TIME-VARYING RISK AVERSION. AN APPLICATION TO EUROPEAN OPTIMAL PORTFOLIOS

Carlos Esparcia Sanchís

Trabajo de investigación 005/016

Master en Banca y Finanzas Cuantitativas

Tutor: Dr. Antonio Díaz Pérez

Universidad Complutense de Madrid

Universidad del País Vasco

Universidad de Valencia

Universidad de Castilla-La Mancha

www.finanzascuantitativas.com

Time-varying risk aversion. An application to European optimal portfolios

Master thesis recited for obtaining the Master degree of Banking and Quantitative Finance

QfB | Master in Banking and Quantitative Finance

VNIVERSITAT DE VALÈNCIA

ABSTRACT

In this research we aim to focus our study in the modelling of risk-aversion parameter so that it changes over time, in order to take into account the variability of investors' expectations. To bring out the above purpose we implement some schemes such as the GARCH-M models. According to the above, we immerse ourselves in the theory of utility and choosing the optimal portfolio for risk averse individuals within the mean-variance context. We begin with the review of the unconditional Markowitz approach to analyse the optimal portfolio in a constant context. Then, we study more in depth how this kind of portfolio changes over time, through conditional models such as GARCH (1, 1) and DCC-GARCH.

Key words

Diversification, screening rules, optimal portfolio, time-varying risk aversion, GARCH models

Author CARLOS ESPARCIA SANCHÍS

Supervisor ANTONIO DÍAZ PÉREZ

Table of Contents

Table of Figures

Table of Results

INTRODUCTION

In accordance with the mean-variance approach, we can partially order the set of investment opportunities, reducing the choice of investors to those portfolios located on the efficient frontier. However, with this approach, the investors cannot compare which alternatives are dominant among themselves, so they are not allowed to select the investment portfolio that best meets their goals.

In order to find this portfolio, we must take a different criterion, incorporating individual risk attitude. Although these preferences are very complex (they depend on the age, gender, education level, income of the individual ...), to make their implementation easier, they are represented by a single parameter that summarizes the personal level of risk aversion, the α parameter.

When an individual decides to invest his money, always chooses an efficient portfolio whose composition depends on his subjective preferences. Some individuals choose to assume greater risks to get higher returns, while more risk averse investors opt for portfolios with less associated risk, but with the promise of lower yields.

As it is well known, an individual is more or less risk averse according to the economic and political circumstances. For instance, nowadays we are in a period in which even the most adventurous investor has had to reduce his optimistic expectations. Given that, it seems reasonable to model the risk aversion parameter so that it changes over time, in order to take into account the variability in agents' expectations.

According to the last paragraphs, there are some studies in financial literature that refer to time-varying risk aversion. For instance, Kim (2014) proposes a consistent indicator of conditional risk aversion in consumption-based CAPM. Other studies have differed widely in their estimates of time-varying risk aversion, such as Dionne (2014), who aim to extend the concept of orders of conditional risk aversion to orders of conditional dependent risk aversion. However, our motivation is in the line of the framework proposed by Cotter and Hanly (2010), which is based on estimating the risk aversion parameter as a derivation of the CRRA[1]. Thus, we estimate the risk aversion parameter through the conditional mean and variance. To bring out the last

[1] This tern refers to the changes in relative risk aversion, which is a way to express the risk aversion attitude of an investor through his utility function.

purpose, we model these conditional moments through several GARCH schemes, such as the GARCH in mean.

Moreover, we propose two novel approaches, based on our own intuition. The first one continues in the same line as previous model and is only based on the application of the European Consumer Confidence Indicator, CCI[2], as a proxy of the European customer sentiment, replacing the individual wealth by this European indicator in the equation of the CRRA. In addition, we propose another approach which has no relation with quadratic functions, but more related with downside risk measures, in order to compare whether is better to work under the quadratic preferences world or according other risk approaches. Thus, the main idea of this research is to build optimal portfolios for different types of investment profiles in order to compare whether is better to use a constant risk aversion parameter or a dynamic one.

Otherwise, the common investor aims to ensure a well-diversified portfolio, in order to be hedged against unfavourable movements in the stock market. According to the above, investors usually include a reasonable number of assets in their portfolios. In addition, they tend to select assets from different sources, such as different sectors of the economy or branches of business. Thus, it seems necessary to set a criterion for choosing the optimal assets that must be include in our portfolio.

Linking to the previous lines, we assign an introductory chapter to review the asset allocation theory, analysing more in depth the screening rules and how can they help us to reduce the investment world to a limited set of assets. To accomplish this purpose, we follow the approach proposed by León, Navarro and Nieto (2015). The main idea is to sort the 50 equities that belong to the EuroStoxx-50 index and choosing the best 10, by the application of some performance measures (PMs) of the Lower partial moment´s family. Then, we explain how to use the principal component analysis (PCA) for reducing the information used in performance ratios. In the same line we are able to find some other researches that have the purpose of adding several performance measures, such as Billio et al.(2012) who build a performance index or Hang and Salmon (2003).

Once we have selected the optimal assets to build a well-diversified portfolio, then we spend the most of this paper reviewing the second chapter, in which we are going to locate our study on the time-varying optimal portfolios. In particular, in this chapter we compare how well may

[2] The survey data contained in this indicator, provide useful information of the current state of the European economy, forecast short-term developments and hence are closely followed by economists, policy-makers and business managers

be the optimal portfolio of different investment profiles in performance terms. For this, we begin studying the optimal portfolio in an unconditional world, to understand how the mean-variance approach works in the context of utility functions.

Then, we focus our study on a conditional approach, modelling the conditional volatilities, correlations and risk aversion parameters through several GARCH schemes. Further, we review several ratios and risk measures to evaluate the performance of our portfolio, such as the Sharpe Ratio and the Certainty Equivalent. In addition, for a better understanding by the reader, we plot the evolution of these performance measures over time to see how change the exposure of our different portfolios throughout the studied period.

Finally, to conclude this research, we present the main results and conclusions that have been obtained from the study. From here we can get some important notions such as the PCA technique is really useful in terms of reducing information and helping us to get a single criterion to select a set of assets. Moreover, analysing more in detail the second chapter, we find that dynamic models could be better than constant ones.

CHAPTER 1: HOW BEST TO SELECT SEVERAL ASSETS. IN SEARCH OF A SINGLE CRITERION

According to the annual report of the National Securities Market Commission (CNMV) on markets and its performance, during 2013 emissions of debt in international markets fell following the large decline in the offer of public debt, but they were partly offset with the emissions of private companies. The forecast for the next years is that fixed income is going to continue losing value worldwide. We are entering in a really difficult time to invest in bonds, in which even those considered to be safest in the world, US bonds, are in free fall.

In recent years, both public and private issuers have been watching as each day passed, was more expensive to issue bonds to finance their projects. In those moments of great European depression, the big winners were the small investors, because the different public and private institutions were offering very high interest rates to be attractive, trying to alleviate the huge risk of insolvency they had associated.

As seen in the market, the investment outlook is changing, although a few years ago they bet more for fixed income, today it seems that this is changing and is focusing radically towards a trend of investment in equities, which grows and grows every day. But, why this change is due in the markets? This is because public debt offer very low interest rates, which makes investors prefer to take the risks of other instruments such as equities due to the promise of higher returns in the future.

Given the above, it seems obvious to think that rational investors aim to focus their portfolios on equities, not in debt. On this basis, in this first chapter, we aim to find the optimum ratio that investors should allocate in portfolios with a number of risky stocks. In particular, for reasons of financial diversification, we analyse this problem for ten risky stocks[3].

However, we must take into account that we are reviewing a theoretical analysis and as a consequence, trying to reach our goals, we assess our portfolios for different time frames, ones related to calm periods and others more related to economic recession. Thus, there are some periods in which the investment in fixed income is more attractive than investing in equities.

[3] We review the diversification problem at MutualFunds (2015)

According to the previous lines and omitting the attractiveness of fixed income in some periods, we immerse ourselves in the equity context. Then, the ten risky shares are selected from the Eurostoxx-50 index, due to it is the most representative indicator in the Stock Europe markets. The main question that should arise is: How could we choose these assets? The answer is not so easy because we should take into account some important financial criteria, in order to ensure a good risk-return ratio of our portfolio and especially, to assure the portfolio diversification.

Otherwise, since the Thesis of Markowitz (1952), asset allocation has been paid a lot of attention by researchers. As a consequence, several studies have been published between 1960-1990 about the relationship between the number of stocks in a portfolio and its variance reduction. One of the papers that has been used as a framework is the research of Evans and Archer (1968). This study show that 10 stocks are enough to reach a well-diversified portfolio. A few years later, some more studies, such as the one proposed by Tapon and Vitali (2013), confirmed the conclusion of Evans and Archer with 8, 9 and 10 stocks. Keep in mind that this is one of the key elements of this work, given that, based on these criteria we are going to choose the optimal number of assets to include in our portfolios.

Moreover, according to León, Navarro and Nieto (2015), the asset selection is an important problem for which the screening rules are useful. These rules aim to reduce the investment set of opportunities to a limited set of assets in order to make the asset allocation easier for financial investors.

Particularly, in this section we focus on ranking the whole assets that belong to the EuroStoxx-50 index and choosing the best 10 (for convenience purposes and given the evidence of the mentioned literature), through the application of some performance measures (PMs) that belong to the same family. Then, we also summarize the information contained in this performance ratios by using the principal component analysis (PCA) method, in order to get a global ranking, based on which, we are able to continue with our study of the time-varying risk aversion in the second chapter.

1.1. DATA: A BRIEF DESCRIPTION OF THE EUROSTOXX-50 INDEX

- **The data sample**

Talking about the EuroStoxx-50 used data, we must make it clear that in this first chapter, we only employ a time frame of 522 days, from 01/01/2002 (after the Dot-com bubble) to 31/12/2003 (approximately 2 years) to obtain the estimations of the PMs for the series of individual asset returns. Particularly, we compute 6 PMs to each individual stock: three measures of the K family for different values of order m and the FT measures for three alternative combinations of m and q. Then, we aim to use the principal component analysis (PCA) for summarizing the above information into a single ratio. After that, we order the 50 stocks and choose the best 10 ones to be included in each one of the time-varying risk aversion portfolios. Lastly, once we have selected the optimum equities, in the second chapter we evaluate the performance of the proposed portfolios during two allocation periods, one related to a calm time frame and another more associated to a stressed one.

Note that we work with continuously compound returns, instead of working with stock prices. We will analyse this procedure more in depth in section (2.1.1).

- **Constituent stocks of EuroStoxx-50 index**

The EuroStoxx-50 index is the most representative indicator in the stock markets of member countries in the Euro zone[4]. This index adds the 50 largest companies by market capitalization. It includes companies from Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal and Spain. This index is calculated by weighting the floating capital of each of the 50 mentioned companies. That is, not all companies have the same weight, but their representation is based on their capitalization.

Currently, companies that have the greatest weight in the EuroStoxx-50 are French, which occupy 35% of the total, followed by German (28%) and Spanish companies (13%). Furthermore, we can affirm that all kind of business sectors belong to this stock index, although it's the sector of financial institutions the one that gives more weight above others. Specifically, 18, 7 % of all companies in the index.

Then, we show the constituent stocks of Eurostoxx-50 index for the period described in the previous lines (from 01/01/2002 to 31/12/2003):

[4] To review that index, we rely on STOXX (2016)

Table 1. Constituent stocks of Eurostoxx-50 index

NUMBER	COMPANIES	ANNUALIZED RETURN
1	SAFRAN	0,1755
2	AIRBUS GROUP	0,1552
3	UNIBAIL-RODAMCO	0,1271
4	LVMH	0,1119
5	ESSILOR INTL.	0,0905
6	SOCIETE GENERALE	0,0517
7	INTESA SANPAOLO	0,0471
8	DEUTSCHE POST (XET)	0,0417
9	IBERDROLA	0,0333
10	BASF (XET)	0,0315
11	ENI	0,0291
12	AIR LIQUIDE	0,0003
13	BANCO SANTANDER	-0,0010
14	VINCI	-0,0015
15	BNP PARIBAS	-0,0032
16	SCHNEIDER ELECTRIC SE	-0,0190
17	UNICREDIT	-0,0251
18	DANONE	-0,0274
19	BMW (XET)	-0,0304
20	TOTAL	-0,0406
21	SAINT GOBAIN	-0,0422
22	SAP (XET)	-0,0452
23	E ON (XET)	-0,0563
24	SIEMENS (XET)	-0,0757
25	ENEL	-0,0771
26	TELEFONICA	-0,0810
27	DEUTSCHE BANK (XET)	-0,0909
28	VOLKSWAGEN PREF. (XET)	-0,0923
29	ASML HOLDING	-0,1039
30	L'OREAL	-0,1050
31	FRESENIUS (XET)	-0,1093
32	BBV.ARGENTARIA	-0,1145
33	UNILEVER CERTS.	-0,1147
34	DAIMLER (XET)	-0,1284
35	INDITEX	-0,1368
36	DEUTSCHE TELEKOM (XET)	-0,1369
37	CARREFOUR	-0,1411
38	AXA	-0,1487
39	SANOFI	-0,1627
40	PHILIPS ELTN.KONINKLIJKE	-0,1756
41	ANHEUSER-BUSCH INBEV	-0,1794
42	ASSICURAZIONI GENERALI	-0,1900
43	BAYER (XET)	-0,2078
44	ING GROEP	-0,2100
45	ORANGE	-0,2591
46	NOKIA	-0,3588
47	ALLIANZ (XET)	-0,4139
48	MUENCHENER RUCK. (XET)	-0,5300
49	VIVENDI	-0,5569
50	ENGIE	-

Source: Compiled by the author based on DataStream database. We ascertain them as an average of the continuously compound returns. Then, we show them in annual terms multiplying the monthly returns by 12

Paying attention to Table.1, we can appreciate that the above companies are sorted according to the annualized yield obtained over the mentioned period. Besides, note that ENGIE firm has been excluded from this analysis due to it is not traded in the index over the described period.

1.2. PERFORMANCE MEASURES BASED ON PARTIAL MOMENTS

First of all, we have to allocate ourselves in the framework of the Partial moments. Generally, we can talk about two different types, the Lower partial moments (LPM) and the Upper partial moments (UPM).

LPM define risk as the negative deviations of the stock returns, R, in relation to the mean return threshold, h, or the minimal acceptable return. In this context, Fishburn (1977), defines the LPM of order m as:

$$LPM(h,m) = E[Max(h-R,0)^m] = \int_{-\infty}^{h} (h-R)^m f(R)dR \tag{1.2.1}$$

Note that in this case, we use the mean of the risk-free rate (3-month US treasury bills) as threshold.

Where $f(R)$ is the probability density function. In contrast to the standard deviation, LPM considers only the negative deviations of returns assuming that investors are especially worried about the losses. The order of the LPM can be interpreted as the investors' risk attitude.

On the other hand, UPM of order q could be defined as:

$$UPM(h,q) = E[Max(R-h,0)^q] = \int_{h}^{\infty} (R-h)^q f(R)dR \tag{1.2.2}$$

In this case, we have to select the proper order q. In addition as in the previous case of LPM, we use the same mean for the h parameter.

Our purpose is to sort a set of assets according to a performance measure (PMs), in order to choose the 10 best assets to build optimal portfolios. Thus, in this case, we have chosen a pair of PMs based on partial moments: The kappa and the Farinelli-Tibiletti ratios.

- **Kappa ratios**

Kaplan and Knowles (2004) introduced the kappa indices which risk measure is estimated by using the Lower Partial Moments to evaluate the properties of returns probability distribution in the left tail. The kappa ratio of order m is defined as:

$$K(h,m) = \frac{E(R) - h}{LPM_{m,h}(R)^{1/m}} \tag{1.2.3}$$

Particularly, in this work we will set the following values for the order of the LPM to consider different risk attitudes: m=10 (defensive investors), m=1.5 (moderate investors) and m=0.5 (aggressive investors).

- **Farinelli-Tibiletti ratios**

Farinelli and Tibiletti (2008) proposed a ratio (FT ratio) that exclusively looks at the upper and lower partial moments by comparing the favorable and the unfavorable events:

$$FT(h,q,m) = \frac{UPM_{q,h}(R)^{1/q}}{LPM_{m,h}(R)^{1/m}} \tag{1.2.4}$$

As in the case of the Kappa ratio, we set some values for the risk attitude parameters. In this case, the values assigned for the two parameters (q,m) are: (0.5, 2) for defensive investors; the Omega ratio (1, 1) for moderate investors; and the Upside Potential ratio (1, 2).

1.3. PRINCIPAL COMPONENT ANALYSIS (PCA)

Otherwise, this first chapter goes on the use of PCA as a way to summarize the information content of some performance measures that belong to the same family (Lower partial moments). Thus, the idea is to summarize the ratings obtained for each one of the 6 performance measures described in the previous section, building a global rating through the application of the PCA technique.

In several studies, analysts often take the greatest number of variables from a data sample. However, if we take too many variables, we have to consider an excess of correlation coefficients which complicates the view of the relationships between variables.

Another problem that occurs in multiple studies, is the strong correlation between variables. By taking too many variables, it is normal that they are related or provide the same information in different views.

Thus, it is necessary to reduce the number of studied variables, being important to highlight the fact that the concept of additional information is related to the higher variability or variance, that is, the greater the variability of the data, then the greater the availability of the information.

To analyse the relations between p variables, we could change the original set of variables into another new set of uncorrelated variables. This new set of assets is known as Principal Components set. Furthermore, we have to mention that resulting variables are linear combinations of the original variables and they are constructed according to the percentage of total variability that they explain from the data sample.

The ideal way is to find $m < \mathrm{p}$ variables which are linear combinations of the p original variables and obviously which are uncorrelated with each other. If the original variables are already uncorrelated, then the principal component analysis has a lower sense. However, in this case our initial variables have a strong relation since the beginning. The above is due to the fact that we are analysing the PCA problem from 6 ratios that belong to the same family. Given that, we can assess the mentioned correlation coefficients through the following matrix:

Table 2. PCA: Correlation Matrix

	KAPPA 0,5	KAPPA 1,5	KAPPA 10	FT(0.5,2)	FT(1,1)	FT(1,2)
KAPPA 0,5	1,0000	0,9964	0,9515	0,7034	0,9985	0,7119
KAPPA 1,5	0,9964	1,0000	0,9678	0,7030	0,9993	0,7098
KAPPA 10	0,9515	0,9678	1,0000	0,5951	0,9597	0,5981
FT(0.5,2)	0,7034	0,7030	0,5951	1,0000	0,7117	0,9328
FT(1,1)	0,9985	0,9993	0,9597	0,7117	1,0000	0,7192
FT(1,2)	0,7119	0,7098	0,5981	0,9328	0,7192	1,0000

Source: Compiled by the author. Each row contains the correlation coefficients between each ratio and the rest of them

Table.2 shows the Pearson correlation coefficients between the 6 analysed ratios. Then, we can observe a rather high lineal relation between the studied variables. The presence of high correlations is a well indicator of the existence of a structure between the mentioned ratios. The above allows us to implement a Principal Component Analysis (PCA), using the above ratios as variables.

1.3.1. How to obtain the Principal Components.

We consider a set of assets characterized as $X_1, X_2, \dots, X_p$. These variables are expressed in deviations from their mean or typified. Then, our purpose is to change these variables into a new set of uncorrelated variables defined as $PC_1, PC_2, \dots, PC_p$ whose variances decrease progressively.

Thus, the first component is denoted by PC_1 and it is defined as a linear combination of $X_1, X_2, \dots, X_p$.

$$PC_{1i} = w_{11}X_{1i} + w_{12}X_{2i} + \dots + w_{1p}X_{pi}\,, \qquad i = 1, 2, \dots, n \tag{1.3.1}$$

Where n represents the data size, w_{1i} is the weight of the X_j variable in the composition of the PC_hcomponent and the variance of PC_1 is given by:

$$V(PC_1) = \frac{\sum_{i=1}^{n} PC_{1i}^2}{n} = \frac{1}{n} PC_2{}'PC_2 = \frac{1}{n}(Xw_1)'(Xw_1) = w_1'\left(\frac{1}{n}X'X\right)w_1 \tag{1.3.2}$$

The pxp matrix $\frac{1}{n}X'X$ is denoted by V, that is the sample covariance matrix, if the variables variables are expressed in deviations from their mean, whereas if the variables are typified, $\frac{1}{n}X'X$ is the correlation matrix.

The first component is constructed so that its variance is maximum, subject to the constraint that the sum of squared weights equals the drive.

$$\sum_{j=1}^{p} w_{1j}{}^2 = w_1{}'w_1 = 1 \tag{1.3.3}$$

The usual way to maximize a multivariable function subject to some constraints is the Lagrange multipliers method. Where $w_1$is the eigenvector associated to the greater eigenvalue of the matrix V. Thus, the first component is defined by $PC_1 = Xw_1$

Then, the second component (like the remaining components) is defined as another linear combination of the original variables:

$$PC_{2i} = w_{21}X_{1i} + w_{22}X_{2i} + \dots + w_{2p}X_{pi}\,, \qquad i = 1, 2, \dots, n \tag{1.3.4}$$

Further, the variance of the second component is obtained through a similar expression:

$$V(PC_2) = \frac{\sum_{i=1}^{n} PC_{2i}^2}{n} = \frac{1}{n} PC_2{}' PC_2 = \frac{1}{n}(Xw_2)'(Xw_2) = w_2'\left(\frac{1}{n}X'X\right)w_2 \tag{1.3.5}$$

Thus, we are in search of the PC_2 vector that maximizes $V(PC_2)$ subject to the constraint $w_2'w_2 = 1$. Furthermore, we are looking that the second component has no correlation with the first one, so we need to include the constraint $w_2'w_1 = 0$.

Again, to solve this optimization problem with two constraints, we use the method of Lagrange multipliers, by which is possible to show that w_2 is the normalized eigenvector of the covariance matrix V, associated with the second largest eigenvalue. Thus, the second component is defined by $PC_2 = Xw_2$.

In the same way, we can calculate the $n'th$ component, which is defined as $PC_h = Xw_h$, where w_h is the eigenvector of the V matrix associated to the greater eigenvalue. However, in this research we build 3 components. In fact, looking for our academic purposes, we only need the first one.

It can be demonstrated, through the PCA, that the variance of the $n'th$ component is equal to the eigenvalue λ_h

$$V(PC_h) = w_h'\left[\frac{1}{n}X'X\right]w_h = w_h'w_h = \lambda_h \tag{2.3.6}$$

1.3.2. Variability Percentages

Given the fact that each eigenvalue corresponds to the variance of PCh component that was defined by the eigenvector x_h, by adding all the eigenvalues we have the total variance of the components:

$$\sum_{h=1}^{p} V(X_h) = trace(V) \tag{1.3.7}$$

Further, since the matrix V is diagonal, it can be shown that:

$$\sum_{h=1}^{p} V(X_h) = trace(V) = trace(D_\lambda) = \sum_{h=1}^{p} \lambda_h \,, \tag{1.3.8}$$

Where D_λ is a diagonal matrix which contains the eigenvalues of the V matrix. The above allows us to talk about the percentage of the total variance which is explained by each one of the principal components:

$$\frac{\lambda_h}{\sum_{h=1}^{p} \lambda_h} = \frac{\lambda_h}{trace(V)} \tag{1.3.9}$$

Thus, we could show the variability percentage explained by the first p components as follows:

$$\frac{\sum_{h=1}^{q} \lambda_h}{\sum_{h=1}^{p} \lambda_h} = \frac{\sum_{h=1}^{q} \lambda_h}{trace(V)} \tag{1.3.10}$$

Then, to obtain the values of the PCh component:

$$PC_h = w_{h1}X_{1i} + w_{h2}X_{2i} + \dots + w_{hp}X_{pi} \,, \qquad i = 1, 2, \dots, n \tag{1.3.11}$$

In addition, by dividing the scores for PCh component between its standard deviation, we obtain the typified components:

$$\frac{PC_h}{\sqrt{\lambda_h}} = \frac{w_{h1}}{\sqrt{\lambda_h}} X_{1i} + \frac{w_{h2}}{\sqrt{\lambda_h}} X_{2i} + \cdots + \frac{w_{hp}}{\sqrt{\lambda_h}} X_{pi} \,, \qquad h = 1, 2, \dots, p; \; i = 1, 2, \dots, n \tag{1.3.12}$$

Then, we show the explained variability according to each one of the ascertained principal components:

Table 3. Explained Variability

component	Initial Eigenvalues			Sum of the squared saturations		
	Total variance	Explained variance	Accumulated explained variance	Total variance	Explained variance	Accumulated explained variance
1	5,1091	85,1517	85,1517	5,1091	85,1517	85,1517
2	0,7767	12,9450	98,0967	0,7767	12,9450	98,0967
3	0,0677	1,1286	99,2253	0,0677	1,1286	99,2253
4	0,0448	0,7464	99,9717			
5	0,0017	0,0275	99,9992			
6	0	0,0008	100			

Source: Compiled by the author. Note that each eigenvalue corresponds to the variance of the PCh component that was defined by the eigenvector x_h

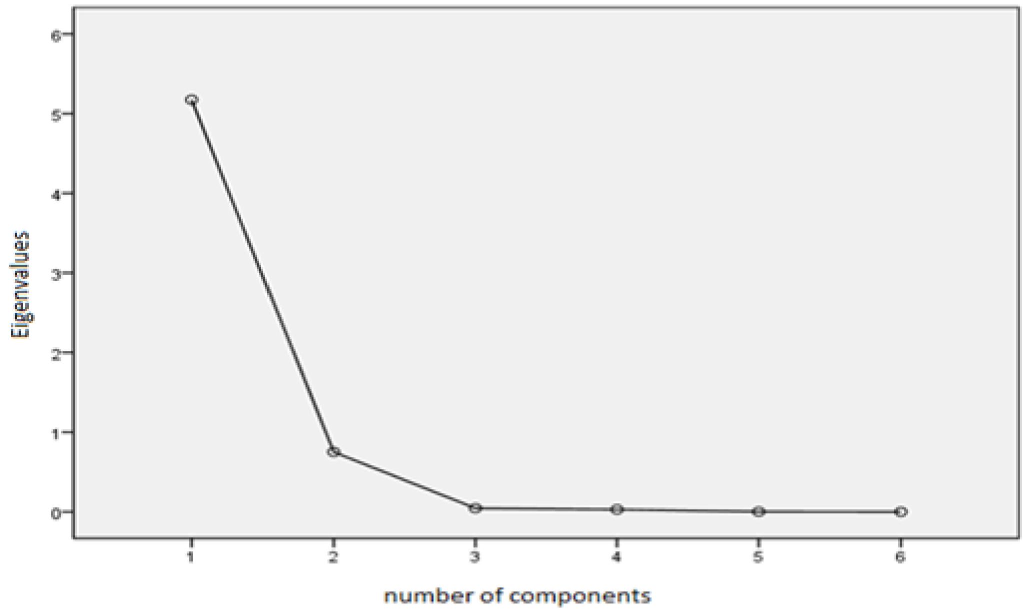
Paying attention to the table of the explained variability (Table.3), we can observe a list of the eigenvalues which comes from the covariance matrix. Note that the first part of the graph is about the initial eigenvalues and the second part is about the 3 searched principal components. Talking about the second part of the table, the first column is showing the eigenvalues, which define the variability explained by each component. In addition, the second column represents the percentage of variance explained by each eigenvalue. The last column shows the accumulated percentage of explained variance.

According to the last table, we can appreciate that the first three components capture the 99.23% of the variability, being the first principal component the one which collect the higher variation in the data sample, reaching the 85.15% of the variability. The last two components only collect 12.95% and 1.13% of the variance, respectively. However, we must make it clear that for academic purposes we are only going to consider de first component, due to we are looking for a single measure, based on which we are going to sort the 50 studied equities.

1.3.3. Identification of the Principal Components

The main purpose of the PCA technique is to reduce the data size. On this basis, the identification of these components seems to be a key element. Usually financial analysts preserved only those components that collect most of the variability, a fact that allows us to represent data in two or three dimensions. Moreover, there are several criteria to determine the number of components that must be considered in the study which is being carried out. In this section we only considerer the simplest one, the scree plot.

Figure 1. Identification of PC. The Scree Plot



Source: Compiled by the author. This graph follows the same idea as Table.3, that is to say, it shows the variability of each component in a decreasing order

The scree plot (Figure.1) is obtained from represent on a Cartesian axis, the magnitude of the eigenvalues in descending order (or their relative position in relation to the sum of the eigenvalues) on the side of ordinates and on the side of abscissas the number of the principal component with which an area chart is generated such as the one illustrated in the last graph.

Both the table of the explained variance and the scree plot shows the eigenvalues sorted from highest to lowest. If an eigenvalue approaches zero is considered as a residual factor and meaningless in the analysis because it is unable to explain a significant amount of the total variance. The scree plot shows the eigenvalues associated with the vectors of the principal components proving that the first three eigenvalues concentrated more information about the data variability.

In particular, paying attention to the scree plot, we choose those components whose eigenvalues are near to the settling zone. In this case, this approach suggests choosing two factors. However, we choose only the first one. This is because our purpose is based on getting a global criteria to select several assets, so we prefer to have one factor instead of two of them.

1.4. MAIN RESULTS AND FINDINGS

In this section we show the key elements and conclusions obtained from our study in the asset allocation area. To begin with the study, we analyse those table related to the Kappa and Farinelli-Tibiletti ratios and then, we review more in depth the PCA technique and the single ratio obtained through that analysis:

First of all, we review the family of those ratios related with the Partial moment family, that is to say, Kappa and Farinelli-Tibiletti ratios. In this case, we sort the 50 assets that belong to EuroStoxx-50 regarding to the two mentioned criteria and choose the best 10 in performance terms. Given that, we can appreciate the greatest equities in the following tables:

Table 4.Kappa ratios

	KAPPA 0,5		KAPPA 1,5		KAPPA 10	
RANKING	FIRM	RATIO	FIRM	RATIO	FIRM	RATIO
1	UNIBAIL-RODAMCO	0,2984	SAFRAN	0,0663	SAFRAN	0,0148
2	SAFRAN	0,2867	UNIBAIL-RODAMCO	0,0660	UNIBAIL-RODAMCO	0,0132
3	ESSILOR INTL.	0,1303	AIRBUS GROUP	0,0372	AIRBUS GROUP	0,0122
4	AIRBUS GROUP	0,1271	ESSILOR INTL.	0,0346	LVMH	0,0104
5	LVMH	0,1173	LVMH	0,0336	ESSILOR INTL.	0,0079
6	IBERDROLA	0,0756	IBERDROLA	0,0188	IBERDROLA	0,0046
7	SOCIETE GENERALE	0,0493	SOCIETE GENERALE	0,0123	INTESA SANPAOLO	0,0030
8	DEUTSCHE POST (XET)	0,0420	DEUTSCHE POST (XET)	0,0115	DEUTSCHE POST (XET)	0,0029
9	INTESA SANPAOLO	0,0360	INTESA SANPAOLO	0,0107	SOCIETE GENERALE	0,0029
10	ENI	0,0355	ENI	0,0095	BASF (XET)	0,0023

Source: Compiled by the author. Note that we use the assets' returns at daily frequency and the 3-month German Treasury bills as a threshold

Table 5. Farinelli-Tibiletti ratios

	FT(0.5,2)		FT(1,1)		FT(1,2)	
RANKING	FIRM	RATIO	FIRM	RATIO	FIRM	RATIO
1	AIRBUS GROUP	0,2377	UNIBAIL-RODAMCO	1,1048	AIRBUS GROUP	0,5889
2	LVMH	0,2291	SAFRAN	1,1024	LVMH	0,5776
3	DEUTSCHE POST (XET)	0,2143	AIRBUS GROUP	1,0534	INTESA SANPAOLO	0,5552
4	VOLKSWAGEN PREF. (XET)	0,2136	ESSILOR INTL.	1,0521	SAFRAN	0,5455
5	BASF (XET)	0,2123	LVMH	1,0487	SAP (XET)	0,5427
6	SAFRAN	0,2107	IBERDROLA	1,0286	DEUTSCHE POST (XET)	0,5412
7	ENI	0,2106	SOCIETE GENERALE	1,0188	VINCI	0,5388
8	TOTAL	0,2106	DEUTSCHE POST (XET)	1,0169	SIEMENS (XET)	0,5375
9	DEUTSCHE BANK (XET)	0,2092	INTESA SANPAOLO	1,0154	ESSILOR INTL.	0,5369
10	SIEMENS (XET)	0,2087	ENI	1,0143	BASF (XET)	0,5353

Source: Compiled by the author. Note that we use the assets' returns at daily frequency and the 3-month German Treasury bills as a threshold

Note that the whole ratios showed in Table.4 and Table.5 come from equation (1.2.3) and equation (1.2.4) for different risk aversion levels.

Following the above tables, we show the 10 selected assets according to the two proposed performance ratios. As well as we can appreciate in the tables, Kappa ratios usually assign the top of the ranking to the same firms (SAFRAN, UNIBAIL-RODAMCO and AIRBUS) while the Farinelli-Tibiletti criterion is more heterogeneous regarding to the asset allocation.

Continuing with our study and applying the PCA technique, we have reduced the information contained in the 6 mentioned ratios, in order to express it in a single one. Then, we take the information contained in this first component and we sort it in the same way as in previous ratios. Thus, we show the results and the main appreciations of this analysis:

Table 6. Global ratio. The optimal set of assets

	GLOBAL RATIO		
RANKING	FIRMS	INDUSTRY	RATIO
1	SAFRAN	Industrial goods	2,2595
2	UNIBAIL-RODAMCO	Real estate	1,9762
3	AIRBUS GROUP	Industrial services	1,8922
4	LVMH	Diversified	1,8280
5	ESSILOR INTL.	Medical equipment	1,7481
6	DEUTSCHE POST	Industrial goods	1,3136
7	INTESA SANPAOLO	Banks	1,3119
8	IBERDROLA	Utilities	1,1130
9	BASF	Chemicals	0,8920
10	ENI	Oil and gas	0,8437

Source: Compiled by the author. Note that this global ratio comes from the first principal component (equation (1.3.1)), but in this case we only show the first ten elements of the component because of academic purposes.

Paying attention to the optimal selected assets (Table.6), we can find some differences regarding to previous lists. Given that, we have started this first chapter following the approach proposed by Tapon,F., Vitali, A. (2013) and looking for a number of assets which, included in a portfolio, ensured us a well-diversified financial one. To bring out the above purpose, we decide to select those equities which have the greatest performance ratio in accordance with the Patial moment's family.

Finally, we must say that the chosen assets meet the previously established requirements at the beginning of this chapter. The reason is, as well as we can appreciate in the last table, the chosen assets meet with requirement of "number" (10 assets) and "different sources" (they come from very different sectors). However, we must make it clear that we do not make any constraints about the different sectors, that is to say, the source they come from is arbitrary.

CHAPTER 2: FROM MARKOWITZ TO OUR DAYS. A CONDITIONAL ANALYSIS OF THE OPTIMAL PORTFOLIO

As it is well-known, investors usually want to ensure a good risk-return trade-off. Financial theory establishes as basic foundation a direct relationship between these two characteristics, return and risk, so those portfolios which incorporate a higher level of risk or more aggressive, tend to generate on average a higher yield, while more conservative portfolios tend to provide lower average yield. Thus, both types of portfolios could be attractive for different investors.

According to the previous context, the most basic financial theory considers that the investor takes his portfolio decisions maximizing his utility function, which depends on two moments of the returns distribution: mean and variance. The investor's utility level depends positively on return and inversely on volatility. The investor supposedly maximizes his utility function subject to the constraints of available resources, obtaining an optimal portfolio solution. This portfolio is defined in terms of the proportion of the investment allocated in each of the considered assets.

Different investors which differ in their utility function, have a different optimal portfolio. This is due to the importance attached to the portfolio returns and its volatility. Of course if the set of selected assets is different, it also makes their optimal portfolios different from each other. Another important criteria to select a portfolio of risky assets is to choose the minimum variance portfolio proposed by Markowitz (1952). Thus, we can observe each investor is really different and chooses an optimal portfolio according to his risk aversion level and his own preferences. For that reason, it seems necessary to use utility functions in order to help us choosing the best portfolio for each kind of investor.

In particular, in this second chapter, we aim to focus our study in the modelling of risk aversion parameter so that it changes over time, in order to take into account the variability in investors' expectations. The concept of time-varying risk aversion is well-understood in the literature under the context of habit formation. This variation in risk aversion raises the correlation between marginal utility and asset returns, while the correlation between consumption and returns remains low. The goal of this second chapter is to extend the idea of time-varying risk aversion making a comparison between different types of investment profiles, ones associated to the unconditional risk aversion and others more related to the modelling of risk aversion parameter over the timeframe.

According to the above, we spend the second chapter analysing the theory of utility and choosing the optimal portfolio for risk averse individuals within the mean-variance context. We begin with the study of the unconditional Markowitz approach to analyse the optimal portfolio in a constant context. Then, we study more in depth how this kind of portfolio changes over time, through conditional schemes such as GARCH (1, 1) and DCC-GARCH.

2.1. DATA ANALYSIS AND PORTFOLIO CONSTRUCTION.

As we have mentioned in the first chapter, the main purpose of this second one is to generate different kind of time-varying portfolios for two sample periods. The first one is the calm period, which runs from 01/01/2004 to 31/12/2007. Moreover, the second timeframe is a more stressed one and comes from 01/09/2008 to 31/12/2012, in order to consider how the portfolios have changed during the crisis period. To show the empirical results of this research, the used data is referred to the daily and monthly closing prices of some European equities and indicators. Then, we analyse them more in detail:

- **Data at daily frequency**

We use data at daily frequency for the cases of Model A, C.1, C.2 and D[5]. These conditional portfolio models are constructed by the application of the EuroSroxx-50 index at daily frequency as a proxy of the time-varying risk aversion. In addition we use the ten equities selected in chapter one at daily frequency, in order to ascertain the different terms and parameters of the conditional portfolio weights. As we have mentioned before, we analyse these portfolios over two different periods. The calm period, which runs from 01/01/2004 to 31/12/2007 has 1043 work days and the second timeframe (stress period) comes from 01/09/2008 to 31/12/2012 (1045 work days).We have been able to find these data at DataStream database.

- **Data at monthly frequency**

Otherwise, we propose a novel approach to analyse the time-varying risk aversion, based on the application of the European Consumer Confidence Indicator, CCI, as a proxy of the customer European sentiment. The CCI is a composite indicator ascertained at monthly frequency and

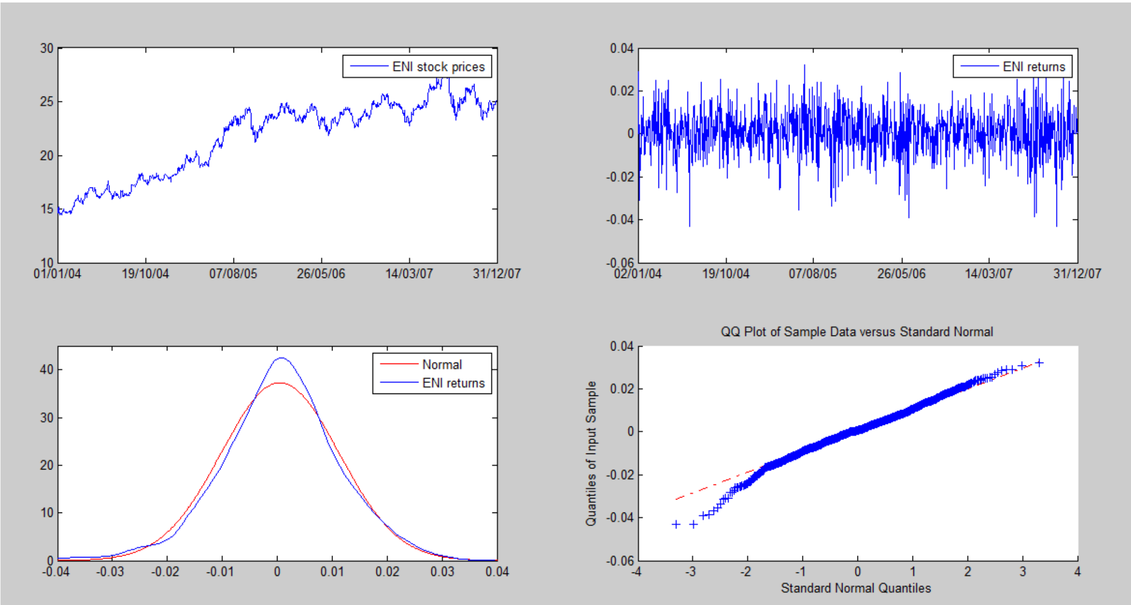
[5] We explain these dynamic models in greater detail in section 2.4

based on answers from several economic questions asked to European consumers. We have been able to find these monthly data at Europa.eu. Thus, we use data at monthly frequency for the case of the mentioned indicator. Note that in this case, we have 48 months for both analysed periods (calm and stress).

Then, we spend the most of this section making a brief analysis about some important statistical characteristics of financial time series. However, we only analyse these attributes in detail for the case of the ENI firm due to the obtained results and appreciations are very similar for the rest of the ten studied equities. In spite of that, note that you can review the obtained results for the rest of the studied stocks in Appendix A.

2.1.1. Key elements of financial time series

Figure 2.Financial time series analysis. ENI



Source: Compiled by the author based on ENI stock prices and returns at daily frequency. In addition, we compare the ENI returns against the normal distribution function.

Paying attention to the first graph (top left of Figure.2), we can observe that stock price changes over time, in other words, stock price returns are being modelled. There are two ways to convert prices into returns, you can either convert to periodically compound returns (See equation 2.1.1) or to continuously compound returns (See equation 2.1.2). Note that in this paper we use the second equation.

$$R_t = \frac{S_t}{S_{t-1}} - 1 \tag{2.1.1}$$

$$R_t = ln\left(\frac{S_t}{S_{t-1}}\right) \tag{2.1.2}$$

, where S_t is the stock price at time t, S_{t-1} is the previous traded stock price in the market and R_t is the stock return. Note that in the last graph we calculate the returns at daily frequency, but for the rest of this paper, we ascertain them in a monthly way, that is to say, taking log-differences in traded stock prices every 22 days.

As we can observe in the first two graphs (located at the top of Figure.2), ENI stocks move like a random walk with stochastic trend or unit root. It would also be unsteady. As for the mean, as in most financial returns, it is constant over time. Moreover, talking about the variance, the serie shows random variability, due to we can observe groupings or “volatility clusters ". This term refers to those moments in time when there is high volatility, they tend to come followed by periods of high volatility. Otherwise, low volatility moments come followed by a succession of low volatility periods. To collect these groupings of volatility, we use models of conditional heteroscedasticity (GARCH, exponential smoothing ...).

Then, if we look at the kernel graph, we can observe ENI returns do not follow a normal distribution, owing to the curve described by them is sharper than the Gaussian one, that is to say, ENI yields are leptokurtic. This is because its kurtosis is higher than normal (kurtosis = 3.98> 3). We could analyse the abnormality of ENI in another way, paying more attention to the distribution tails, due to ENI tails are wider than normal ones.

Finally, the fourth graph, draw the qq-plot of returns by comparing the number of returns to a normal distribution. The red line emerges to face the quantiles of a normal distribution with the quantiles of a normal distribution. Otherwise, the blue line represents our performance, and comes to face the normal quantiles against the ENI quantiles. Analysing this graph, we could say that blue line is not completely over red line, as we can affirm again that its tails are wider than normal.

2.1.2. Jarque-Bera normality test

The Jarque-Bera test is a goodness of fit test that examines whether a data sample has the skewness and kurtosis of a normal distribution. The above measures is a parametric test, which is defined as:

$$JB = \frac{n-k+1}{6}\left(S^2 + \frac{1}{4}(K-3)^2\right) \tag{2.1.3}$$

, where S represents the sample skewness and K is the sample Kurtosis of the time serie. Otherwise, n are the degrees of freedom and k is the number of regressors.

The Jarque-Bera statistic is asymptotically distributed as chi-squared distribution,χ^2, with two degrees of freedom. In this case, we use it to test the null hypothesis that ENI yields belong to a normal distribution. The null hypothesis is a joint hypothesis that skewness and kurtosis are nil.

Then, we make this contrast for the chosen yield:

Table 7. Jarque-Bera statistic.

ENI	
JB-Statistic	**P-Value**
59,1933	1,40E-13

Source: Compiled by the author

Then, according to Table.7, we can appreciate the two ways by which we can analyse whether we reject the null hypothesis or not. That is to say, the JB-statistic and the P-value. Note that in this research we assess this test according to the P-value. Remember that the p-value is the probability of observing an statistical evidence as or more extreme than the observed value under the null hypothesis (this hypothesis is assumed true at the beginning).

Given the above, we reject the null hypothesis at a significance level of 95%, since the P-value is less than α (0.05). Thus, it is not possible to affirm that ENI returns follow a normal distribution.

$$H_0 = Normality\ in\ returns \tag{2.1.4}$$

Linking to the above, we must mention that in spite of knowing that assets' returns have the problem of heavy tails, we are going to continue this research with the assumption of normality in returns.

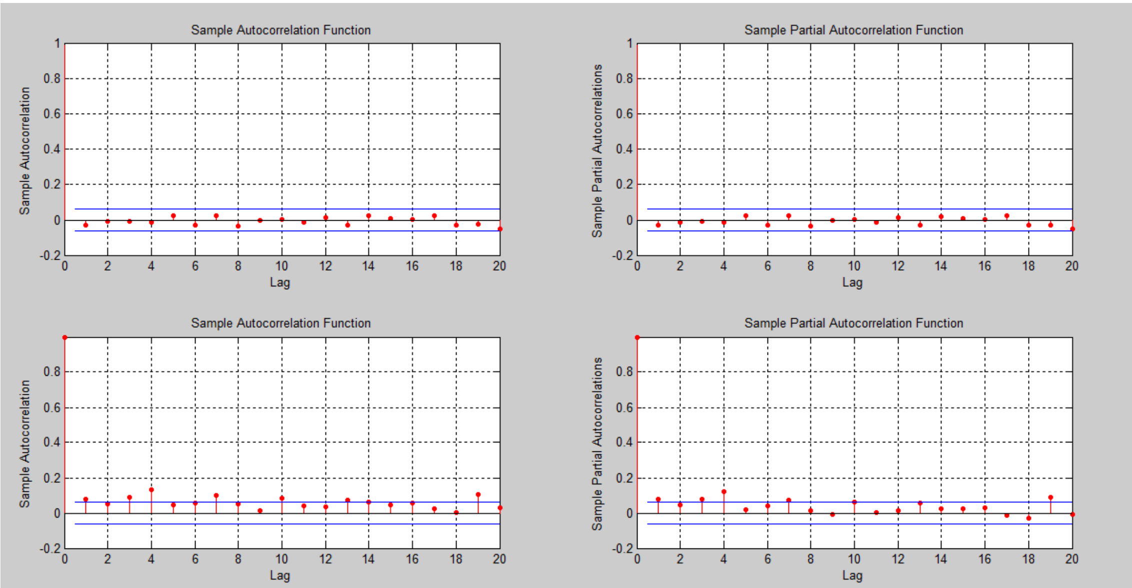
2.1.3. Simple and partial autocorrelation functions (FAS and FAP)

The analysis of persistence in performance is an interesting line of research because of the controversy over whether this phenomenon occurs and if so, whether that persistence exists only in short-term time periods or in longer time horizons. Moreover, the existence of this phenomenon can be considered a useful information tool for participants in financial markets when making their investment decisions.

Correlation refers to the persistence in returns or correlation between current and past values of the studied variable. The autocorrelation of a stochastic process is measured by simple and partial autocorrelation functions. The ρ parameter is generally known as the persistence of the process. An increase or decrease in the actual return of assets, take effect in their own future performance, although the influence of the current value gradually grows over time, in accordance with the decrease of the ρ coefficient. A ρ value closes to 1 is introducing high persistence in the process, and conversely, a value of ρ close to 0 is not introducing persistence in the process.

Then, we plot the simple and partial autocorrelation functions of ENI firm. To ascertain these functions, we have used 20 lags and the application of MatLab functions:

Figure 3. Simple and partial autocorrelation functions



Source: Compiled by the author

In accordance with Figure.3, top graphs show the simple and partial autocorrelation function of asset returns, while graphs located at the bottom represent the same functions, but for squared returns. Through that, it is intended to see the possible existence of serial correlation and heteroscedasticity.

As shown in the graph of both simple and partial autocorrelation functions, they appear to be among the bands in the case of the studied equity, suggesting the absence of autocorrelation or that it is weak. On this basis, we can conclude that a clear persistence in returns is not appreciated, though we will confirm it using statistical tests.

Moreover, talking about the serial correlation of squared returns, we can appreciate that ENI correlations are out of bounds, which indicates heteroscedasticity or autocorrelation in second moments, that is to say, it is changing over time. Given that, it seems advisable to model volatilities and correlations over time, by the application of several GARCH schemes.

However, the autocorrelation functions are merely qualitative tools for analysing the presence of autocorrelation in yield lags. Therefore, to evaluate the combined autocorrelation of some lags in a more quantitative way, we will use the statistical test of Box-Pierce.

2.1.4. Box-Pierce statistical test

Box and Pierce (1970) developed a statistic based on the squares of the first autocorrelation coefficients of residual yields, to analyse whether there is autocorrelation. The statistic is defined as a cumulative sum of squares of the correlation coefficients, that is:

$$Q_p = n \sum_{j=1}^{p} \hat{\rho}_j^2 \qquad (2.1.5)$$

, where $\hat{\rho}_j = \frac{\sum_{t=j+1}^{n} \varepsilon_t \varepsilon_{t-j}}{\sum_{t=1}^{n} \varepsilon_t^2}$ and $\varepsilon_t$is the residual.

Under the null hypothesis of no autocorrelation, Q statistic is asymptotically distributed as a chi-squared, χ^2 with degrees of freedom equal to the difference between the accumulated number of coefficients (p) and the number of parameters estimated by adjusting the considered process.

Table 8. Box-Pierce statistic

	ENI	
LAGS	Q	P-Value
1	1,0275	0,3107
2	1,1412	0,5652
3	1,1824	0,7572
4	1,3162	0,8586
5	1,8899	0,8642

Source: Compiled by the author

Note that, as in the JB-test, we analyse this test according to the associated P-value. In addition, we must mention that we have used 5 lags in the calculus of this test.

Then, we evaluate the results obtained from Table.8. For a significance level of 95%, we affirm that there is no autocorrelation between returns, since the p-value is not less than 0.05 for neither of the studied cases, so we accept the null hypothesis in such cases.

$$H_0 = No\ Autocorrelation \tag{2.1.6}$$

Following the above, this equity has no persistence in returns. Given that, this equity follows a white noise structure.

Linking to the above, we must mention that in spite of knowing that there are a few cases in which a clear persistence in returns is appreciated (see Appendix. A), we are going to continue this research without adjusting the equities' returns by autocorrelation.

2.2. MEAN-VARIANCE APPROACH. THE UNCONDITIONAL MARKOWITZ THEORY

It has always said that we can divide investment history in two clearly defined parts, before and after 1952, when the economist Harry Markowitz published his doctoral thesis on the Portfolio Selection. Markowitz (1952) was the first on paying attention to the practice of portfolio diversification, as well as we could appreciate in his publishing of 1959, "Portfolio Selection: Efficient Diversification of Investments". This is the base where investors generally prefer to keep asset portfolios rather than individual assets, because they do not take into account only the returns of these assets but also the risk thereof.

Within this framework, Markowitz proposed the minimum variance portfolio, which is a combination of risky assets that has the lowest level of risk between the different possible combinations of risky assets. Formally, in the original problem, the author fix a specific expected return of the portfolio as a constraint. In other words, the original Minimum-Variance Markowitz problem is as follows:

$$\left\{ \begin{array}{l} \min\ w'Vw \\ restricted\ to\text{:} \sum_{i=1}^{n} w_i \,\mathrm{E} = E_p \\ \qquad\qquad\qquad \sum_{i=1}^{n} w_i = 1 \end{array} \right. \tag{2.2.1}$$

However, in this paper we face the construction of the minimum-variance portfolio in a different way, without restricting short positions, that is to say, those positions which probably would take on the assets of higher risk (volatility). Thus, the proposed problem is:

$$\left\{\begin{array}{l} min\ \ w'Vw \\ restricted\ to\colon \sum_{i=1}^{n} w_i = 1; \end{array}\right. \tag{2.2.2}$$

Through this minimization problem, we can obtain the following expression for the minimum variance portfolio:

$$W_{mv} = \frac{V^{-1}1n}{1n'V^{-1}1n} \tag{2.2.3}$$

, where V^{-1} is the inverse of the covariances matrix and $1n$ is a $10x1$ ones vector

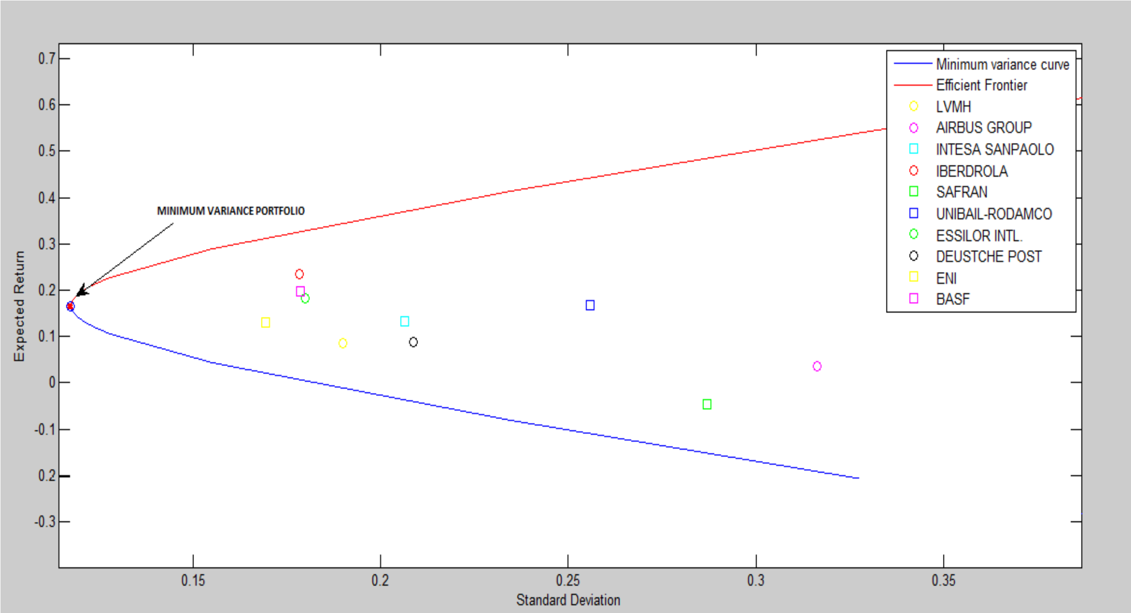
Then, we can ascertain the minimum variance expected return and volatility thorough the following expression:

$$\begin{gathered} E_{mv} = W'_{mv}E \\ \sigma_{mv} = \sqrt{W'_{mv}VW_{mv}} \end{gathered} \tag{2.2.4}$$

, where V is the $10x10$ covariance matrix, E is the $1x10$ expected return vector and W_{mv} is the $10x1$ minimum-variance weight vector. Note that both arrays are obtained from the individual assets set for the whole period, that is to say, they are constant over the mentioned time frame.

According to the minimum-variance approach, the set of portfolios we could build in the case of n risky assets can be displayed as a cloud of points showing the set of investment opportunities which is given by the market. Possible portfolios entirely cover a region of the mean-variance space and this region is convex. There are actually many possible combinations of assets, but for simplicity, we only represent the surround set, called "the minimum variance curve". Based on the above, the unconditional portfolios can be formed from contributions of the ten risky stocks are represented in the next figure. As an example of the traditional Markowitz's approach, we only plot these combinations for the calm period (01/01/2004-31/12/2007):

Figure 4. Investment Opportunity Set. The Efficient Frontier



Source: Compiled by the author. Note that expected returns and standard deviations are expressed in annual terms. It is possible by multiplying the monthly returns by 12 and their associated standard deviations by $\sqrt{12}$.

According to Figure.4, we can observe that in general terms, the whole studied assets have a rather high standard deviation. To solve the above problem, with the construction of the minimum-variance portfolio and its associated diversification effect, we can get this deviation greatly reduced. Moreover, talking about yield terms, the minimum-variance portfolio has only been surpassed by IBERDROLA, BASF, ESSILOR and UNIBAIL-RODAMCO. The above means that we have reached a portfolio with fairly good results on average (Mean=16%, Stdv=2%).

However, currently have emerged numerous empirical studies that have trashed the Markowitz theorem. The reason is that the last theory analysed the portfolio management at a particular moment in time, while new approaches are based on the possibility of introducing different statistical moments that are changing over time. Therefore, in section 2.4., “How to build time-varying portfolios. A conditional approach”, we introduce this possibility of change by applying conditional correlations models (DCC GARCH). For this, previously, we model the changing volatilities over time by developing models of conditional variances such as GARCH (1, 1). In addition, we add the possibility of time-varying risk aversion in the following sections.

2.3. UTILITY FUNCTIONS. LOOKING FOR THE INVESTORS' OPTIMAL PORTFOLIO

In uncertainty contexts it is possible to reach the preferences representation of economic agents through the expected utility. The use of expected utility although, on the one hand, excludes some behaviours that would be rational as a representation by ordinal utility functions, on the other hand, provides a greater degree of detail and understanding in decision-making.

In short, it is suggested that financial theory have developed utility functions to assess how good an investment is, according to its expected utility. When we represent investor preferences through utility functions, we are assuming that the decision maker has a well-defined utility function of his wealth, $U\ (W)$. It is also assumed that each individual chooses among different alternatives maximizing the expected utility of his wealth.

Within this area, we must make a very important distinction. While risk depends on the specific characteristics of financial assets, the risk attitude depends on the individual preferences and therefore, may be different for each kind of agent. In fact, in accordance with the shape of the utility function, we can distinguish three types of attitudes toward risk: risk aversion, risk neutrality and risk appetite. To get a feel for the risk attitude, it is essential to study the risk aversion coefficients of Arrow-Pratt (1971).

On the one hand, Arrow developed the absolute risk aversion coefficient for an individual with an associated utility function:

$$ARA = -\frac{U''(W)}{U'(W)} \tag{2.3.1}$$

This aversion coefficient is positive if and only if the individual is risk averse, that is to say, if the individual shows a concave utility function $U''(W) < 0$. Actually, is the concavity (second derivative) which reflects the risk aversion level, but it is necessary to adjust this measure by the first derivative of the utility function, to ensure that it does not change under linear transformations.

On the other hand, we can talk about the relative risk aversion coefficient, which measures aversion in percentage terms:

$$CRRA = -W\frac{U''(W)}{U'(W)} = W\,ARA \tag{2.3.2}$$

Intuitively, it seems clear that absolute risk aversion should be decreasing for most individuals, whereas relative aversion is generally decreasing.

In general, we assume that investors are risk-averse. For that reason, in this work we only focus on the analysis of two risk-averse utility functions, such as quadratic and CARA functions. Given that, we use the first ones to obtain explicit forms for optimal portfolios and the second ones in order to help us modelling the risk aversion parameter over time.

2.3.1. Negative exponential utility functions

Analytically, the negative exponential utility function (CARA) is represented as follows:

$$U(W) = -e^{-\alpha w},\ \ \alpha > 0 \tag{2.3.3}$$

, where an increase in wealth (W) produces an equal diminishing utility level.[6]Then, calculating the first and the second derivative of this expression, we can see that it is an increasing and concave function. Which makes sense, due to this is a function which represents the preferences of risk averse individuals.

$$\begin{gathered} U'^{(W)} = \alpha e^{-\alpha w} > 0 \quad increasing \text{ function} \\ U''^{(W)} = -\alpha^2 e^{-\alpha w} < 0 \quad concave\ function \end{gathered} \tag{2.3.4}$$

Although this function exhibits constant absolute risk aversion, it is widely used. This is because this function combined with the assumption of normality in returns of financial assets, allows to obtain explicit forms for optimal portfolios. We show the above through the following expressions:

[6] This is often a feature of institutional investors.

$$
ARA = -\frac{(-\alpha^2 e^{-\alpha w})}{\alpha e^{-\alpha w}} = \alpha
$$

$$
\frac{\delta ARA}{\delta W} = 0 \qquad ARA = constant
$$

$$
CRRA = -W\frac{(-\alpha^2 e^{-\alpha w})}{\alpha e^{-\alpha w}} = W\alpha \tag{2.3.5}
$$

$$
\frac{\delta CRRA}{\delta W} > 0 \qquad CRRA = increasing
$$

Then, to show the CARA function graphically, we assign the values 10,1 and 0.1 to the risk aversion parameter, in order to assess how the function changes according to the aversion level of the studied individual (from more averse investors to less averse ones). In addition, to make the graph analysis easier for the reader, we assign some arbitrary convenience values to the investor wealth. Given that, the mentioned function is as follows:

Figure 5. Individual preferences. CARA utility function

Source: Compiled by the author

As we can appreciate in Figure.5, the CARA function is becoming flatter as we are decreasing the risk aversion level from 10 to 0.1.

2.3.2. The optimal portfolio construction. An extension of the CARA function

Formally, in this research we will analyse optimal portfolio using the negative exponential utility function (CARA), which has been described in detail in the previous lines (equation 2.3.3). Given that, if the final investor's wealth follows a normal distribution with an associated mean μ and variance σ^2, then using the moment-generating function of a normal distribution:

$$E[U(W)] = E[-e^{-\alpha w}] = -e^{\left[-\alpha w\left(\mu - \frac{1}{2\alpha\sigma^2}\right)\right]} = U\left[E(W) - \frac{\alpha}{2}\sigma^2(W)\right] \qquad (2.3.6)$$

On the other hand, analytically, the investor's problem is based on determine the weights of the risky assets that maximize the expected utility, given the constraint that these weights sum the unity[7]:

$$\begin{cases} \max\limits_{w} U(E_P, \sigma_P^2) = \left(w'E - \frac{\alpha}{2} w'Vw\right) \\ s.a. \sum_{i=1}^{3} w_i = 1 \end{cases} \qquad (2.3.7)$$

, and the Lagrangian function associated would be as follows:

$$\max. L = \left(w'E - \frac{1}{2} w'Vw\right) - \lambda(w'1_n - 1)$$

Then, we calculate the partial derivatives conditioning to the assets weights and to the multiplier lambda. After that, we equate this equation to zero and we obtain:

$$\frac{\partial L}{\partial w} = 0; \qquad E - \alpha V w - \lambda 1_n = 0_n$$

$$\frac{\partial L}{\partial \lambda} = 0; \qquad w'1_n - 1 = 0$$

, based on the above, we solve w:

$$w = \frac{1}{\alpha} V^{-1} E - \lambda \frac{1}{\alpha} V^{-1} 1_n$$

Finally calculating and replacing values in above equations, we get an expression for the optimal portfolio weights:

[7] To analyse that problem, we rely on: Gómez (2011).

$$W_o = \frac{V^{-1}1n}{1n'V^{-1}1n} + \frac{1}{\alpha}\left(V^{-1}E - \frac{1n'V^{-1}E}{1n'V^{-1}1n}V^{-1}1n\right) \tag{2.3.8}$$
$$W_o = W_{mv} + \frac{1}{\alpha}\left(V^{-1}E - \frac{1n'V^{-1}E}{1n'V^{-1}1n}V^{-1}1n\right)$$

, where V^{-1} is the $10x10$ inverse covariance matrix, $1n$ is a $10x1$ ones vector, E is the $1x10$ expected return vector and α is the individual level of risk aversion.

Then, the optimal expected return and volatility can be ascertained as follows:

$$E_o = W_o'E \tag{2.3.9}$$
$$\sigma_o = \sqrt{W_o'VW_o}$$

, where V is the $10x10$ covariance matrix, E is the $1x10$ expected return vector and W_o is the $10x1$ minimum-variance weight vector. Note that both arrays are obtained from the individual assets set for the whole period, that is to say, they are constant over the mentioned time frame.

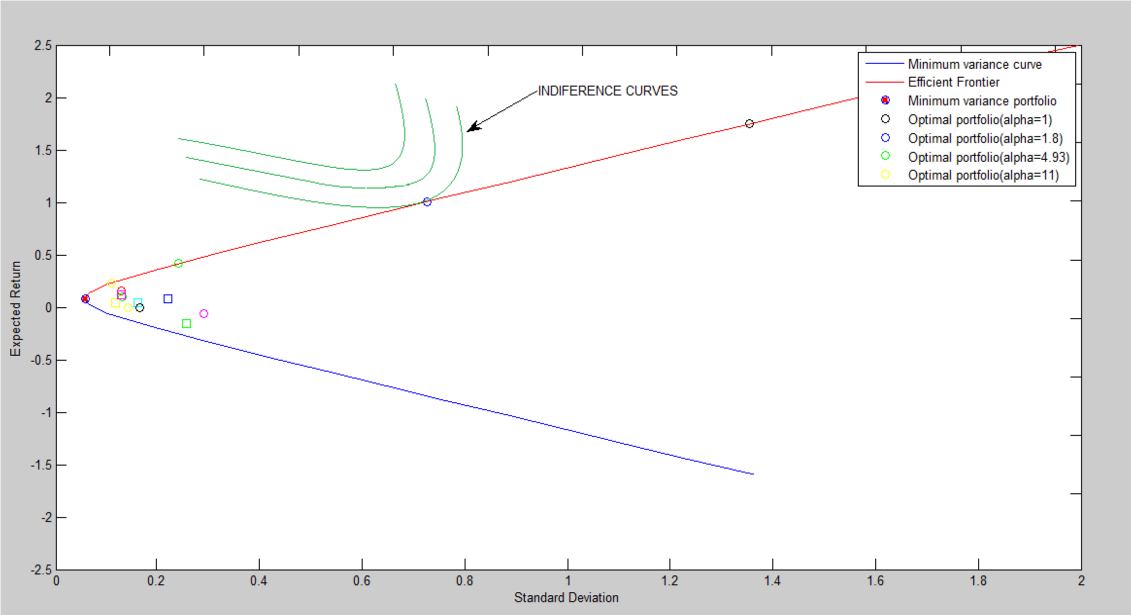
Moreover, in this section we take into account the way in we can introduce the individual risk attitude within the mean-variance world. Obviously, to introduce the risk preferences of an investor we need to use an economic tool, the indifference curves. Specifically, these curves show the investor arrangement of exchanging risk by return. We could build these curves equalizing the expected utility function calculated above to a constant parameter, K:

$$U\left[E(W) - \frac{\alpha}{2}\sigma^2(W)\right] \tag{2.3.10}$$
$$E_P - \frac{1}{2}\alpha\sigma_P^2 = K$$
$$E_P = K + \frac{1}{2}\alpha\sigma_P^2$$

Then, by applying these equations we can obtain the following indifference curves (the green ones) for different k values[8]:

[8] Each one of the indifference curves has the same k associated parameter. Along the curves, the only changing parameters are the standard deviation and the expected return of the portfolio.

Figure 6. The indifference curves and the optimal portfolio.



Source: Compiled by the author. Note that expected returns and standard deviations are expressed in annual terms. It is possible by multiplying the monthly returns by 12 *and their associated standard deviations by* $\sqrt{12}$*.*

Paying attention to Figure.6, we can observe that we are moving away from the minimum-variance portfolio, as we are decreasing the risk aversion level from 11 to 1. According to the above, those optimal portfolios with a higher risk aversion level also has a greater expected return but a rather high standard deviation. Otherwise, the above occurs inversely.

As in the case of the minimum-variance approach, for simplicity, we have only plot the mentioned figure for the calm period. According to this chart, we can obtain the optimal portfolio of risky assets. This kind of portfolio is an efficient alternative which represents the optimal risk-return ratio that an investor should take into account, given his individual preferences, which are represented by a utility function that depends on the investor's risk-aversion level. In other words, it is the portfolio of risky assets that maximize the expected utility of the investor.

Graphically, the investor's optimal portfolio is represented as the intersection of the efficient frontier and the indifference curve (utility function). At this point the slopes of both curves are equal, so the rate at which we can exchange return for market risk is equal to the ratio at which the investor is willing to do it personally. Thus, the optimal portfolio represents the combination of assets that supports the efficient frontier and also is in the highest indifference curve.

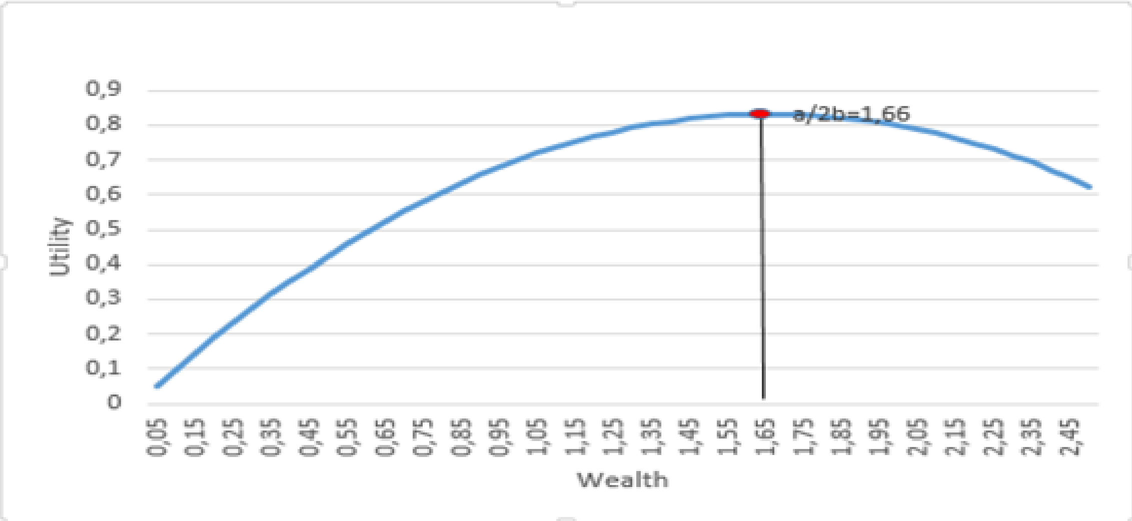
2.3.3. Quadratic utility functions

Formally, the quadratic utility function is represented by the next expression:

$$U(W) = \mathrm{aW} - \mathrm{b}W^2, \;\; b > 0, W < \frac{a}{2b} \tag{2.3.11}$$

To make its graph implementation easier, we assign some arbitrary values to the individual wealth (we fix the same values as in the case of CARA utility function) and we also make the function parameters (a, b) be fixed in 1 and 0.3, respectively. Given the above values, this kind of utility function looks like:

Figure 7. Individual preferences. Quadratic utility function



Source: Compiled by the author

Then, calculating the first and second derivative we can show the main characteristics of this kind of function:

$$U'^{(W)} = \mathrm{a} - 2bW \quad \left(> 0 \;\; if \;\; W < \frac{a}{2b}\right) \;\; \text{not always an } increasing\ function$$

$$U''^{(W)} = -2b < 0 \quad concave\ function$$

$$ARA = -\frac{(-2b)}{\mathrm{a} - 2\mathrm{bW}}$$

$$\frac{\delta ARA}{\delta W} > 0 \qquad ARA = increasing \tag{2.3.12}$$

$$CRRA = -W\frac{(-2b)}{\mathrm{a} - 2\mathrm{bW}} = \frac{2b}{\left(\frac{a}{W}\right) - 2\mathrm{b}}$$

$$\frac{\delta CRRA}{\delta W} > 0 \qquad CRRA = increasing$$

These functions have some important drawbacks. On the one hand, utility is not always increasing. On the other, the absolute risk aversion is always increasing. However, working with this utility function, the mean-variance approach is consistent with the criteria of maximizing the investor's expected utility, which is something relevant in the CAPM context. The above is a key element in this work, as it relates the quadratic utility functions with the market risk premium. Given the importance of this element, we analyse it more in depth in the following section.

2.3.4. The risk aversion parameter and the quadratic utility framework

Despite the influence of risk aversion in the optimal portfolio context, there are not many studies which have explicitly estimated the risk aversion of an investor. Instead of that, they choose random values to reflect the common levels of risk aversion. The equity literature on risk aversion has developed around the review of Arrow (1971), who affirmed that the risk aversion parameter should be around 1. Otherwise, in the equity context have appeared several studies which differ in their estimations of risk aversion. For instance, Mehra and Prescott (1985) argued that this parameter should be greater than 10. Moreover, Ghysels et al (2005) have affirmed that the risk attitude should be between 1.5 and 2 on average, while Guo and Whitelaw (2006) established the mentioned parameter in 4.93.

However, the common sense tells us that the use of arbitrary values for this parameter could yield us optimal portfolios that do not reflect the actual investor´s attitude towards risk. Given that, one of the goals of this paper is to propose the modelling of the risk aversion parameter in order to make it changing over time. To carry out this key proposal, we focus our attention on the review of the quadratic utility functions framework.

Within this context, we must make a very important distinction. The ARA parameter presented in the last section, refers to the changes in absolute risk aversion, that is to say, it is a measure of investor reaction to euro changes in wealth. On the other hand, we have to talk about the CRRA. This term is more related to the measuring of changes in relative percentages invested in risky and risk free assets. Moreover, as we could intuit, this expression is a really useful tool in financial contexts in the sense of helping us with the calculus of the risk aversion parameter.

Thus, we could affirm it is possible to represent the risk aversion attitude of an investor in a single number by the CRRA expression. As we have mentioned in the previous lines, we only review the CRRA within the context of quadratic utility, following two different approaches. The first one, published by Cotter and Hanly (2010), is based on the estimation of the CRRA through the market risk premium. In this context of asset pricing, the size of the risk premium is determined by the aggregate risk aversion of investors and by market volatility, which is usually represented by the variance. Particularly, since we are analysing the performance in European stock markets, we use the EuroStoxx-50 index as a proxy of the market. The proposed formula is as follows:

$$E(R_m) - Rf = \alpha\sigma_m^2$$

$$\alpha(CRRA) = \frac{E(R_m) - Rf}{\sigma_m^2} \tag{2.3.13}$$

Moreover, we propose a novel approach, based on our own intuition. It is based only on the application of the European Consumer Confidence Indicator, CCI, as a proxy of the customer European sentiment, replacing the individual wealth by this European indicator in the expression of the CRRA presented above (the fifth equation of expression 2.3.11)[9]. We have chosen this indicator because of its economic transcendence, as it reflects the customers' opinions about past, current and future economic developments. The CCI is a composite indicator ascertained at monthly frequency and based on answers from several economic questions asked to European consumers[10]. It is generally viewed as a timely pointer of developments in private consumption.

In other way, we propose another approach which has no relation with quadratic functions, but more related with downside risk measures, in order to compare whether is better to work under the quadratic preferences world or according other risk approaches. We analyse it more in depth in section 2.4.4.

[9] This and other procedures will be explained in detail in section 2.4.
[10] We have been able to find these monthly data at Europa.eu

2.4. HOW TO BUILD TIME-VARYING PORTFOLIOS. A CONDITIONAL APPROACH

First of all, we must remember that one of the main ideas of this paper was the time-modelling of probability distribution moments, in order to make our optimal portfolios changing over time. To reach the above, and focusing on the optimal portfolio equation (2.3.8), we propose the application of conditional variance and correlation schemes such as GARCH (1, 1) and DCC-GARCH, to model the conditional moments included in the mentioned formula.

Otherwise, according to the previous section, the CRRA allows us to obtain the risk aversion attitude of an investor in a single number. However, in this research we are more interested in the time-varying risk aversion, not in a constant parameter. Thus, what we will do is to model the market mean and variance through conditional models such as GARCH-M, GARCH (1, 1) or EWMA schemes. Once we have done this, we get a number of risk aversion parameters that fluctuate over time, due to the movements in conditional market means and volatilities. In addition, we propose another approach, based on the implementation of the CCI.

Further, we aim to assess whether it is better to work with a constant or changing risk aversion parameter. Thus, the idea is to build optimal portfolios for different types of investment profiles, the conditional ones associated to the CRRA and other one based in constant risk aversion.

2.4.1. Optimal portfolio construction. Modelling variances and correlations

This section describes how to obtain the optimal portfolio weights, given a set of assets. According to Clements and Silvennoinen (2013), we assume that the expected return of our portfolio follows a normal distribution function with a constant mean and a time-varying variance:

$$E_{o,t} \sim N(E, V_t) \tag{2.4.1}$$

, where $E_{o,t}$, is the expected return of the optimal portfolio at a given moment of time, N is the multivariate normal distribution function, E is the fixed expected return vector and V_t is the dynamic covariance matrix, calculated by modelling conditional correlations and volatilities

Focusing our study on the construction of dynamic portfolios, to obtain the optimal conditional weights, we must adapt the equation (2.3.8) to a time-varying context:

$$W_{o,t} = \frac{V_t^{-1}1n}{1n'V_t^{-1}1n} + \frac{1}{\alpha}\left(V_t^{-1}E - \frac{1n'V_t^{-1}E}{1n'V_t^{-1}1n}V_t^{-1}1n\right)$$
$$W_o = W_{mv,t} + \frac{1}{\alpha}\left(V_t^{-1}E - \frac{1n'V_t^{-1}E}{1n'V_t^{-1}1n}V_t^{-1}1n\right) \quad (2.4.2)$$

, where V^{-1} is the $10x10$ inverse dynamic covariance matrix, $1n$ is a $10x1$ ones vector, E is the $1x10$ fixed expected return vector and α is the time-varying individual level of risk aversion. Note that the expected return vector and the covariance matrix are expressed in annual terms.

Then, once we known the amounts invested in each one of the selected assets, we can calculate the expected return and the volatility of the optimal portfolio:

$$E_{o,t} = W'_{o,t}R_t$$
$$\sigma_{o,t} = \sqrt{W'_{o,t}V_tW_{o,t}} \quad (2.4.3)$$

, where $W_{o,t}$ is the vector which contains the dynamic weights invested in each of the studied equities, R_t is a vector composed by the assets' returns for each month of the market and V_t is the conditional covariance matrix.

Moreover, to carry out the above purpose, we need to ascertain the previous calculus of the time-varying covariance's matrix. As well as we describe in section 2.3.2., "The optimal portfolio construction. An extension of the CARA function", this matrix is given by the next expression:

$$V_t = \begin{pmatrix} \sigma_{1t}^2 & \cdots & \sigma_{1,10t} \\ \vdots & \ddots & \vdots \\ \sigma_{10,1t} & \cdots & \sigma_{10t}^2 \end{pmatrix} \quad (2.4.4)$$

, where the main diagonal elements are the variances of each one of the selected assets and the rest of the elements are the covariances between these equities. Note that, in spite of having the assets' returns at monthly frequency, we must annualize them multiplying the covariance matrix by$\sqrt{12}$, in order to ascertain the optimal portfolio weights. In addition, we must multiply the expected return vector by 12.

Given that, the question that arises in this context is: How can we make these probability distribution moments changing over time? The answer is so easy, by applying conditional variances and correlation moments. In particular, in the context of optimal portfolio's construction, we only model these moments through the GARCH (1, 1) schemes for the case of the variance and DCC-GARCH for correlation terms[11].

Once we have obtained the conditional correlations, the conditional covariance matrix is obtained from the next expression:

$$V_t = D_t \Gamma_t D_t$$

$$V_t = \begin{pmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{10,t} \end{pmatrix} \begin{pmatrix} 1 & \cdots & \rho_{1,10t} \\ \vdots & 1 & \vdots \\ \rho_{10,1t} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{10,t} \end{pmatrix} \tag{2.4.5}$$

, where D_t is the conditional deviation matrix, which main diagonal is composed by the conditional deviations obtained from the GARCH (1, 1) model and the rest of the matrix is composed by zeros. On the other hand, Γ_t is the dynamic correlation matrix obtained through the application of DCC-GARCH schemes. This matrix has its main diagonal composed by ones and conditional correlations out of the mentioned diagonal.

2.4.2. Model A. An application of the market risk premium

As we have been discussing, the risk aversion attitude is a key input in the estimation of optimal portfolios based on Expected-Utility maximization. However, in the equity literature there are not many studies that has explicitly calculated the risk aversion parameter. For that reason, we follow the approach proposed by Cotter and Hanly (2010) which is based on estimates the observed risk aversion through a derivation of the CRRA and applies it to generate utility maximizing based on the unleaded gasoline market. Specifically, as we have described in the last section, the derivation of the CRRA is based on the market risk premium. In this case, we use the mentioned estimation of the market risk premium, but for the EuroStoxx-50 data[12]. In addition, we adjust the equation (2.3.12) to our purpose, trying to avoid negative values of the parameter when the market yields less than the risk-free asset. For that reason, our proposal is based on fixing the numerator of the mentioned formula as the maximum between the excess market return at each month of the market and its mean for the whole data period. The formula would be as:

[11] These and other models are explained more in detail in section 2.5. , “Methodology. Conditional probability distribution moments”.

[12] We use daily closing prices and we transform them into returns by applying logarithms, following the same procedure described in section 2.1.

$$\alpha = \frac{Max\left(E(R_{m,t}) - Rf_t, h\right)}{\sigma_{m,t}^2}$$

$$where, \qquad h = \frac{\sum_{t=1}^{n}\left(E(R_{m,t}) - Rf_t\right)}{n} \tag{2.4.6}$$

Talking about the formula terms, $E(R_m) - Rf$ represents the market risk premium (the excess return on the market), α is the coefficient of relative risk aversion (CRRA) and σ_m^2 is the market variance. In particular, we use daily closing prices of EuroStoxx-50 index as market portfolio and 3-month German Treasury Bills at daily frequency as risk free rate. Then, we ascertain the returns in a monthly way.

In this case, we use the GARCH in mean schemes to estimate the α parameter. We have chosen these kind of models because of their good statistical properties, that is to say, these schemes are well-known for modelling the mean and the variance simultaneously. In particular, we estimate the risk parameter through the GARCH-M (1, 1) specification. To sum up, the mentioned model is as follows:

$$E(R_{m,t}) = \delta + \lambda\sigma_{m,t}^2$$

$$\sigma_{m,t}^2 = \omega + \alpha R_{m,t-1}^2 + \beta\sigma_{m,t-1}^2 \tag{2.4.7}$$

2.4.3. Model B. The CCI as a proxy of the customers' preferences

In the previous lines we have explained and adapted the approach proposed by Cotter and Hanly (2010). However, there are many ways to model the risk attitude which are not specifically based on the market risk premium. In this case, we propose to continue working within the context of quadratic utility functions and replacing the investor's wealth by the Consumer Confidence Indicator at monthly frequency in the expression of the quadratic CRRA (equation 2.3.11):[13]

[13] In this case, we work with the CCI indicator at levels, that is to say, without transforming it by applying logarithms.

$$CRRA = \frac{2b}{\left(\frac{a}{W}\right) - 2\text{b}}$$

$$\alpha(CRRA) = \frac{2b}{\left(\frac{a}{CCI}\right) - 2\text{b}} \tag{2.4.8}$$

Note that in this case, we must calculate the whole terms (mean and variance) of the optimal portfolio weights in a monthly way, because we are introducing the risk aversion level at monthly frequency too.

Otherwise, we have to make it clear that in this case we do not have to estimate any parameters. Conversely, in this case, we assign some arbitrary values to a and b parameters. Obviously, according to the last equation, the risk aversion parameter is going to be smaller if we reduce the numerator or increase the denominator. Given that and following the approach proposed by Dybvig (1983) and Grinblatt and Titman (1983), we give some economic coherence to this review, setting the values of a and b parameters according to the studied period. Thus, we fix the values 30 and 0.1 for the parameters a and b, respectively, in the case of the calm period. Moreover, in the case of the stressed period, we assign the values 25 and 0.1, respectively.

2.4.4. Model C. Risk aversion attitude and the downside risk measures

In performance literature, it is usually assumed that risk aversion cannot be expressed exclusively in terms of expected value and standard deviation, as there are many examples in which the Sharpe ratio violates the criterion of stochastic dominance. This ratio does not take into account those investors which are not concerned about the large deviations above the mean or threshold. For that reason, we propose another approach to model the risk aversion parameter based on the implementation of the downside risk measures. In particular, we use the first lower partial moment, fixing the threshold as a risk-free rate.

The higher the standard deviation, the greater risk aversion, but this is especially worrying for investors when they are on the side of the losses. We aim to find a different way to model the risk aversion parameter, which depends on the volatility and which takes into account the fact that market moves up or down, penalizing the negative trends with an increase of the parameter. Given that, our proposal is as follows:

$$\alpha = \sigma_{m.t}\left(1 + Max\left(Rf_t - R_{m,t}, 0\right)\right) \tag{2.4.9}$$

We use the 3-month German Treasury Bills as risk-free rate and the EuroStoxx-50 index as a proxy of the market (we take both prices at daily frequency and then we change them into monthly returns). On the other hand, talking about the market volatility, our purpose is to model the EuroStoxx-50 volatility by the application of dynamic schemes, such as EWMA and GARCH. Based on the above, we analyse this model in two different ways:

- **Model C.1 (EWMA)**

$$\sigma_{m,t}^2 = \lambda\sigma_{m,t-1}^2 + (1-\lambda)R_{m,t-1}^2 \tag{2.4.10}$$

- **Model C.2 (GARCH(1, 1))**

$$\sigma_{m,t}^2 = \omega + \alpha R_{m,t-1}^2 + \beta\sigma_{m,t-1}^2 \tag{2.4.11}$$

2.4.5. Model D. Constant risk aversion as a derivation of the Sharpe ratio

Focusing on the case of constant risk aversion, we must set a criterion for choosing an appropriate parameter according to the risk aversion attitude in Europe. In particular, our proposal is based on choosing several values according to each one of the mentioned literatures (section 2.3.3.). The chosen values are 1 (Arrow), 11 (Mehra and Prescott), 1.8 (Ghysels et al) and 4.93 (Guo and Whitelaw).

Once we have selected these values, we keep the ones that make the optimal portfolio has better performance at each month according to the Sharpe ratio. We ascertain this ratio monthly, to avoid the noise frequency of this type of data, and we use the 3-month German Treasury Bills as risk-free rate[14].

Lately, we calculate the average of the optimal parameters obtained at each month in the market, in order to reach a single risk parameter. Finally, the obtained average parameter is 8.51, and it is the one that will be used to model the optimal portfolios at each time of the market.

[14] Remember that we are working with daily equity prices, but we can evaluate the performance of our portfolio in a monthly way by ascertaining the returns and the variance at monthly frequency too.

2.5. METHODOLOGY. CONDITIONAL PROBABILITY DISTRIBUTION MOMENTS

In this section we show how to implement the conditional distribution models. Paying attention to the whole written formulas, we can appreciate that the modelization of the current variance or correlation is expressed as a function which depends on the residuals of the previous period, among other parameters. However, we aim to make it clear that in this research we do not apply the autocorrelation adjustment. The above means that we work with returns, in spite of having the whole models expressed in terms of residuals.

We estimate the model parameters through the application of the maximum log likelihood. Note that you can review the application of this technique for univariate and multivariate models in Appendix B.

2.5.1. Conditional variance/mean models

Knowing the volatility is really important in financial markets. Investors obviously are interested in the volatility of stock prices, as high volatility can mean huge losses or potential profits, and consequently lead to greater uncertainty. Given the above, the question is, how could we model the volatility of time series?

Talking about levels, a characteristic of most time series is that they are random walkers, ie, they are not stationary. Moreover, in the form of first differences, usually, they are stationary. As a result, the models are built with first differences. However, these differences often show wide variations or "volatility", which makes us think that variance of time series changes over time. In these cases, it is very useful to use the “Autoregressive conditional heteroscedasticity model” (ARCH), developed by Engle (1982). In this model, the unequal variance, may have an autoregressive structure, in which we observed that heteroscedasticity over different periods could be autocorrelated.

Since its discovery in 1982, the development of ARCH models has become a booming area, with all kinds of variations from the original model. One of the most popular is the Generalized Autoregressive Conditional Heteroscedasticity, proposed by Bollerslev (1986). The GARCH model in its simplest version is the GARCH (1, 1) and is the one we use in this paper to model the conditional volatility over time. In addition, we model de volatility through other conditional schemes, such as the EWMA and GARCH-M models. Note that the application of the last one is really important in this research, due to this scheme allows us to model the variance and the mean simultaneously.

- **Exponential smoothing, EWMA model**

To model the volatility structure through EWMA schemes, we assume the returns are following a simple stochastic process:

$$R_t = \gamma + \varepsilon_t$$
$$\varepsilon_t = \sigma_t \eta_t \qquad (2.5.1)$$
$$\text{Where} \;\; \eta_t \sim N(0, 1).$$

Thus, the EWMA model is defined as follows:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda)\varepsilon_{t-1}^2 \qquad (2.5.2)$$
$$s.a \;\; \lambda > 0$$

Where R_t denotes the assets' returns, η_t are the standardized returns, ε_t are the residuals and σ_t^2 are the assets' variances.

- **GARCH (1, 1) model**

This model states that current conditional variance depends not only on squared return of the previous period (as in ARCH (1)), but also on its conditional variance of the previous period. In fact, the GARCH (1, 1) model is much more like an ARCH (2).

Thus, the GARCH (1, 1) model looks like:

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \qquad (2.5.3)$$

With the following constraints

$$\alpha + \beta < 1$$
$$\omega > 0 \qquad (2.5.4)$$
$$\alpha \geq 0$$
$$\beta \geq 0$$

For regression purposes, the model can be rewritten as:

$$R_t = \gamma + \varepsilon_t \tag{2.5.5}$$
$$\varepsilon_t = \sigma_t \eta_t \quad \eta_t \sim N(0,1)$$

Where R_t denotes the assets' return, ε_t is the residual and σ_t^2 is the assets' variance. Moreover, following this scheme, the long-term variance can be calculated as follows:

$$\sigma^2 = \omega/(1-\alpha-\beta) \tag{2.5.6}$$

- **GARCH-M (1,1)**

One of the most important statements of financial theory is the relationship between risk and return. The CAPM model, for instance, implies a linear relationship between the expected return of the market portfolio and its variance. If the variance is not constant over time, then the conditional expected return of the market is a linear function of the conditional variance. Engle(1987) proposed the estimation of conditional variances by GARCH schemes and then these estimations will be used in the conditional means' estimation. This is well-known as the GARCH-in-Mean (GARCH-M) model.

The GARCH-M scheme models the mean by making it dependent on the variance. In addition, the variance is modelled by a GARCH (1, 1) scheme, so we have to estimate simultaneously the conditional mean and variance of the process.

The variance is modelled according to a GARCH (1, 1) scheme, as we describe before. As for the mean, it will look like this:

$$\mu_t = \delta + \lambda \sigma_t^2 \tag{2.5.7}$$

Where δ is a constant and λ is a parameter to be estimated.

The GARCH regression model, adding an extra regressor as the standard deviation, for this scheme is:

$$R_t = \delta + \lambda \sigma_t + \varepsilon_t$$
$$\varepsilon_t = \sigma_t \eta_t \quad \eta_t \sim N(0,1)$$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad (2.5.8)$$

Where R_t denotes the Eurostoxx-50 return, ε_t is the residual, σ_t^2 is the Eurostoxx-50 variance and λ is the CRRA. Further, in this model the $\lambda\sigma_t$ term could be interpreted as the risk premium.

2.5.2. Conditional correlation models

- **DCC GARCH model**

The dynamic conditional correlations GARCH model is defined as:

$$q_{ij,t+1} = \omega + \alpha(\eta_{i,t}\eta_{j,t}) + \beta q_{ij,t} \qquad (2.5.9)$$

With constraints

$$\begin{gathered} \overline{\omega} = (1-\alpha-\beta)\rho_{ij} \\ \alpha > 0 \\ \beta > 0 \end{gathered} \qquad (2.5.10)$$

, where $\eta_{i,t}$ y $\eta_{j,t}$ are the standardized returns of the chosen assets, obtained from the GARCH (1, 1) model.

In addition, to normalize the conditional correlation, we use the following expression:

$$\rho_{ij,t+1} = \frac{q_{ij,t+1}}{\sqrt{q_{ii,t+1}}\sqrt{q_{jj,t+1}}} \qquad (2.5.11)$$

Furthermore, to initialize the calculus of the correlation coefficient by DCC GARCH model, we must impose two initial conditions:

$$q_{ij,1} = \frac{\sum_{i=1}^{T} \eta_{i,t}\eta_{j,t}}{T}$$
$$q_{ii,1} = q_{jj,1} = 1 \tag{2.5.12}$$

2.6. EXPERIMENTATION. THE USE OF TIME-VARYING OPTIMAL PORTFOLIOS

In this section, we make an overview of the main results and findings obtained by the application of the different studied models. In particular we analyse the estimated parameters, the monthly evolution of the different portfolio's weights and the conditional evolution of the risk aversion parameter over the two selected periods (calm and stress).

2.6.1. Parameter estimation

First of all, we want to make it clear that in this section, we only analyse in detail the estimated parameters for each one of the risk aversion models described in section 2.4, while the estimated parameters for the construction of conditional optimal portfolios (GARCH and DCC GARCH schemes), are shown in Appendix E. In addition, we can review the dynamic volatilities and correlations obtained from the GARCH models in Appendices C and D.

Them, we show the main results and parameters obtained for each of the proposed time-varying risk aversion models. In addition, we must remember that the whole estimations are based on the EuroStoxx-50 returns at monthly frequency.

Note that in this case, we exclude Model B and Model D because we do not have to estimate any parameters in these schemes. The above is because of we fixed the parameters (a, b) in the case of Model B. Moreover, talking about Model D, we have assigned a constant risk aversion parameter (8.51) obtained as a derivation of the Sharpe Ratio, so we do not have to show any results.

- **Parameters of Model A (GARCH-M(1, 1))**

The results of the estimation procedure for Model A are shown in the next table:

Table 9. Estimated parameters. Model A

CALM PERIOD	
Parameter	**Value**
ω	0,00001
α	0,7266
β	0,2724
λ	0,8999
δ	0,000005

STRESS PERIOD	
Parameter	**Value**
ω	0,00001
α	0,8919
β	0,0755
λ	0,9000
δ	0,000001

Source: Compiled by the author

Then, we can appreciate an extra value in Table.9, the λ parameter, showing the effect of the variance in the mean model. Substituted into the model, the equations will look as follows:

$$\mu_t = 0.000005 + 0.8999\sigma_t^2$$
$$\sigma_t^2 = 0.00001 + 0.7266R_{t-1}^2 + 0.2724\sigma_{t-1}^2 \tag{2.6.1}$$

$$\mu_t = 0.000001 + 0.9\sigma_t^2$$
$$\sigma_t^2 = 0.00001 + 0.8919R_{t-1}^2 + 0.0755\sigma_{t-1}^2 \tag{2.6.2}$$

In this model and for both studied periods, the previous variance has a low impact on the current variance, while the most of the effects come from the returns of the previous period. Even though an extra parameter was added in the equation of the mean, it has no impact in the variance in mean model, that is to say, it is not significantly different from 0.

- **Parameters of Model C.1 (EWMA)**

The estimated parameters for EWMA model are represented in the next figures:

Table 10.Estimated parameters. Model C.1

CALM PERIOD	
Parameter	**Value**
λ	0,9478

STRESS PERIOD	
Parameter	**Value**
λ	0,9650

Source: Compiled by the author

$$\sigma_t^2 = 0.9478\sigma_{t-1}^2 + 0.0522R_{t-1}^2$$
$$\sigma_t^2 = 0.965\sigma_{t-1}^2 + 0.035R_{t-1}^2 \qquad (2.6.3)$$

As we can observe in Table.10, in this fist model and for both studied cases, the previous returns have a low impact on the current variance, while the most of the effects come from the variance of the previous period, that is to say, the previous variance has a high impact on the current variance.

- **Parameters of Model C.2 (GARCH(1, 1))**

The estimated parameters for the case of the GARCH (1, 1) model are as follows:

Table 11. Estimated parameters. Model C.2

CALM PERIOD	
Parameter	**Value**
ω	0,000001
α	0,9172
β	0,0756

STRESS PERIOD	
Parameter	**Value**
ω	0,000005
α	0,8944
β	0,0899

Source: Compiled by the author

$$\sigma_t^2 = 0.000001 + 0.9172R_{t-1}^2 + 0.0756\sigma_{t-1}^2$$
$$\sigma_t^2 = 0.000005 + 0.8944R_{t-1}^2 + 0.0899\sigma_{t-1}^2 \qquad (2.6.4)$$

As can be appreciated in Table.11, the constants are very low and have a little effect on the current variance. The hypothesis that ω is not significantly different from 0 cannot be accepted.

The most of the effects, however, come from the variance and the returns of the previous period. Whereby the variance seem to say relatively a lot less than the returns for both periods.

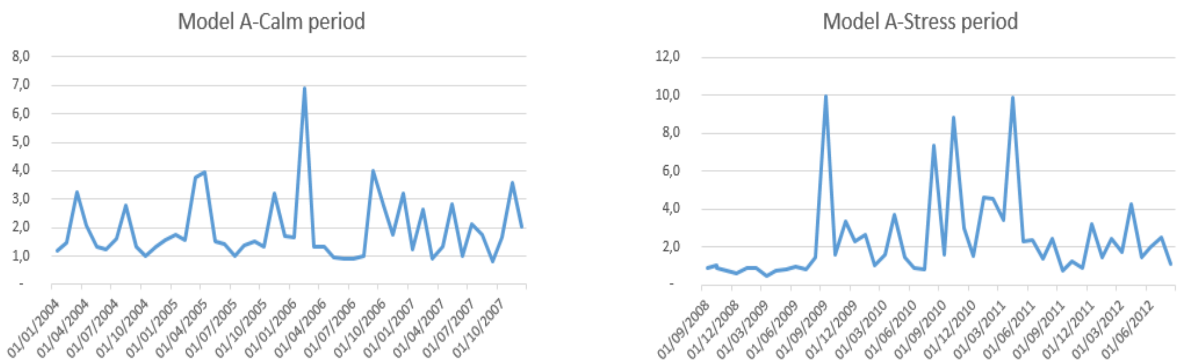
2.6.2. Monthly evolution of the risk aversion parameter

Once we have estimated the parameters associated to each one of the previous models, then we ascertain the risk aversion attitude. We assess the results and findings for the mentioned models (we have exclude Model D because the parameter is constant over the time frame) and for the two selected scenarios (calm and stress), in order to appreciate which one of the models does better for the whole period.

- **Model A. GARCH-M(1, 1)**

The conditional risk aversion parameters ascertained according model A are shown in the next figure:

Figure 8. Monthly evolution of the risk aversion parameter. Model A



Source: Compiled by the author based on the CRRA ascertained at monthly frequency. We use the GARCH-M schemes, in order to model the conditional variance and return simultaneously. We use the EuroStoxx-50 data, available at DataStream database.

According to Figure.8, the risk aversion parameter is greater on average in the stress period, which make sense, due to the common investor is more risk averse when the market is having a negative trend (bearish market). In addition, according to this model, we can assess that risk-aversion time series present a random variability, due to we can observe groupings or “clusters".

Table 12. Key elements of Model A

Calm period	
MEAN	1,9328
MIN	0,8116
MAX	6,8949
STDEV	1,1420

Stress period	
MEAN	2,4284
MIN	0,4657
MAX	9,9864
STDEV	2,2914

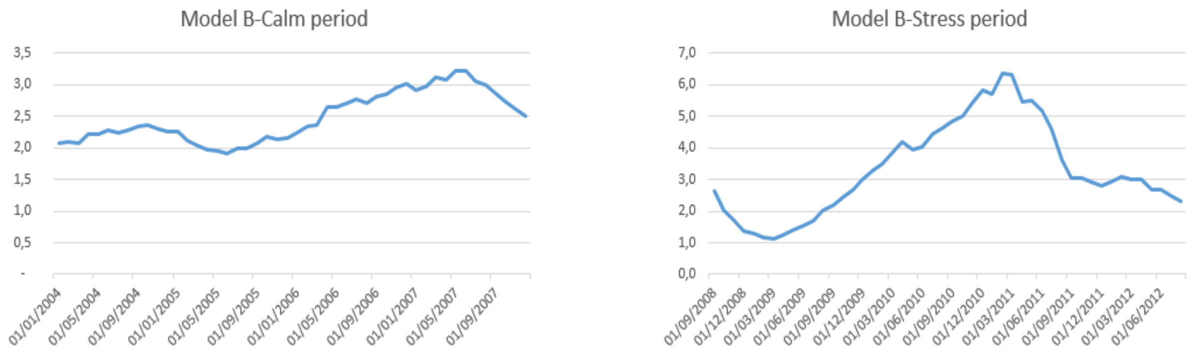
Source: Compiled by the author

Moreover, as we can appreciate in the previous tables (Table.12), the maximum values are greater and the minimum values are lower in the case of the stress period (the values range between 0.5 and 10). In addition, the above can also be seen reflected in the conditional parameters' standard deviation, due to it is higher in the second case.

- **Model B. CCI**

The estimated risk aversion parameters for the case of model B are as follows:

Figure 9. Monthly evolution of the risk aversion parameter. Model B



Source: Compiled by the author based on the CRRA ascertained at monthly frequency. We use the CCI data. These data are available at Europa.Eu

As in the previous model, the risk aversion parameter is greater on average in the stress period, as can be seen in Figure.9. However, this case is a bit different from the last model because the new one exhibits a more stable trend over the studied period, without the presence of clusters. The above could be due to the fact that in this case we do not follow a conditional volatility scheme to model the risk aversion parameter.

Table 13. Key elements of Model B

Calm period	
MEAN	2,4774
MIN	1,9183
MAX	3,2254
STDEV	0,3927

Stress period	
MEAN	3,3183
MIN	1,1222
MAX	6,3529
STDEV	1,4807

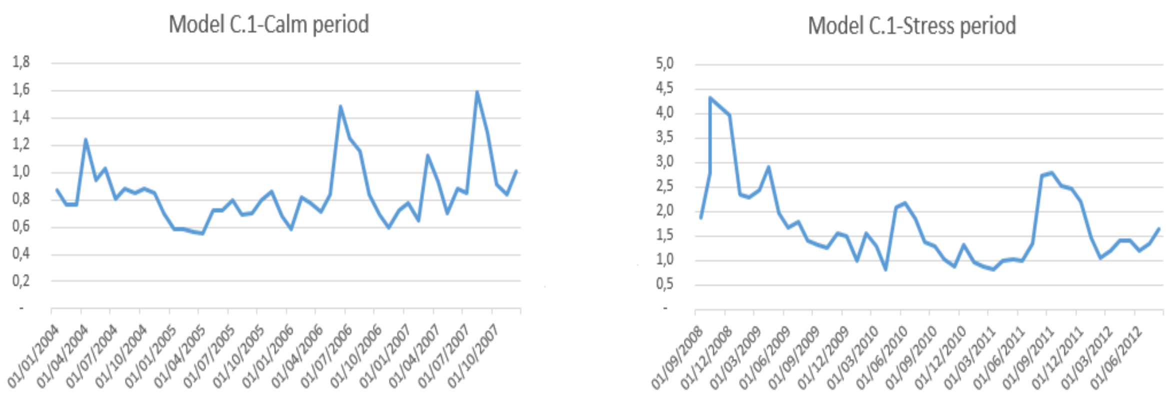
Source: Compiled by the author

Furthermore, we can also assess, following Table.13, that maximum and minimum values are more extreme in the case of stress period, reaching values close to 7.

- **Model C.1. EWMA**

Then, we show the conditional risk aversion parameters ascertained according Model C.1:

Figure 10. Monthly evolution of the risk aversion parameter. Model C.1



Source: Compiled by the author based on Downside risk measures ascertained at monthly frequency. We implement EWMA schemes, in order to model the conditional volatility. We use the EuroStoxx-50 data, available at DataStream database.

According to Figure.10, as for the case of Model A, the risk aversion parameter is higher on average in the stress period. Then, we evaluate the main characteristics of this model through the following table:

Table 14. Key elements of Model C.1

Calm period	
MEAN	0,8505
MIN	0,5527
MAX	1,5904
STDEV	0,2281

Stress period	
MEAN	1,7208
MIN	0,8066
MAX	4,3359
STDEV	0,7792

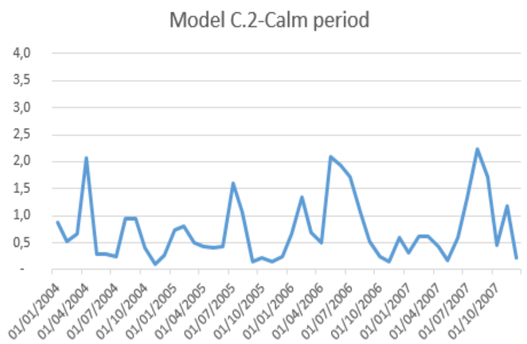
Source: Compiled by the author

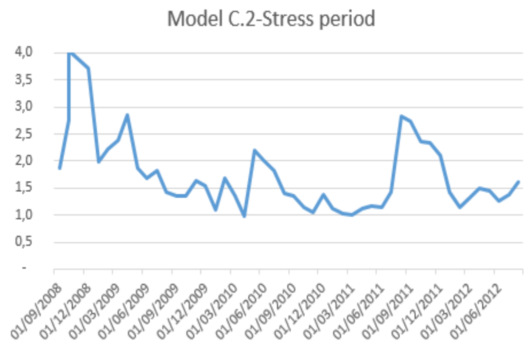
However, paying attention to Table.14, we can appreciate that Model C.1 is also different because in this case, even though the minimum and the maximum parameters follow the same line as in model A, they are much lower. Further, as in the mentioned model, we can assess that the parameter series present a random variability, due to we can observe groupings or “clusters".

- **Model C.2. GARCH(1, 1)**

The estimated parameters for the case of Model C.2 are as follows:

Figure 11. Monthly evolution of the risk aversion parameter. Model C.2





Source: Compiled by the author based on Downside risk measures ascertained at monthly frequency. We implement GARCH (1, 1) schemes, in order to model the conditional volatility. We use the EuroStoxx-50 data, available at DataStream database.

As we can deduct from Figure.11, this model is more similar to the previous one than any other of the assessed cases. Its parameters are essentially the same on average. Then, analysing the following table:

Table 15. Key elements of Model C.2

Calm period	
MEAN	0,7456
MIN	0,0961
MAX	2,2254
STDEV	0,5798

Stress period	
MEAN	1,7387
MIN	0,9738
MAX	4,0416
STDEV	0,6868

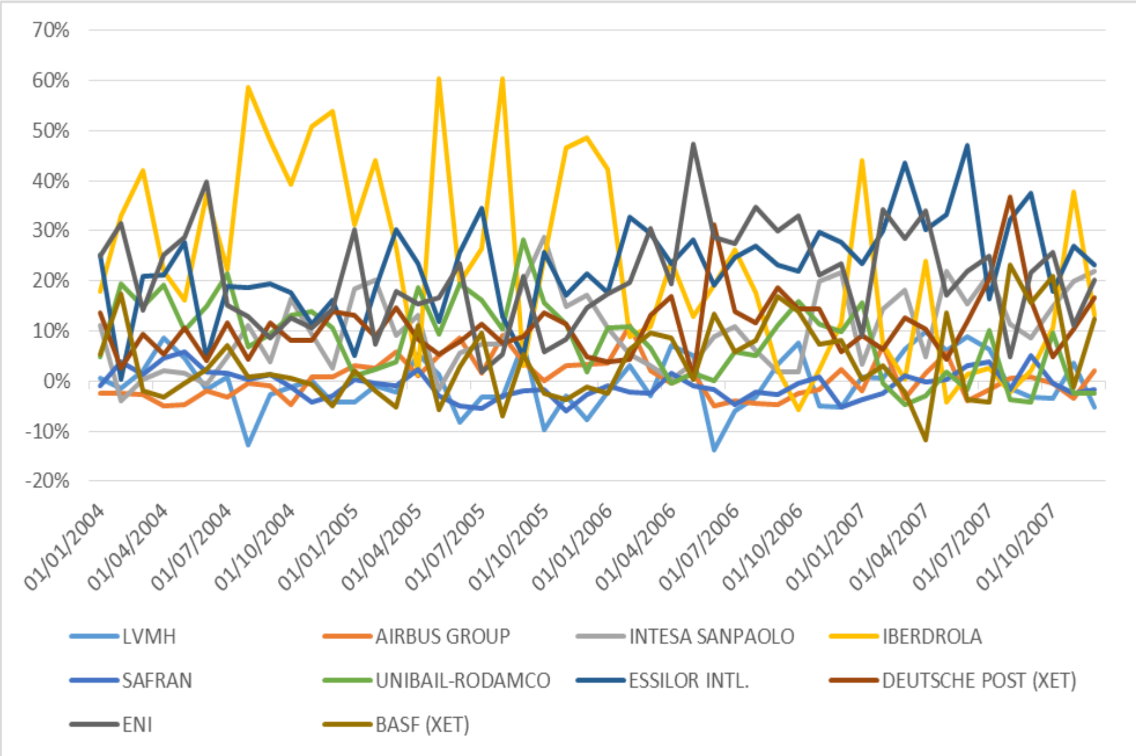
Source: Compiled by the author

In accordance with Table.15, the minimum values are lower in the calm period and greater in the stressed one. However, in occurs inversely for the case of the maximum values.

2.6.3. Monthly evolution of the optimal portfolio weights

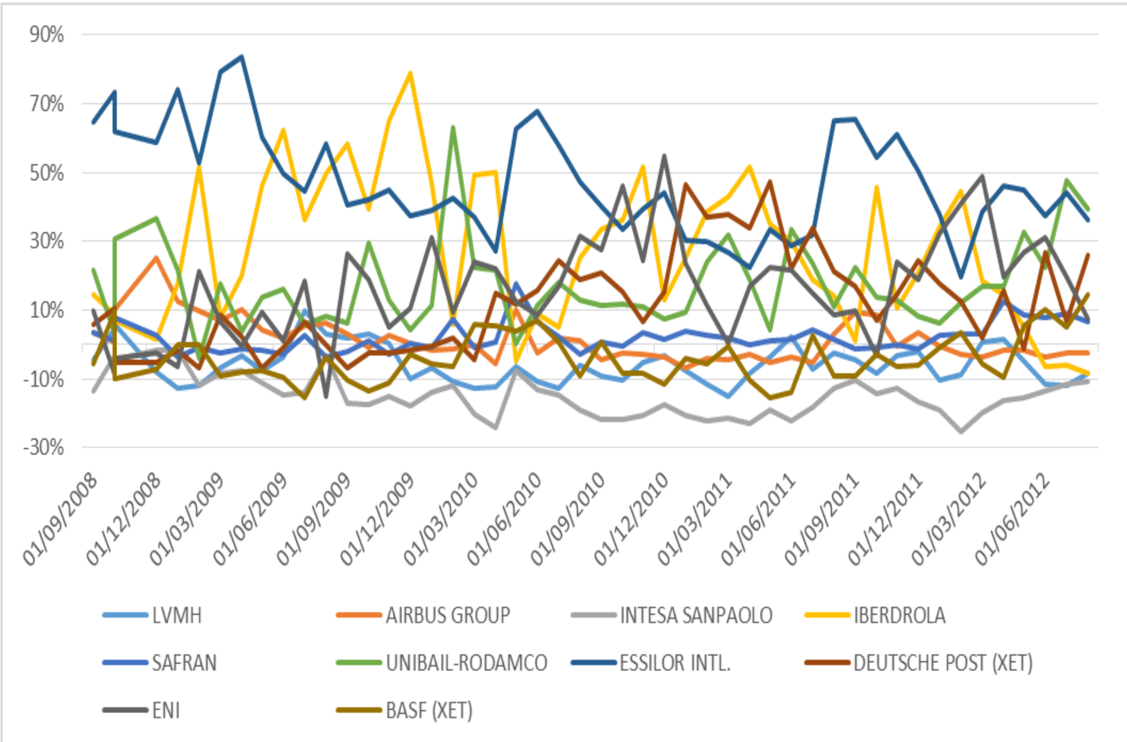
Then, we ascertain the optimal portfolio weights of each one of the whole mentioned models (A, B, C.1, C.2, D), that is to say, those weights that maximize the investor´s expected utility. As in the last section, we analyse the portfolio evolution for the two selected periods. Moreover, we only assess in detail the evolution of the constant risk aversion attitude, Model D. The above is because the conditional evolution trend is really similar for all the studied cases. In spite of that, note that you can review the rest of the conditional optimum models in Appendix F.

Figure 12. Monthly evolution of the optimal portfolio for the calm period. Model D



Source: Compiled by the author

Figure 13. Monthly evolution of the optimal portfolio for the stress period. Model D



Source: Compiled by the author

According to Figure.12, we can assess the evolution of the optimal weights that an investor has to assign to each one of the previously selected equities, in order to maximize his expected utility function at each time of the market. In particular, in this chart we are analysing the evolution of the optimal portfolio in the case of the calm period.

Paying attention to the last graph of the portfolio evolution over the stress period (Figure.13), we can appreciate that assets' trend has several differences regarding to the calm period analysis. Firstly, the optimal weights are higher on average. In addition, we can observe that weights distribution is a bit different.

- **Summary table**

Then, we show a summary table of the analysed weights according to each one of the mentioned models and for the two studied periods:

Table 16. Summary of the optimum percentage weights/ calm period

	LVMH	AIRBUS	INTESA	IBERD	SAFRAN	UNIBAIL	ESSILOR	D. POST	ENI	BASF
Model A	- 1,00	0,13	10,09	24,91	- 1,41	8,43	23,15	10,61	20,31	4,78
Model B	- 0,75	0,24	10,13	24,45	- 1,14	8,38	23,04	10,77	20,46	4,42
Model C.1	- 1,75	- 0,32	9,99	26,25	- 2,18	8,60	23,46	10,21	19,91	5,83
Model C.2	- 3,33	- 1,16	9,73	28,83	- 3,49	8,94	24,08	9,35	19,10	7,94
Model D	- 0,40	0,43	10,18	23,80	- 0,77	8,31	22,90	10,97	20,66	3,93

Source: Compiled by the author

Following Table.16, which shows the average of the conditional portfolio weights expressed as a percentage, we can evaluate more in detail how each one of the assessed portfolios behaves over the selected period. Thus, we can remark the conditional evolution of IBERDROLA (23.80%) and ESSILOR (22.90%), which are the firms that have captured the greatest weights on average over the considered period. Moreover, the lowest weights have been assigned to LVMH (-0.40%) and SAFRAN (-0.77%), which are the only two companies by which, we introduce the short selling strategy.

Table 17. Summary of the optimum percentage weights/ stress period

	LVMH	AIRBUS	INTESA	IBERD	SAFRAN	UNIBAIL	ESSILOR	D. POST	ENI	BASF
Model A	- 5,24	1,43	-15,04	26,47	2,32	17,62	47,70	12,02	16,32	- 3,60
Model B	- 5,51	1,21	-15,72	26,77	2,32	17,78	46,71	12,58	17,35	- 3,50
Model C.1	- 5,05	1,42	-15,20	25,76	2,44	17,83	47,91	12,04	16,25	- 3,40
Model C.2	- 5,10	1,42	-15,16	25,91	2,41	17,79	47,86	12,04	16,28	- 3,45
Model D	- 5,62	1,37	-14,83	27,43	2,11	17,48	47,28	12,15	16,57	- 3,94

Source: Compiled by the author

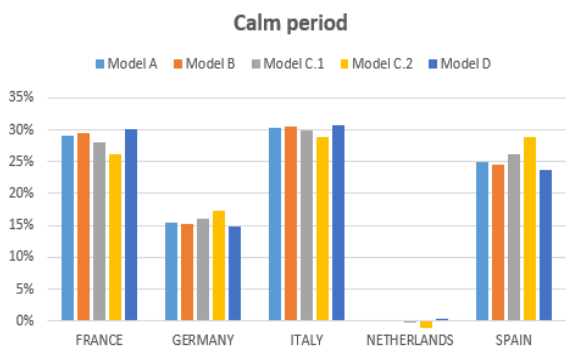
As in the last case, Table.17 shows the weights of the 10 firms expressed as a percentage. In this case, we must highlight the conditional evolution of ESSILOR (47.28%), because this firm have captured the highest weights on average over the considered period, standing well above the rest of companies. Moreover, the lowest weights have been assigned to INTESA (-14.83%), LVMH (-5.62%) and BASF (-3.94%) which are the base of the short selling strategy.

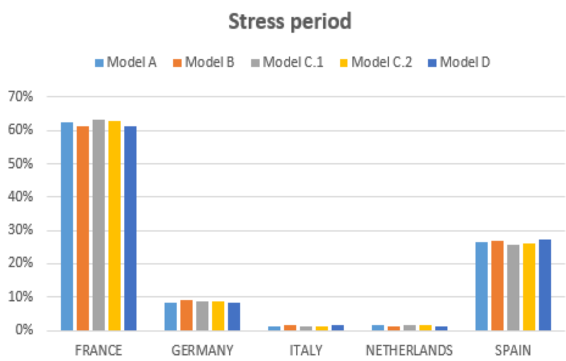
2.7. ANALYSIS OF PORTFOLIO MANAGEMENT. SOME PERFORMANCE RATIOS

In this section, we show a number of ratios and risk measures to analyse the performance of our portfolio. In addition, we assess the exposures of the different studied funds. Note that we spend the most of this section with the analysis of the Sharpe Ratio and as a consequence it is the one that we explain in greater detail.

2.7.1. Portfolio exposures. Geographical and sector analysis

Figure 14. Geographical Exposures





Source: Compiled by the author as an average of the different weights showed by each one of the models over the mentioned periods. Note that in this case we group these weights by Countries.

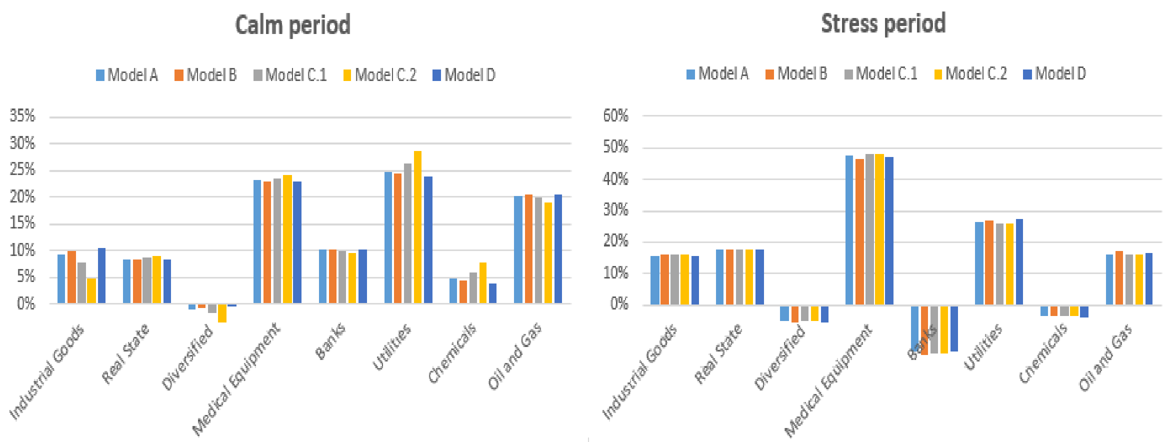
The above graphs (Figure.14) show the geographical exposure of our portfolios, that is to say, the assigned weight to each one of the main European countries. Thus, this figure represents a comparative analysis of the averaged weights that we have assigned for the whole studied models and for both studied periods.

On the one hand, talking about the calm period we can appreciate that imposed weights are really similar for the case of France, Italy and Spain, ranging between 25% and 30%, being perhaps Italy the country which has the higher weights on average. In another way, our portfolios have little exposure to Netherlands, showing weights that become negative in the case of some models. The above means that in this case, we are harmed by the increases in Dutch market and conversely, we are rewarded by downward trends in that market.

On the other hand, in the recession period, the largest averaged percentage is located in France (more than 60%). Moreover, the rest of the countries have a much lower associated percentage, remarking the case of Spain (25%)

Then, we evaluate the Sector exposure of our portfolios for both time frames:

Figure 15. Sector Exposures



Source: Compiled by the author as an average of the different weights showed by each one of the models over the mentioned periods. Note that in this case we group these weights by sectors

Paying attention to Figure.15 and talking about the calm scenario, we can assess that the largest averaged weights are assigned to Medical and Utilities sectors (25%), while the Diversified sector is the only one that has assigned a negative exposure.

Otherwise, as in the case of geographical exposures, the greatest percentages have been assigned to the Medical Equipment sector (50%), which agrees with the highest percentage assigned by geographical areas (France: 60%). Furthermore, the negative exposure of our portfolios is better distributed: Banks (-10%), Real State (-5%) and Chemicals (-3%). The above negative weights make sense. For instance, it seems reasonable to avoid the exposure to the European Banking sector, because of its huge decline.

2.7.2. Unconditional distribution moments. Individual assets versus composed portfolios

Then, we show the first two unconditional moments of each one of the assets and the appropriate moments for the five studied portfolios:

Table 18. Unconditional moments. Individual assets

	INDIVIDUAL ASSETS. CALM PERIOD									
MEASURES	LVMH	AIRBUS	INTESA	IBERD	SAFRAN	UNIBAIL	ESSILOR	D. POST	ENI	BASF
Mean	0,0863	0,0353	0,1334	0,2342	-0,0459	0,1679	0,1816	0,0871	0,1289	0,1973
Std.deviation	0,1899	0,3163	0,2065	0,1784	0,2870	0,2557	0,1801	0,2088	0,1693	0,1787
	INDIVIDUAL ASSETS. STRESS PERIOD									
MEASURES	LVMH	AIRBUS	INTESA	IBERD	SAFRAN	UNIBAIL	ESSILOR	D. POST	ENI	BASF
Mean	0,1376	0,1574	-0,2456	-0,1899	0,1932	0,0600	0,1554	-0,0088	-0,0523	0,1077
Std.deviation	0,3585	0,3983	0,5690	0,3682	0,4112	0,3081	0,2232	0,3790	0,3494	0,3898

Source: Compiled by the author. *Note that expected returns and standard deviations are expressed in annual terms. It is possible by multiplying the monthly returns by* 12 *and their associated standard deviations by* $\sqrt{12}$.

Table 19. Unconditional moments. Composed portfolios

	COMPOSED PORTFOLIOS									
	CALM PERIOD					STRESS PERIOD				
MEASURES	Model A	Model B	Model C.1	Model C.2	Model D	Model A	Model B	Model C.1	Model C.2	Model D
Mean	0,1727	0,1711	0,1773	0,1860	0,1690	0,0570	0,0546	0,0598	0,0591	0,0523
Std.deviation	0,1068	0,1068	0,1070	0,1081	0,1067	0,1676	0,1675	0,1676	0,1676	0,1675

Source: Compiled by the author. *Note that expected returns and standard deviations are expressed in annual terms. It is possible by multiplying the monthly returns by* 12 *and their associated standard deviations by* $\sqrt{12}$.

Paying attention to Table.18 and Table.19, we can observe that mean and standard deviation of individual assets and composed portfolios are expressed in annual terms, due to this is the preferred approach in finance. In addition, talking about the assets' moments, we need to annualize them because this is a requirement to ascertain the optimal portfolio weights. Otherwise, we must make it clear that skewness and kurtosis of the proposed portfolios are assumed to be 0 and 3, respectively, as we have assumed normality when calculating such portfolios[15].

Then, talking about the calm period we can assess that SAFRAN firm has a negative return on average and a rather high standard deviation. However, with portfolio construction, through the diversification effect, we can get this deviation greatly reduced, as can be seen in Table 19. Given that, we have finally achieved a well-diversified portfolio for each one of the studied models. The above means that we have reached some portfolios with fairly good results on average (Mean=17%, Stdv=10%).

Moreover, analysing more in detail the stress period, we can observe that in general terms, the set of assets have a lower mean and a greater deviation, regarding to the calm period. In fact, as can be seen at the bottom of table 18, we can appreciate four firms which have a negative

[15] Although we know that the probability distribution of assets' returns has the problem of heavy tails (leptokurtosis), we assumed normality in returns

performance in terms of returns (INTESA, IBERDROLA, ENI and DEUSTCHE POST). In addition, as in the last period, we have finally reached a well-diversified portfolio for each one of the studied models. However, due to the above reasons, the results of the stress portfolios are worse on average (Mean=5%, Stdv=16%).

2.7.3. The Sharpe Ratio

- **How to ascertain the Ratio**

The Sharpe ratio measures the excess return per unit of risk[16]. This ratio allows prioritizing the different investment options based on return and risk. In addition, the Sharpe ratio should be only used when normality is assumed, due to the standard deviation of the portfolio only makes sense if we have a stable probability distribution over the sample period. The higher the value of this ratio is, the best performance of our portfolio, that is to say, it indicates that we are getting higher returns relative to their associated risk.

We can ascertain it as follows:

$$SR = \frac{E(R_p) - Rf}{\sigma(R_p)} \tag{2.7.1}$$

, where $E(R_p)$ is the expected portfolio return, Rf is the risk free rate and $\sigma(R_p)$ is the volatility approximated as the standard deviation of the portfolio. For the allocation period of our portfolios, we show how this ratio evolves for each one of the studied portfolio models. We calculate it at monthly frequency and we use the 3-month German Treasury Bills as risk-free asset return.

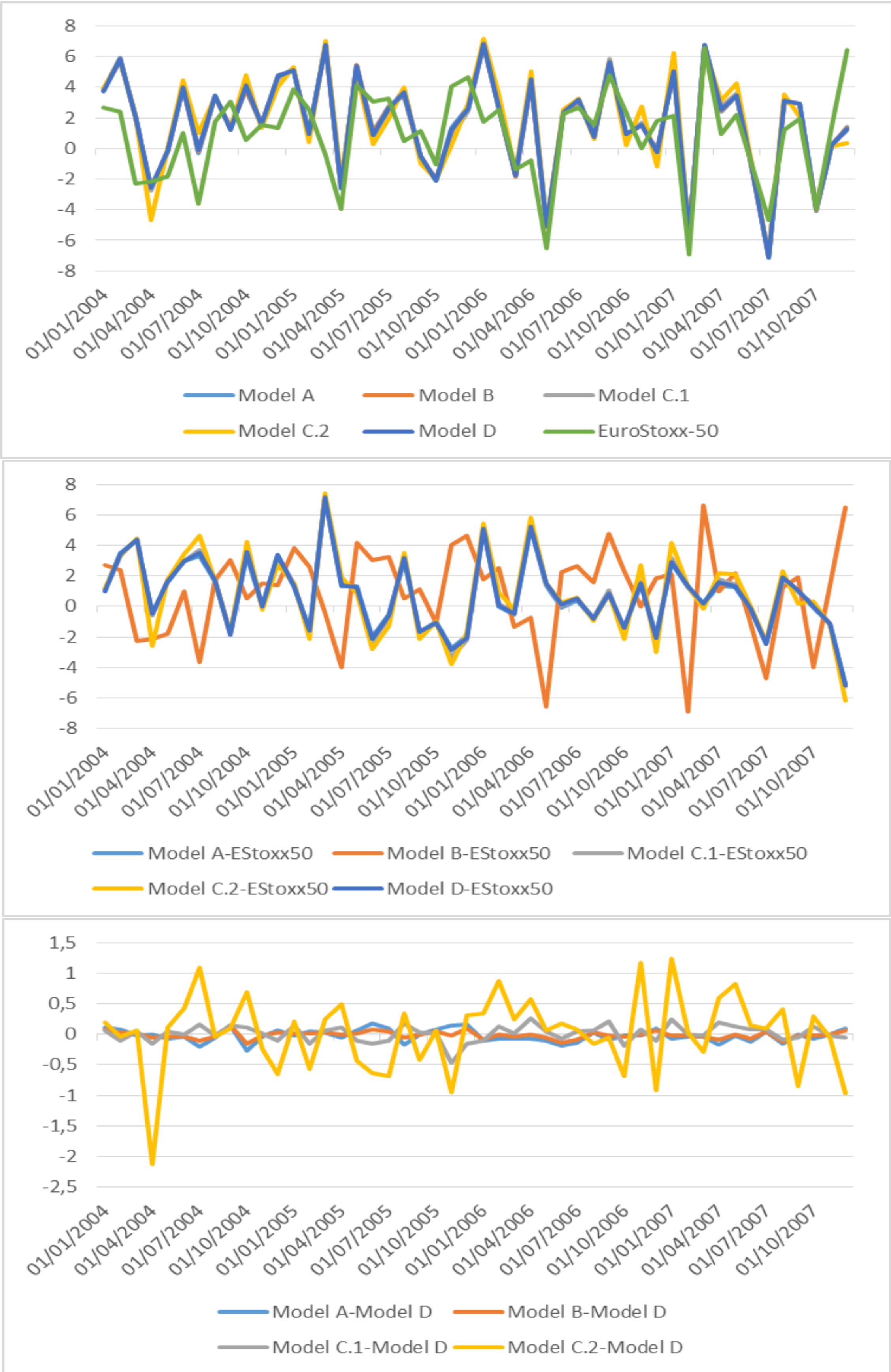
In spite of calculation this ratio monthly, we expressed it in annual terms for all the mentioned portfolios. To carry out this process, we have annualized the risk-free return because we have previously ascertained the expected return and the variance of each one of the portfolios in annual terms. Further, as in previous sections, we analyse this ratio evolution for the two selected periods.

- **Conditional evolution of the Sharpe Ratio**

First of all, we begin with the study of the calm period:

[16] To review this section, we rely on Sharpe (1994) and Sharpe (1966)

Figure 16. The Sharpe Ratio. Calm period review



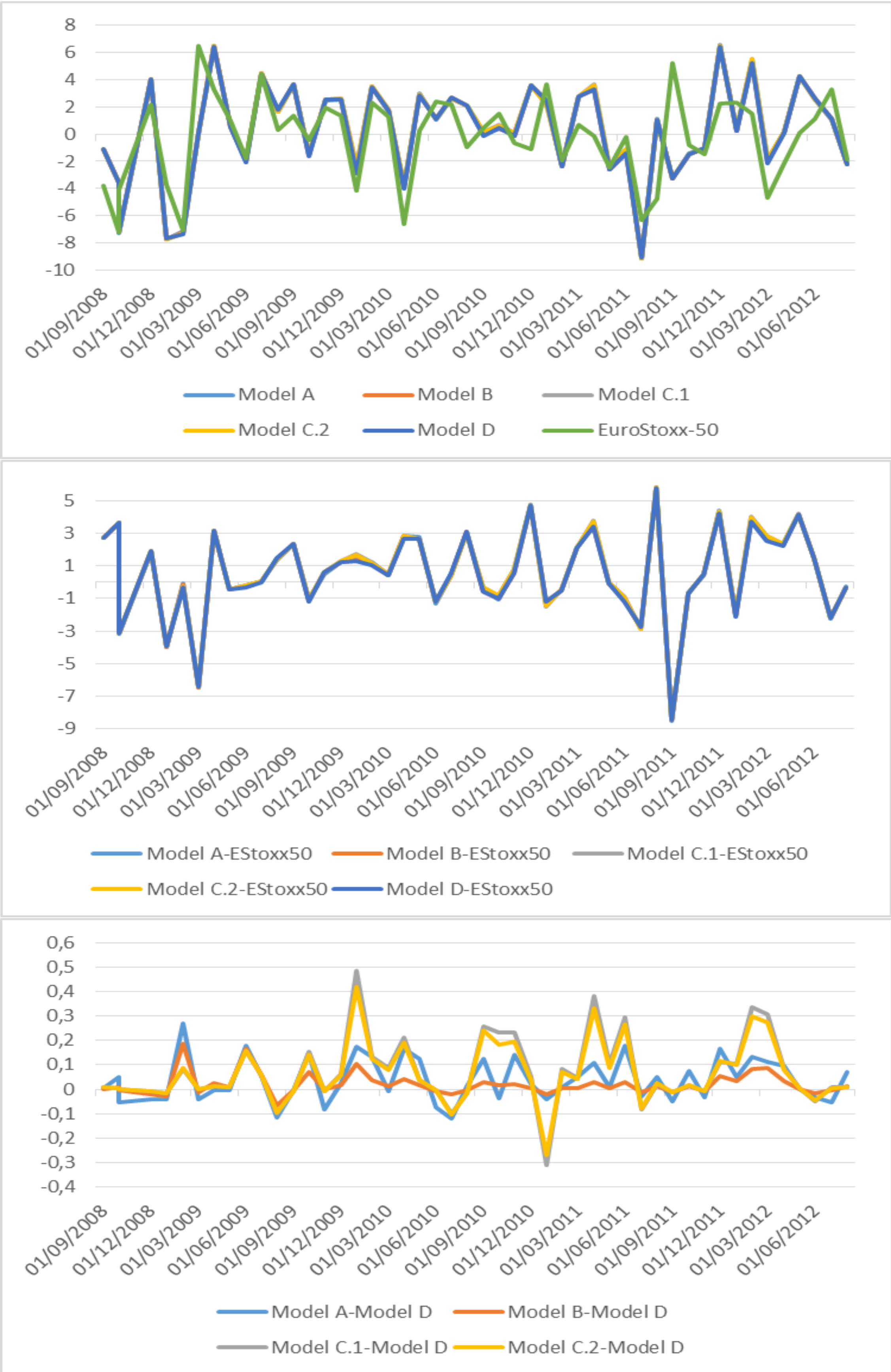
Source: Compiled by the author. Expressed in annual terms

According to Figure.16, we can observe the evolution of the Sharpe Ratio over the calm time frame for the whole studied models. Note that for a well understanding by the reader, we have divided the graph in three clear-defined parts. The first part of the graph compares the selected models against the benchmark (market performance). The second part follows the same line as the first one, but in this case we are plotting the whole models in excess of the market index. The key of the above, is to appreciate more in detail how well we are performing with an active managing of our equities portfolios regarding to the passive management (be invested in the EuroSroxx-50 index over the time frame). The bottom of the graph is about the models in excess of the naïve one (Model D), in order to visualize how well are doing the dynamic models regarding to the constant one.

Following the previous graphs, firstly, we can appreciate that the performance of our portfolio over the calm period is quite acceptable for the whole analysed models as it ranges between -6 and 7, been the Model C.2 the best model on average. According to the above, as can be seen in the graphs, we are outperforming the benchmark (EuroStoxx-50) for much of the time frame and for the case of the Models C.1, C.2 and D. Further, we can observe a greater frequency of negative Sharpe values at the end of the period, that is to say, when we are really close to the beginning of the Economic Crisis. Note that negative values are not too high in this calm period. The above seems reasonable, because we are talking about a calm period in which the equity investment offered more attractive returns than fixed income, so it is logical to assume that the Sharpe Ratio is going to be almost always positive.

Then, we continue our study analysing the stress period:

Figure 17. The Sharpe Ratio. Stress period review



Source: Compiled by the author. Expressed in annual terms

According to Figure.17, as in the last case, we can observe the evolution of the Sharpe Ratio over the stress period for the whole mentioned models. In this case, we follow the same structure as for the calm period. Given that, the first part of the graph compares the selected models against the benchmark (market performance). The second part follows the same line as the first one, but in this case we are plotting the whole models in excess of the market index. The bottom of the graph is about the models in excess of the naïve one (Model D), in order to visualize how well are doing the dynamic models regarding to the constant one.

Otherwise, talking about the stress time frame, our portfolios continue making a rather good performance regarding to the benchmark (in fact, the EuroStoxx-50 index has a negative stress ratio on average). However, in this case, the Sharpe Ratio shows negative values with a higher frequency than the observed in the previous period, ranging between -8 and 7. According to the previous lines, in those periods in which we observe negative values of the ratio, we can intuit that invest our money in fixed income would be more profitable than keep it invested in our portfolio models. The above comes from the fact that negative values of the Sharpe Ratio are caused by high values of the risk free rate, that is to say, the risk free asset (fixed income) offers a greater return regarding to our portfolios.

Then, we show a summary table of the averaged ratios described above for the two analysed periods:

Table 20. Sharpe Ratio average. Calm period

	PORTFOLIOS					
SHARPE AVERAGE	**Model A**	**Model B**	**Model C.1**	**Model C.2**	**Model D**	**EuroStoxx-50**
Sharpe Standard	1,6535	1,6617	1,6888	1,6913	1,6769	0,9048
portfolios vs EStoxx50	0,7487	0,9048	0,7840	0,7865	0,7720	-
Dynamic vs Constant	-0,0234	-0,0152	0,0119	0,0144	-	-

Source: Compiled by the author. Expressed in annual terms

Table 21. Sharpe Ratio average. Stress period

	PORTFOLIOS					
SHARPE AVERAGE	**Model A**	**Model B**	**Model C.1**	**Model C.2**	**Model D**	**EuroStoxx-50**
Sharpe Standard	0,2437	0,2258	0,2770	0,2698	0,2052	-0,3263
portfolios vs EStoxx50	0,5700	0,5521	0,6033	0,5961	0,5315	-
Dynamic vs Constant	0,2437	0,2258	0,2770	0,2698	-	-

Source: Compiled by the author. Expressed in annual terms

Note that both tables (Table.20 and Table.21) are expressed in annual terms. The first row describes the standard Sharpe Ratio reached on average for each one of the mentioned periods. Moreover, the second row is showing the mean of the Sharpe Ratio on excess of the EuroStoxx-

50 index, reached for the whole models. Lately, the third row is about the averaged differences between the dynamic portfolios and the constant one (Model D)

According to the last tables, we continue the analysis of the Sharpe Ratio assessing the differences in performance terms between the different selected models. Firstly, talking about the calm period, the fund which exhibits the highest ratio over this time frame is Model C.2. However, paying attention to the stress period, the best fund in Sharpe terms is Model C.1, although it is followed closely by Model C.2. Thus, we can conclude that Model C.2 is the best one according to the Sharpe Ratio, because this fund has the best performance on average for the two considered periods.

- **Hypothesis testing for the Ratio average**

In this case, we implement a mean-difference test in two independent samples, that is to say, we are comparing whether the differences between the averaged ratios of dynamic models and the constant one are significant or not. The above is well-known as a parametric test.

We can decide whether we reject the null hypothesis or not in two different ways. That is to say, the t-statistic and the associated P-value. Given that, we can ascertain the t statistic, which is defined as:

$$t = \frac{\mu_1 - \mu_2}{\sigma_{1,2}\sqrt{2/n}} \tag{2.7.2}$$

, where $\mu_1 - \mu_2$ represents the mean difference between each one of the models and the constant one (Model D). Otherwise, $\sigma_{1,2}$ is the joint deviation and n represents the data size. Then, the null hypothesis of this parametric test is as follows:

$$H_0\colon\ \mu_1 - \mu_2 = 0 \tag{2.7.3}$$

Then we show the statistics and p-values for the different studied models in the following tables:

Table 22. Mean test. Calm period

Model A		Model B		Model C.1		Model C.2	
t-statistic	P-value	t-statistic	P-value	t-statistic	P-value	t-statistic	P-value
-0,0351	0,4861	-0,0228	0,4909	0,0179	0,4929	0,0211	0,4916

Source: Compiled by the author

Table 23. Mean test. Stress period

Model A		Model B		Model C.1		Model C.2	
t-statistic	P-value	t-statistic	P-value	t-statistic	P-value	t-statistic	P-value
0,0526	0,4791	0,0281	0.4888	0,0981	0.4611	0,0882	0,4649

Source: Compiled by the author

According to Table.22 and Table.23, we accept the null hypothesis at a significance level of 95% for the whole studied models and for the two mentioned periods. This is due to the $pval$ is not less than α (0.05). Thus, it is not possible to affirm that dynamic models are better than the static one, that is to say, there are not significant evidences.

According to the previous lines, if we had to invest some money in a risky portfolio, we would choose the one associated to Model C.2. However, if we observe the results obtained in the last test, we can appreciate that there are not many differences between the different selected models. In fact, if we prefer not to make our life complex (by ascertaining the conditional risk-aversion attitude through different mathematical equations), we can select the constant risk-aversion scheme (Model D). As we have mentioned before, this is because the differences between the best model (Model C.2) and the worst one (Model D) are not really significant.

2.7.4. The Certainty Equivalent

In addition, we discuss another measure to assess our portfolio management, the certainty equivalent. This analysis tool can be calculated as follows:

$$CE = E(R_p) - \frac{1}{2}\alpha\sigma_{Rp}^2 + \frac{\tau(R_p)}{6}\alpha^2\sigma_{rp}^3 - \frac{k(R_p) - 3}{24}\alpha^3\sigma_{rp}^4 \qquad (2.7.4)$$

Since we are assuming normality in returns with expected return $E(R_p)$, variance σ_{Rp}^2, skewness $\tau(R) = 0$ and kurtosis $k(R_p) = 3$, we can reach a new expression:

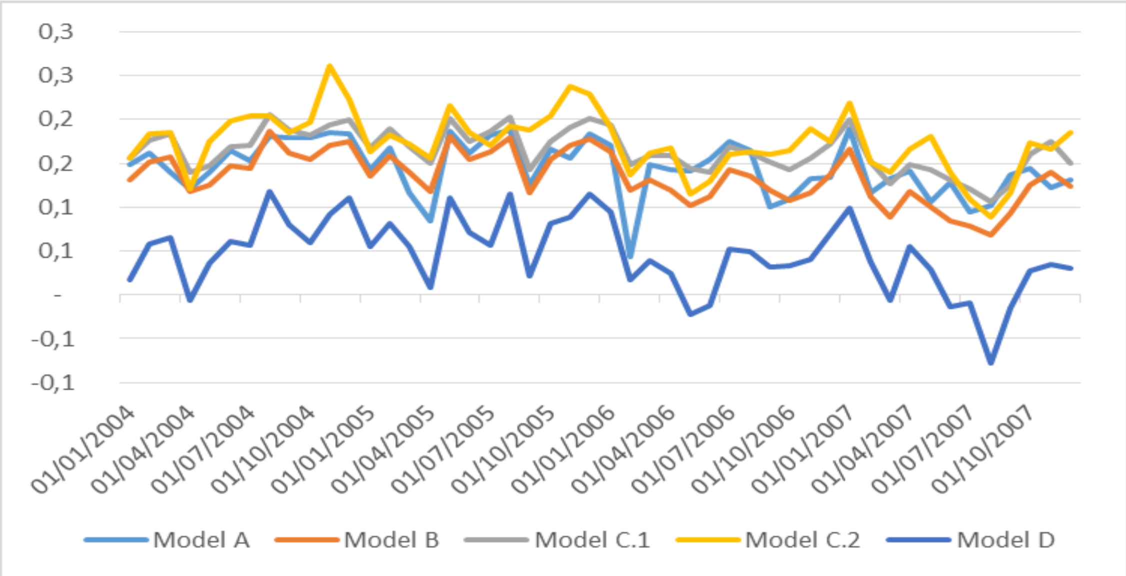
$$CE = E(R_p) - \frac{1}{2}\alpha\sigma_{Rp}^2 \qquad (2.7.5)$$

This expression refers to the amount of money due to which, an investor would be willing to give up keep his portfolio invested under uncertainty. Obviously, as in the case of the Sharpe Ratio, the highest the Certainty Equivalent is, the best performance of our portfolio model.

Note that we ascertain it in annual terms, as in the previous section.

Firstly, we begin with the analysis of the calm period:

Figure 18. The Certainty Equivalent. Calm period analysis

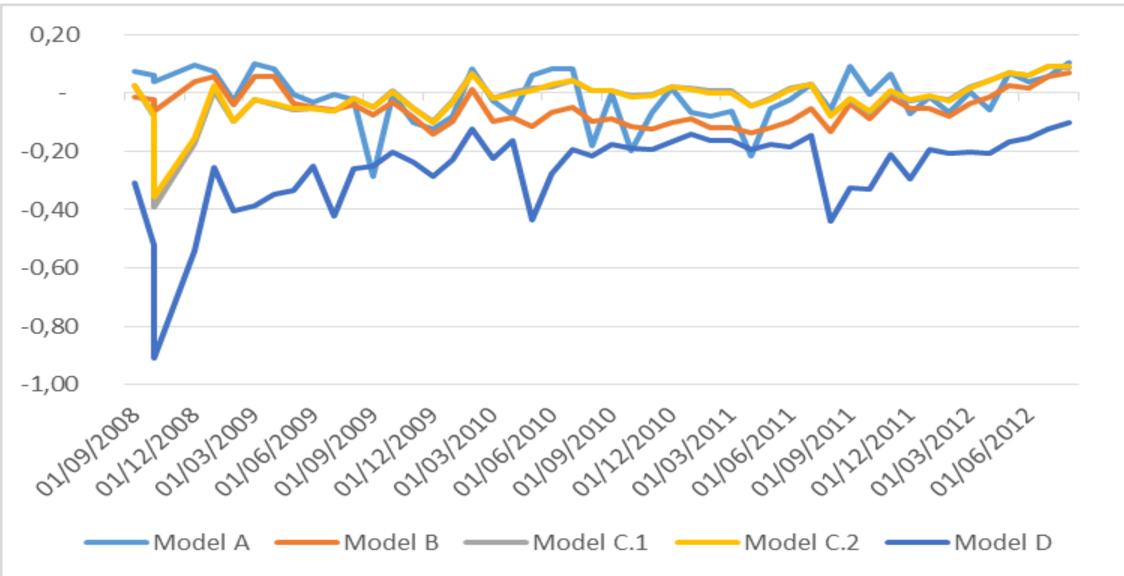


Source: Compiled by the author. Expressed in annual terms

Paying attention to the last chart (Figure.18), we can appreciate that the whole models show the same trend along the calm time frame, but having many differences in terms of magnitudes. Thus, the best performing fund in this period is Model C.2. Note that in this period the Certainty Equivalent is always positive.

Then, we continue reviewing about the stress period:

Figure 19.The Certainty Equivalent. Stress period analysis



Source: Compiled by the author. Expressed in annual terms

Analysing the stress period (figure.19), we can observe the trend is not the same as in the previous time frame. In this period, as for the Sharpe Ratio, the best performing fund is Model C.1. Furthermore, the Certainty Equivalent is not always positive. In fact, for the case of Model D, it is almost always negative, which means that is the worst model in this context.

To sum up, according to the last figures, we can conclude that the highest risk premium offered on average to exchange our portfolio, is the one showed by Model C.2, since it is the one that usually offers us the greatest relationship over time[17]. As a consequence, we can enounce the following statement:

"Those portfolios with better performances based on Certainty-Equivalent ratio are associated with the time-varying risk aversion attitude, while those with a constant risk aversion parameter (Model D), have a negative risk-return relationship and a very unstable trend throughout the whole studied period".

2.7.5. Lower and Upper Partial Moments Family

Then, we come back to the first chapter (section 1.2.), in order to rescue the performance measures based on partial moments. In this case, we assess some popular ratios, such as the Kappa of order 1 and its associated Omega statistic. In addition we analyse the Sortino Ratio (kappa of order 2). According to the above, in this section we fix the following values for the order of the LPM to consider different risk attitudes: $m = 2$ (moderate investors-Sortino Ratio) and $m = 1$ (aggressive investors-Kappa(1)). Moreover, we set the value $q = 1$ for the case of Omega index. The three mentioned ratios are ascertained as follows:

$$K(R_f, 1) = \frac{E(R) - R_f}{E[Max(R_f - R, 0)]} \tag{2.7.6}$$

$$\Omega(R_f, 1,1) = \frac{E[Max(R - R_f, 0)]}{E[Max(R_f - R, 0)]} = K(R_f, 1) + 1 \tag{2.7.7}$$

$$K(R_f, 2) = \frac{E(R) - R_f}{E[Max(R_f - R, 0)^2]^{1/2}} \tag{2.7.8}$$

[17] In this context, the average means, taking into account the performance of the models over the whole period, that is to say, the sum of the calm and the stress period.

Note that we have fixed the value of the threshold as the risk-free rate for the whole studied ratios.

Table 24. Performance measures based on partial moments

	CALM PERIOD					STRESS PERIOD				
RATIOS	Model A	Model B	Model C.1	Model C.2	Model D	Model A	Model B	Model C.1	Model C.2	Model D
Kappa 1	0,0755	0,0748	0,0775	0,0810	0,0739	0,0128	0,0124	0,0132	0,0131	0,0121
Omega	1,0755	1,0748	1,0775	1,0810	1,0739	1,0128	1,0124	1,0132	1,0131	1,0121
Sortino	0,3347	0,3318	0,3435	0,3593	0,3277	0,0899	0,0874	0,0929	0,0922	0,0850

Source: Compiled by the author. Expressed in annual terms

As can be seen in Table.24, the results follow the same trend as in the previous performance measures. The highest average ratio, is the one showed by Model C.2, since it is the one that usually offer us the greatest relationship over time, that is to say, the greatest ratio on average for the two considered periods[18].

In addition, we can appreciate that those portfolios with the lowest associated risk, also have the lowest associated performance ratios.

[18] Note that Model C.1 offers better results (according to the Partial Moments) than Model C.2 in the stress period. However, Model C.2 is the best one on average.

RESEARCH LIMITATIONS

The main limitation of this study is that the optimal assets, which were previously selected in chapter 1, were assessed in an economic expansion period (2002-2003). However, we have used these assets to build portfolios in subsequent periods, which are located in other different economic contexts, such as the calm period (2004-2008) and the stress one (2008-2012). As a possible extension, we could choose the appropriated equities for both periods, by ascertaining the screening rules twice, one for the calm time frame and other for the stress one.

Moreover, talking about the second chapter, although we know that the probability distribution of assets' returns has the problem of heavy tails, we assumed normality in returns. In this case, we can model the returns according to other distributions such as the student's t distribution.

Otherwise, another limitation of this study is that we have worked with returns unadjusted by autocorrelation, so the obtained results may be biased. In this case, as a possible extension, we can correct the results by autocorrelation. In addition, the above allows us to estimate the conditional volatility and correlation models using the residual instead of the assets' return.

However, linking to the above, we must remember the results obtained in section 2.1, when we use the simple and partial autocorrelation functions and the Box-Pierce test. The above tools suggested the absence of autocorrelation or that it is weak. On this basis, we can conclude that a clear persistence in returns is not appreciated. Given that, we can conclude that is not really significant to work with data unadjusted by autocorrelation.

CONCLUSIONS

As it is well known, when an individual decides to invest an amount of money in a risky portfolio, always choose an efficient one whose composition depends on his subjective preferences. Analysing the market more in depth, we can appreciate that investor preferences are heterogeneous, that is to say, there are some individuals that prefer to take some risks but there are others more cautious. However, it is assumed in financial literature that investors are traditionally risk averse individuals.

Given the above, at first, we planned to build a well-diversified portfolio from the set of assets listed in the EuroStoxx-50 index. Then, the following question arises: What and how many assets we should include in our portfolio? To answer the last question, we spent the first chapter proposing several performance ratios, which belong to the Lower Partial Moments (LPM) risk measures. In particular, we proposed 3 Kappa ratios of different orders and 3 ratios of the Farinelli-Tibiletti family. In addition, we proposed the use of the Principal Component Analysis (PCA) in order to help us summarizing the information contained in the last ratios.

Once we have finished the last research, we can assess (based on previous studies) that we can obtain a well-diversified portfolio including 10 assets that belong to different sectors or branches of business. The key of this proposal is to compensate the adverse movements in some assets with the earnings obtained in others. It is well known as “diversification effect”. Moreover, the 10 risky assets have been selected based in a single criterion given by the application of the PCA technique. This single criterion is an acceptable way to select several assets because it contains the information referred to 6 different performance ratios related with different risk aversion attitudes. Thus, the ten selected assets are: SAFRAN, UNIBAIL-RODAMCO, AIRBUS GROUP, LVMH, ESSILOR INTL., DEUSTCHE-POST, INTESA SANPAOLO, IBERDROLA, BASF and ENI.

Otherwise, an investor is more or less risk averse according to the economic and political circumstances, that is to say, the investment attitude depends on the market trend. As an example of the last statement, nowadays even the most adventurous investor has had to reduce his optimistic expectations due to we are in an economic recession period. Given that, we proposed to spend the second chapter looking for the optimal portfolio that best meets with the customer behaviour, taking into account the variability of the market. To bring out the last proposal, we planned the possibility of introducing some different models of time-varying risk aversion attitude and compare them against the constant risk aversion. In addition, we analysed this study in two different periods (calm and stress) due to we aimed to observe whether is better to fix a single parameter to build the optimal portfolio over the whole period or make it changes over the time frame.

Given that, in particular in the second chapter, trying to reach the proposed terms, we have immersed ourselves in the theory of utility and choosing the optimal portfolio for risk averse individuals. We began with the study of the unconditional Markowitz approach to analyse how we can build the optimal portfolio in a constant context. After that, we have studied more in depth how this portfolio changes over time, through conditional schemes such as GARCH (1, 1) and DCC-GARCH. Lately, we have spent the most of the second chapter focusing our study in the modelling of risk aversion parameter so that it changes over time.

Despite the influence of risk aversion in the optimal portfolio context, there are not many studies which have explicitly estimated the risk aversion of an investor. Instead of that, they choose random values to reflect the common levels of risk aversion. However, the common sense tells us that the use of arbitrary values for this parameter could yield us optimal portfolios that do not reflect the actual investor´s attitude towards risk. Given that, one of the goals of this paper has been to propose the modelling of the risk aversion parameter in order to make it changing over time. To carry out this key proposal, we have focused our attention on the review of the quadratic utility functions framework, analysing the CRRA in two different ways. One way associated to the market risk premium and other more related to the Consumer Confidence Indicator (CCI). Further, we have considered another approach to model the risk aversion parameter, which is based on the application of the downside risk measures.

Thus, the key of this paper was to assess whether is better to work with a constant or a time-varying risk aversion parameter. Then, analysing the performance results for the whole proposed models and for both studied periods, we have tested that in general, those models related to time-varying risk aversion showed a better performance on average. This is so, both from the point of view of Sharpe Ratio as the Certainty Equivalent. Furthermore, more specifically, the best way to model the risk aversion parameter in performance terms is the one associated to Model C.2. This scheme was based on modelling the risk aversion parameter in a new way, depending on the volatility and taking into account the fact that the market goes up and down, penalizing the decreases with an increase in the parameter. We have brought out the above, through the application of GARCH (1, 1) schemes, in order to model the conditional variance.

REFERENCES

Arrow, K., 1971. Essays in the theory of risk bearing, Markham, New York.

Billio, M., Caporin, M., Costola, M., 2012. Backward/forward optimal combination of performance measures for equity screening, Ca' Foscari University of Venice, Volume. 13.

Bollerslev,T., 1986. "Generalized Autoregressive Conditional Heteroscedasticity", Journal of Econometrics, volume.31, 307-326.

Box, G.E.P. and Pierce D. A. 1970. "Distribution of Residual Autocorrelations in Autoregresive-Integrated Moving Average Time Series Models".Journal of the American Statistical Association, volume. 65, 1509-1526.

Campbell, J., Cochrane, J., 1999. By force of habit: a consumption-based explanation of aggregate stock market behaviour. The Journal of Political Economy 2, 205-251.

Clements, A., and Silvennoinen, A., 2013. Volatility timing: How best to forecast portfolio exposures. Journal of Empirical Finance, 108-115

Cotter, J., Hanly, J., 2010. Time varying risk aversion: An application to energy hedging. Journal of Empirical Finance, 432-441.

Dionne, G., 2014. When can expected utility handle first-order risk aversion?. Journal of Economic Theory, volume 154, 403-422.

Dybvig, P.H., 1983. An explicit bound on individual assets deviations from APT Pricing in finite economy. Journal of Financial Economics.

Engle, R.F., 1982. "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation", Econometrics, volume. 50, num. 1, 987-1007.

Engle, R.F., Lilien, D.M. and Robins, R. P., 1987. Estimating time varying risk premium in the term structure: The ARCH-M model, Econometrics 55, 391-407.

Evans, J., & Archer, S., 1968. Diversification and the Reduction of Dispersion: An Empirical Analysis. Journal of Finance XXIV, 761-769.

Farinelli, S., Tibiletti, L., 2008. Sharpe thinking in asset ranking with one-sided measures, European Journal of Operational Research 185, 1542-1547.

Fishburn, P.C., 1977. Mean-risk analysis with risk associated with below-target returns, The American Economic Review 67, 116-126.

Gómez, J.C., 2011. Dirección Financiera I. Editorial Club Universitario.

Guo, H., Whitelaw, R., 2006. Uncovering the risk–return relation in the stock market. The Journal of Finance 3, 1433 – 1463.

Ghysels, E., Santa-Clara, P., Valkanov, R., 2005. There is a risk-return trade off after all. Journal of Financial Economics 76, 509-548.

Grinblatt, M. and Titman, S., 1983. Factor pricing in a finite economy. Journal of Financial Economics.

Hwang, S., Salmon, M., 2003. An analysis of performance measures using copula, in Knight, J., and Satchell, S. (eds), Performance measurement in finance: firms, funds and managers, Butterworth-Heinemann Finance, Quantitative Finance Series.

Kaplan, P.D. and Knowles, J.A., 2004. Kappa, a generalized downside riskadjusted performance measure. Journal of Performance Measurement, 8(3), 42–54

Kim, K.H., 2014. Counter-cyclical Risk Aversion. Journal of Empirical Finance 29, 384-401.

León, A., Navarro, L., Nieto, B., 2015. Don't stand so close to Sharpe. XXIII Finance Forum.

Marin J. "Análisis de Componentes Principales"

Markowitz, H. M., 1952. Portfolio selection. The Journal of Finance, March.

Markowitz, H. M., 1959. Portfolio Selection: Efficient Diversification of Investments. Wiley, Yale University Press, 1970, Basil Blackwell, 1991

Mehra, R., Prescott, E., 1985. The equity premium: A puzzle. Journal of Monetary Economics 15, 145-161.

Peña D., 2002. “Análisis de datos multivariantes”, McGRAW-HILL/ Interamericana de España, S.A.U., 133-165.

Sharpe, W. F., 1966. Mutual Fund Performance. Journal of Business, Volume 39, 119-138.

Sharpe, w. F., 1994. The Sharpe Ratio. Journal of Portfolio Management, Volume 21, 49-58.

Tapon,F., Vitali, A., 2013. Equity portfolio diversification: How many stocks are enough? Evidence from five developed markets. Tasmanian School of Business and Economics, N 2013-16.

WEBSITES

COMISIÓN NACIONAL DEL MERCADO DE VALORES (2013): Informe anual de la CNMV sobre los mercados de valores y su actuación,
http://www.cnmv.es/DocPortal/Publicaciones/Informes/IA2013_Web.pdf

EUROPA.EU (2016): Economic and Financial Affairs,

http://ec.europa.eu/economy_finance/db_indicators/surveys/time_series/index_en.htm

MUTUALFUNDS (2015): Portfolio Diversification: How Many Stocks Are Enough?,
http://mutualfunds.com/expert-analysis/portfolio-diversification-how-many-stocks-are-enough/

STOXX (2016): Innovative Global Indices, https://www.stoxx.com/index-details?symbol=SX5E

APPENDIX.A: NORMALITY AND AUTOCORRELATION TESTS

JARQUE-BERA NORMALITY TEST									
SAFRAN		UNIBAIL-RODAMCO		AIRBUS GROUP		LVMH		ESSILOR INTL.	
JB-Statistic	P-Value	JB-Statistic	P-Value	JB-Statistic	P-Value	JB-Statistic	P-Value	JB-Statistic	P-Value
459,7206	0	13.658,71	0	118.135,8	0	70,4538	5,55E-16	390,3702	0
DEUTSCHE POST		INTESA SANPAOLO		IBERDROLA		BASF		ENI	
JB-Statistic	P-Value	JB-Statistic	P-Value	JB-Statistic	P-Value	JB-Statistic	P-Value	JB-Statistic	P-Value
180,4343	0	457,0526	0	19.512,7	0	21,0186	2,73E-05	59,1933	1,40E-13

	BOX-PIERCE STATISTICAL TEST									
	SAFRAN		UNIBAIL-RODAMCO		AIRBUS GROUP		LVMH		ESSILOR INTL.	
LAGS	Q	P-Value	Q	P-Value	Q	P-Value	Q	P-Value	Q	P-Value
1	13,4735	0,0002	0,1260	0,7226	1,5391	0,2148	0,4388	0,5077	45,4881	1,54E-11
2	14,3712	0,0008	0,1266	0,9387	2,1335	0,3441	0,9460	0,6231	54,0483	1,83E-12
3	15,0863	0,0017	0,6722	0,8797	5,8912	0,1170	4,0549	0,2556	54,8063	7,55E-12
4	15,4449	0,0039	6,3676	0,1733	5,9200	0,2052	4,0896	0,3940	57,1724	1,14E-11
5	16,2532	0,0062	6,4031	0,2689	6,5949	0,2526	6,6643	0,2468	63,5873	2,20E-12

	BOX-PIERCE STATISTICAL TEST									
	DEUTSCHE POST		INTESA SANPAOLO		IBERDROLA		BASF		ENI	
LAGS	Q	P-Value	Q	P-Value	Q	P-Value	Q	P-Value	Q	P-Value
1	0,4132	0,5203	2,1736	0,1404	2,2396	0,1345	3,5330	0,0602	1,0275	0,3107
2	9,2235	0,0099	6,7424	0,0343	2,8568	0,2397	5,6139	0,0604	1,1412	0,5652
3	16,1833	0,0010	7,0930	0,0690	3,3744	0,3374	8,4131	0,0382	1,1824	0,7572
4	16,1893	0,0028	7,6667	0,1046	3,7070	0,4471	8,6544	0,0703	1,3162	0,8586
5	22,3782	0,0004	7,6670	0,1756	4,0125	0,5476	9,2274	0,1003	1,8899	0,8642

APPENDIX.B: THE LOG LIKELIHOOD. UNIVARIATE AND MULTIVARIATE MODELS

The parameters have been estimated by the application of the maximum likelihood method. Further, we have assumed normality in returns. Thus, the procedure for obtaining the covariance matrices through the previous calculus of dynamic correlations, requires consideration of univariate and multivariate models. Univariate models are used in modelling the volatility of each one of the ten selected assets. Otherwise, Multivariate models are used to model the conditional correlations between the mentioned assets. Note, we have also used the univariate schemes to model the volatility of Eurostoxx-50 index.

B.1. Univariate Normal Distribution

First of all, we consider that asset returns follow a univariate normal distribution. Given that, its associated density function is as follows:

$$f(x/\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

Where the likelihood function is obtained as the multiplication of the density function form 1 to n:

$$L(\mu,\sigma^2/x_i) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}\, exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$L(\mu,\sigma^2/x_i) = 2\pi^{-\frac{n}{2}}\,\sigma^{-\frac{2n}{2}}\,\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}\right)$$

Then, the log likelihood is defined as:

$$logL(\mu, \sigma^2/x_i) = -\frac{n}{2}\log(2\pi\sigma^2) \; - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i - \mu)^2}{\sigma^2}$$

B.2. Multivariate Normal Distribution

Moreover, we assume multivariate probability distributions for estimating conditional correlations models, that is to say, it is assumed that asset returns are distributed by a given multivariate distribution function. Then, we show the distribution and likelihood functions:

$$f(x_1 \dots x_n/\mu, V) = \frac{1}{(2\pi)^{\frac{n}{2}}(V)^{\frac{1}{2}}} \; exp\left(-\frac{1}{2}(x - \mu)'V^{-1}(x - \mu)\right)$$

, where $x = [x_1 \dots x_n]$ are the standardized returns and $V^{-1}$is the 10x10 inverse covariance matrix

The likelihood function is defined as:

$$L(\mu, V/x_i) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{\frac{n}{2}}|V|^{\frac{1}{2}}} \; exp\left(-\frac{1}{2}(x_i - \mu)'V^{-1}(x_i - \mu)\right)$$

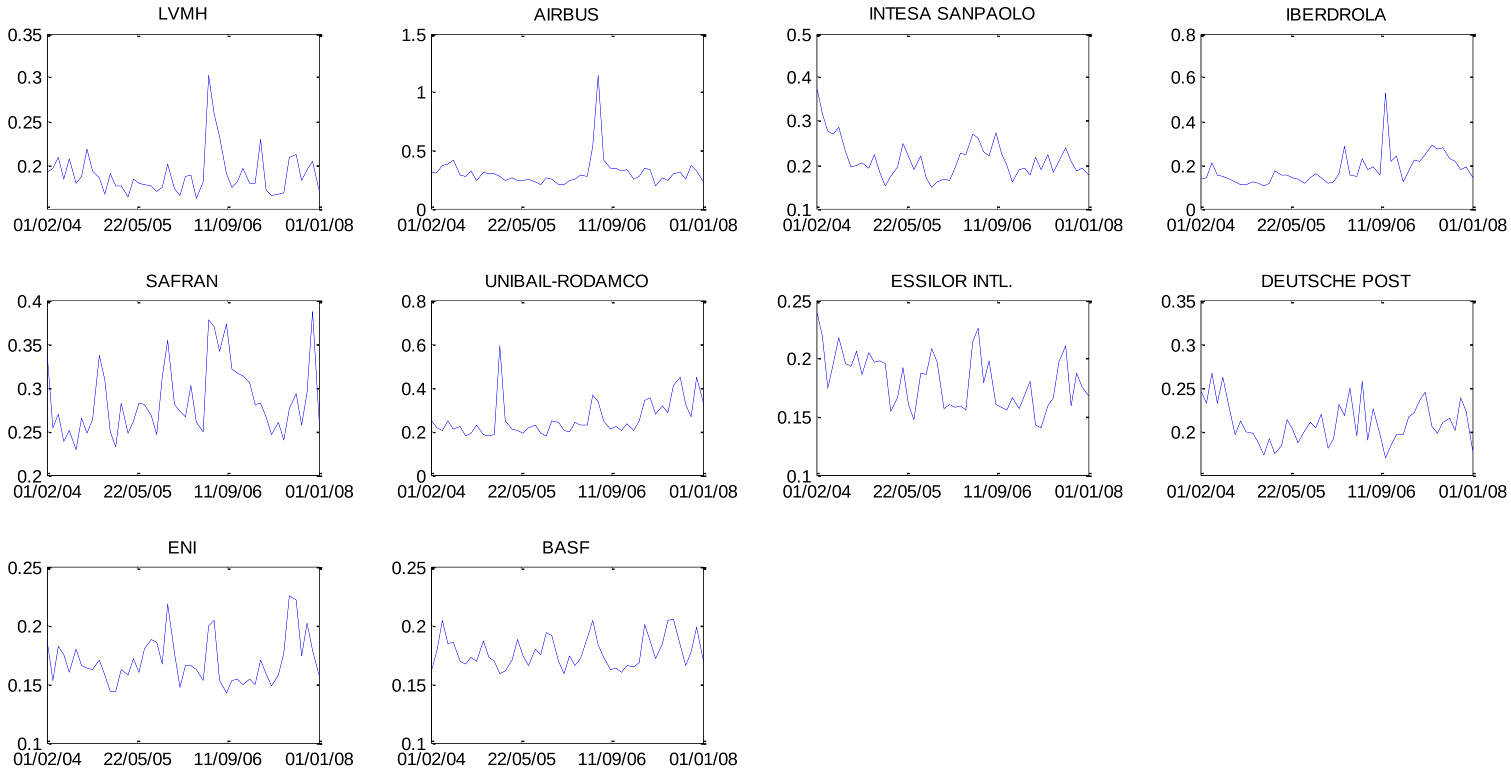
$$L(\mu, V/x_i) = 2\pi^{-\frac{n}{2}}\,|V|^{-\frac{2n}{2}} \; exp\left(-\frac{1}{2}(x_i - \mu)'V^{-1}(x_i - \mu)\right)$$

Then, we show the log likelihood:

$$logL(\mu, V/x_i) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log|V| \; - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)'V^{-1}(x_i - \mu)$$

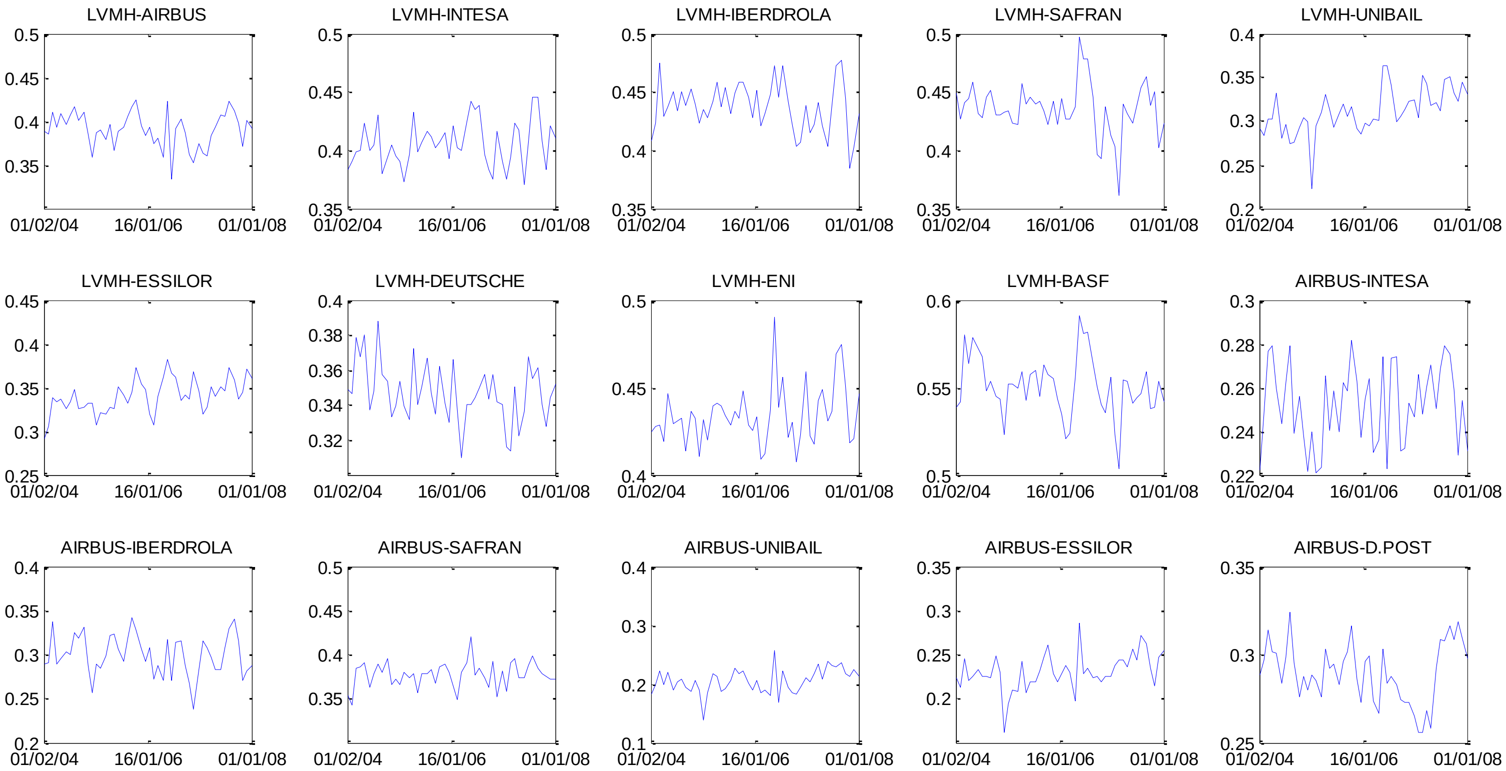
APPENDIX.C: OPTIMAL PORTFOLIO CONSTRUCTION. THE CALM PERIOD

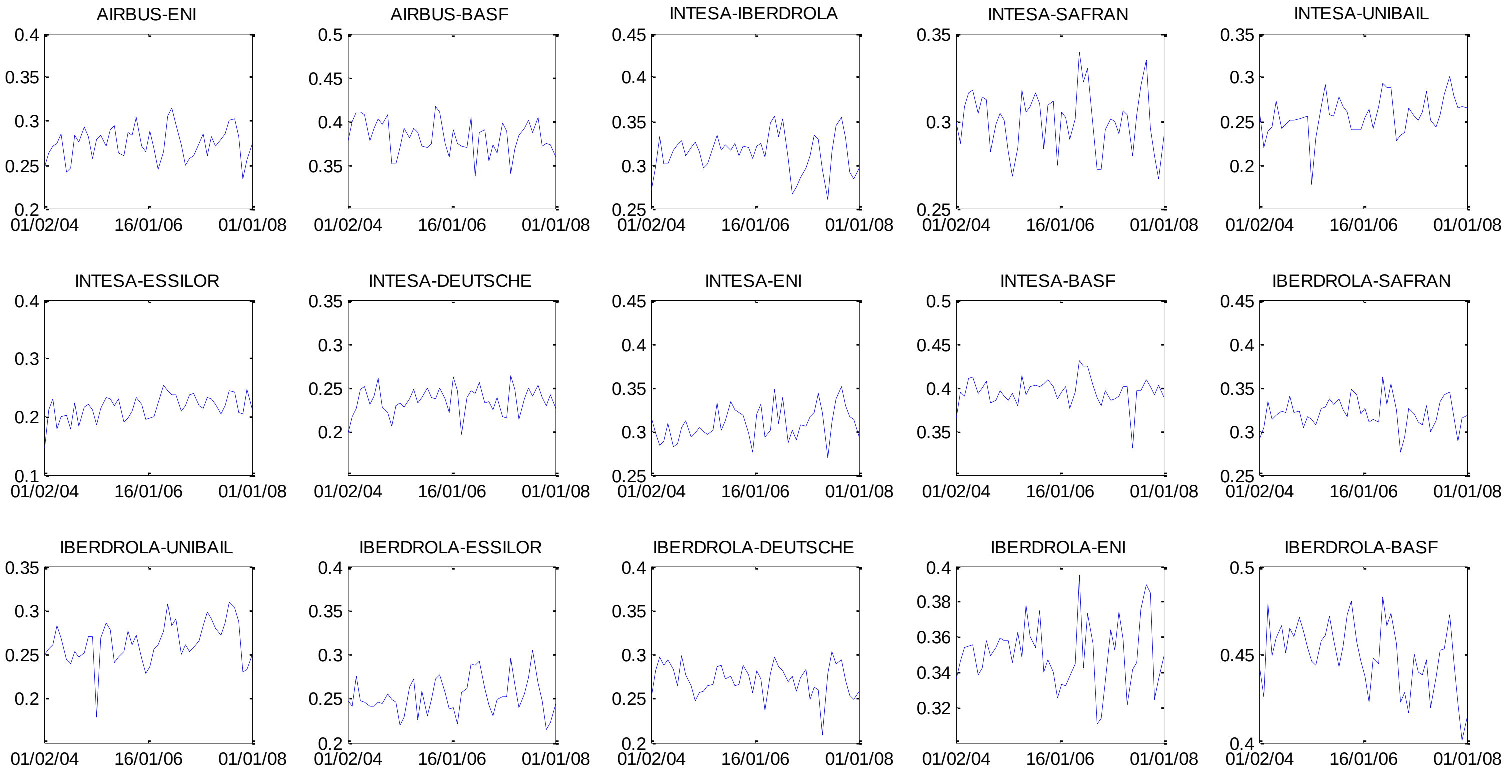
C.1 Monthly conditional volatilities/ GARCH (1, 1). The calm period

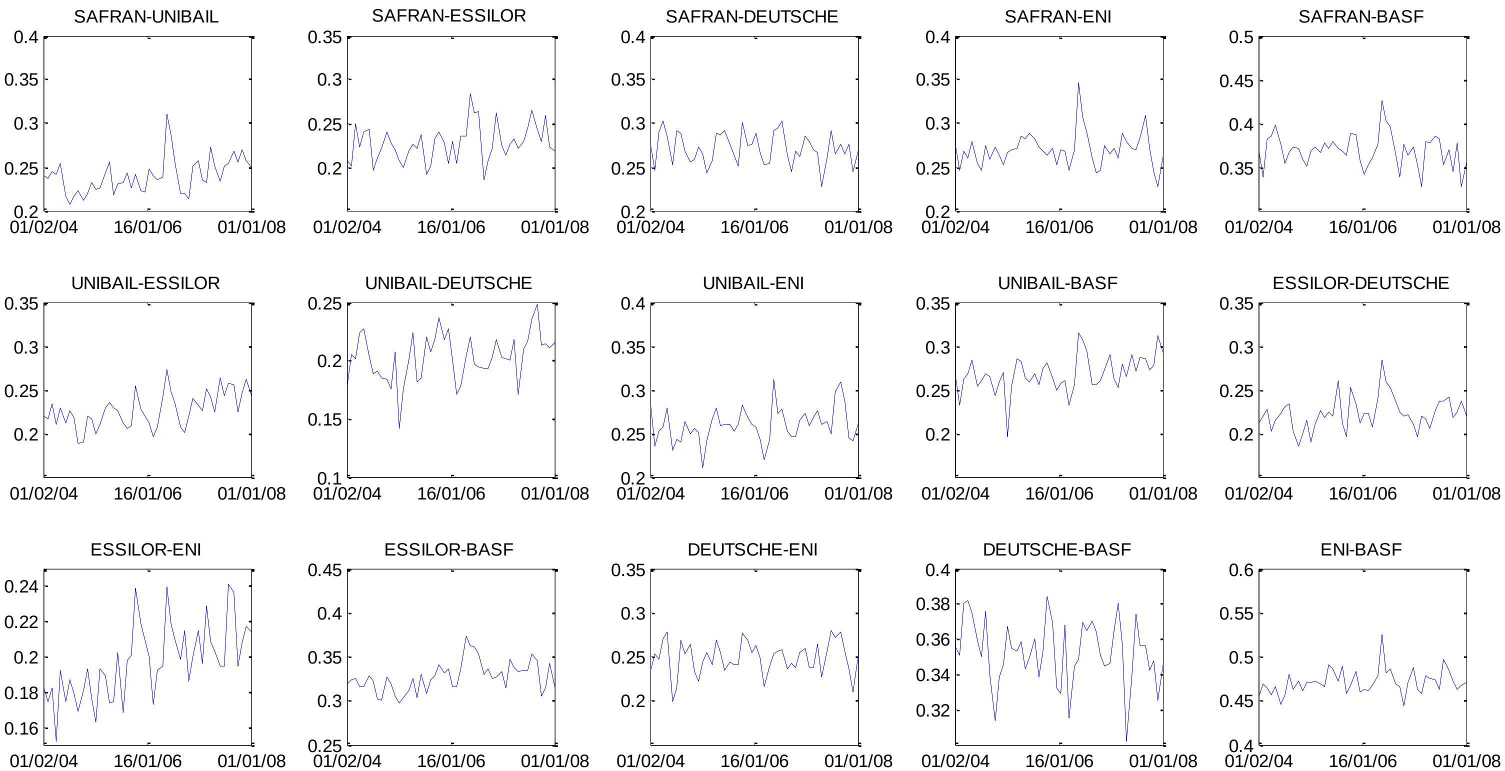


Note that monthly conditional volatilities are expressed in annual terms, multiplying them by $\sqrt{12}$

C.2 Monthly conditional correlations/ DCC-GARCH. The calm period

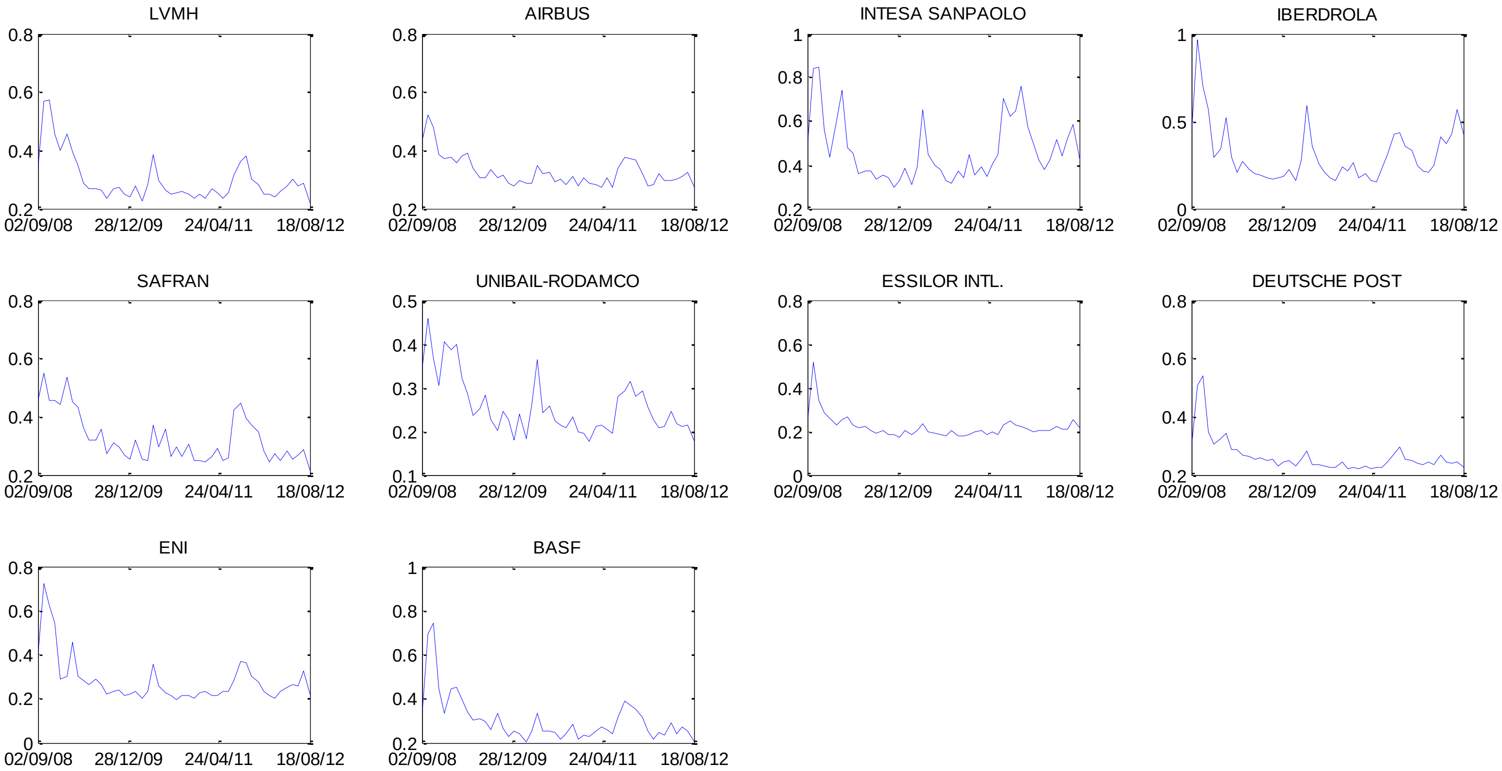



AIRBUS-ENI
AIRBUS-BASF
INTESA-IBERDROLA
INTESA-SAFRAN
INTESA-UNIBAIL
INTESA-ESSILOR
INTESA-DEUTSCHE
INTESA-ENI
INTESA-BASF
IBERDROLA-SAFRAN
IBERDROLA-UNIBAIL
IBERDROLA-ESSILOR
IBERDROLA-DEUTSCHE
IBERDROLA-ENI
IBERDROLA-BASF


SAFRAN-UNIBAIL
SAFRAN-ESSILOR
SAFRAN-DEUTSCHE
SAFRAN-ENI
SAFRAN-BASF
UNIBAIL-ESSILOR
UNIBAIL-DEUTSCHE
UNIBAIL-ENI
UNIBAIL-BASF
ESSILOR-DEUTSCHE
ESSILOR-ENI
ESSILOR-BASF
DEUTSCHE-ENI
DEUTSCHE-BASF
ENI-BASF
01/02/04
16/01/06
01/01/08

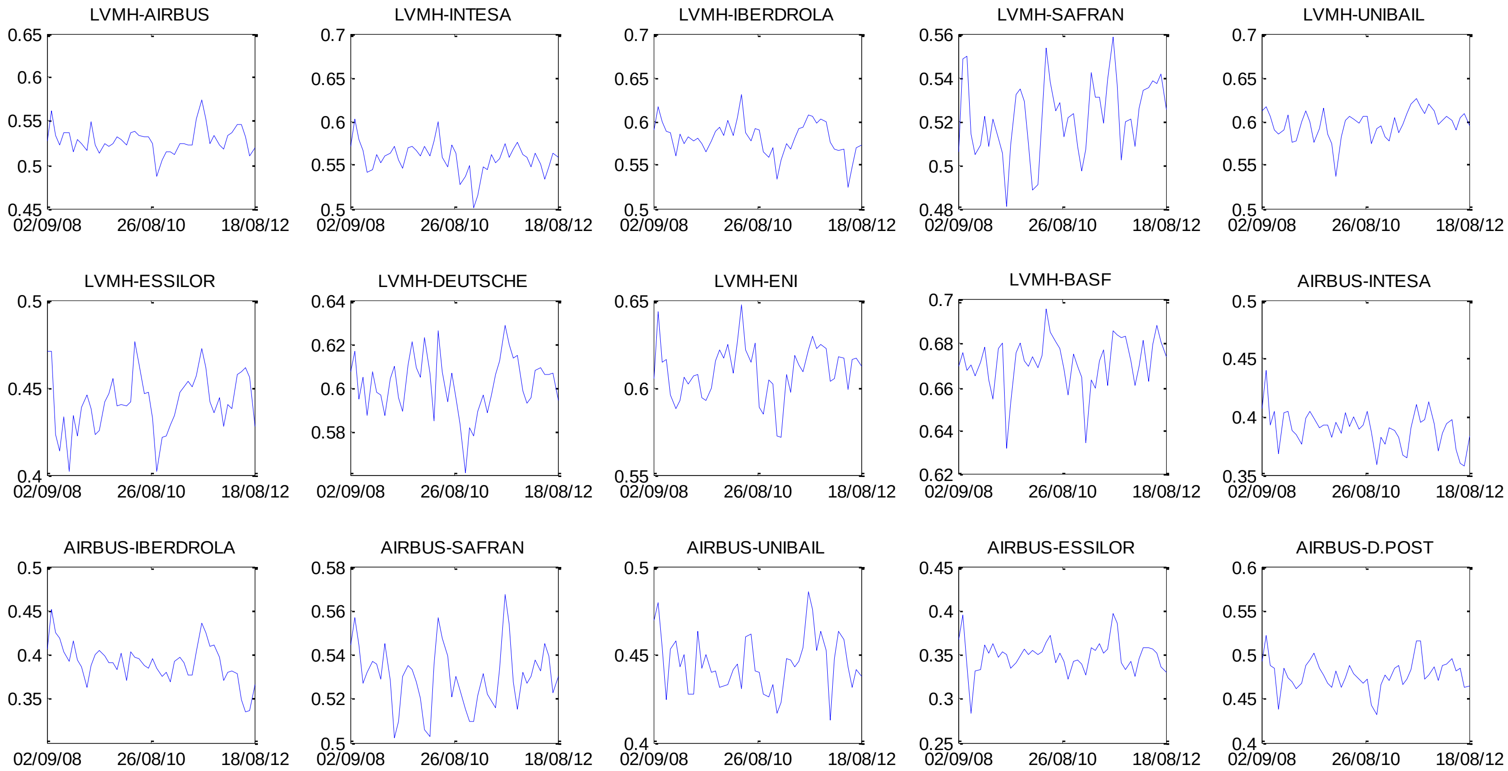
APPENDIX.D: OPTIMAL PORTFOLIO CONSTRUCTION. THE STRESS PERIOD

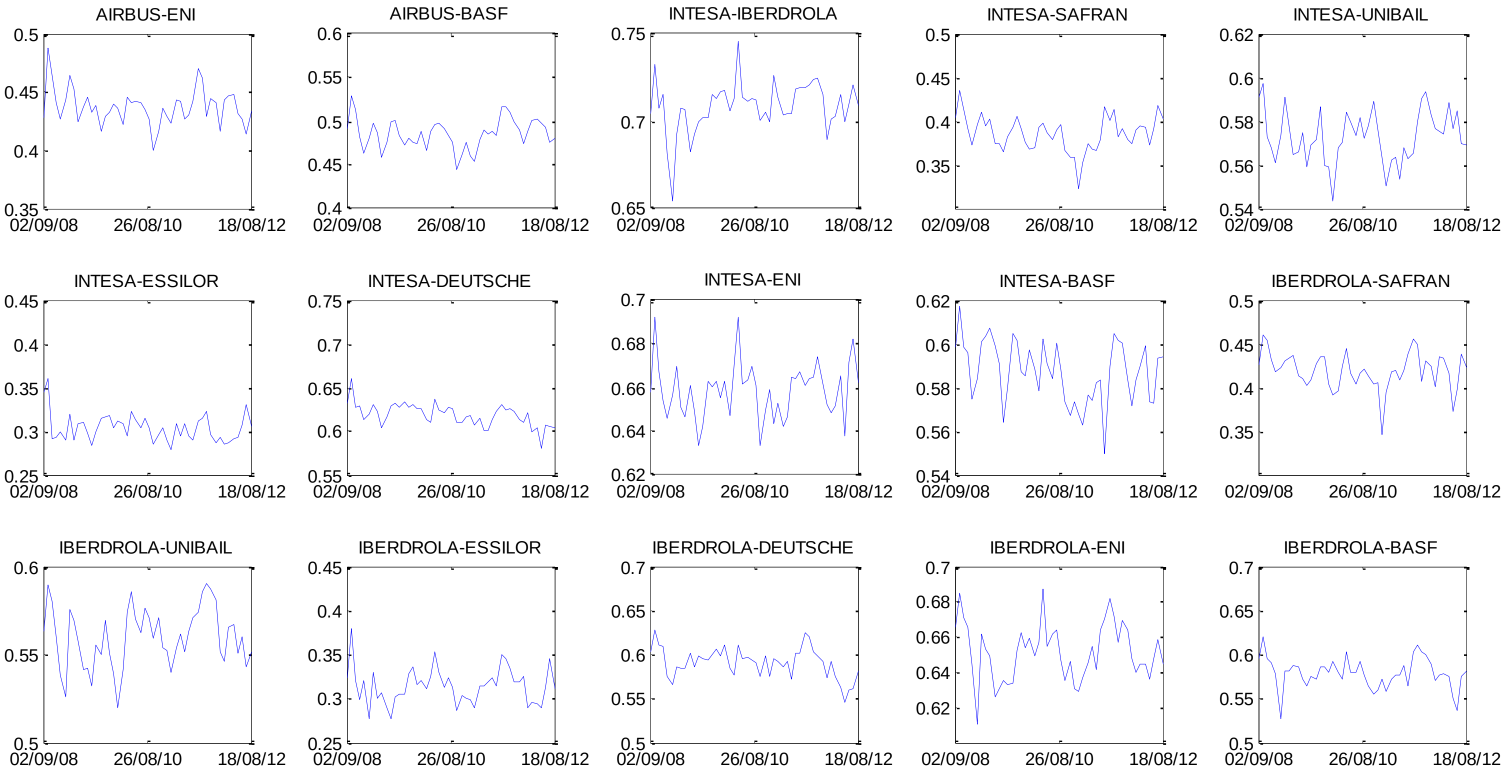
D.1 Monthly conditional volatilities/ GARCH (1, 1). The stress period

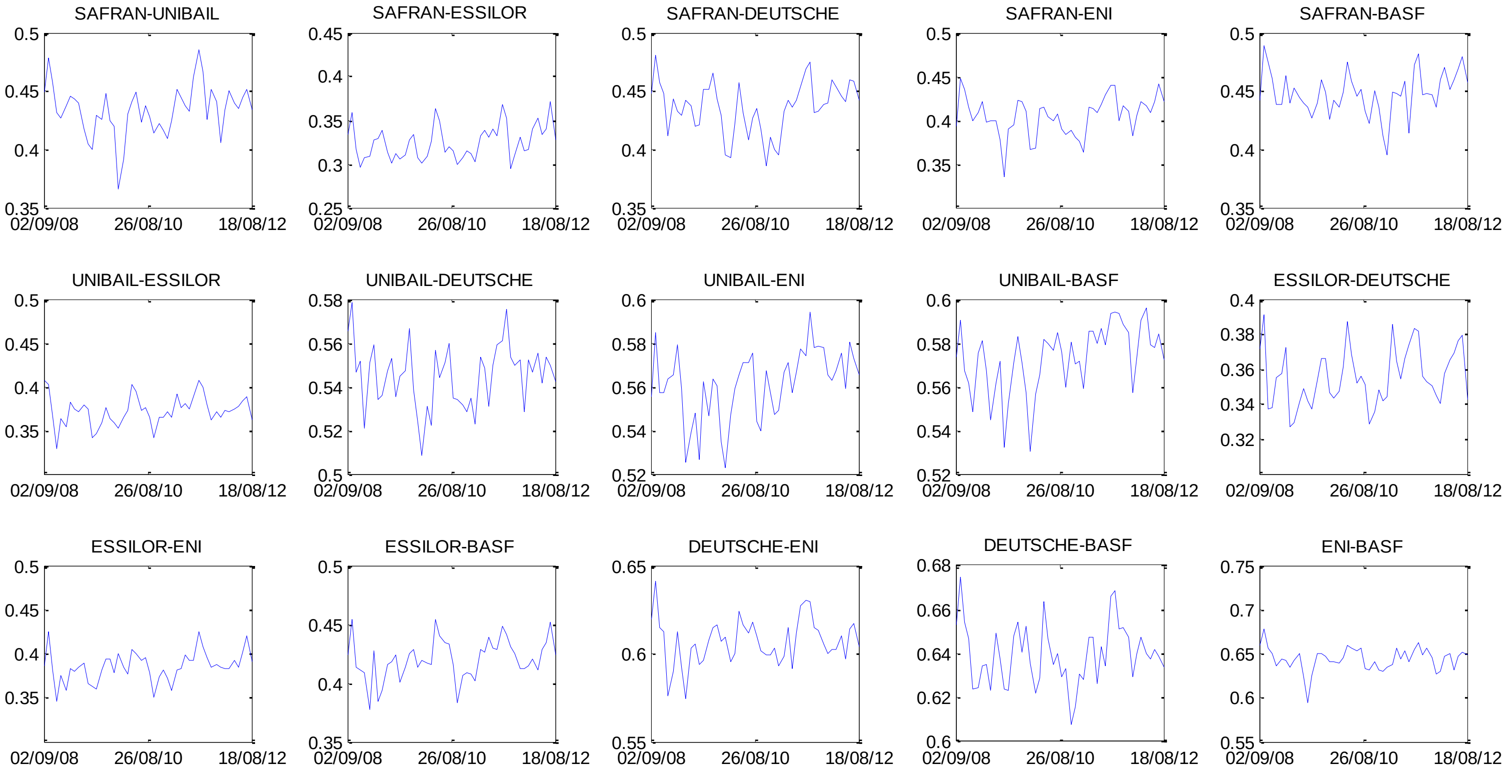


Note that monthly conditional volatilities are expressed in annual terms, multiplying them by $\sqrt{12}$

D.2 Monthly conditional correlations/ DCC-GARCH. The stress period




AIRBUS-ENI
0.5
0.45
0.4
0.35
02/09/08
26/08/10
18/08/12
AIRBUS-BASF
0.6
0.55
0.5
0.45
0.4
INTESA-IBERDROLA
0.75
0.7
0.65
INTESA-SAFRAN
0.5
0.45
0.4
0.35
INTESA-UNIBAIL
0.62
0.6
0.58
0.56
0.54
INTESA-ESSILOR
0.45
0.4
0.35
0.3
0.25
INTESA-DEUTSCHE
0.75
0.7
0.65
0.6
0.55
INTESA-ENI
0.7
0.68
0.66
0.64
0.62
INTESA-BASF
0.62
0.6
0.58
0.56
0.54
IBERDROLA-SAFRAN
0.5
0.45
0.4
0.35
IBERDROLA-UNIBAIL
0.6
0.55
0.5
IBERDROLA-ESSILOR
0.45
0.4
0.35
0.3
0.25
IBERDROLA-DEUTSCHE
0.7
0.65
0.6
0.55
0.5
IBERDROLA-ENI
0.7
0.68
0.66
0.64
0.62
IBERDROLA-BASF
0.7
0.65
0.6
0.55
0.5


SAFRAN-UNIBAIL
SAFRAN-ESSILOR
SAFRAN-DEUTSCHE
SAFRAN-ENI
SAFRAN-BASF
UNIBAIL-ESSILOR
UNIBAIL-DEUTSCHE
UNIBAIL-ENI
UNIBAIL-BASF
ESSILOR-DEUTSCHE
ESSILOR-ENI
ESSILOR-BASF
DEUTSCHE-ENI
DEUTSCHE-BASF
ENI-BASF
02/09/08
26/08/10
18/08/12

APPENDIX.E: ESTIMATED PARAMETERS

E.1 Conditional volatility model/ GARCH (1, 1)

- **Calm period**

	LVMH	AIRBUS	INTESA	IBERD	SAFRAN	UNIBAIL	ESSILOR	D. POST	ENI	BASF
ω	0,00001	0,00001	0,00000	0,00000	0,00003	0,00002	0,00001	0,00002	0,00001	0,00002
α	0,0991	0,1759	0,0546	0,1952	0,1002	0,1647	0,1001	0,0983	0,1000	0,0677
β	0,8000	0,8231	0,9444	0,8000	0,8000	0,8000	0,8000	0,8000	0,8000	0,8000

- **Stress period**

	LVMH	AIRBUS	INTESA	IBERD	SAFRAN	UNIBAIL	ESSILOR	D. POST	ENI	BASF
ω	0,00003	0,00005	0,0001	0,00001	0,00002	0,00002	0,00002	0,00003	0,00002	0,00002
α	0,0983	0,0588	0,1042	0,1653	0,1004	0,1000	0,0989	0,0394	0,1001	0,0989
β	0,8000	0,8000	0,8000	0,8000	0,8000	0,8000	0,8000	0,8079	0,8000	0,8000

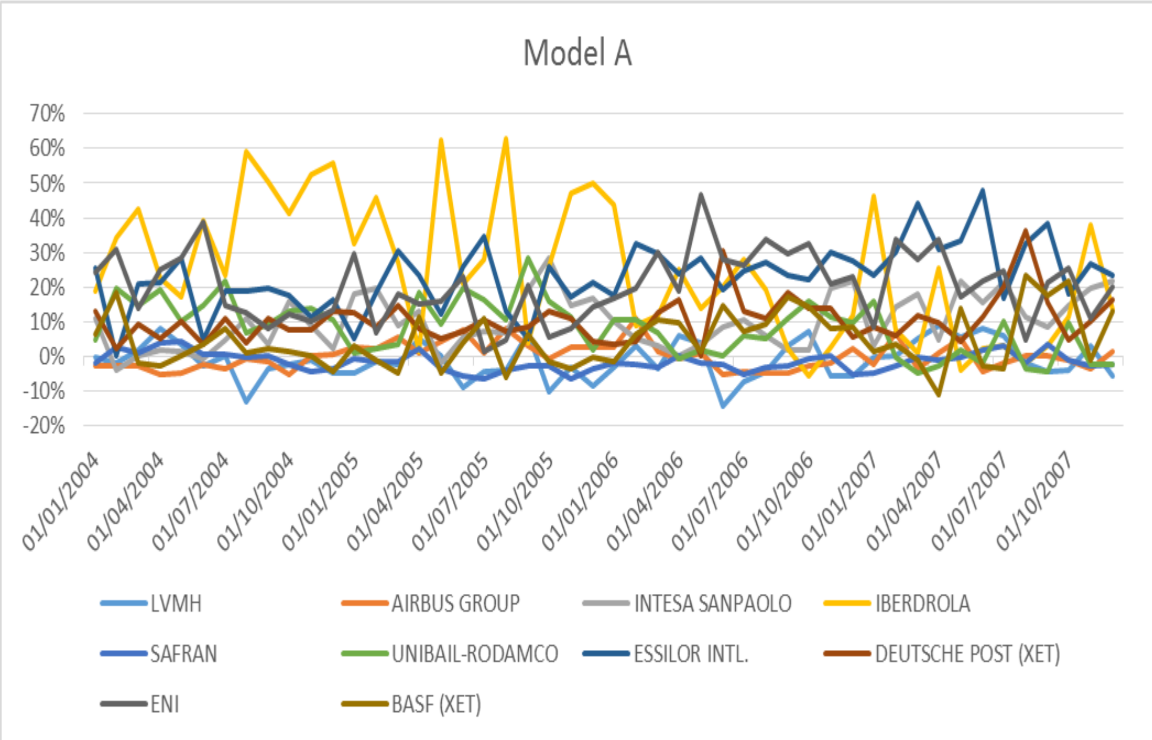
E.2 Conditional correlations model/ DCC-GARCH

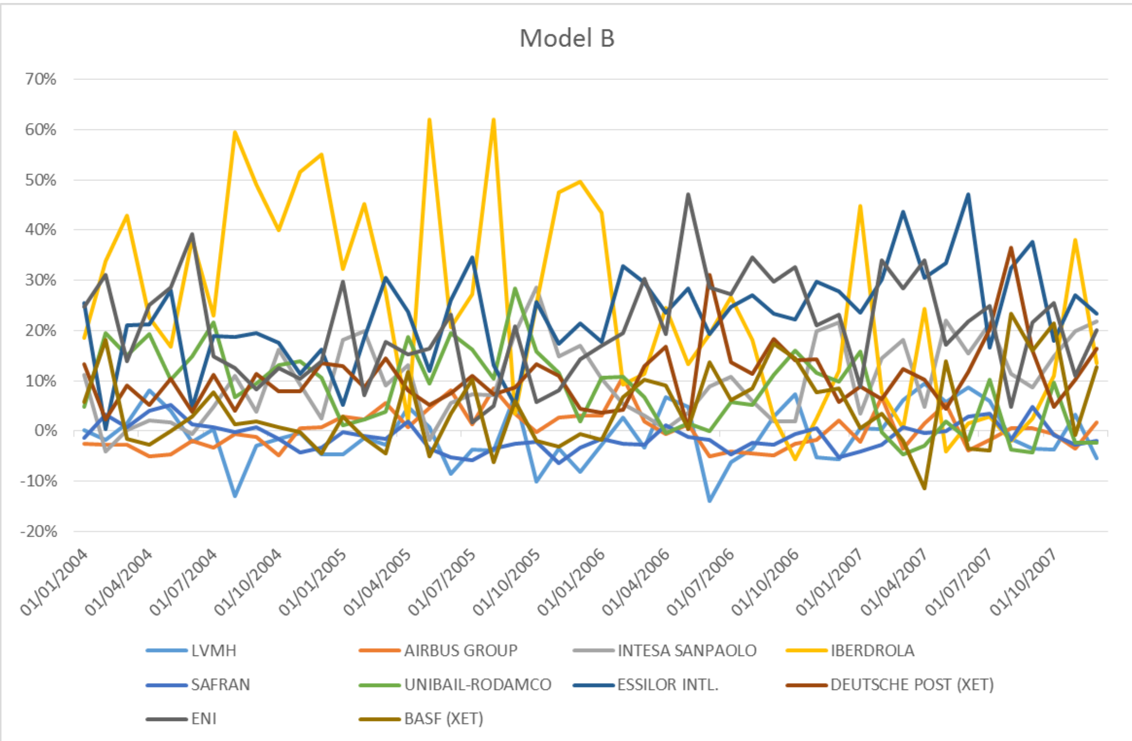
Calm period	
parameter	Value
ω	0,00001
α	0,0107
β	0,8863

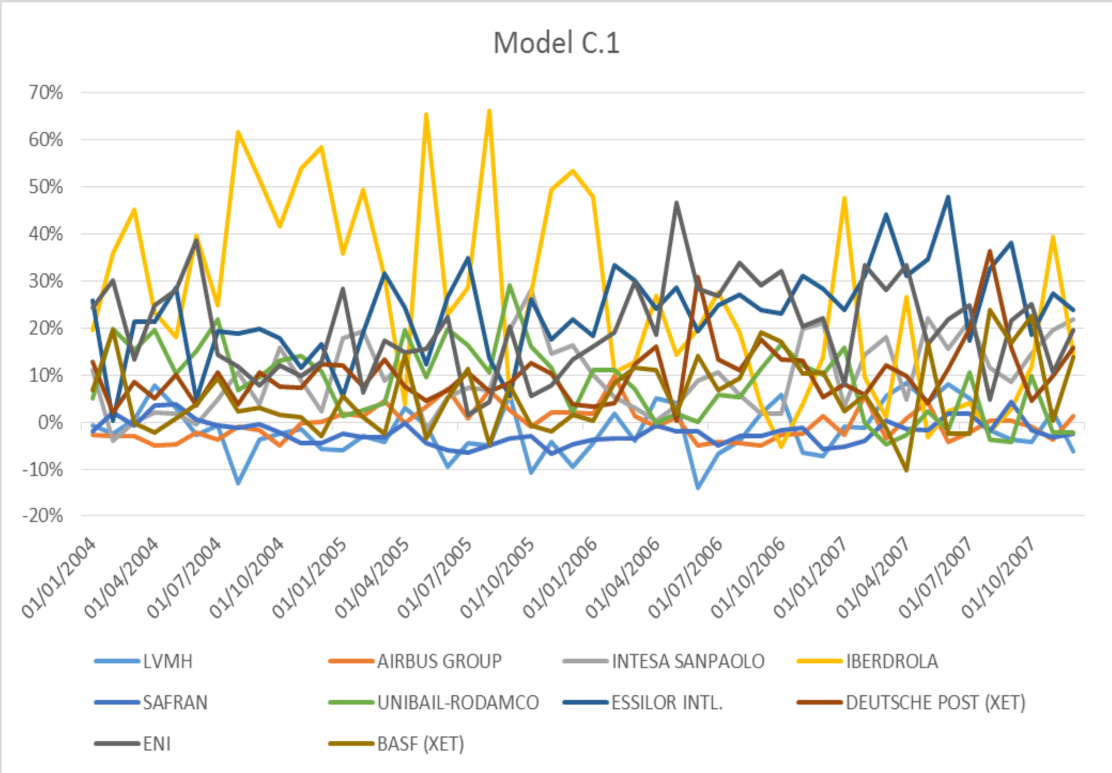
Stress period	
parameter	Value
ω	0,00003
α	0,0117
β	0,8874

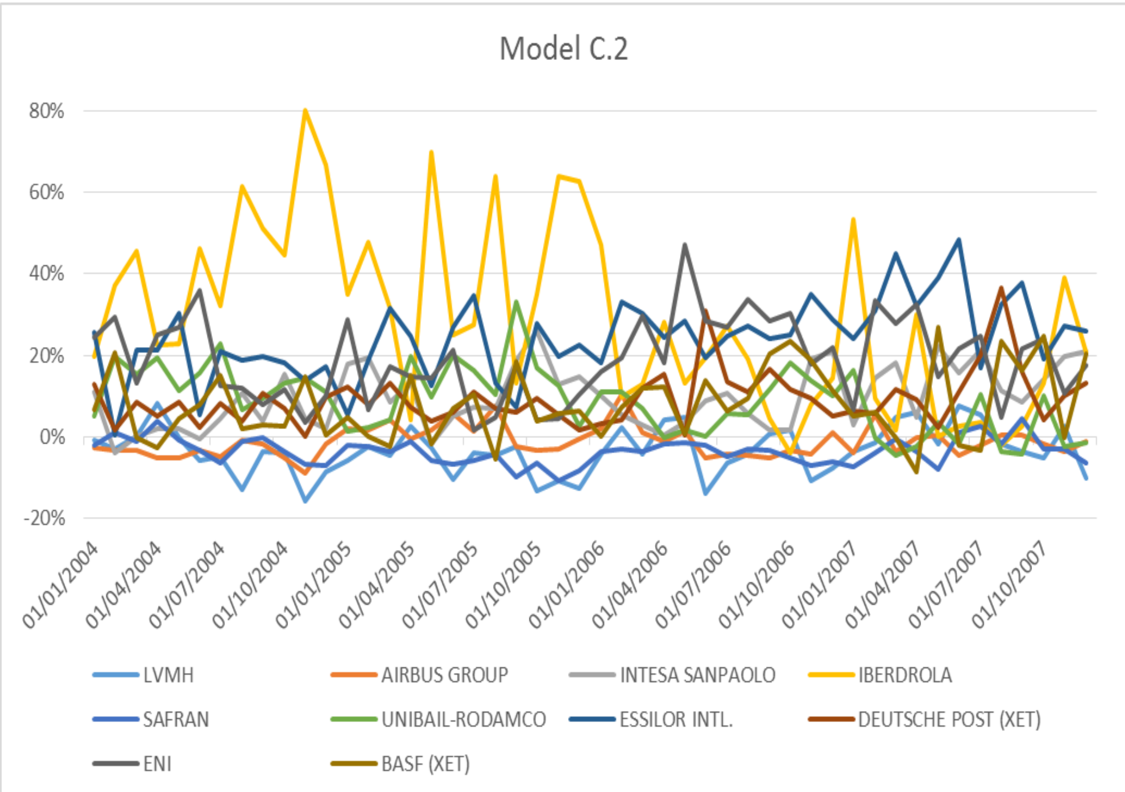
APPENDIX F: MONTHLY EVOLUTION OF THE OPTIMAL PORTFOLIO

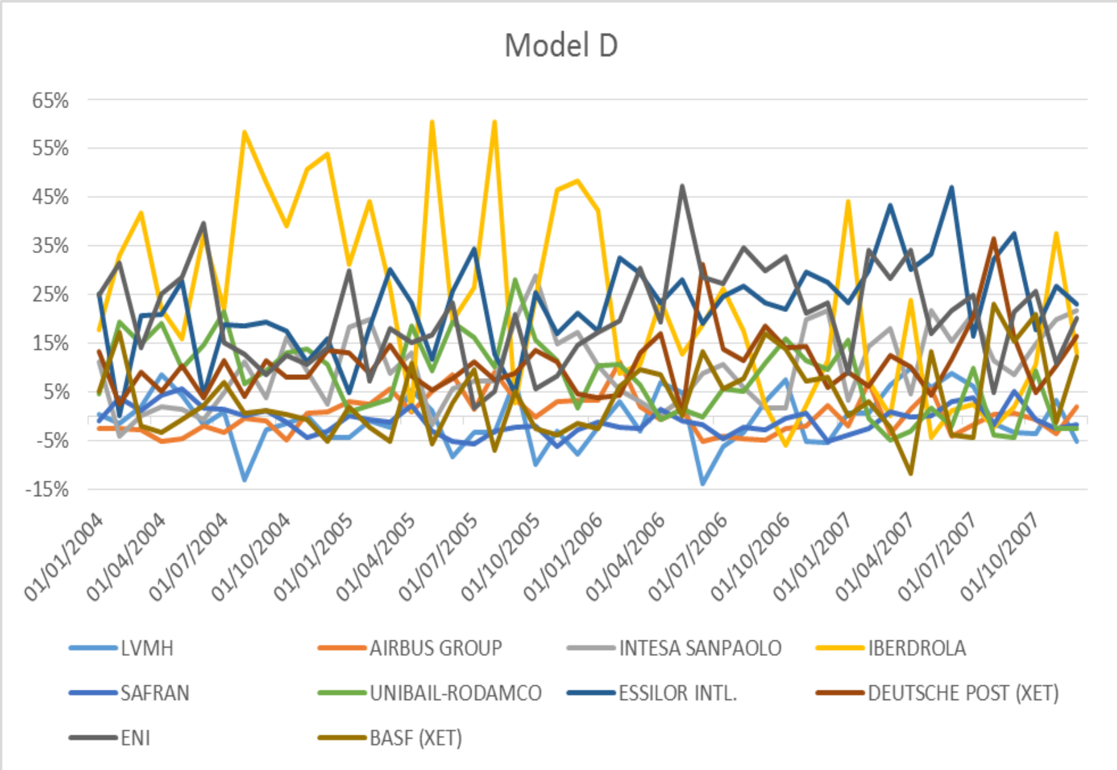
F1. Calm period weights



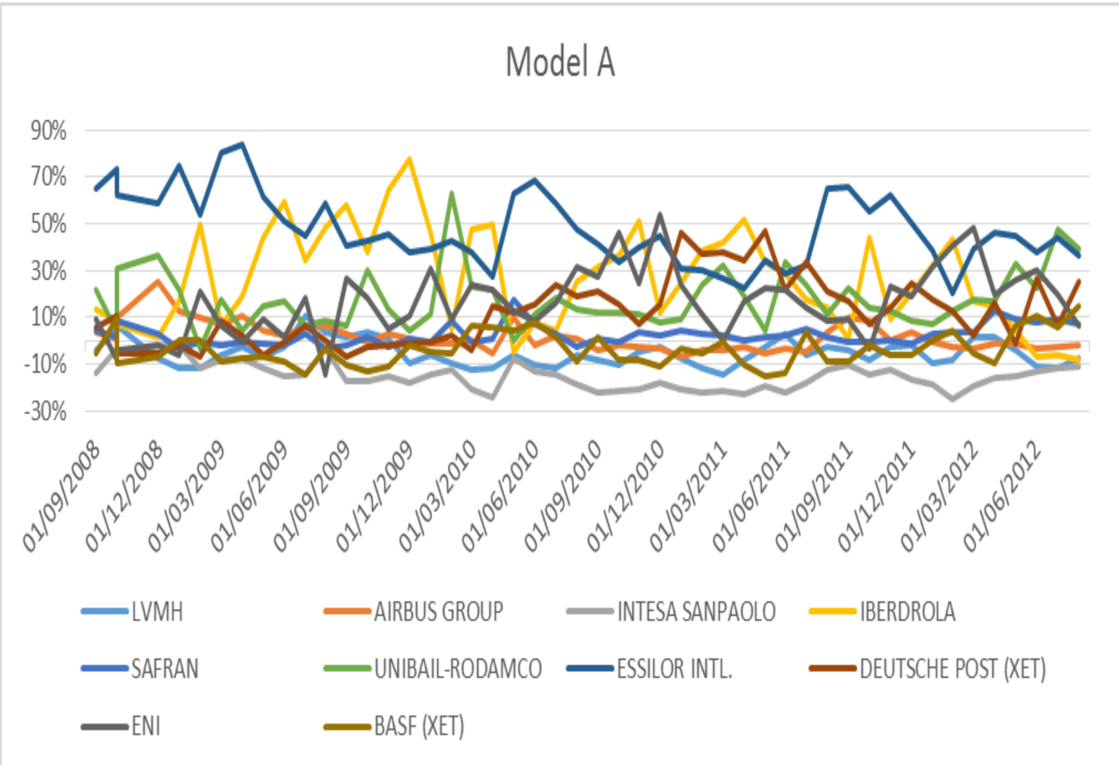


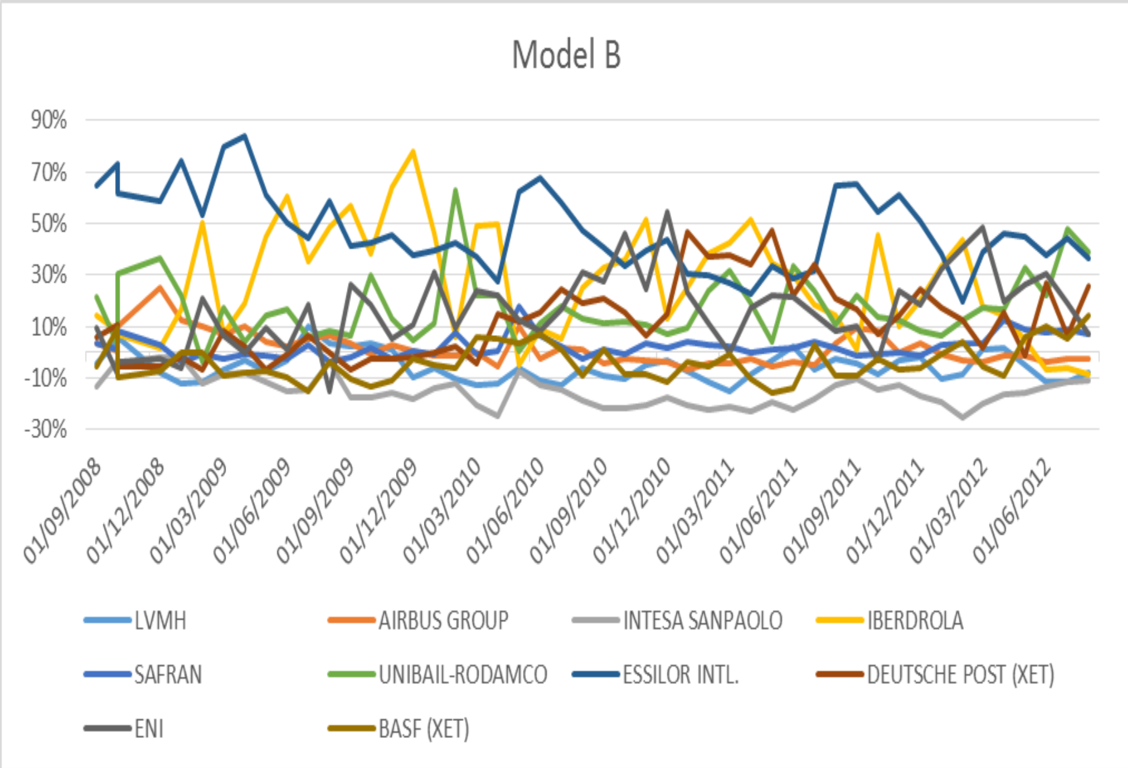

Model C.1
70%
60%
50%
40%
30%
20%
10%
0%
-10%
-20%
01/01/2004
01/04/2004
01/07/2004
01/10/2004
01/01/2005
01/04/2005
01/07/2005
01/10/2005
01/01/2006
01/04/2006
01/07/2006
01/10/2006
01/01/2007
01/04/2007
01/07/2007
01/10/2007
LVMH
AIRBUS GROUP
INTESA SANPAOLO
IBERDROLA
SAFRAN
UNIBAIL-RODAMCO
ESSILOR INTL.
DEUTSCHE POST (XET)
ENI
BASF (XET)

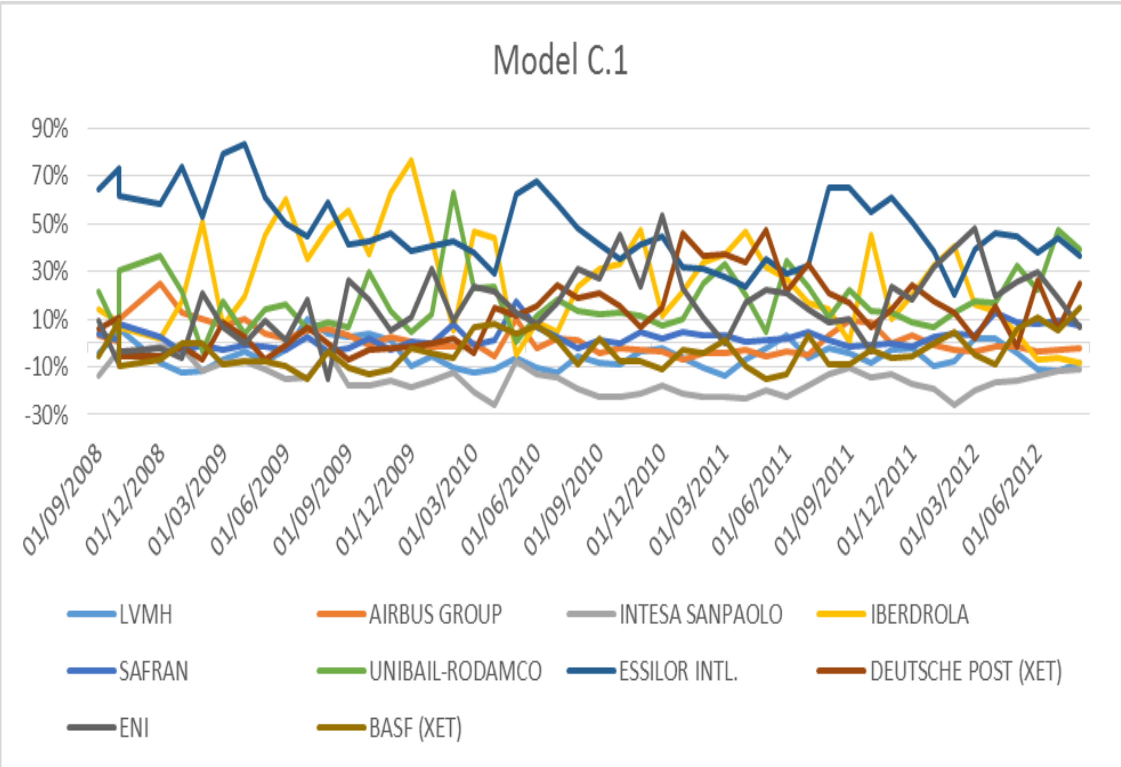

Model C.2
80%
60%
40%
20%
0%
-20%
01/01/2004
01/04/2004
01/07/2004
01/10/2004
01/01/2005
01/04/2005
01/07/2005
01/10/2005
01/01/2006
01/04/2006
01/07/2006
01/10/2006
01/01/2007
01/04/2007
01/07/2007
01/10/2007
LVMH
AIRBUS GROUP
INTESA SANPAOLO
IBERDROLA
SAFRAN
UNIBAIL-RODAMCO
ESSILOR INTL.
DEUTSCHE POST (XET)
ENI
BASF (XET)

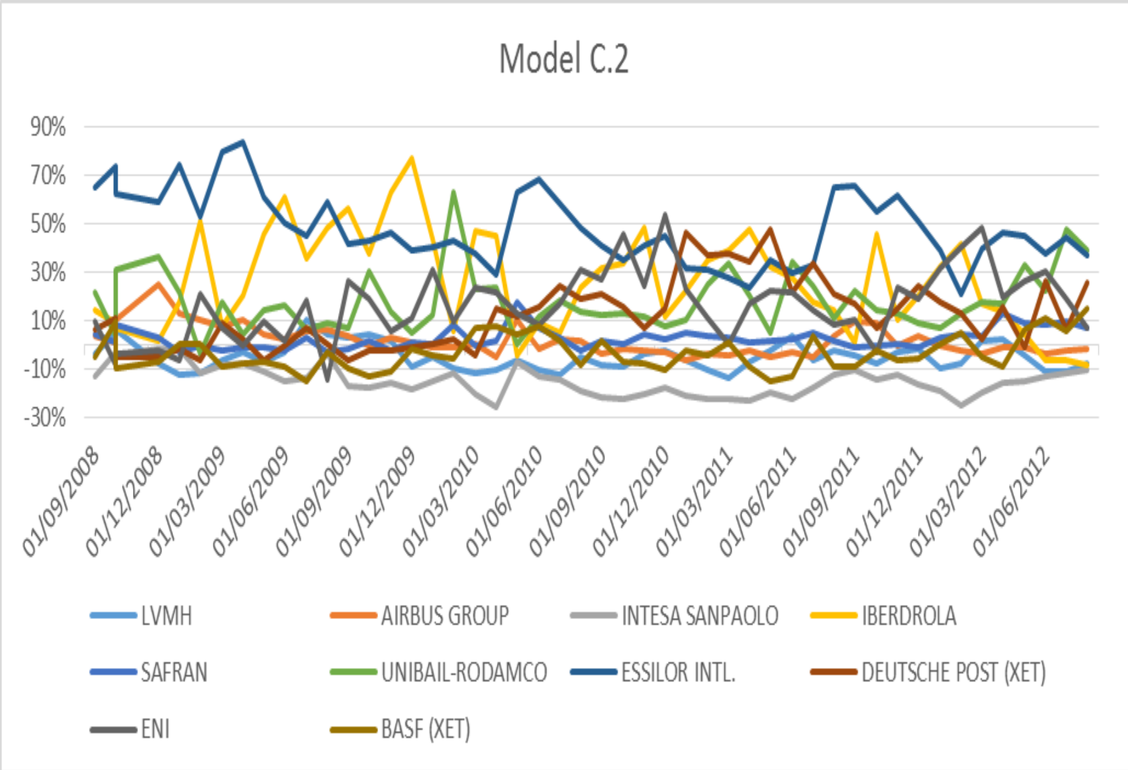


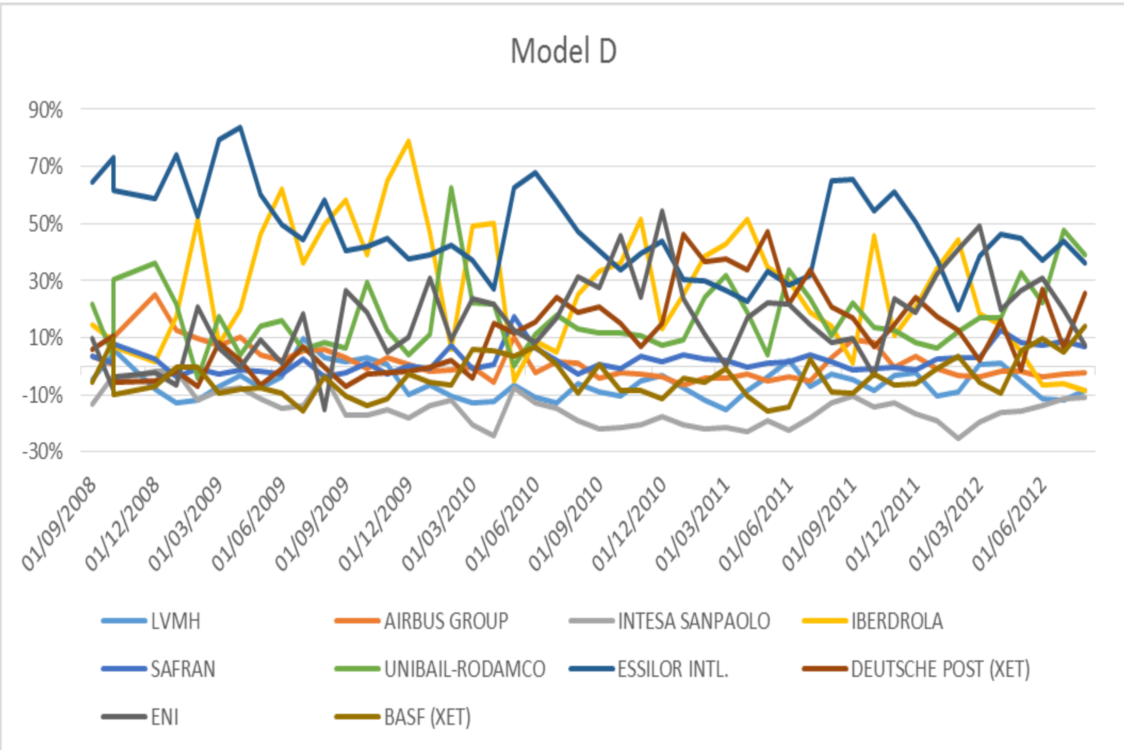
F.2. Stress period weights




Model B
90%
70%
50%
30%
10%
-10%
-30%
01/09/2008
01/12/2008
01/03/2009
01/06/2009
01/09/2009
01/12/2009
01/03/2010
01/06/2010
01/09/2010
01/12/2010
01/03/2011
01/06/2011
01/09/2011
01/12/2011
01/03/2012
01/06/2012
LVMH
AIRBUS GROUP
INTESA SANPAOLO
IBERDROLA
SAFRAN
UNIBAIL-RODAMCO
ESSILOR INTL.
DEUTSCHE POST (XET)
ENI
BASF (XET)


Model C.1
90%
70%
50%
30%
10%
-10%
-30%
01/09/2008
01/12/2008
01/03/2009
01/06/2009
01/09/2009
01/12/2009
01/03/2010
01/06/2010
01/09/2010
01/12/2010
01/03/2011
01/06/2011
01/09/2011
01/12/2011
01/03/2012
01/06/2012
LVMH
AIRBUS GROUP
INTESA SANPAOLO
IBERDROLA
SAFRAN
UNIBAIL-RODAMCO
ESSILOR INTL.
DEUTSCHE POST (XET)
ENI
BASF (XET)


Model C.2
90%
70%
50%
30%
10%
-10%
-30%
01/09/2008
01/12/2008
01/03/2009
01/06/2009
01/09/2009
01/12/2009
01/03/2010
01/06/2010
01/09/2010
01/12/2010
01/03/2011
01/06/2011
01/09/2011
01/12/2011
01/03/2012
01/06/2012
LVMH
AIRBUS GROUP
INTESA SANPAOLO
IBERDROLA
SAFRAN
UNIBAIL-RODAMCO
ESSILOR INTL.
DEUTSCHE POST (XET)
ENI
BASF (XET)


Model D
90%
70%
50%
30%
10%
-10%
-30%
01/09/2008
01/12/2008
01/03/2009
01/06/2009
01/09/2009
01/12/2009
01/03/2010
01/06/2010
01/09/2010
01/12/2010
01/03/2011
01/06/2011
01/09/2011
01/12/2011
01/03/2012
01/06/2012
LVMH
AIRBUS GROUP
INTESA SANPAOLO
IBERDROLA
SAFRAN
UNIBAIL-RODAMCO
ESSILOR INTL.
DEUTSCHE POST (XET)
ENI
BASF (XET)