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Fundamental nonlinear equations in physics and their fundamental solutions

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The structure of the talk

(1) Introduction

(2) The Gross-Pitaevskii (**GP**) / nonlinear Schrödinger (**NLS**) equations

(3) The discrete **NLS** equation

(4) The Korteweg – de Vries (**KdV**) equation

(5) Two-dimensional (**2D**) equations: Kadomtsev-Petviashvili (**KP**) of types **I** and **II**

(6) Dissipative models: complex Ginzburg-Landau (**CGL**) equations

(7) Conclusion

(8) Addition: some recent results

At the fundamental (quantum) level, physics is governed by ***linear*** equations. In particular, the Schrödinger equation for wave function **Ψ** of a quantum particle in the three-dimensional space is ***linear*** :

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi + U\left(x, y, z\right)\Psi.$$

The ***nonlinearity***, which is common in ***classical mechanics***, emerges from the Schrödinger equation in the ***semi-classical limit***, which formally corresponds to treating the Planck's constant, $\hbar$, as a small parameter. In this limit, the ***semi-classical approximation*** is used to seek for a solution for the wave function as

$$\Psi(x,y,z,t) = f(x,y,z,t)\exp\left(-\frac{iS(x,y,z,t)}{\hbar}\right),$$

where real function $S(x,y,z,t)$ actually is the *classical action*. In the lowest approximation with respect to $\hbar$, it follows from the substitution into the Schrödinger equation that real phase $S(x,y,z,t)$ obeys the nonlinear *Hamilton - Jacobi equation*:

$$\frac{\partial S}{\partial t} = \frac{1}{2m}(\nabla S)^2 + U(x,y,z).$$

This result was obatined at order $\hbar^0$.

At the next orer, $\hbar^1$, one can derive the equation for the real pre-exponential amplitude:

$$\frac{\partial f}{\partial t} = \frac{1}{m} \nabla S \cdot \nabla f \, .$$

On the other hand, ***effective nonlinearity*** is possible in ***many-body macroscopic quantum states***. An important example is provided by ***Bose-Einstein condensates*** (**BEC**). In this case, all boson atoms in a rarefied ultracold gas fall into a **ground state**, and are described by a **single-atom wave function**. However, the corresponding Schrödinger equation does not take into account ***collisions*** between atoms.

The collisions may be effectively taken into regard, in the ***mean-field approximation*** (which is a very accurate approach for **rarefied BEC** gases), by adding a ***cubic nonlinear local term*** to the respective linear Schrödinger equation. Thus one arrives at the ***Gross-Pitaevskii equation***, **GPE** (alias the ***nonlinear Schrödinger***, **NLS**, ***equation***), which was derived in 1961 in the context of the liquid-helium theory:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U\left(x,y,z\right)\Psi + \frac{4\pi\hbar^2}{m}a_s\,|\,\Psi\,|^2\,\Psi.$$

Here a_s is the *scattering length* which characterizes collisions between two atoms considered as *classical* particles.

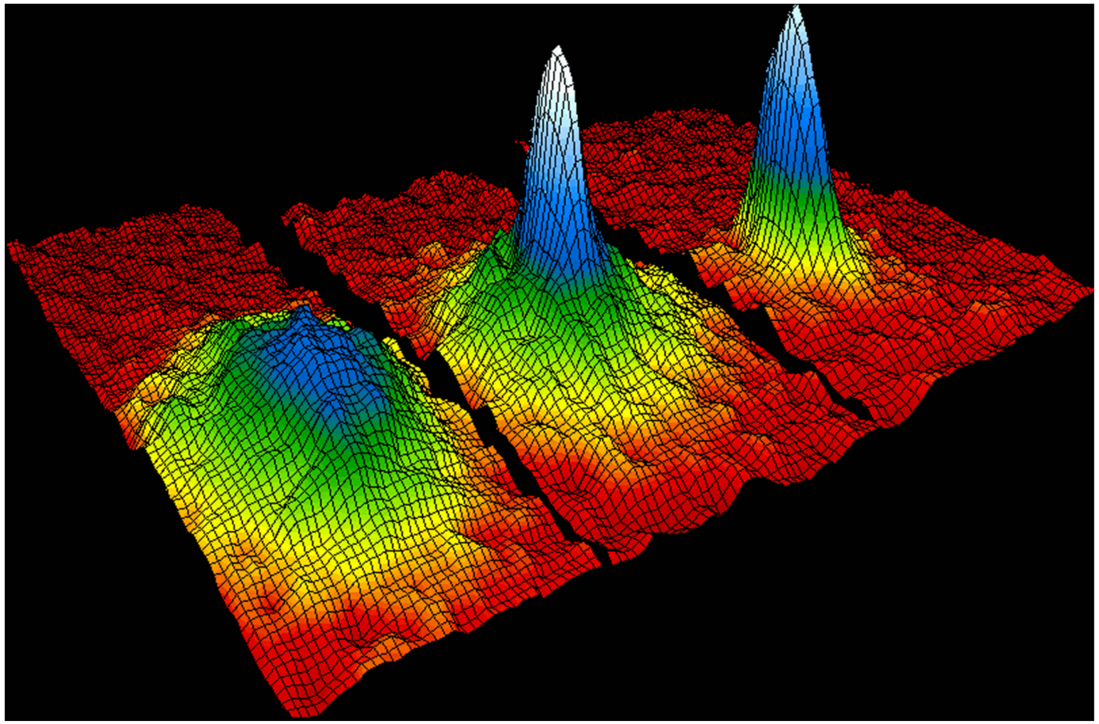
In this notation, the total number of atoms in the **BEC** is given by the ***norm*** of wave function, which is a ***dynamical invariant*** (conserved quantity) of the Gross-Pitaevskii equation:

$$N = \iiint |\Psi(x, y, z)|^2 \, dxdydz \, .$$

In most cases, interactions between atoms are ***repulsive***, which corresponds to $\boldsymbol{a_s} > 0$. However, the interaction may sometimes be ***attractive***, which corresponds to $\boldsymbol{a_s < 0}$. Accordingly, ***rescaling*** leads to ***two different NLS equations***, with the repulsive (+) and attractive (**-**) cubic nonlinearity, respectively:

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\nabla^2\Psi + U(x,y,z)\Psi \pm |\Psi|^2\Psi.$$

BEC as a ***quantum state of matter*** was predicted by Bose and Einstein in 1924-1925, and created experimentally in a vapor of atoms of **Rb-87** in 1995 (by the group of E. Cornell and C. Wieman), at temperature $T = 1.7 \times 10^{-7}$ K:



The inter-atomic interactions may be switched from repulsion to attraction by means of the ***Feshbach resonance***, in an external dc magnetic field (predicted in 1958) – in particular, in condensates of **Li-7** and **Rb-85** atoms. The condensate can be created in ***reduced*** one- and two-dimensional (**1D** and **2D**) geometries, using ***trapping fields*** which act in the transverse direction(s). The corresponding **1D NLS** equations are, in the scaled form:

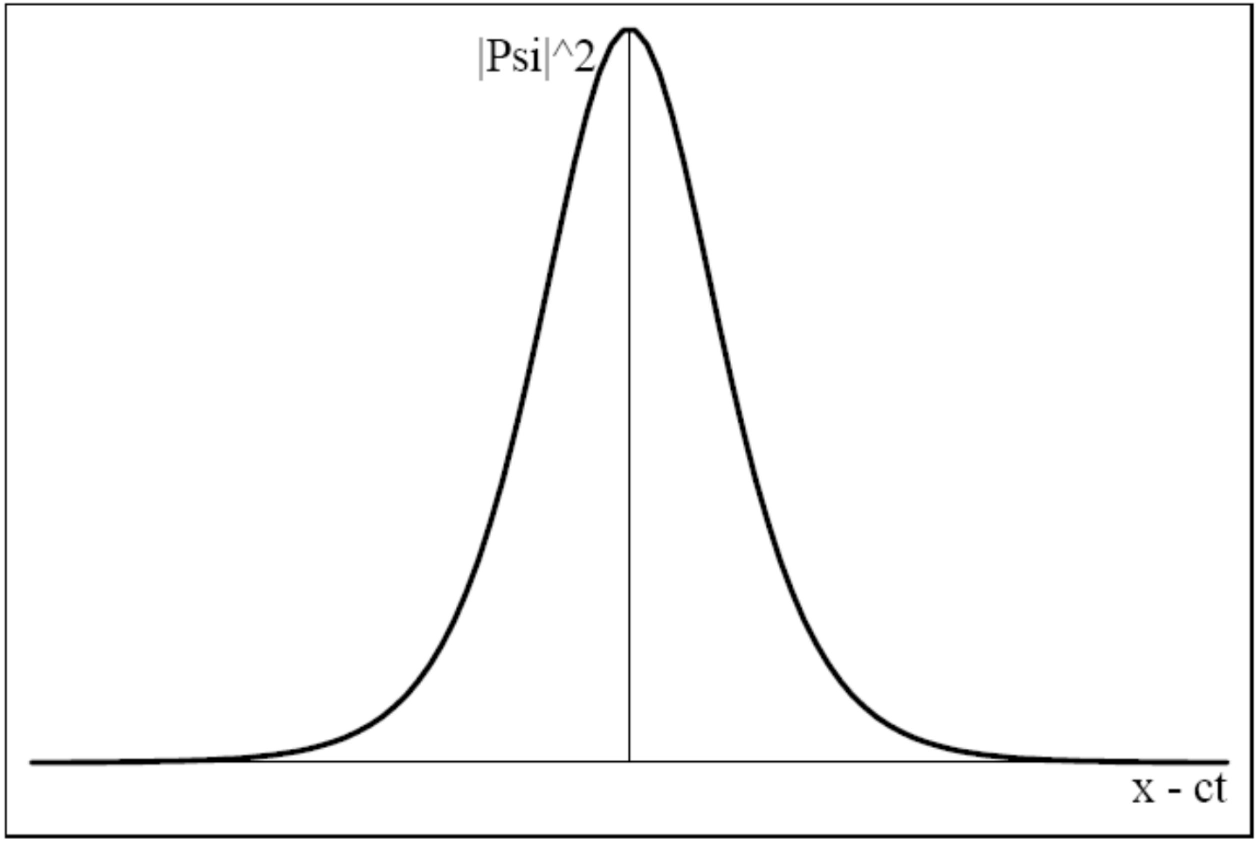
$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi \pm |\Psi|^2\Psi.$$

In the free space (no external potential, ***U* = 0**), the **1D NLS** equation with the ***attractive*** cubic term gives rise to a family of ***stable*** elementary solutions in the form of ***solitary waves***, alias ***fundamental solitons***. The soliton family depends on two arbitrary parameters - amplitude ***η*** and velocity ***c***:

$$\Psi(x,t) = e^{i\left(\eta^2 - c^2\right)t/2} \frac{\eta}{\cosh(\eta(x-ct))};$$

reminder: $\cosh z \equiv (1/2)\left(e^{z} + e^{-z}\right).$

The localized shape of the ***fundamental soliton*** [$\mathbf{1/\cosh^2(\mathit{x} - \mathit{ct})}$]:



In experiments with **BEC** loaded into nearly one-dimensional ("cigar-shaped") trapping potentials, ***matter-wave solitons*** (single ones and chains of several solitons) have been created in **Li-7**:

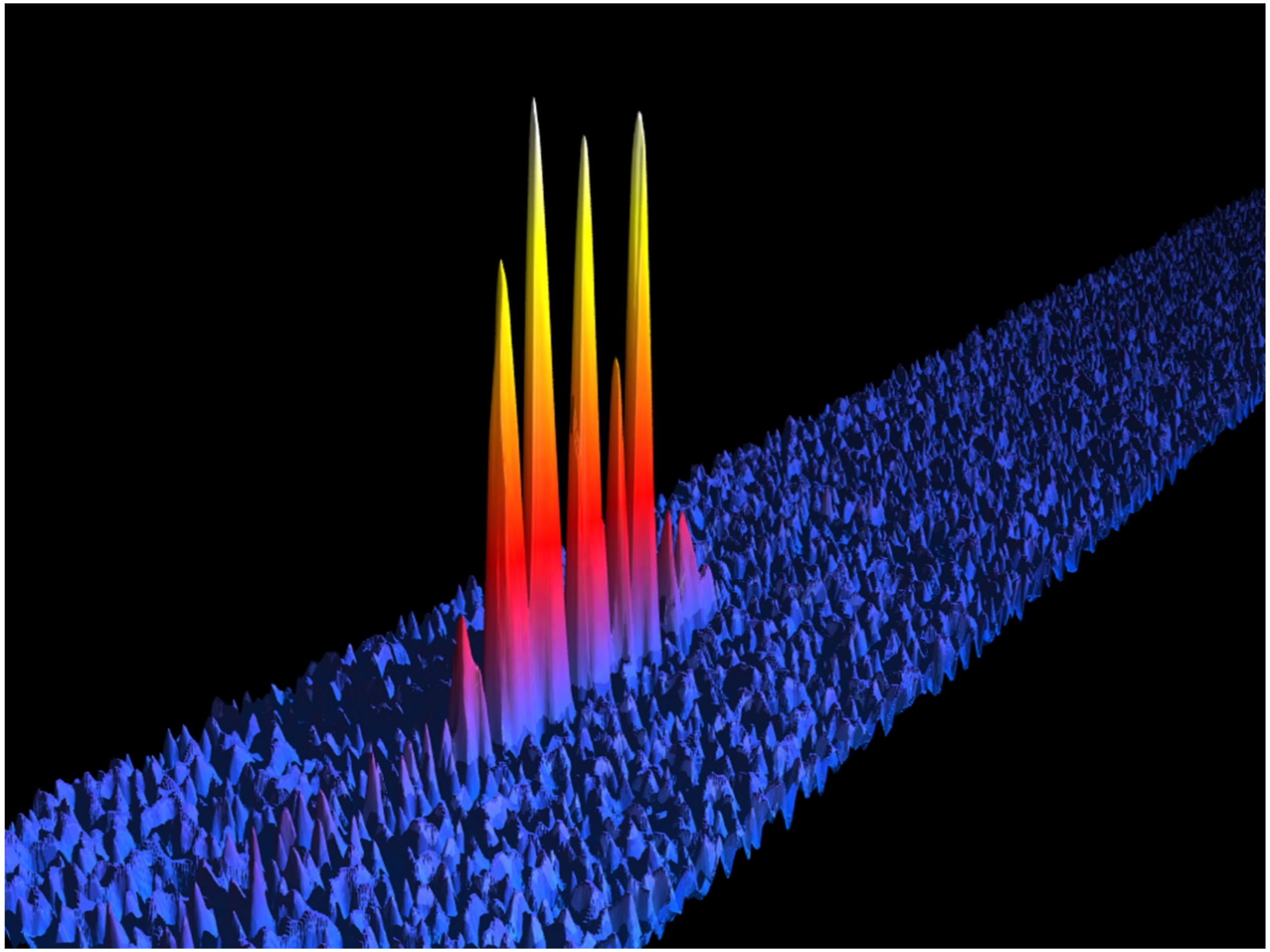
K.E. Strecker, G.B. Partridge, A.G. Truscott, and R. G. Hulet, Nature **417**, 150 (2002);

L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L.D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).

Then, solitons in a less anisotropic trap (with aspect ratio **2.5**) were created in **Rb-85**:

S.L. Cornish, S.T. Thompson, and C.E. Wieman, Phys. Rev. Lett. **96**, 170401 (2006).

The famous experimental picture of the density distribution in a ***chain of 7 solitons*** in **Li-7** (produced by the group of *R. Hulet*):



Further experimental results for bright matter-wave solitons

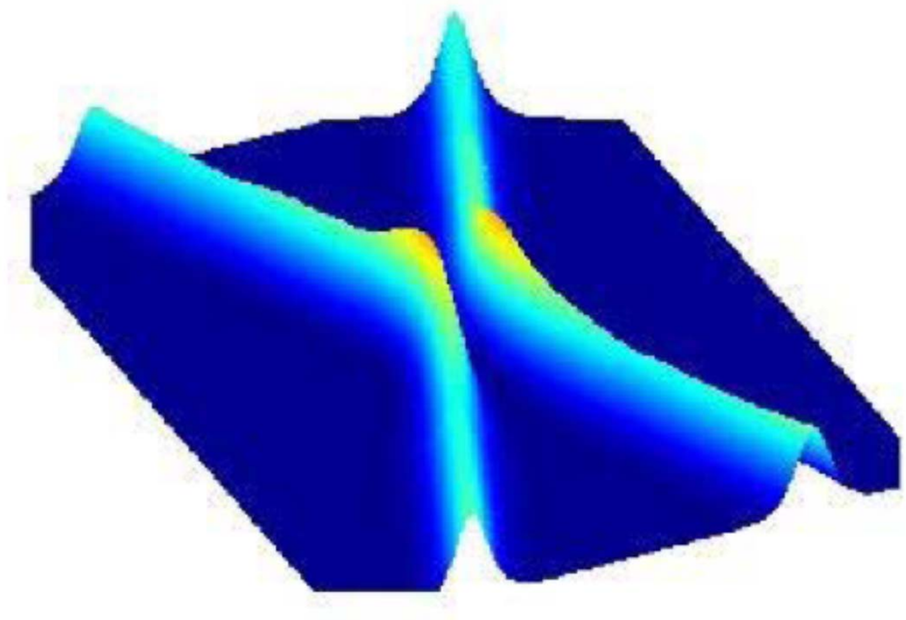
Controlled formation and ***reflection*** of a soliton from a potential barrier in **Rb-85** (the soliton traveled a **macroscopic** distance 1.1 mm in the course of 150 ms):

A.L. Marchant, T.P. Billam, T.P. Wiles, M.M.H. Yu, S.A. Gardiner, and S.L. Cornish, Nature Comm. **4**, 1865 (2013).

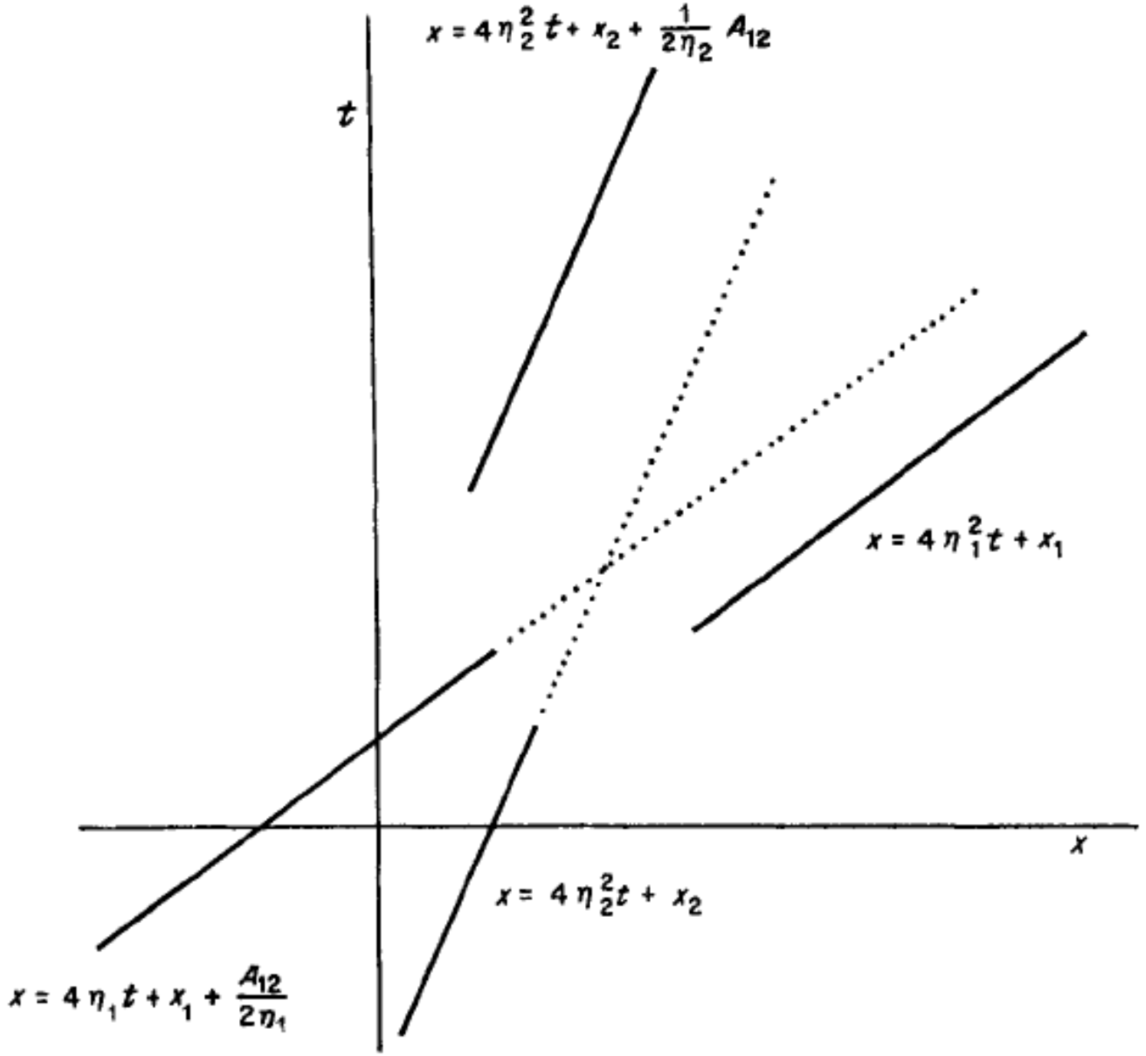
Creation of single and paired solitons in a magnetic waveguide by means of the evaporative technique in **Li-7** (the Stanford-University group):

P. Medley, M. A. Minar, N. C. Cizek, D. Berryrieser, and M. A. Kasevich, Phys. Rev. Lett. **112**, 060401 (2014).

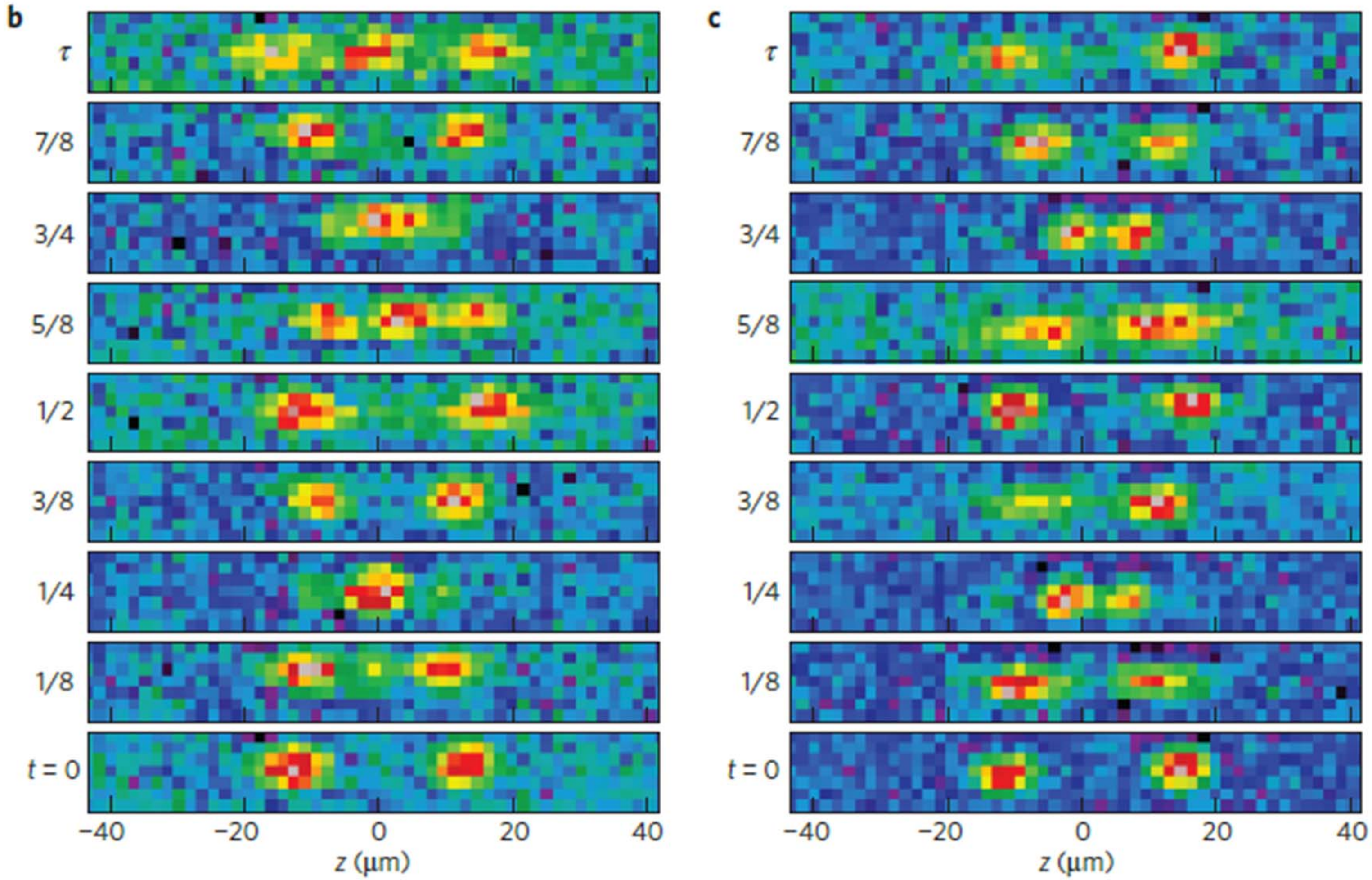
In 1971, ***Zakharov*** and ***Shabat*** had discovered that the **1D NLS** equations (with the ***attractive*** and ***repulsive*** nonlinearities ***alike***) are ***integrable*** equations. The ***integrability*** is revealed by a mathematical technique called "***inverse scattering transform***". In particular, collisions between moving solitons are ***completely elastic***, i.e., they reappear after the collisions, either ***bouncing back from***, or ***passing through*** each other, with ***precisely the same*** shapes, amplitudes, and velocities as they had before the collision:



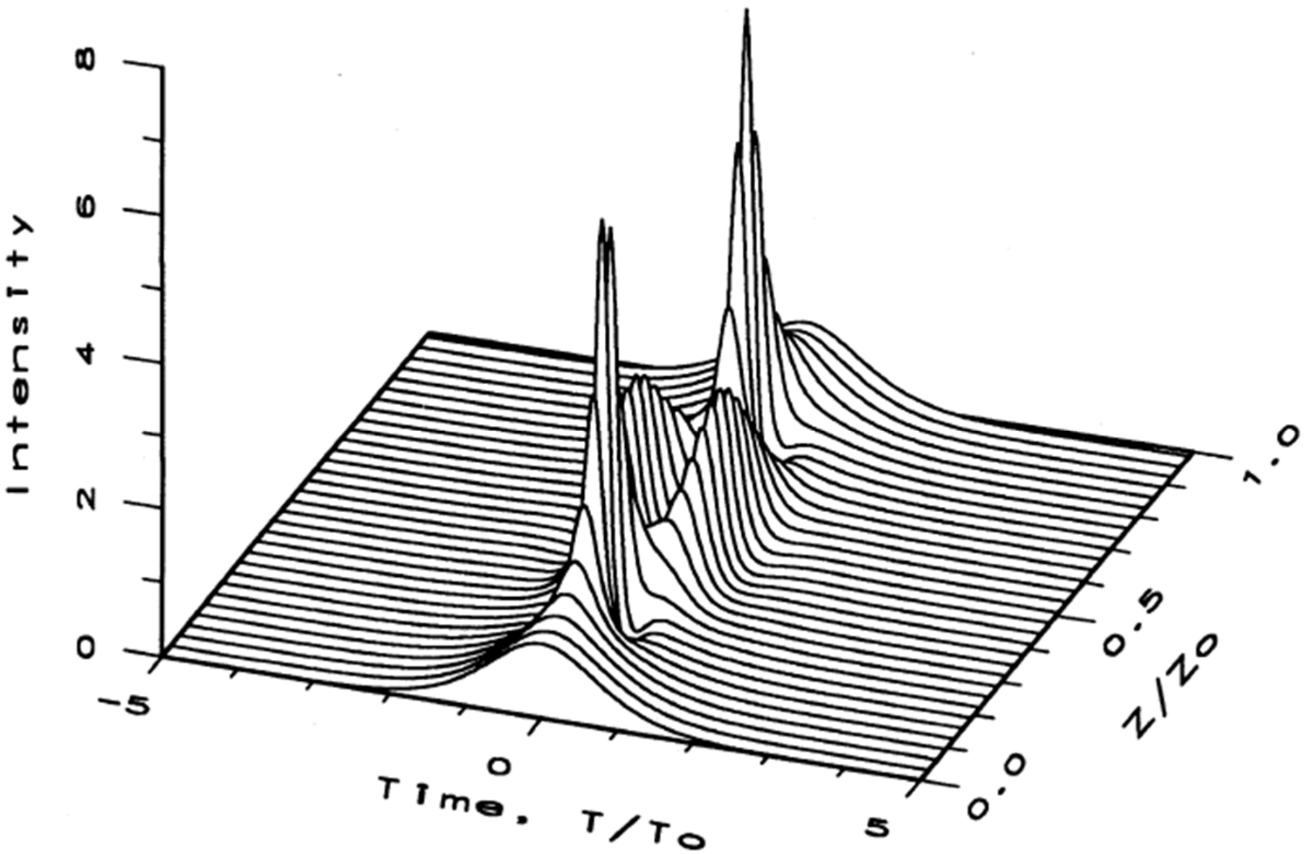
The only dynamical effect of the collision is a ***shift*** of trajectories of both solitons, without any change in their shapes:



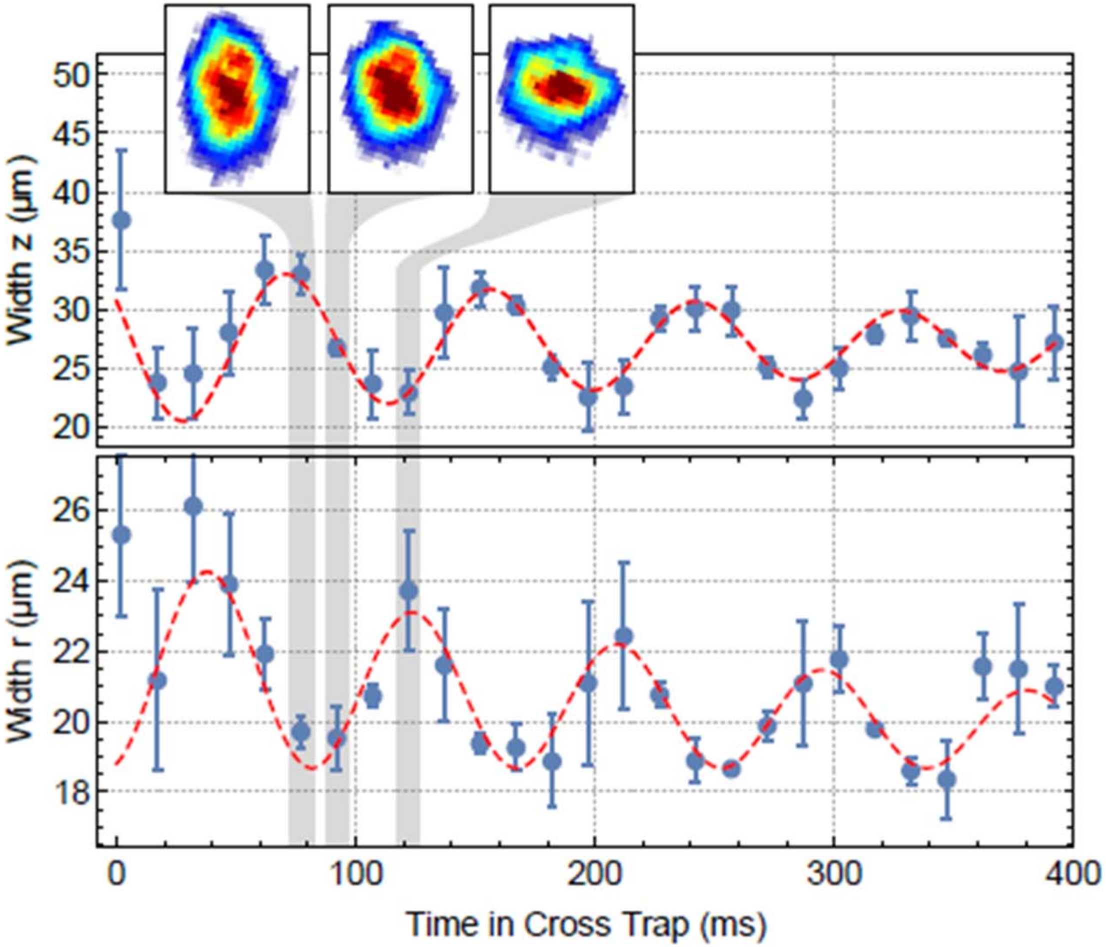
Experimentally, collisions of **nearly-1D** matter-wave ***quasi-solitons*** were studied in detail only recently, in **Li-7**:
J. H. V. Nguyen, P. Dyke, D. Luo, B. A. Malomed, and R. G. Hulet, Nature Phys. **10**, 918-922 (2014).
Images of **in-phase** (left) and **out-of-phase** (right) collisions:



In addition to the fundamental solitons, the ***inverse-scattering*** technique allows one to find ***exact analytical solutions*** for ***higher-order solitons***, generated by initial conditions $\boldsymbol{\Psi(x,t=0) = N\eta\,\mathrm{sech}(\eta x)}$, with ***integer*** $\boldsymbol{N}$. An example: the ***third-order soliton*** ($\boldsymbol{N} = \mathbf{3}$), which oscillates, periodically splitting and recombining back into a single peak:



An attempt of observation of ***breathers*** (bright solitons featuring regular oscillations) in **Rb-85**:
P.J. Everitt, M.A. Sooriyabandara, G.D. McDonald1 K.S. Hardman, C.Quinlivan, P.Manju, P.Wigley, J.E. Debs, J.D. Close, C.C.N. Kuhn, and N.P. Robins, **arXiv:1509.06844**.



The **NLS** equations also emerge as universal models of nonlinear-wave propagation in numerous ***classical*** settings. A famous example is provided by solitons in ***nonlinear optical fibers***. The **real** electric field in the electromagnetic wave (with fixed polarization **e**), coupled into the fiber, is approximated by the product of the rapidly oscillating ***carrier wave***, $\mathbf{exp}(ikz - i\omega t)$, ***transverse eigenmode***, $f((x^2+y^2)^{1/2})$, and a ***slowly varying complex envelope***, $U(z,\tau)$:

$$\mathbf{E}(x,y,z,t) = \exp\left(ikz - i\omega t\right)\mathbf{e}\, f\left(\sqrt{x^2+y^2}\right)U(z,\tau) + \mathrm{c.c.},$$

$$\tau \equiv t - z/V_{\mathrm{gr}}\,.$$

The substitution of this *ansatz* in the Maxwell's equations leads, after the separation of rapidly and slowly varying functions of *z* and *τ*, to the **NLS** equation in the following scaled form:

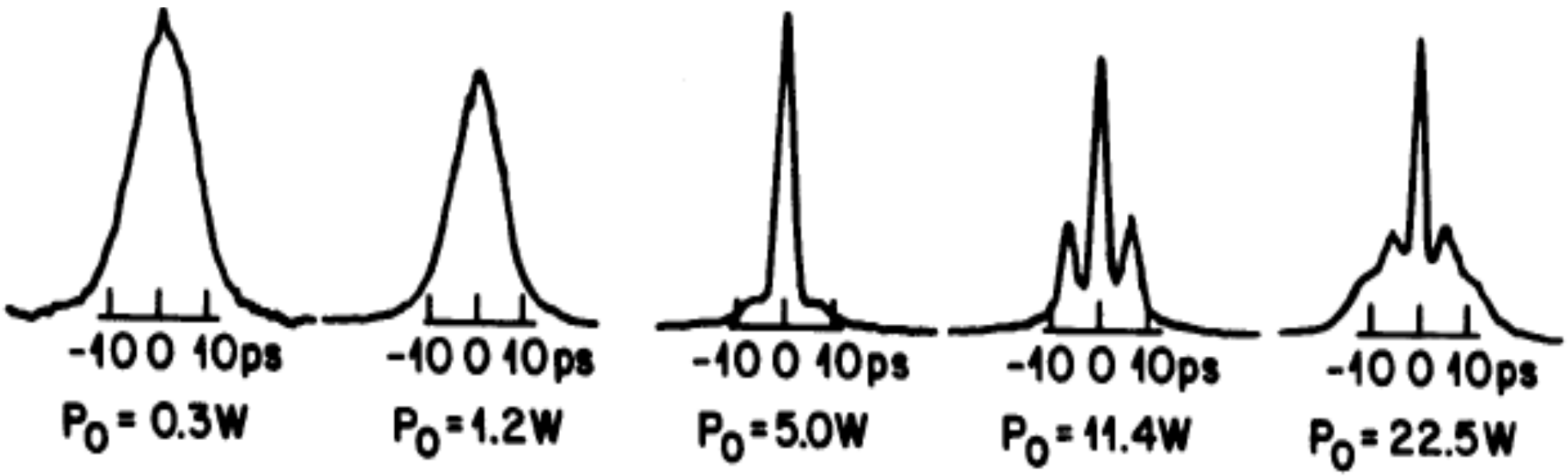
$$i\frac{\partial U}{\partial z} \pm \frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} + |U|^2 U = 0,$$

with + and - corresponding to the *anomalous* and *normal* group-velocity dispersion (alias chromatic dispersion) in the fiber.

Thus, **bright solitons** may be expected in nonlinear fibers featuring the *anomalous* dispersion.

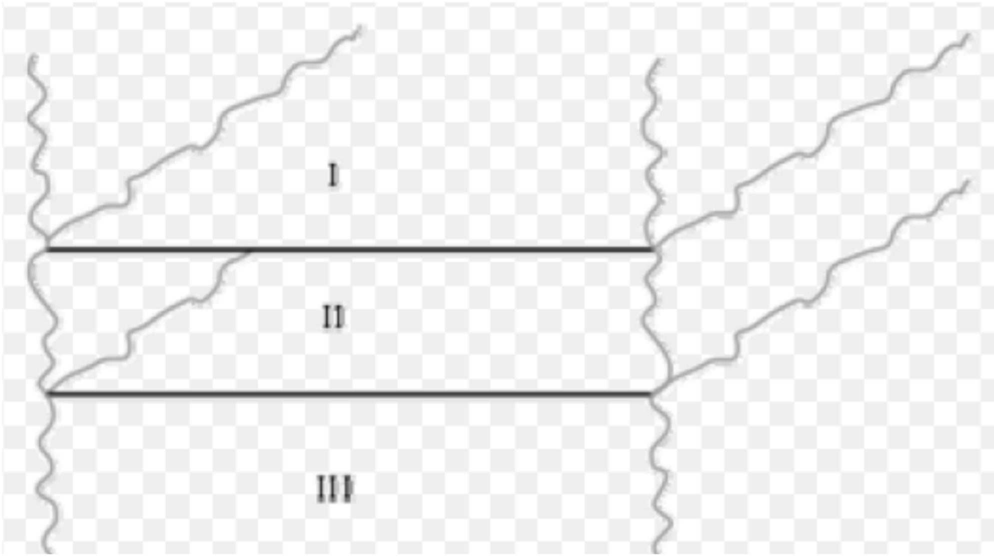
These ***temporal solitons*** in optical fibers were predicted by ***Hasegawa*** and ***Tappert*** in 1973, and experimentally created by ***Mollenauer, Stolen*** and ***Gordon*** in 1980.

The observed ***self-trapping*** of an input pulse into a ***fundamental*** or ***higher-order soliton*** (***breather***) with the increase of the peak power:



Standard telecommunications fibers feature ***anomalous dispersion***, hence they can carry ***soliton streams***, which may be used to transmit data in fiber-optical telecom networks. The bit-rate of up to **100 GB/s per channel** can be easily achieved, using currently available soliton technologies. The ***single*** so far built soliton-based ***commercial*** telecom link, about **3,000 km long**, was installed in Australia (between Adelaide and Perth) in 2003.

Another possibility to realize the effectively **1D *NLS equation*** in nonlinear optics, and create solitons in the ***spatial domain***, is offered by the light transmission in a planar waveguide (a ***thin slab*** **II**), placed between materials (***claddings***) with a lower refractive index, **I** and **III**):



The ***ansatz*** for the ***monochromatic*** electromagnetic field (with the single value of the frequency) and fixed polarization **e** in this case is

$$\mathbf{E}(x,y,z,t) = \exp\left(ikz - i\omega t\right)\mathbf{e}\, f\left(y\right)U(x,z) + \mathrm{c.c.}$$

Substituting this in the Maxwell's equations and assuming that $U(x,z)$ is a slowly varying function in comparison with $\exp\left(ikz\right)$ (the *paraxial approximation*), one arrives at the **NLS** equation in the *spatial domain*:

$$iU_z + \frac{1}{2}U_{xx} + |U|^2 U = 0.$$

Thus, the *spatial diffraction* is formally equivalent (as concerns the sign in front of the term) to the *anomalous* group-velocity dispersion in the temporal-domain **NLS** equation, hence **bright solitons** should be expected in this case too.

In a similar way, one can consider the transmission of an **optical beam** in the ***bulk*** (**3D**) nonlinear optical medium. In this case, the ***ansatz*** for the monochromatic wave is

$$\mathbf{E}(x,y,z,t) = \exp\left(ikz - i\omega t\right)\mathbf{e}U(x,y,z).$$

The substitution into the Maxwell's equations and assuming that $U(x,y,z)$ is a slowly varying function in comparison with $\exp(ikz)$, in the 2D paraxial approximation, one arrives at the **two-dimensional NLS** equation (again, in the **spatial domain**):

$$iU_z + \frac{1}{2}\left(U_{xx} + U_{yy}\right) + |U|^2 U = 0.$$

Unlike its 1D counterpart, this equation is *not* integrable. It gives rise to **2D** *axially symmetric* solitons, in the form of

$U(z,x,y) = e^{iqz}V(r),\ r \equiv \sqrt{x^2+y^2}$ (*Townes' solitons*),

but they *all* are *unstable* against the *collapse* (formation of a singularity in the solution after a *finite* propagation distance).

The **2D NLS** equation also gives rise to solitons with ***embedded vorticity***, in the form of

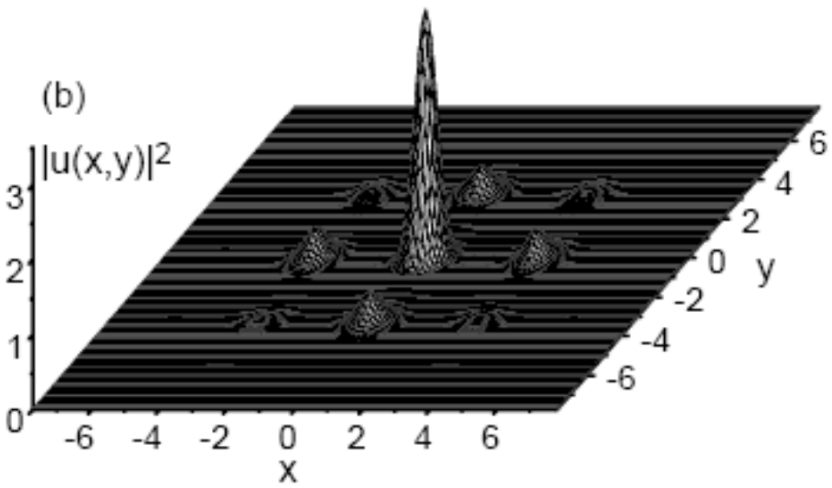
$$U(z,x,y)=e^{iS\theta}f(r),$$

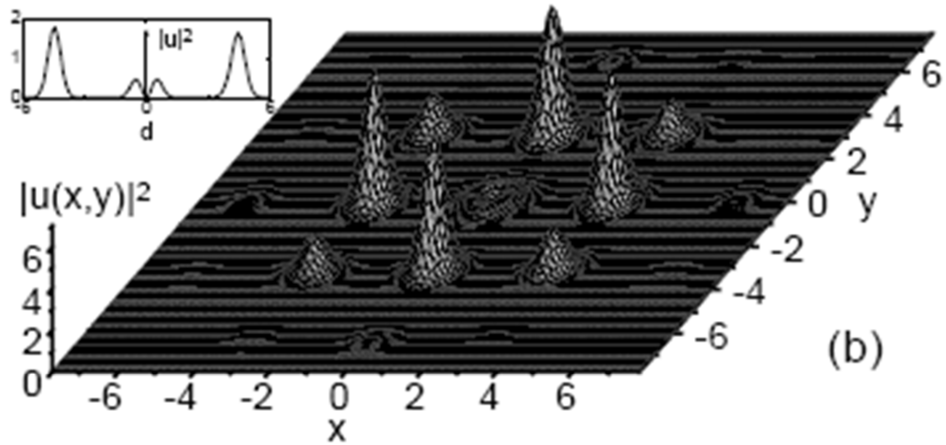
where $S=\pm1,\pm2,\pm3,\ldots$ is the vorticity (topological charge of the vortex), r and θ are the polar coordinates in the (x,y) plane, and at $r\to0$ the amplitude must decay as $r^{|S|}$. However, the *vortical solitons* are still more unstable than the fundamental ones, as, prior to the onset of the collapse, the vortices *split* into fragments, which actually are fundamental solitons (subsequently, they collapse by themselves).

Both the fundamental and vortical solitons can be *stabilized*, against the collapse and splitting alike, by means of a transverse *grating* (an effective periodic potential), with the **2D NLS** equation taking the accordingly modified form:

$$iU_z+\frac{1}{2}\left(U_{xx}+U_{yy}\right)-\varepsilon\left[\cos(2kx)+\cos(2ky)\right]U+|U|^2U=0.$$

Examples of ***fundamental*** (***S* = 0**) and ***vortical*** (***S* = 1**) **2D** solitons ***stabilized*** by the grating (the same mechanism was predicted to stabilize **2D** ***matter-wave solitons*** and ***vortices*** in **BEC**, where the periodic potential is readily provided by the ***optical lattice***), as per the paper: B.B. Baizakov, B.A. Malomed, and M. Salerno, *Multidimensional solitons in periodic potentials*. Europhys. Lett. **63**, 642 (2003) [see also J. Yang, Z.H. Musslimani, Opt. Lett. **28**, 2094 (2003)]:





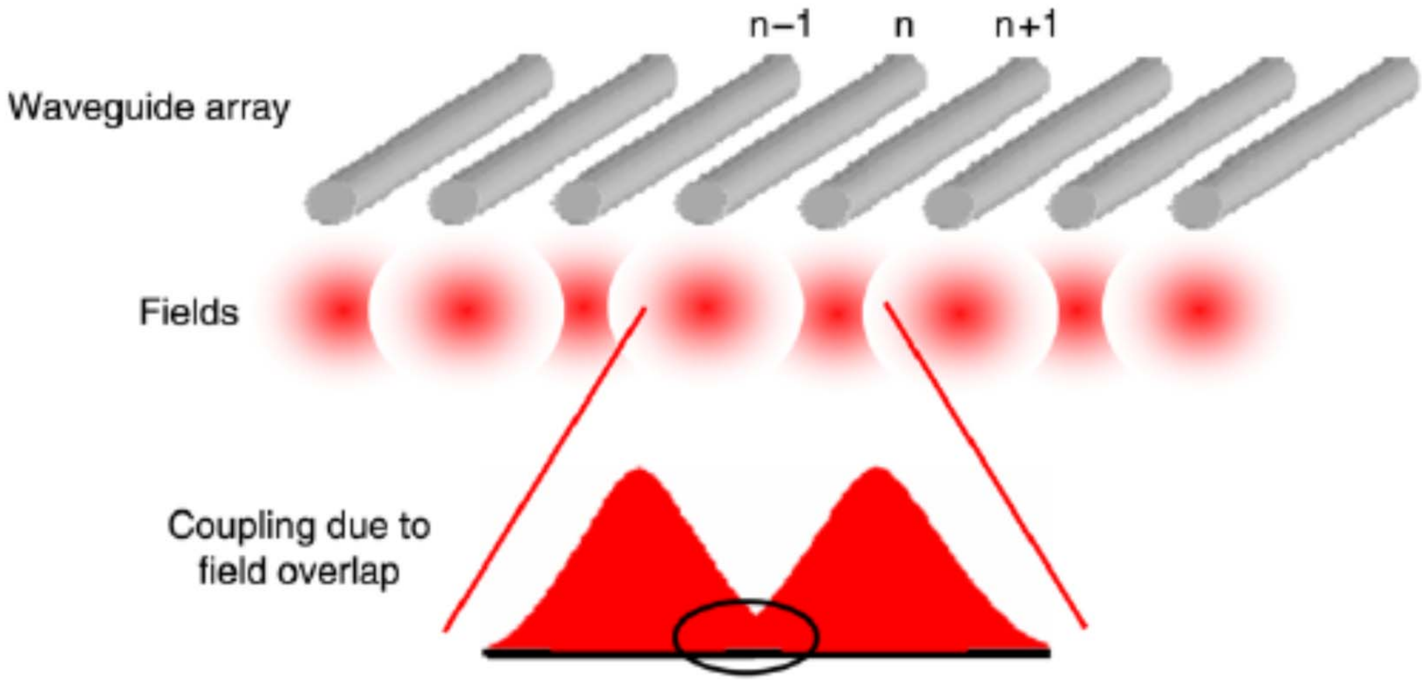
Experimentally, effectively **1D spatial solitons** in ***planar waveguides*** made of materials with the Kerr (cubic self-focusing) nonlinearity were first created in a liquid medium:

S. Maneuf, R. Dassailly, and C. Froehly, Opt. Commun. **65**, 193 (1988),

and then in a planar silica-glass waveguide:

J. S. Aitchison, A. M. Weiner, Y. Silberberg, M. K. Oliver, J. L. Jackel, D. E. Leaird, E. M. Vogel, and P. W. E. Smith, Opt. Lett. **15**, 471 (1993).

Another physically important generalization of the **NLS** equation is its ***discrete*** version. It directly describes, in particular, ***arrays*** of weakly coupled nonlinear optical waveguides, as well as **BEC** ***fragmented*** in a ***deep optical-lattice potential***:



The discrete NLS equation

In one dimension it is

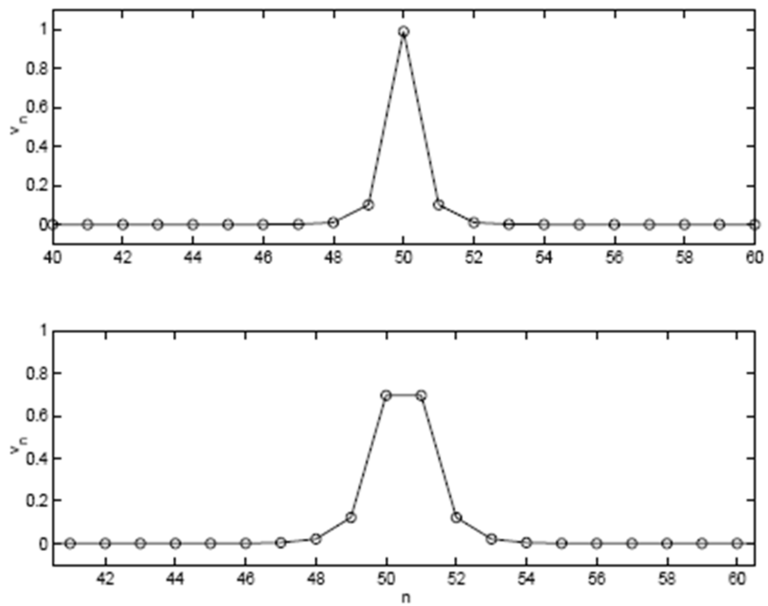
$$i\frac{du_n}{dz}+\frac{1}{2}\left(u_{n+1}+u_{n-1}-2u_n\right)+|u_n|^2 u_n=0,$$

and in two dimensions

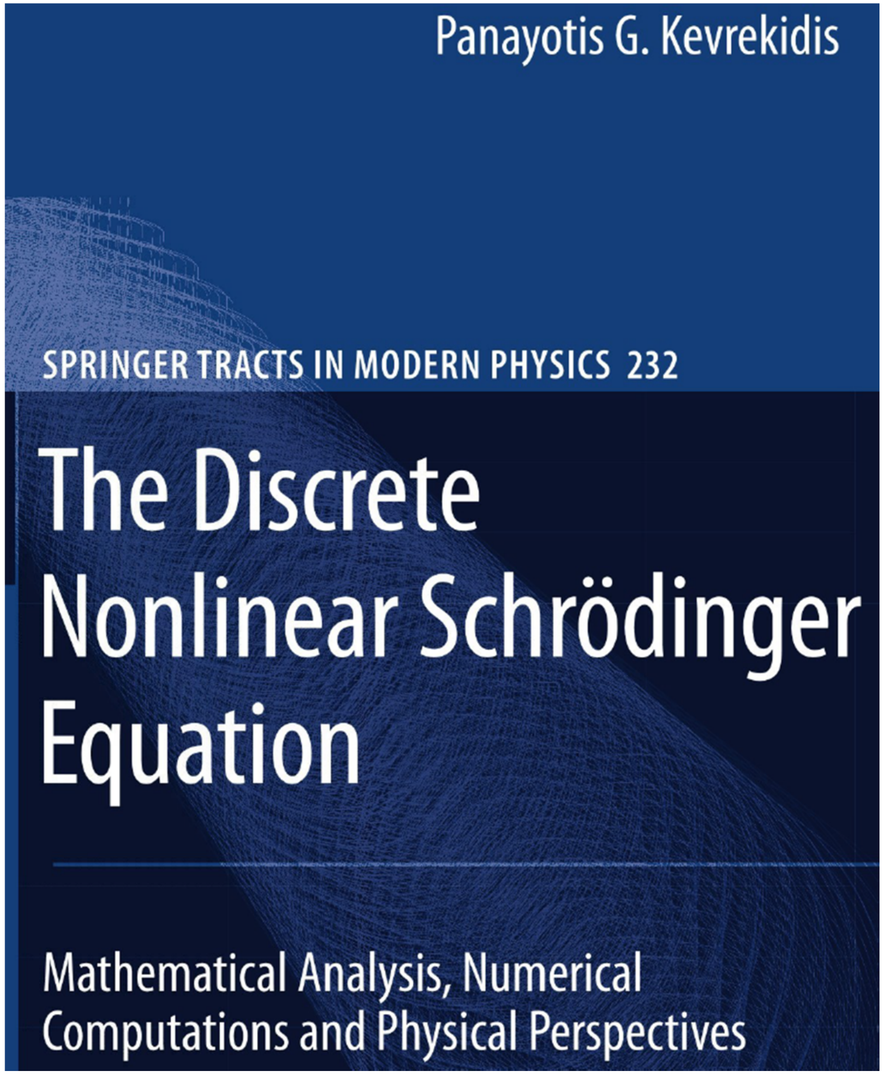
$$i\frac{du_{m,n}}{dz}+\frac{1}{2}\left(u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}\right)+|u_{m,n}|^2 u_{m,n}=0.$$

Typical examples of *discrete solitons* generated by the **1D** equation:

onsite-centered (stable) and intersite-centered (unstable):



Many results for discrete **NLS** equations and respective discrete solitons were collected in a book (Springer, 2009):



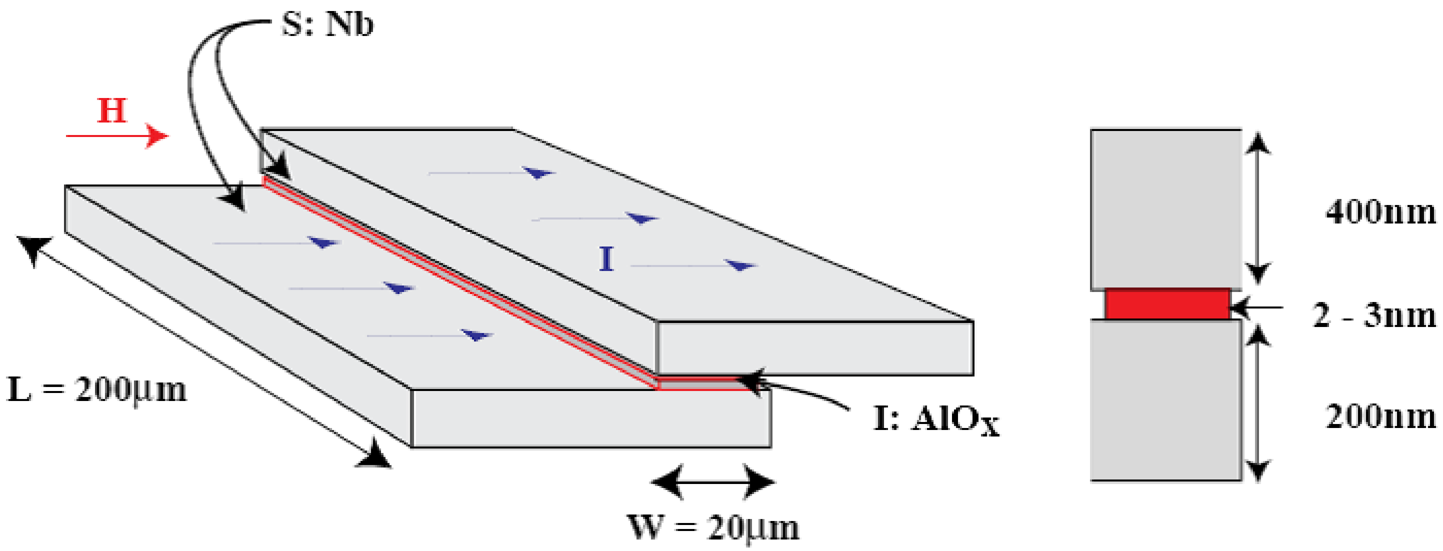
Other fundamental nonlinear-wave equations of modern physics.

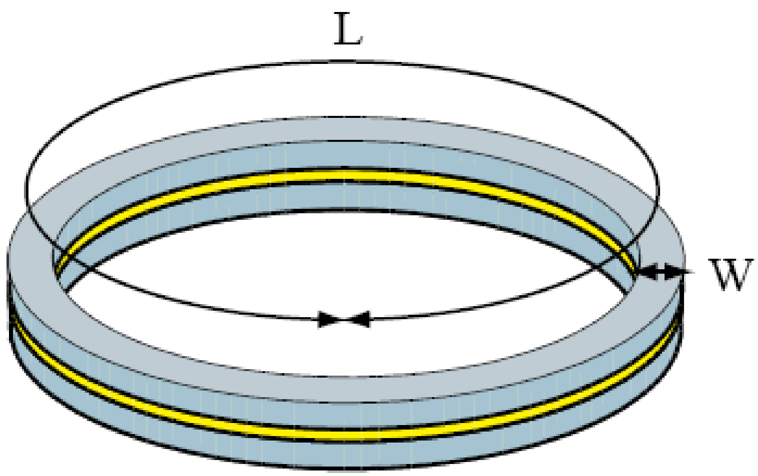
The ***sine-Gordon*** (**SG**) equation:

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\sin\phi=0.$$

Its important realization in physics is provided by *long Josephson junctions* (two bulk superconductors separated by a narrow layer of a dielectric material), where ϕ is the jump of the phase of the wave function of superconducting electrons across the junction. This equation is **integrable** too (actually, its integrability was known since *the nineteenth century*, in terms of the *Bäcklund transformation*). It gives rise to solitary waves of two types: stationary topological solitons (*kinks* and *antikins*), and localized oscillatory solutions (*breathers*), that may be considered as kink-antikink bound states.

Typical experimentally implemented structures of ***linear*** and ***circular*** long Josephson junctions:





The **topological charge** of kinks and antikinks is represented by the difference of their fields at $\boldsymbol{x = \pm\infty}$, normalized to $\boldsymbol{2\pi}$:

$$[\phi(x=+\infty)-\phi(x=-\infty)]/(2\pi)=\pm 1.$$

Explicit analytical solutions for kinks and antikinks:

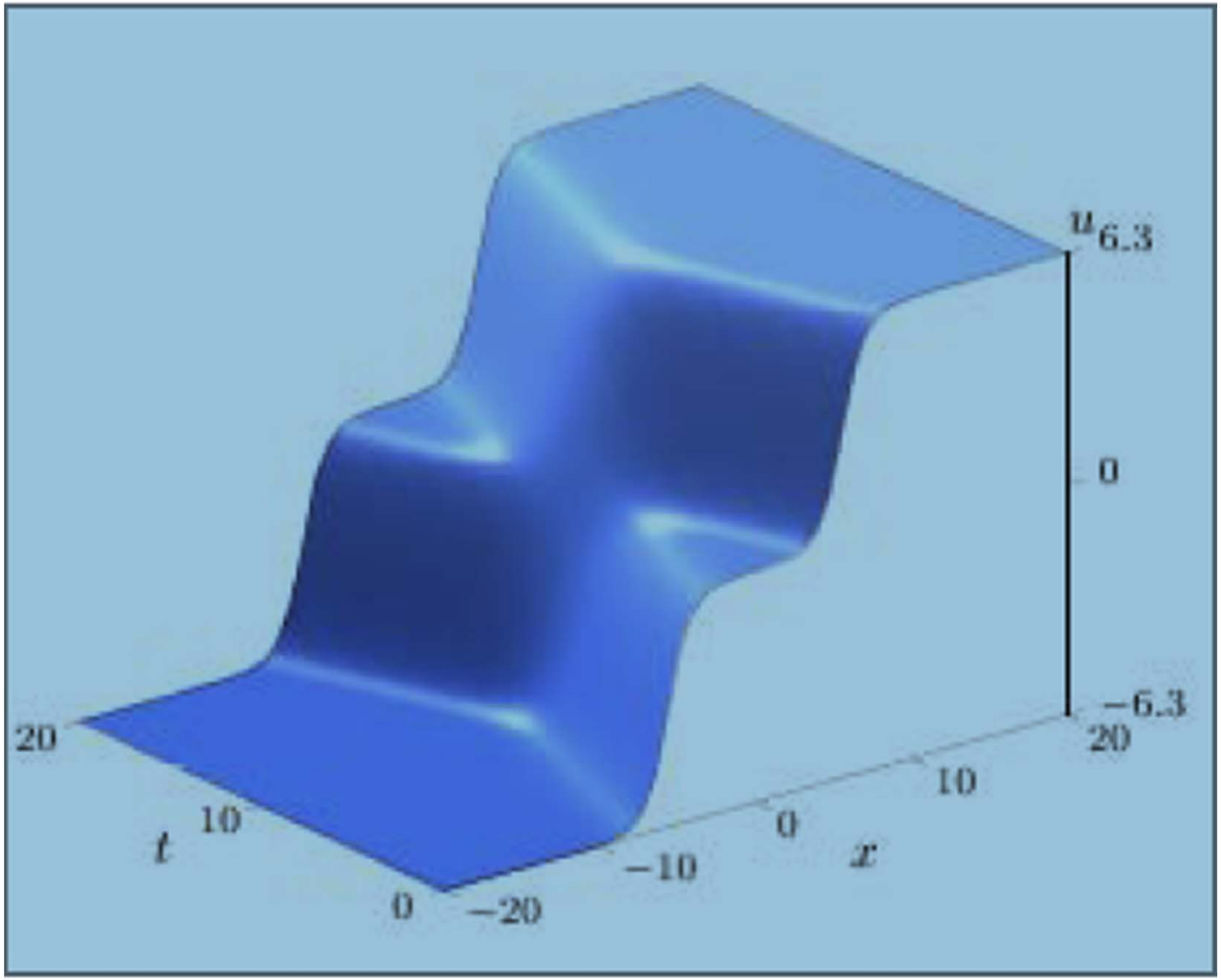
$$\phi(x,t)=4\arctan\left(\exp\left(\sigma\frac{x-ct}{\sqrt{1-c^2}}\right)\right),$$

where c is the velocity, which may take values

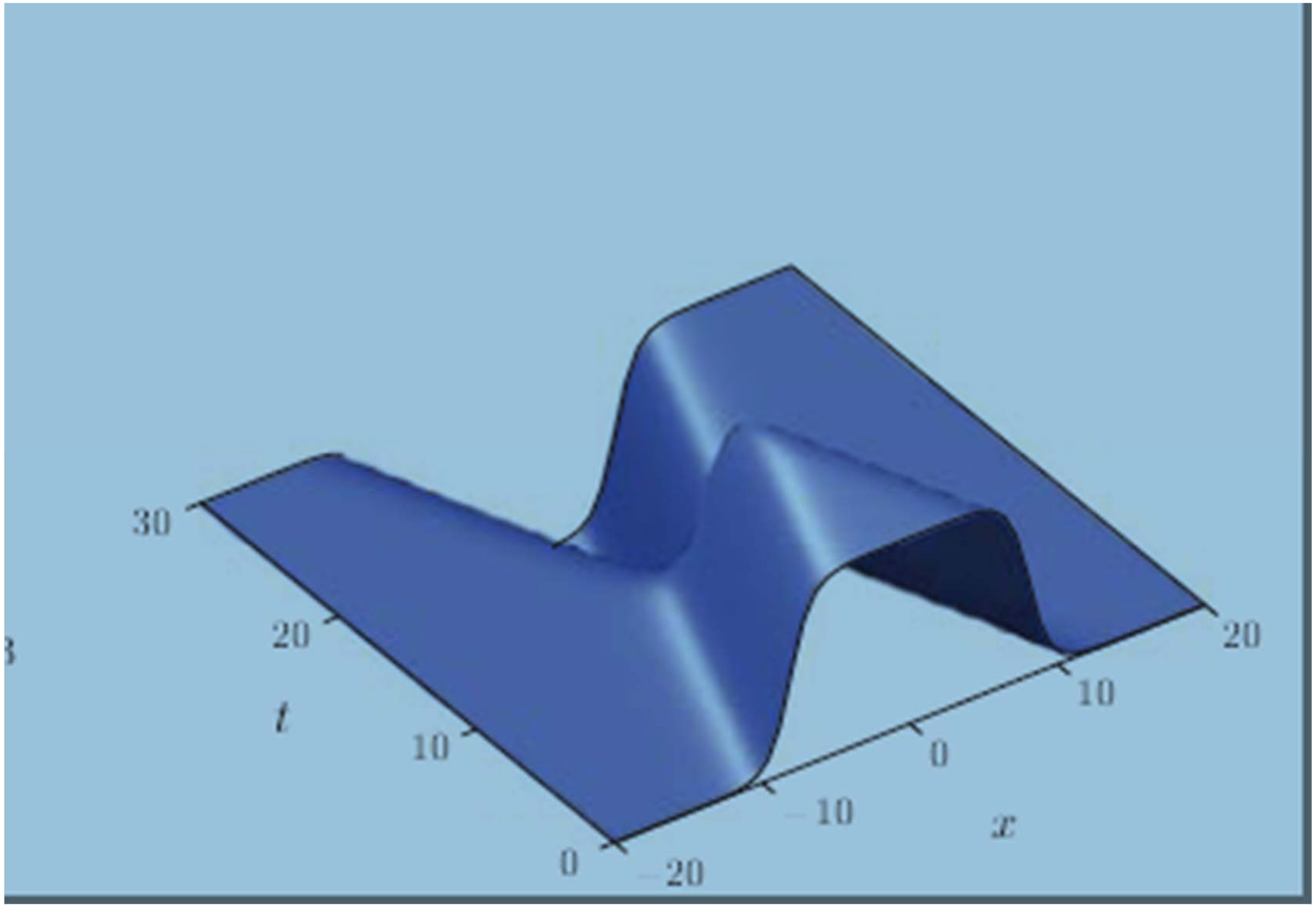
-1 $< c <$ $+1$, and $\sigma=\pm 1$ is the topological charge.

Like in other integrable equations, collisions between kinks and (anti)kinks are completely elastic (they are represented by rather cumbersome but exact analytical solutions).

The collisions of a two sine-Gordon kinks (in fact, they ***bounce back*** from each other):



The elastic collision between a kink and an antikink, which ***pass*** through each other:



The exact analytical solution for a sine-Gordon breather:

$$\phi(x,t) = 4\arctan\left[\frac{(\tan\mu)\sin((\cos\mu)t)}{\cosh((\sin\mu)x)}\right],$$

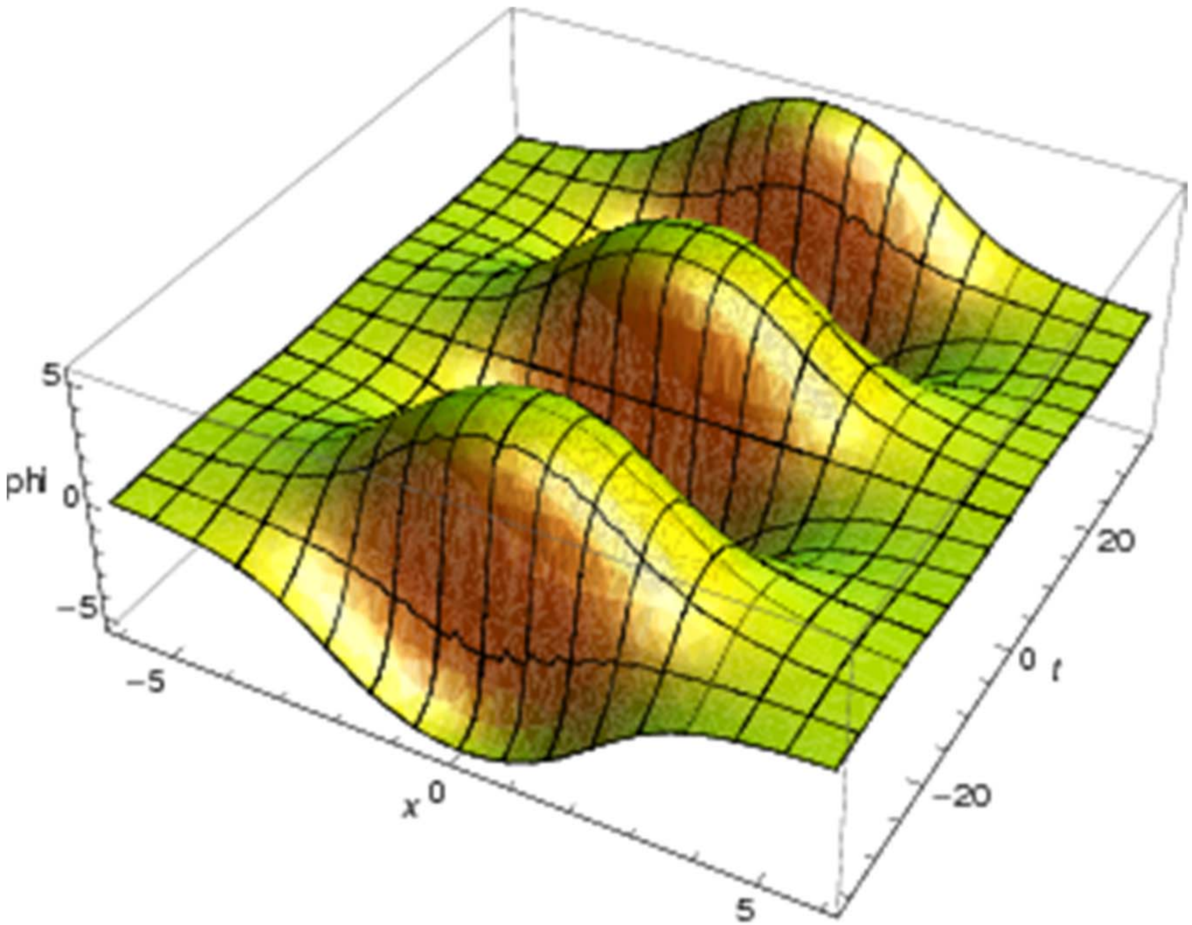
$0 < \mu < \pi/2.$

The limit case of $\mu \to \pi/2$:

$$\phi(x,t) = 4\arctan\left(\frac{t}{\cosh x}\right)$$

– an exact solution for a slowly separating kink-antikink pair.

An image of the periodically oscillating sine-Gordon breather:



The **Korteweg - de Vries** (**KdV**) equation (actually, for the first time derived by **Boussinesq**) was the equation for which the ***inverse-scattering transform*** was discovered (*Gardner*, *Kruskal*, *Greene*, *Miura*, 1967):

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

It applies to the description of small-amplitude surface waves on the surface of a shallow water, ion-acoustic waves in plasma, etc. The **KdV** equation gives rise to a family of stable solitons traveling at an *arbitrary velocity*, $c > 0$:

$$u = \frac{c}{2\cosh^2\left(\left(\sqrt{c}/2\right)\left(x - ct\right)\right)}.$$

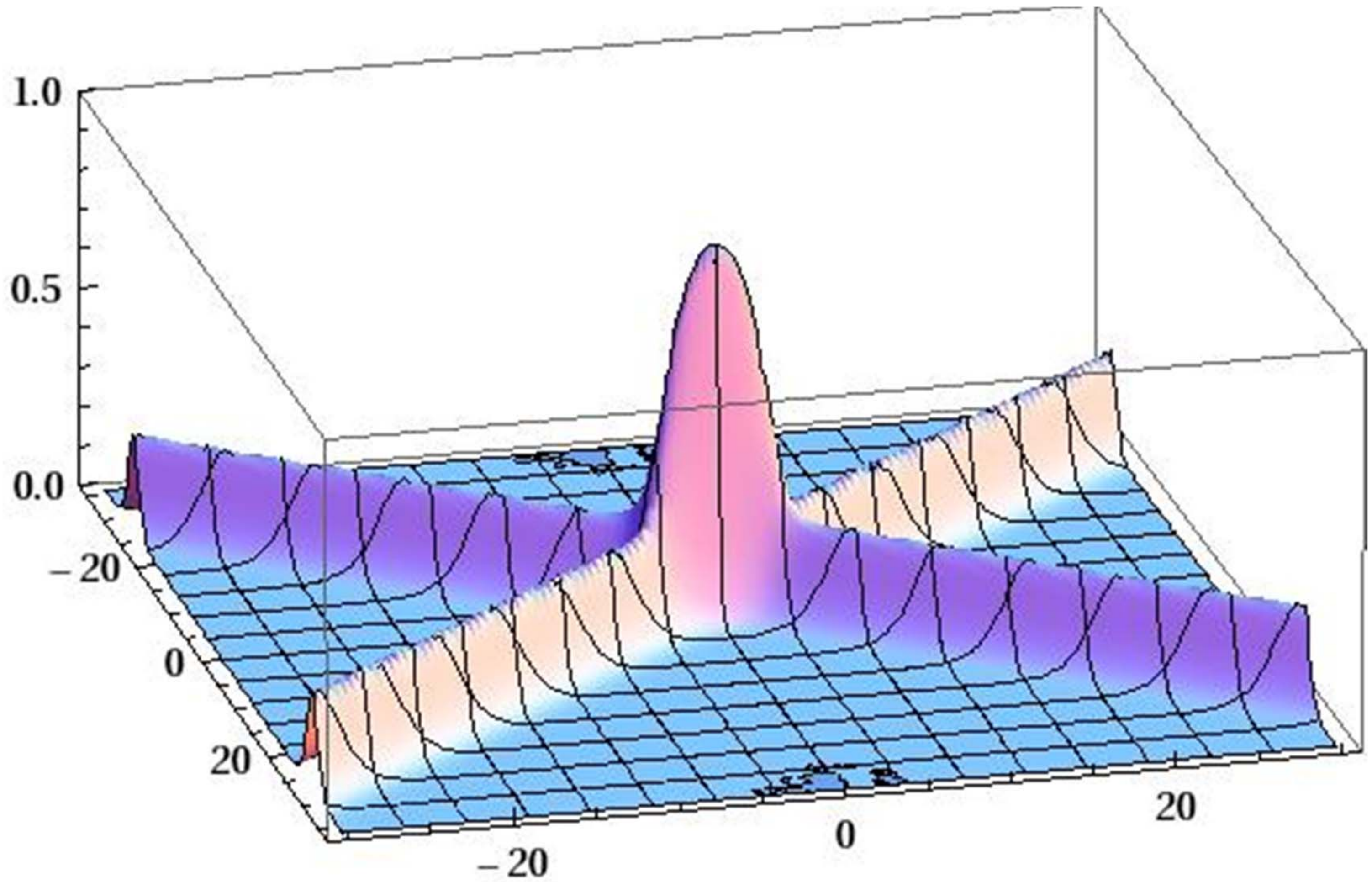
Integrable **two-dimensional** (**2D**) equations are known too. Most important are the ***Kadomtsev-Petviashvili*** equations (**2D** extensions of the **KdV** equation):

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)=\frac{\partial^2 u}{\partial y^2} \quad \text{(KP-I);}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)=-\frac{\partial^2 u}{\partial y^2} \quad \text{(KP-II).}$$

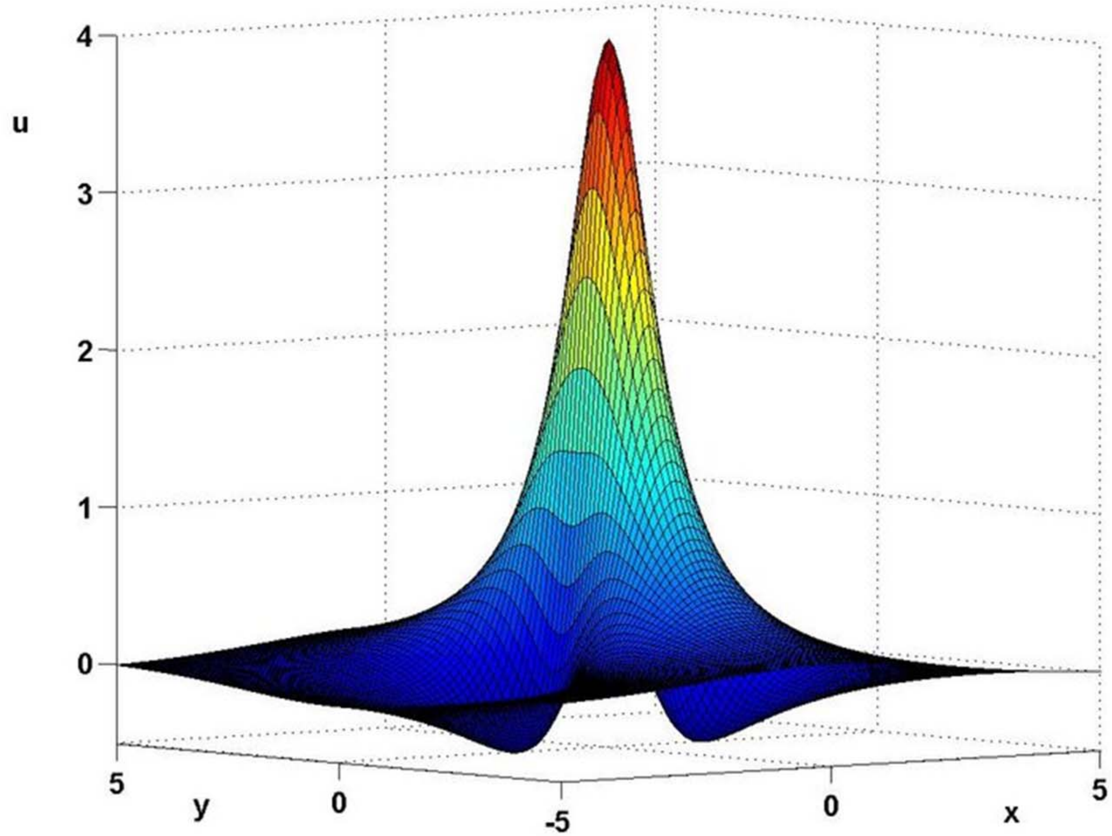
Both equations apply to the description of **2D** surface waves on a layer of shallow water, **KP-I** being relevant for small-scale waves dominated by the *surface tension*, while **KP-II** applies to long *gravity waves* on shallow water. **KP-II** has only **quasi-1D** soliton solutions (in fact, exactly the same as the **KdV** solitons), which are *stable*. In the case of **KP-I**, all the **quasi-1D** solitons are *unstable*, but this equation gives rise to *stable* **2D** weakly localized solitons (**lumps**).

An exact solution for the collision of two stable quasi-1D solitons of **KP-II**:



A stable ***lump soliton*** of **KP-I**:

$$u(x,y,t) = 4\frac{b^2(y+6at)^2 - \left(x+ay+3(a^2-b^2)t\right)^2 + 1/b^2}{\left[\left(x+ay+3(a^2-b^2)t\right)^2 + b^2(y+6at)^2 + 1/b^2\right]^2}.$$



Another ramification of the studies: equations for ***dissipative*** nonlinear media.

The corresponding generalization of the **NLS** equation is its counterpart with ***complex coefficients***, alias the ***complex Ginzburg-Landau*** (**CGL**) equation, which, in particular, is the simplest model of ***fiber lasers*** in optics:

$$i\frac{\partial U}{\partial z}+\left(\frac{D}{2}-i\gamma_2\right)\frac{\partial^2 U}{\partial \tau^2}+\left(1+i\gamma_1\right)|U|^2 U=i\gamma_0 U,$$

with $\gamma_1, \gamma_2 \geq 0$, and $\gamma_0 > 0$.

The **CGL** equation, unlike its **NLS** counterpart, is ***not*** integrable; nevertheless, it admits a (***single***) exact solitary-pulse solution (a "***dissipative soliton***"), as found by ***Pereira and Stenflo*** (1977):

$$U(\tau, z) = \frac{Ae^{ikz}}{\left[\cosh(\eta\tau)\right]^{1+i\mu}},$$ where real coefficient μ

is called *chirp*.

However, this solitary pulse is definitely ***unstable***, because of the obvious instability of its **zero background**, in the presence of the linear gain ($\gamma_0 > 0$).

An extended version of the **CGL** equation, which admits ***stable*** solutions for solitary pulses (although they are not available in an exact analytical form), features a combination of linear ***loss*** (instead of the linear gain) with ***cubic gain*** and ***quintic loss.*** This ***cubic-quintic*** (**CQ**) **CGL** equation was first introduced by ***Petviashvili and Sergeev*** (1984) (in fact, in a **two-dimensional** form):

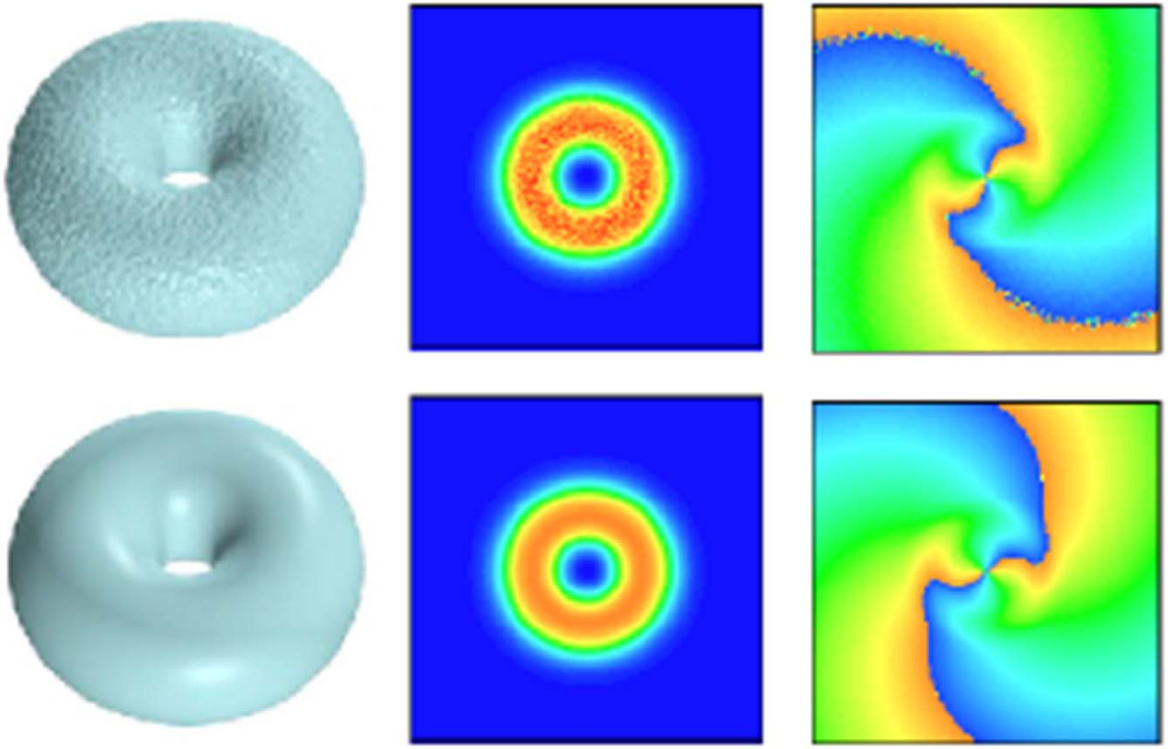
$$i\frac{\partial U}{\partial z}+\left(\frac{D}{2}-i\gamma_2\right)\frac{\partial^2 U}{\partial \tau^2}+\left(1+i\gamma_1\right)|U|^2 U+i\gamma_3|U|^4 U=i\gamma_0 U,$$

with $\gamma_2,\gamma_3 \geq 0$, and $\gamma_0,\gamma_1 < 0$.

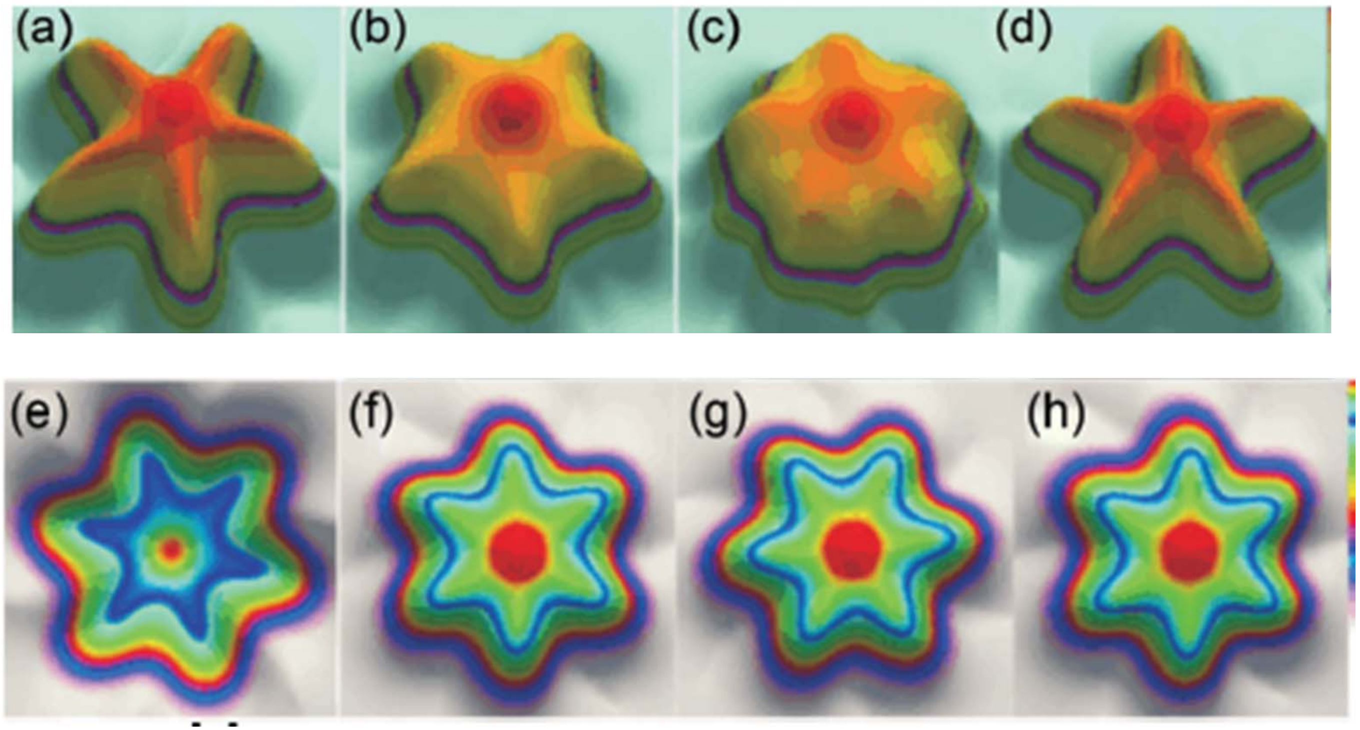
Stable solitary-pulse solutions to this equation (***dissipative solitons***) can be found numerically, and also in an approximate analytical form, in the limit when the dissipative coefficients are either ***very small*** or ***very large***.

The **CQ CGL** equation is a model of fiber lasers which include a combination of linear amplifiers and ***saturable absorbers***, that gives rise to the effective ***nonlinear amplification***. Stable solitary pulses in real fiber lasers have been created in many experimental works. ***Stable multidimensional dissipative solitons*** have also been predicted in **2D** and **3D** versions of the **CQ CGL** equation.

An example of a ***stable*** **3D** dissipative ***vortex soliton*** with ***embedded vorticity*** ***S*** **= 1** (taken from *Stable Vortex Tori in the Three-Dimensional Cubic-Quintic Ginzburg-Landau Equation*, by D. Mihalache, D. Mazilu, F. Lederer, Y.V. Kartashov, L.-C. Crasovan, L. Torner, and B. A. Malomed, Phys. Rev. Lett. **97**, 073904 (2006)):



An example of ***periodic metamorphoses*** of robust five- and six-point star patterns in the 2D **CQ CGL** equation with **localized linear gain** placed at the center [as per *V. Skarka, N. B. Aleksić, M. Lekić, B. N. Aleksić, B. A. Malomed, D. Mihalache, and H. Leblond*, Formation of complex two-dimensional dissipative solitons via spontaneous symmetry breaking, Phys. Rev. A **90**, 023845 (2014)]:



CONCLUSION

Systematic analysis of the nonlinear wave propagation in various physical media, both classical and quantum ones, makes it possible to identify several fundamentally important ***nonlinea***r equations that emerge in a very broad range of applications: ***two different*** **NLS** equations (with the positive and negative signs of the nonlinearity and/or dispersion), the **SG** equation, the **KdV** equation, ***two different*** **KP** equations in the **2D** geometry, and others.

Not only are these equations universal, but also their fundamental solutions – ***solitons*** (including ***multi-soliton complexes***) are profoundly important in all physical realizations where they can be predicted. In the most fundamental form (without additional terms), all the above-mentioned equations are ***integrable***. In the presence of small terms which break the integrability, the solitons can be studied by means of the appropriate perturbation theory.

In **1D**, both the theoretical analysis and experimental studies of solitons – in optics, **BEC**, long Josephson junctions, fluid flows, plasmas, etc. – are close to the completion. A challenge is the study of ***two***- and ***three-dimensional*** solitons, especially in the ***experiment***, which lags far behind the theoretical results for multidimensional results.

Some review articles on the topic of multidimensional solitons and solitary vortices:

B. A. Malomed, D. Mihalache, F. Wise, and L. Torner, Spatiotemporal optical solitons. J. Optics B: Quant. Semicl. Opt. **7**, R53-R72 (2005);

Viewpoint (update): On multidimensional solitons and their legacy in contemporary Atomic, Molecular and Optical physics, J. Phys. B: At. Mol. Opt. Phys. **49**, 170502 (2016).

B. A. Malomed, Multidimensional solitons: Well-established results and novel findings, Eur. Phys. J. Special Topics **225**, 2507-2532 (2016)

Addition: some recent results (obtained by the present speaker and his coauthors).

(1) Not solitons but something similar: stable bound states of the **3D** bosonic gas of atoms with **repulsive** inter-atomic interactions between them, pulled to the attractive center with potential $\boldsymbol{- U_0 /r^2}$. The respective **3D Gross-Pitaevskii** equation, in the scaled form:

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\nabla^2\Psi + \frac{1}{2}\left(-\frac{U_0}{r^2} + \Omega^2 r^2\right)\Psi + |\Psi|^2\,\Psi.$$

The ***quantum collapse***, generated by the single-particle ***linear*** Schrödinger equation if $\boldsymbol{U_0}$ exceeds a critical value, $(\boldsymbol{U_0})_{cr}$ **= ¼**, is replaced, in the framework of the mean-field **Gross-Pitaevskii** equation, by a newly formed ***ground state***:

H. Sakaguchi and B. A. Malomed, Suppression of the quantum-mechanical collapse by repulsive interactions in a quantum gas, Phys. Rev. A **83**, 013607 (2011).

The same setting, considered in the framework of the **many-body quantum theory**, gives rise, instead of the ground state, to a ***metastable state*** (the quantum collapse cannot be completely suppressed, but a ***stable bound state*** emerges):

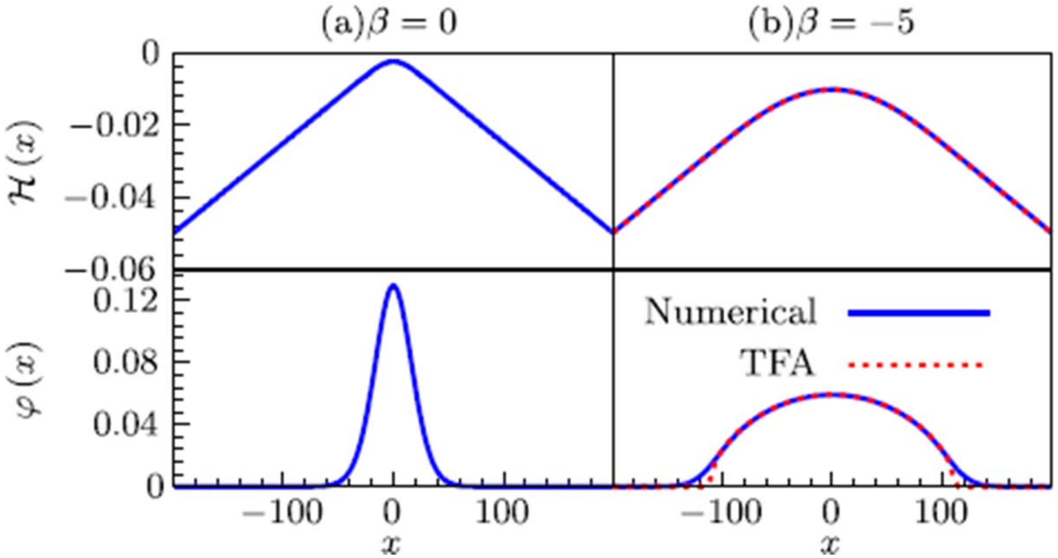
G. E. Astrakharchik and B. A. Malomed, Quantum versus mean-field collapse in a many-body system, Phys. Rev. A **92**, 043632 (2015).

(2) Very robust solitons mixing matter waves and an electromagnetic field can be created in a **two-component BEC** with or without contact interactions, if the components are linearly coupled by a ***radio/microwave field*** (the **Rabi coupling**):

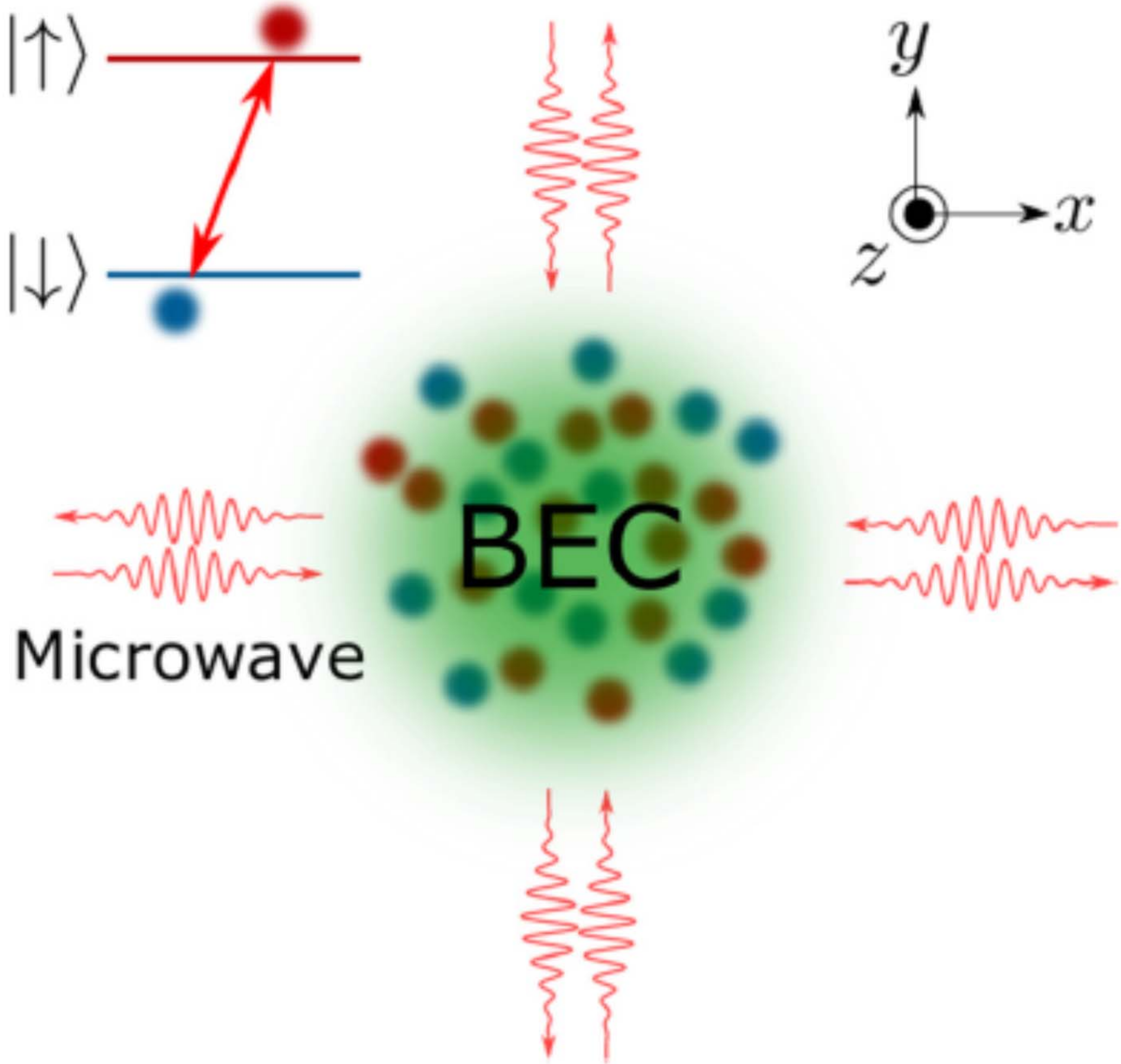
J. Qin, G. Dong, and B. A. Malomed, Hybrid matter-wave-microwave solitons produced by the local-field effect, Phys. Rev. Lett. **115**, 023901 (2015):

$$i\frac{\partial|\phi\rangle}{\partial\tau} = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \eta\sigma_3 - \begin{pmatrix} \beta|\phi_\uparrow|^2 & \mathcal{H}^* \\ \mathcal{H} & \beta|\phi_\downarrow|^2 \end{pmatrix}\right]|\phi\rangle,$$

$$\partial_x^2\mathcal{H} + \kappa^2\mathcal{H} = -\gamma\phi_\downarrow^*\phi_\uparrow,$$

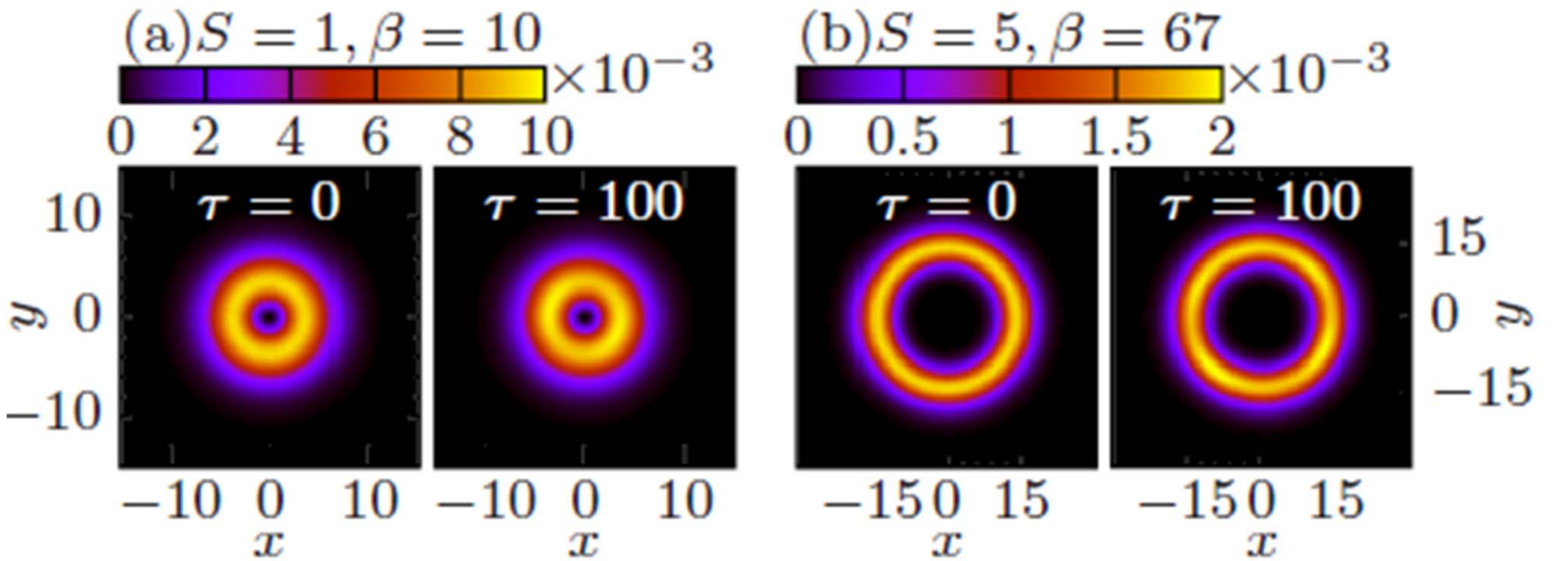


The illustration of the interaction between the two-component **BEC** state and the microwave state:



An extension of the model to two dimensions the creation of stable giant vortex solitons – for instance, with topological charge ***S* = 5**:

J. Qin, G. Dong, and B. A. Malomed, Stable giant vortex annuli in microwave-coupled atomic condensates, Phys. Rev. A **94**, 053611 (2016).



(3) ***Stable*** **2D** and **3D** solitons can be created in two-component (***spinor***) **BEC** with ***attractive interactions*** and ***spin-orbit coupling*** in the free space

H. Sakaguchi, B. Li, and B. A. Malomed, Creation of **two-dimensional** composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space, Phys. Rev. E **89**, 032920 (2014).

Y.-C. Zhang, Z.-W. Zhou, B. A. Malomed, and H. Pu, Stable solitons in three dimensional free space without the ground state: Self-trapped Bose-Einstein condensates with spin-orbit coupling, Phys. Rev. Lett. **115**, 253902 (2015).