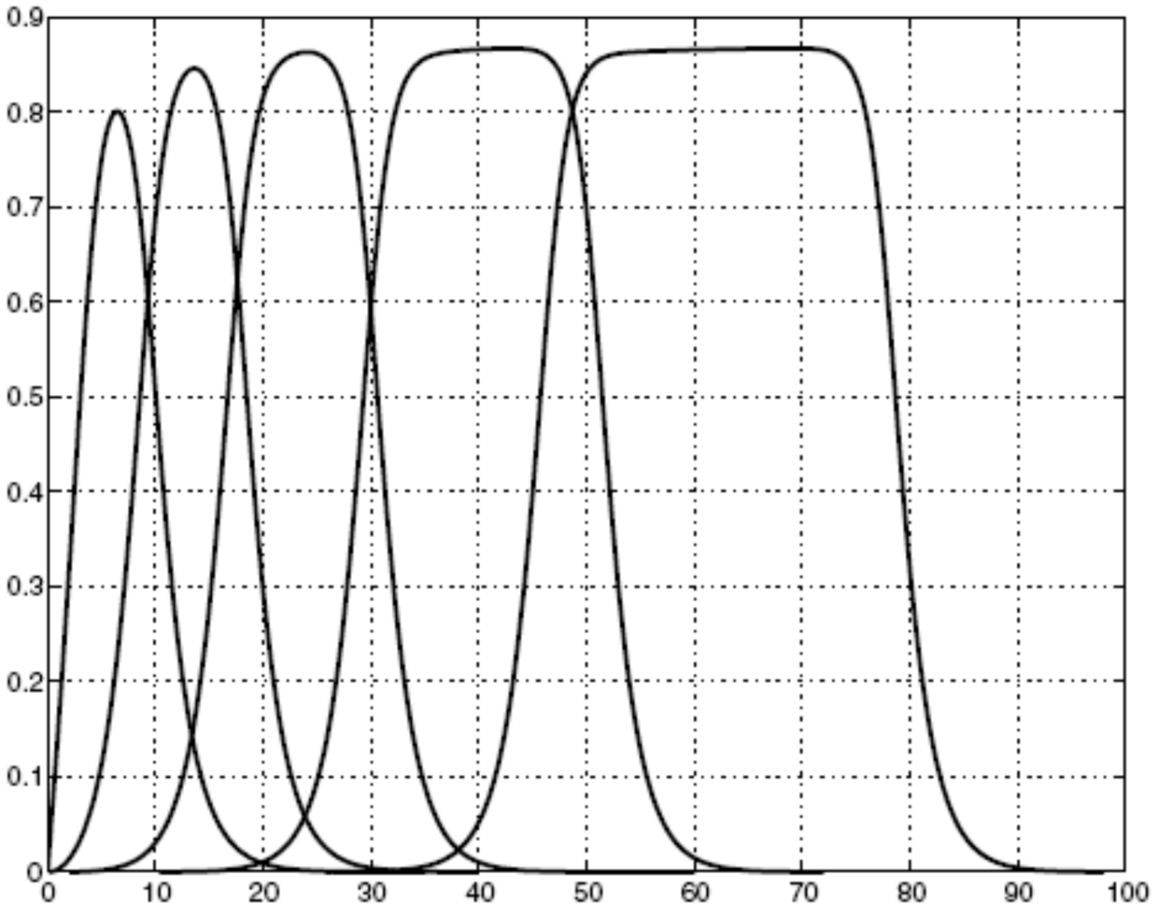

Fig. 9. Profiles at stability transition for $m = 1, 2, 3, 4, 5$.

Thus far, no experimental observation of ***stable or quasi-stable*** **2D** soliton with embedded vorticity has been reported. Experimental demonstration of such (effectively) ***stable spatial vortex solitons*** would be a ***great achievement***.

Part C: localized vortices in dissipative media described by ***complex Ginzburg-Landau equations*** (**CGLEs**).

1. The 2D model in free space

First, we consider the stability of ***vortex*** (***spiral***) **2D** ***solitons*** in the framework of the **CGLE** with the ***cubic-quintic*** nonlinearity. This class of models was introduced in:

V. I. Petviashvili and A. M. Sergeev, “Spiral solitons in active media with excitation thresholds,” Dokl. AN SSSR (Sov. Phys. Dokl.) **276**, 1380–1384 (1984).

The equation may be generalized as a model of ***laser cavities***. In this case, it includes ***linear loss***, **$\delta > 0$** (to secure the stability of the zero background around the soliton), ***cubic gain***, **$\varepsilon > 0$**, and ***quintic loss***, **$\mu > 0$** (to secure the overall stability of the system).
The equation also includes the ***diffusion term*** **$\sim\beta$**, which actually does not occur in optical models:

$$iA_z + i\delta\cdot A + (1/2 - i\beta)(A_{xx} + A_{yy}) + (1 - i\varepsilon)|A|^2 A - (\nu - i\mu)|A|^4 A = 0.$$

The presentation of results for the 2D model will follow a rather old paper on this topic:

PHYSICAL REVIEW E, VOLUME 63, 016605

Stable vortex solitons in the two-dimensional Ginzburg-Landau equation

L.-C. Crasovan,[1] B. A. Malomed,[2] and D. Mihalache[1]
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[2]*Department of Interdisciplinary Sciences, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
(Received 19 June 2000; revised manuscript received 27 September 2000; published 20 December 2000)

Examples of the self-trapping of ***stable*** vortical solitons with vorticities ***S* = 1 (a)** and ***S* = 2 (b)**:

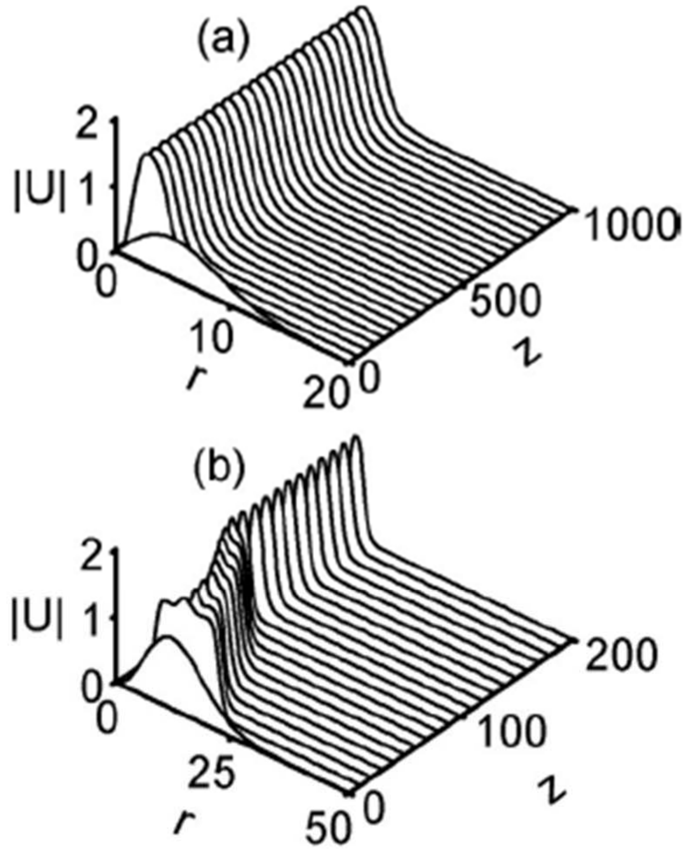


FIG. 3. Formation of spinning solitons from real initial field configurations: (a) $S=1$ and (b) $S=2$. The parameters are $\beta=0.5$, $\delta=0.5$, $\nu=0.1$, $\mu=1$, and $\varepsilon=2.5$. The initial field distributions are: (a) $U(r;z=0)=0.2r\exp[-(r/7)^2]$, and (b) $U(r;z=0)=0.02r^2\exp[-(r/12)^2]$.

The illustration of the *spiral form* of the phase field in the stable vortex soliton (s = 1 and 2):

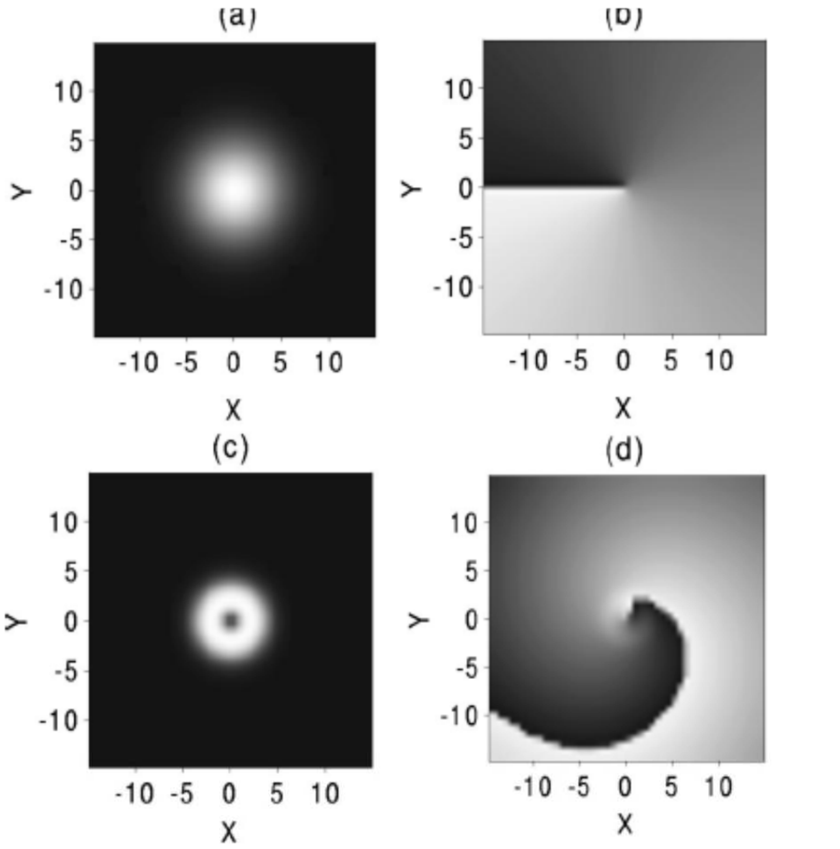


FIG. 7. Gray-scale plots of the input Gaussian field with the vorticity $S=1$: (a) amplitude and (b) phase. The output field at $z=150$: (c) amplitude and (d) phase. The parameters are $\beta=0.5$, $\delta=0.5$, $\nu=0.1$, $\mu=1$, and $\varepsilon=2.5$.

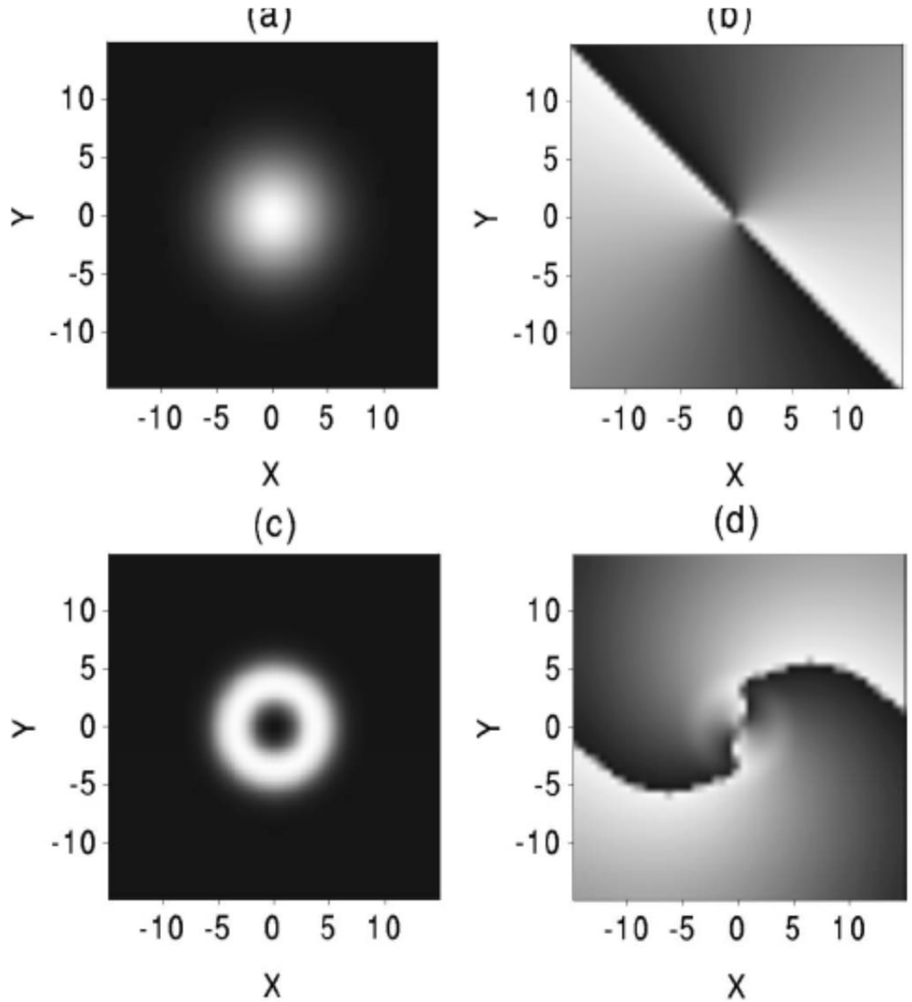


FIG. 8. The same as in Fig. 6, but for the vorticity $S=2$. The output field was taken at $z=200$.

2. Stabilization of 2D vortices in the *cubic-quintic* CGLE with the *cellular* (*lattice*) *potential*

The stabilization of dissipative ***vortex complexes***, built of **4 peaks**, of both the ***on-site-centered*** and ***off-site-centered*** types, in the **2D** ***spatial-domain*** model of **laser cavities**, was demonstrated in:

H. Leblond, B. A. Malomed, and D. Mihalache, *Stable vortex solitons in the Ginzburg-Landau model of a two-dimensional lasing medium with a transverse grating*, Phys. Rev. A **80**, 033835 (2009).

The **2D CGLE** with the ***cubic-quintic*** nonlinearity and ***cellular potential*** is

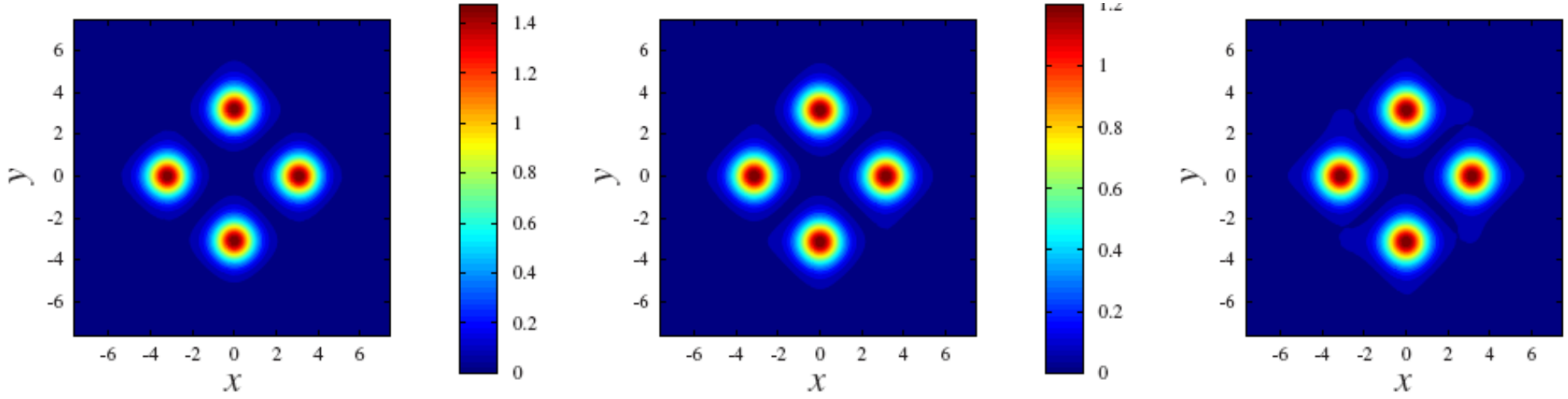
$$\frac{\partial u}{\partial z} = \left[-\delta + \left(\frac{i}{2} + \boldsymbol{\beta} \right) \nabla^2 + (i + \varepsilon) |u|^2 - (i\nu + \mu) |u|^4 \right] u$$

$$+ iV_0 \left[\cos(2x) + \cos(2y) \right] u,$$

with $\delta > 0,\ \varepsilon > 0,\ \mu > 0.$

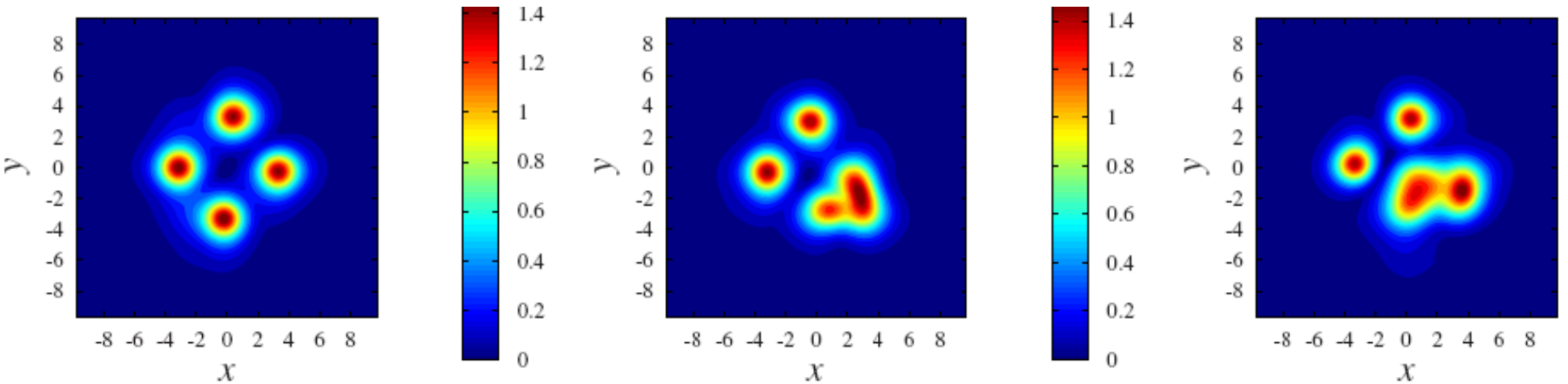
As said above, in laser media there is no "diffusion of light", hence $\boldsymbol{\beta = 0}$ will be fixed.

A typical set of **stable 4-peak vortex complexes**, found by varying the **linear-loss factor δ**, while the other parameters are fixed: $\varepsilon = 1.85, \mu = 1, \nu = 0.1,$ and $\mathbf{V_0 = 1}$:



(a) $\delta = 0.31$ (b) $\delta = 0.595$ (c) $\varepsilon = 1.525$

On the other hand, if the lattice potential is ***too weak*** (***V_0* = 0.15**), the 4-peak complexes are ***unstable***:



(a) $z = 2567.3$ (b) $z = 2576.7$ (c) $z = 2586.2$

Another possibility: ***stabilization*** of various types of ***vortical modes*** in the **2D CGLE** ***without diffusion*** and ***without trapping potentials***, but with ***spatial modulation of the linear loss***:

PRL **105,** 213901 (2010) PHYSICAL REVIEW LETTERS week ending 19 NOVEMBER 2010

Varieties of Stable Vortical Solitons in Ginzburg-Landau Media with Radially Inhomogeneous Losses

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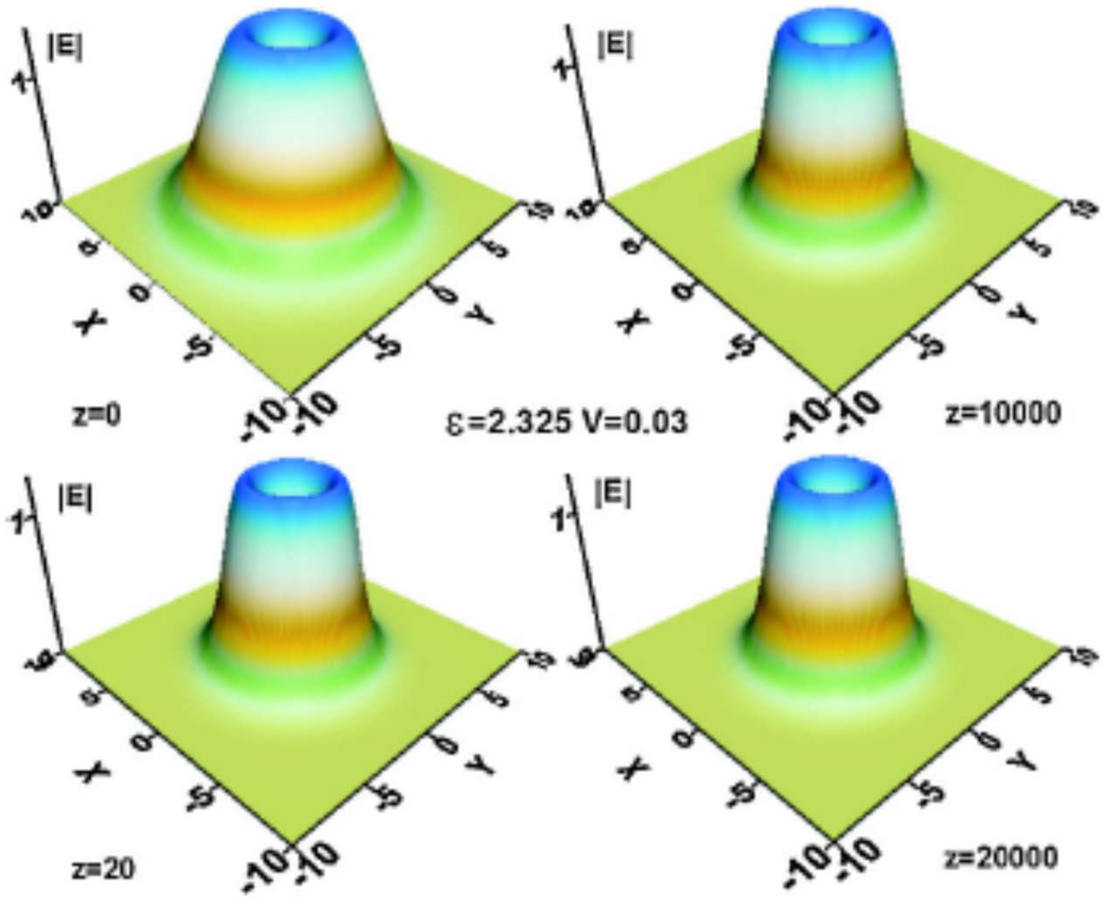
The model equation (which can also be realized in laser cavities):

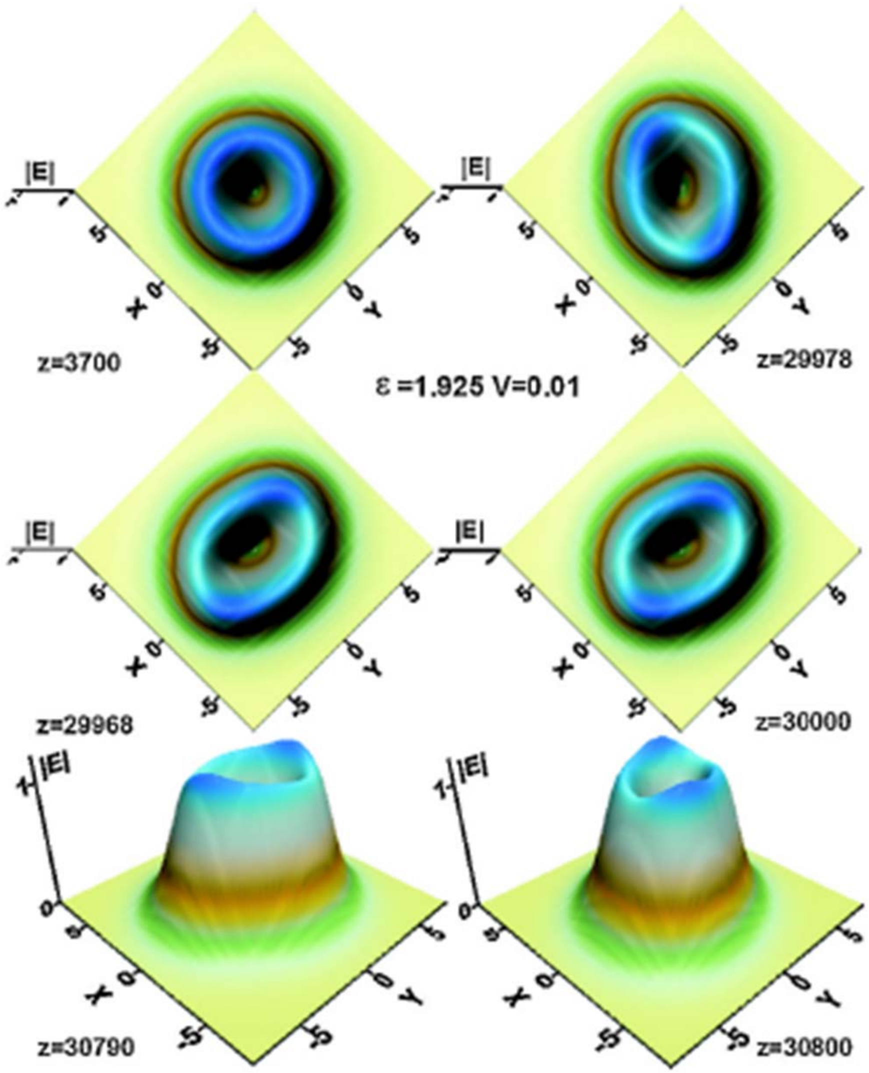
$$iE_z + (1/2)(E_{xx} + E_{yy}) + (1 - i\varepsilon)|E|^2 E - (\nu - i\mu)|E|^4$$
$$\times E = -ig(r)E, \qquad r \equiv \sqrt{x^2 + y^2}, \qquad (1)$$

Here the profile of the ***modulation of the local linear loss*** is given by $\boldsymbol{g(r) = \gamma + Vr^2}$, with $\boldsymbol{\gamma > 0}$ and $\boldsymbol{V > 0}$.

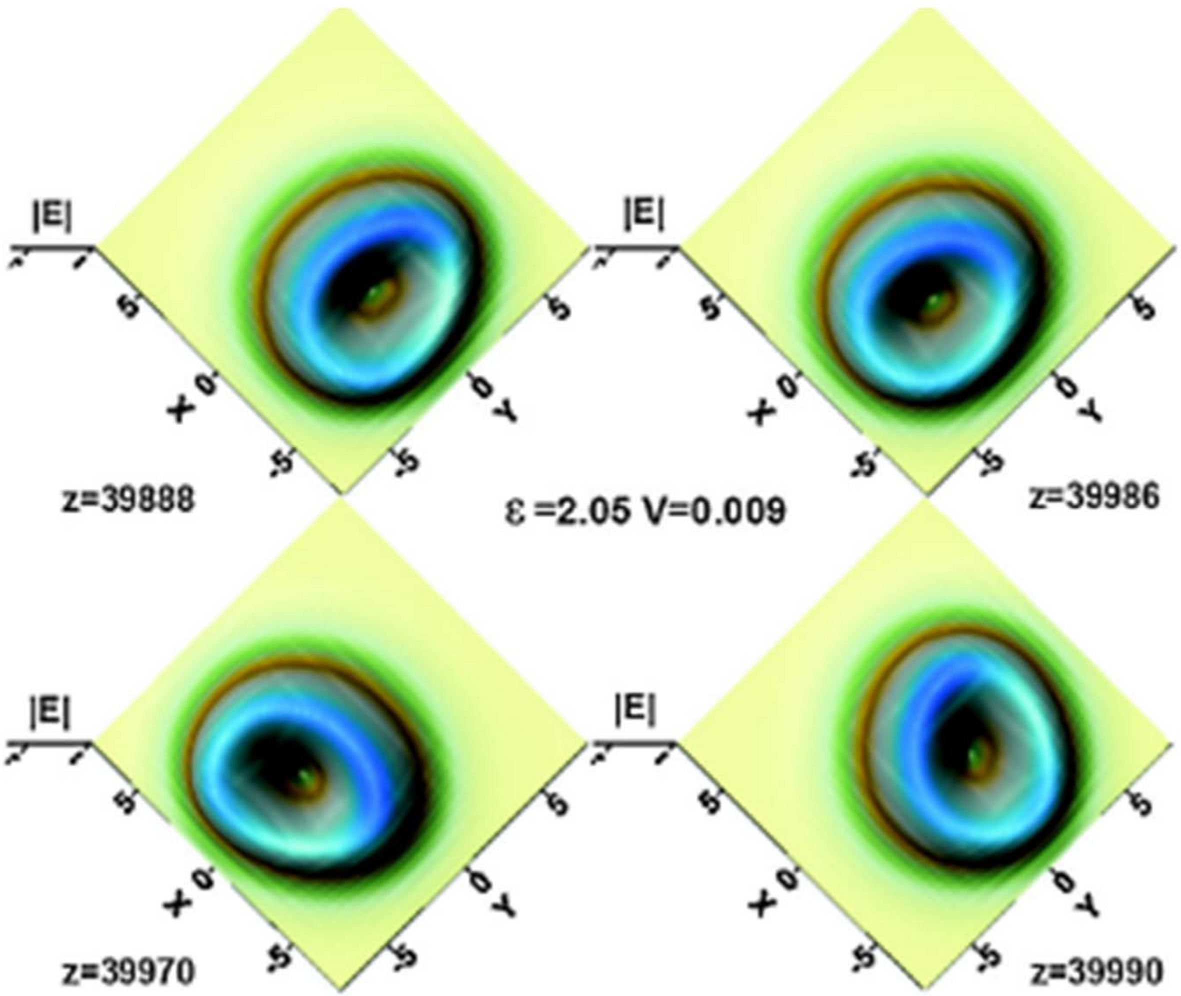
The model gives rise to a ***great variety*** of ***stable*** vortex solitons. An example of self-trapping of a ***simple stable vortex soliton***:



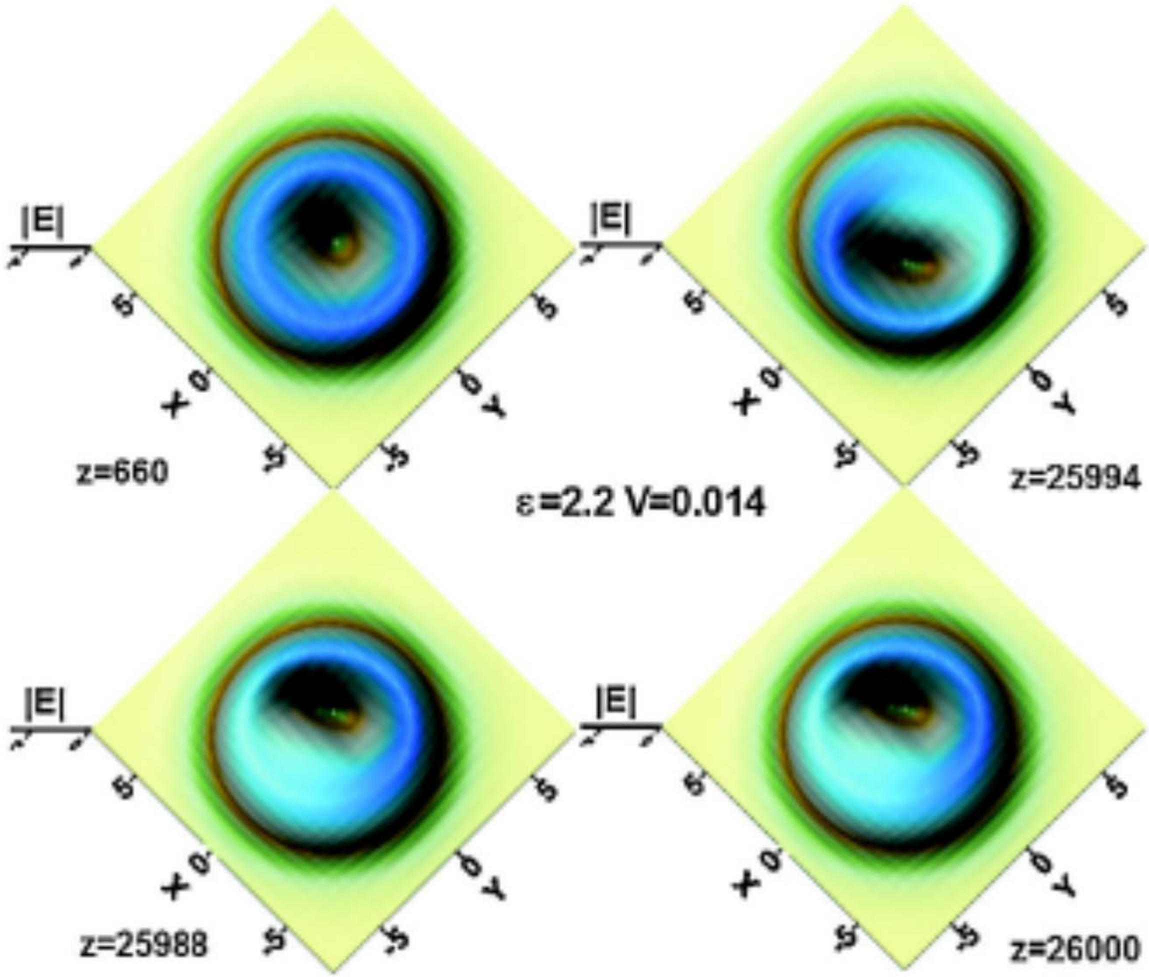
An example of another species of ***stable*** vortical modes, *viz*., a ***rotating elliptically deformed vortex***:



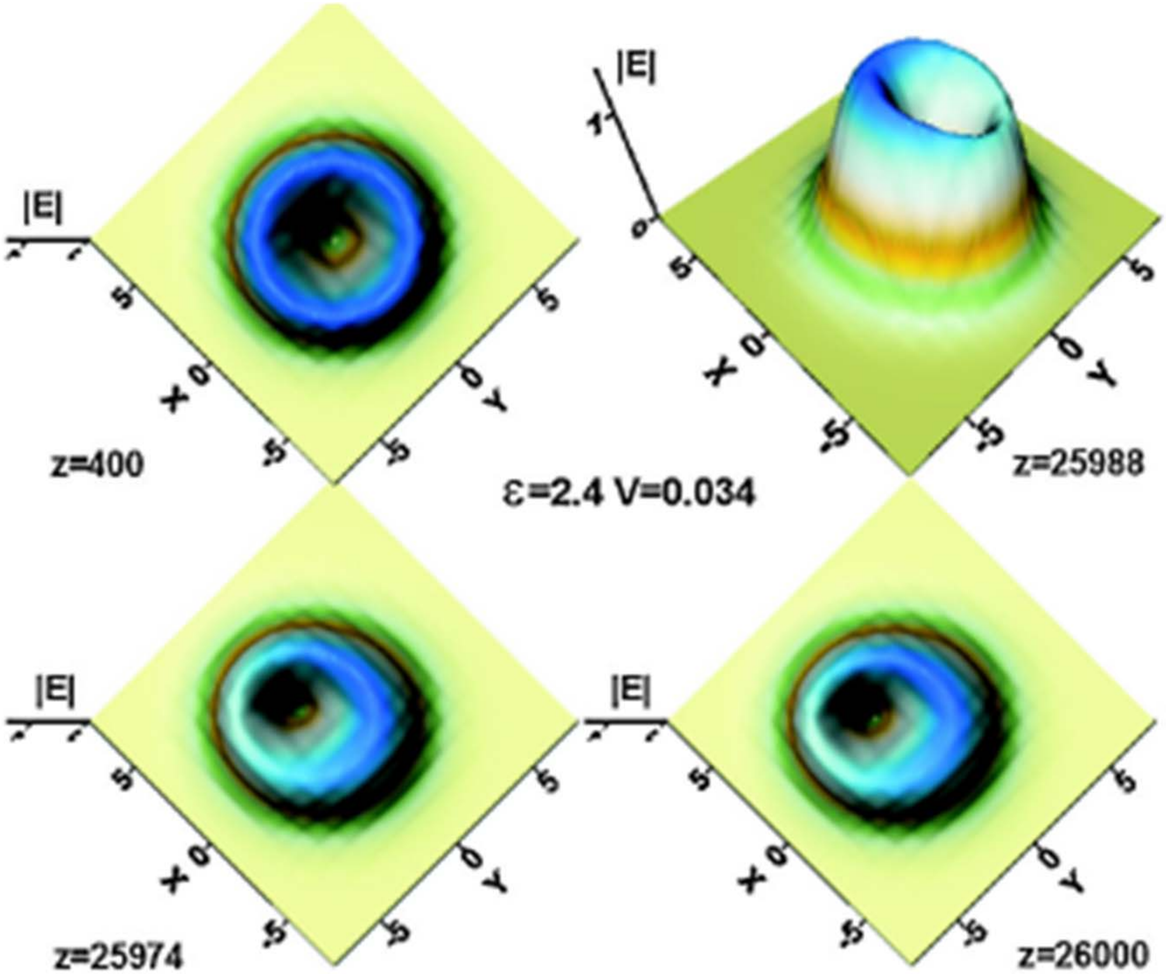
Another ***stable*** species: an ***eccentric spinning vortex*** periodically orbiting around the center:



Still another ***stable*** species – a ***rotating*** deformed ***crescent-shaped vortex***:



The last ***stable*** species – an ***elliptically-shaped*** "***slanted crater***":



Part D: Stable two-dimensional composite solitons in spin-orbit (SO)-coupled self-attractive Bose-Einstein condensates (BEC) in free space

A result of a collaborative work with:

Hidetsugu Sakaguchi and Ben Li

Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Fukuoka, Japan

A paper reporting basic results to be presented in this part:

PHYSICAL REVIEW E **89**, 032920 (2014)

Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

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(Received 12 December 2013; published 26 March 2014)

It is commonly known that two-dimensional mean-field models of optical and matter waves with cubic self-attraction cannot produce stable solitons in free space because of the occurrence of collapse in the same setting. By means of numerical analysis and variational approximation, we demonstrate that the two-component model of the Bose-Einstein condensate with the spin-orbit Rashba coupling and cubic attractive interactions gives rise to solitary-vortex complexes of two types: semivortices (SVs, with a vortex in one component and a fundamental soliton in the other), and mixed modes (MMs, with topological charges 0 and ± 1 mixed in both components). These two-dimensional composite modes can be created using the trapping harmonic-oscillator (HO) potential, but remain stable in free space, if the trap is gradually removed. The SVs and MMs realize the ground state of the system, provided that the self-attraction in the two components is, respectively, stronger or weaker than the cross attraction between them. The SVs and MMs which are not the ground states are subject to a drift instability. In free space (in the absence of the HO trap), modes of both types degenerate into unstable Townes solitons when their norms attain the respective critical values, while there is no lower existence threshold for the stable modes. Moving free-space stable solitons are also found in the present non-Galilean-invariant system, up to a critical velocity. Collisions between two moving solitons lead to their merger into a single one.

(1) Introduction and objectives

The concept of ***emulation*** (alias ***simulation***) of complex physical effects, known in condensed-matter physics, by much simpler settings available in **BEC** (***matter waves***) and **photonics** (***optical waves***), has drawn a great deal of interest:

P. Hauke, F. M. Cucchietti, L. Tagliacozzo, I. Deutsch, and M. Lewenstein, Rep. Prog. Phys. **75**, 082401 (2012).

A new topic has emerged in the framework of this approach: the emulation of **spin-orbit** (**SO**) **interactions** in semiconductors, such as those accounted for by the ***Rashba*** and ***Dresselhaus*** terms, by ***mapping*** the spinor wave function of electrons into the pseudo-spinor two-component wave function of a binary **BEC** gas:

Y. J. Lin, K. Jimenez-Garcia, and I. B. Spielman, Nature **471**, 83 (2011);

Y. Zhang, L. Mao, and C. Zhang, Phys. Rev. Lett. **108**, 035302 (2012);

A brief review: *H. Zhai*, Int. J. Mod. Phys. B **26**, 1230001 (2012).

The **SO** coupling is a ***linear feature*** of the system. Its combination with the natural self-repulsive ***cubic nonlinearity*** of the atomic **BEC** gives rise to nonlinear effects, such as delocalized ***vortices***:

C. J. Wu, Mod. Phys. Lett. B **23**, 1 (2009);
X.-Q. Xu and J. H. Han, Phys. Rev. Lett. **107**, 200401 (2011);
J. Radic', T. A. Sedrakyan, I. B. Spielman, and V. Galitski, Phys. Rev. A **84**, 063604 (2011);
X.-J. Liu, H. Pu, P. D. Drummond, and H. Hu, Phys. Rev. A **85**, 023606 (2012);
H. Sakaguchi and B. Li, Phys. Rev. A **87**, 015602 (2013);

A. Fetter, Phys. Rev. A **89**, 023629 (2014).

The objective here is to construct ***self-trapped*** (localized) ***stable*** **2D** ***vortical modes*** in the **SO-coupled BEC** with ***attractive*** **SPM** and **XPM** nonlinearities, in the ***free space*** (without any trapping potential).

At the first glance, this objective seems ***absolutely impossible***. Formal **2D** ***vortex-soliton*** solutions (alias the above-mentioned "***vortex Townes' solitons***") of the **NLSE** with the ***self-attractive*** cubic term are well known:

JOURNAL OF MODERN OPTICS, 1992, VOL. 39, NO. 11, 2277–2291

The theory of spiral laser beams in nonlinear media

V. I. KRUGLOV, YU. A. LOGVIN

Institute of Physics, Byelorussian Academy of Sciences, 220602 Minsk, Republic of Belarus

and V. M. VOLKOV

Institute of Mathematics, Byelorussian Academy of Sciences, 220602 Minsk, Republic of Belarus

However, such solitons are subject to the above-mentioned ***strong instability***, against ***splitting*** and ***collapse***.

Therefore, as already discussed previously, a problem of fundamental interest is to introduce ***physically meaningful*** settings, in which both the ***fundamental and vortical solitons*** can be ***stabilized***, against the collapse and splitting.

(2) The model

The system of **GPES** for the (pseudo-) spinor wave function **(*Φ*₊, *Φ*₋)** of the binary **BEC** coupled by the **SO** terms of the ***Rashba type*** with strength ***λ* ≡ 1**, coefficient of the **SPM** self-attraction **≡ 1**, coefficient of the **XPM** inter-component attraction ***γ* ≥ 0**, and (***for the time being***) the strength of the **HO** trapping potential **Ω**:

$$
\begin{aligned}
i\frac{\partial \phi_+}{\partial t} &= -\frac{1}{2}\nabla^2\phi_+ - (|\phi_+|^2 + \gamma|\phi_-|^2)\phi_+ \\
&\quad + \lambda\left(\frac{\partial \phi_-}{\partial x} - i\frac{\partial \phi_-}{\partial y}\right) + \frac{1}{2}\Omega^2(x^2+y^2)\phi_+, \\
i\frac{\partial \phi_-}{\partial t} &= -\frac{1}{2}\nabla^2\phi_- - (|\phi_-|^2 + \gamma|\phi_+|^2)\phi_- \\
&\quad - \lambda\left(\frac{\partial \phi_+}{\partial x} + i\frac{\partial \phi_+}{\partial y}\right) + \frac{1}{2}\Omega^2(x^2+y^2)\phi_-,
\end{aligned}
$$

(3) Semi-vortex states

The coupled GPEs admit a family of solutions for ***semi-vortices***, with vorticities ***m*₊ = 0** in one component, and ***m*₋ = 1** in the other. The ***exact ansatz*** for these solutions is

$$\phi_+\left(x,y,t\right)=e^{-i\mu t}f_1(r^2),\phi_-\left(x,y,t\right)=e^{-i\mu t+i\theta}rf_2(r^2),$$

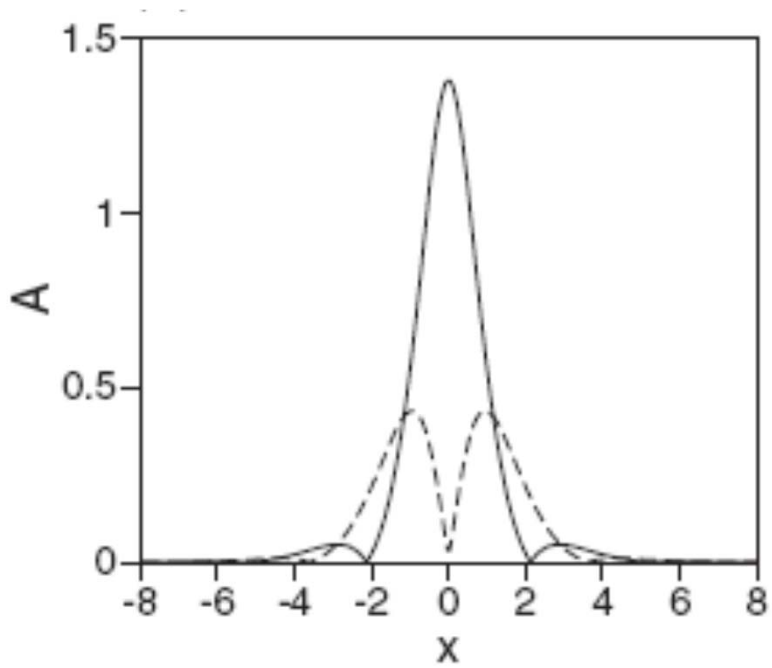
where μ is the chemical potential, $\left(r,\theta\right)$ are the polar

coordinates, and real functions $f_{1,2}(r^2)$ obey the following equations:

$$\mu f_1+2\left[r^2\frac{d^2 f_1}{d(r^2)^2}+\frac{df_1}{d(r^2)}\right]+\left(f_1^2+\gamma r^2 f_2^2\right)f_1 \;-2\lambda\left[r^2\frac{df_2}{d(r^2)}+f_2\right]-\frac{\Omega^2}{2}r^2 f_1=0,$$

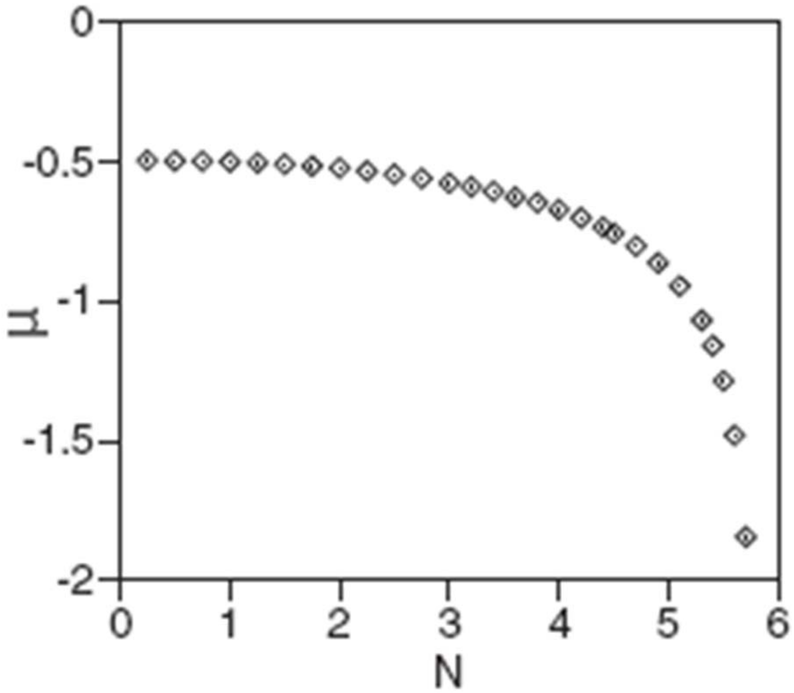
$$\mu f_2+2\left[r^2\frac{d^2 f_2}{d(r^2)^2}+2\frac{df_1}{d(r^2)}\right]+\left(r^2 f_2^2+\gamma f_1^2\right)f_2 \;+2\lambda\frac{df_1}{d(r^2)}-\frac{\Omega^2}{2}r^2 f_2=0.$$

A numerically found cross-section (along ***y* = 0**) of a stable semi-vortex, at ***γ* = 0** and **Ω = 0**, obtained by means of the imaginary-time integration, as a stationary soliton in the ***free space***:



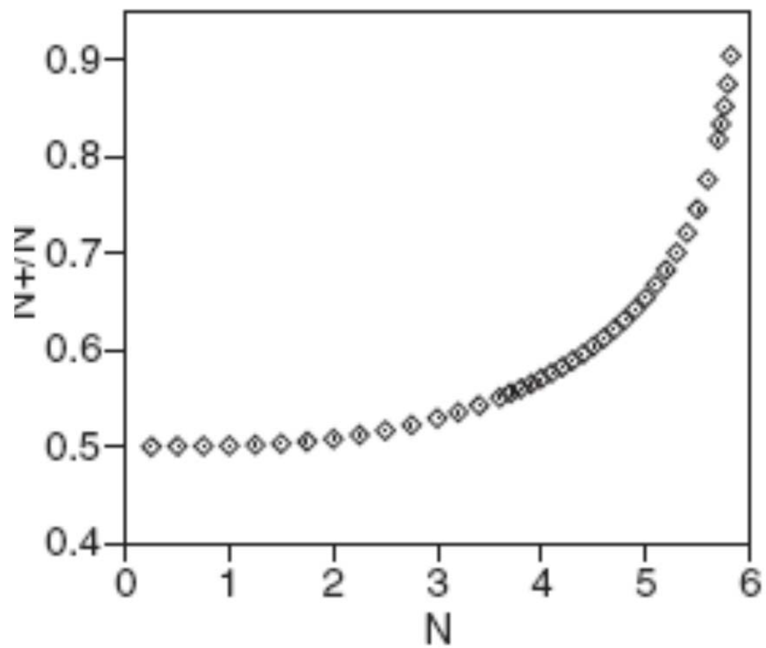
The numerically found dependence between the total norm of the semi-vortices and their chemical potential demonstrates that **(1)** the norm of the semi-vortex indeed falls ***below the critical value*** and **(2)** there is ***no finite minimum (threshold) value*** of the norm necessary for the existence of the semi-vortex; **(3)** the norm is ***bounded from above*** precisely by the ***critical value***; (4) the dependence satisfies the Vakhitov-Kolokolov (VK) criterion,

dμ/dN < 0, which is a necessary condition for the stability:



Direct simulations demonstrate that the semi-vortices are ***completely stable*** at $\gamma < 1$ (**XPM/SPM < 1**), but they are ***unstable*** at $\gamma > 1$.

In the limit of $\mathbf{N \to N_{critical} \approx 5.85}$, the semi-vortex ***degenerates*** into the usual (unstable) ***Townes' soliton*** with an ***infinitely large chemical potential***, in the first component, leaving the second (formerly vortical) component ***empty***:



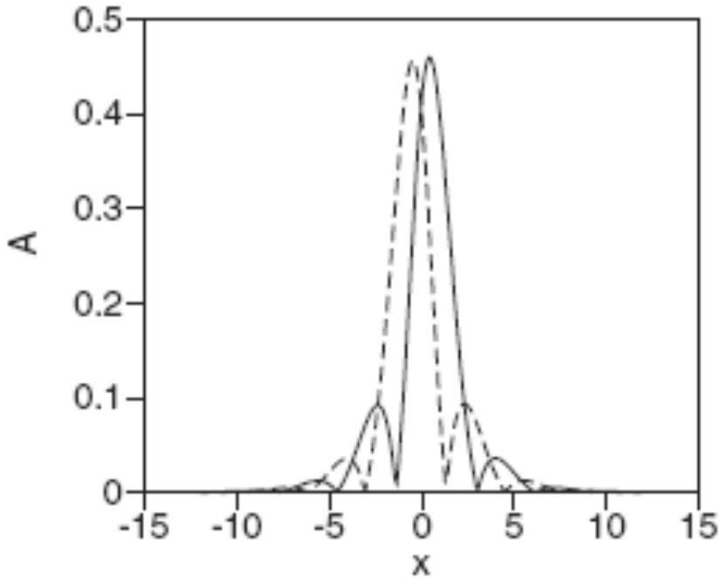
(4) Mixed modes

Another class of localized states can be constructed in the form of **mixed modes**, so called because they **mix** fundamental and vortical terms in each component, namely, $m_1 = (0, -1)$ and $m_2 = (0, +1)$, as per the following ansatz (initial guess):

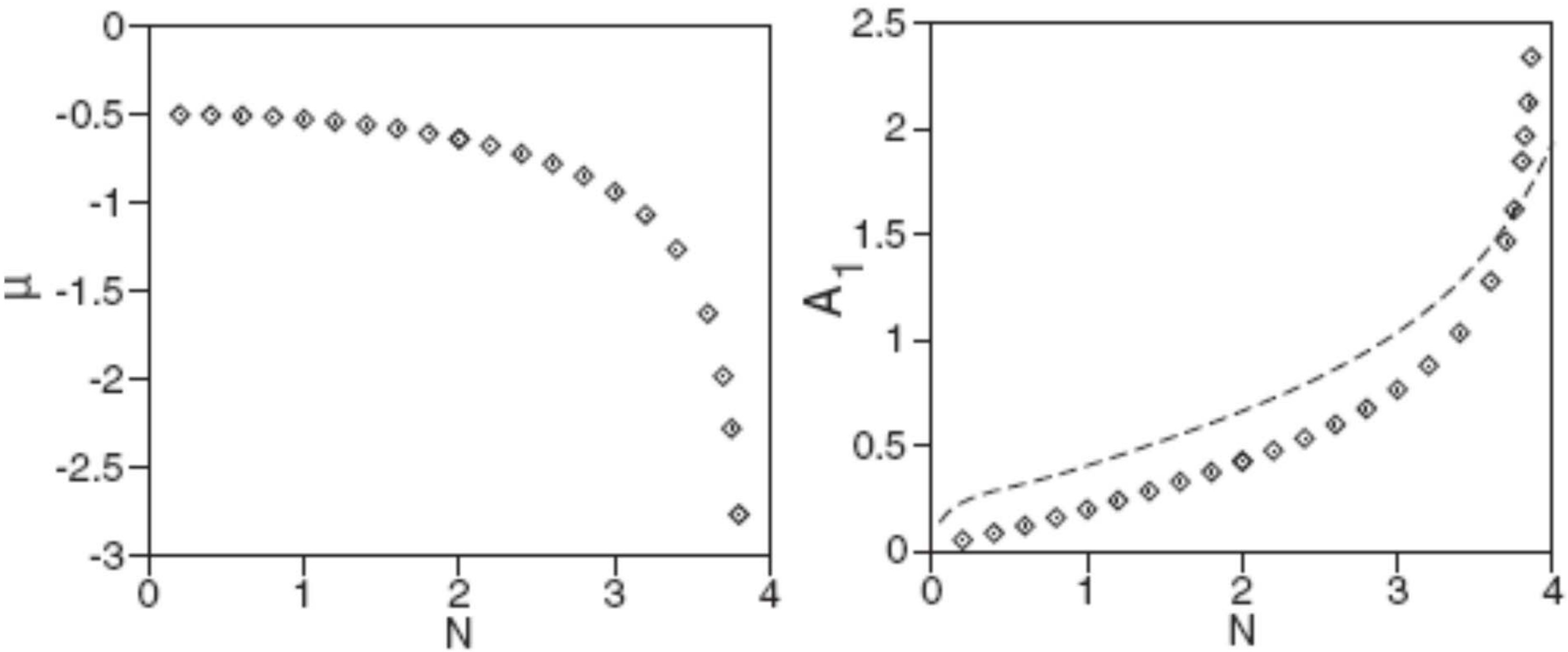
$$\phi_+ = A_1 \exp\left(-\alpha_1 r^2\right) - A_2 r \exp\left(-i\theta - \alpha_2 r^2\right),$$

$$\phi_- = A_1 \exp\left(-\alpha_1 r^2\right) + A_2 r \exp\left(+i\theta - \alpha_2 r^2\right).$$

A typical example of the cross-section of the mixed mode:



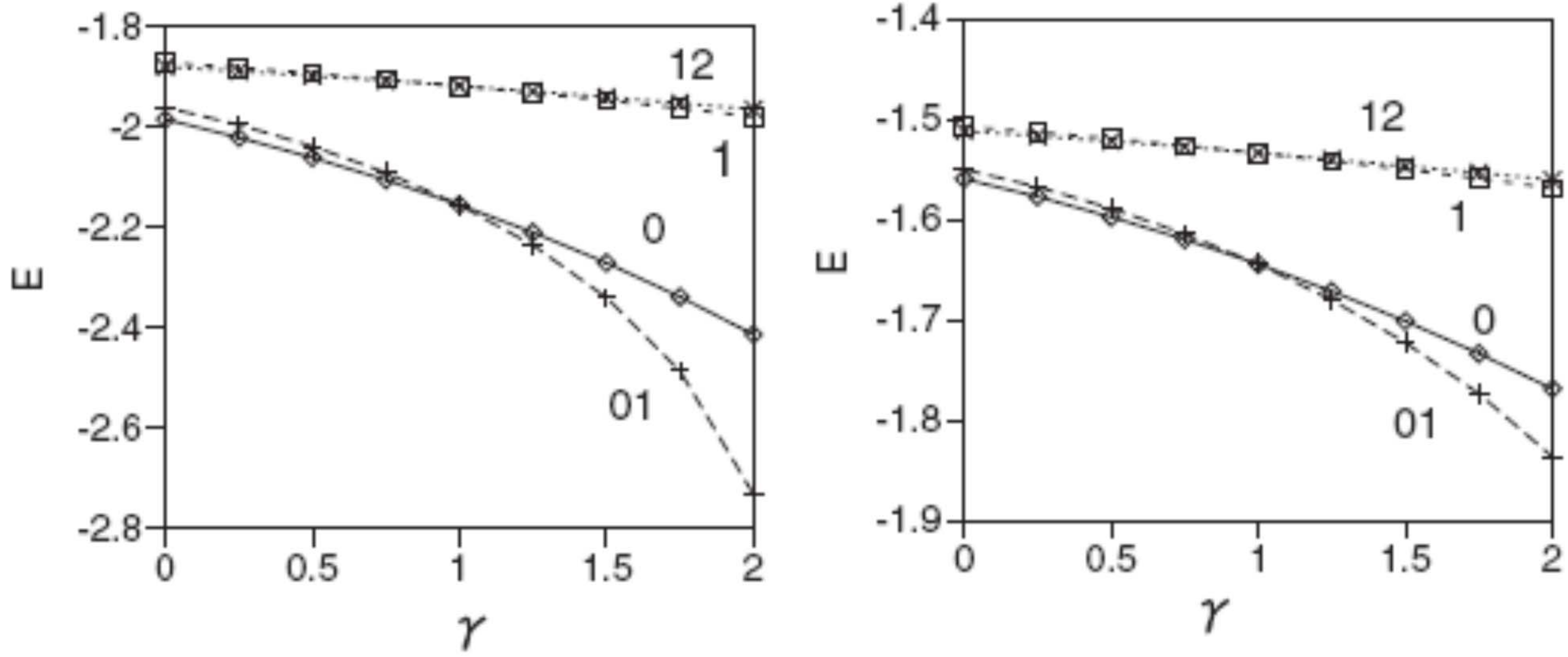
The dependence between the chemical potential and norm demonstrates that the **mixed-mode** family also complies with the **VK criterion**, hence it may be stable. In the limit of **$N \rightarrow N_{critical} \approx 2 \cdot 5.85/(1 + \gamma)$**, the mixed mode ***degenerates*** into a two-component ***Townes' soliton***, while the vortical terms in both components ***vanish***.



Direct simulations demonstrate that the mixed mode is ***unstable*** at ***γ* < 1**, and ***stable*** at ***γ* > 1**, i.e., exactly where the semi-vortex is ***stable*** or ***unstable***, respectively. This ***stability switch*** between the **semi-vortex** and **mixed mode** with the increase of ***γ* ≡ XPM/SPM** is explained by the fact that the **semi-vortex** and **mixed mode** are ***ground states***, which realize the ***minimum*** of the system's energy, ***E***, precisely at ***γ* < 1** and ***γ* > 1**, respectively.

$$E = \iint \left\{ \frac{1}{2}(|\nabla\phi_+|^2 + |\nabla\phi_-|^2) - \frac{1}{2}(|\phi_+|^4 + |\phi_-|^4) - \gamma|\phi_+|^2|\phi_-|^2 + \frac{\lambda}{2}\left[\phi_+^*\left(\frac{\partial\phi_-}{\partial x} - i\frac{\partial\phi_-}{\partial y}\right) + \phi_-^*\left(-\frac{\partial\phi_+}{\partial x} - i\frac{\partial\phi_+}{\partial y}\right)\right] + \text{c.c.}\right\} dx dy,$$

The dependence of the **energy** of the semi-vortex ("**0**") and mixed mode ("**01**") on ***γ* ≡ XPM/SPM,** for two different fixed values of the total norm, **N = 3.7**, and **N = 3.0** (precisely at ***γ* = 1**, **both** the semi-vortex and mixed mode are **stable**):



Conclusions of Part D

The main result is that the system of two **2D GPEs** with the self-attracting nonlinearity, coupled by the linear **SO** terms of the ***Rashba type***, gives rise to two families of ***composite*** (half-fundamental, half-vortical) solitons: **semi-vortices**, which are ***stable***, and realize the ***ground state*** of the system, at $\gamma \equiv$ **XPM/SPM < 1**, and **mixed modes**, which do the same at $\gamma > 1$.

This is the ***first example*** of a model in which **2D** solitons, supported by the ***cubic self-focusing*** in the free space, are ***stable***. This may be explained by the fact that the solitons exist with values of the total norm falling ***below the critical value*** necessary for the onset of the ***collapse***. In the limit of the norm approaching the critical value, the solitons degenerate into unstable ***Townes' solitons***.