THE JOINT DISTRIBUTION OF DOMESTIC INDEXES. AN APPROACH USING CONDITIONAL COPULAS

Jone Ascorbebeitia Bilbatua

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Tutores: Dra. Eva Ferreira

Dra. Susan Orbe-Mandaluniz

Universidad Complutense de Madrid

Universidad del País Vasco

Universidad de Valencia

Universidad de Castilla-La Mancha

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The Joint Distribution of Domestic Indexes. An Approach Using Conditional Copulas

Jone Ascorbebeitia Bilbatua
Master's Degree in Banking and Quantitative Finance

Supervisors:
*Eva Ferreira**
*Susan Orbe-Mandaluniz **

Sarrera

Finantza-aktiboak askotan erlazionatuta egoten dira eta erlazio horreek garrantzi handikoak izan daitekez. Izan ere, finantza-aktiboek euren artean daukiezan erlazioen analisia oso interesgarria da Ekonomia eta Finantza arloetan. Zorroek, aktiboek eta baita indizeek ere aldaketak jasotzen dabez egunero euren balioetan eta ondorioz aktiboen arteko erlazioak ere aldaketen eragina izan daikie. Aldaketak edozein unetan eta arrazoi desberdinengatik gertatu daitekez. Baliteke zorroen arteko mugimenduak denborarekin aldatzearen ondorioz emotea. Literaturan badagoz gertakizun hau balioztatzen daben lan desbardinak, esaterako Krishnan, Petkova eta Ritchken (2009), Ferreira, Gil eta Orbe (2011) eta Ferreira eta Orbek ("Why are there time-varying comovements in the European stock market?"(2015)) argitaratutako lanak. Guk mugimenduen aldakuntza zenbait arrisku faktore globalen bitartez azaldu daitekeen aurkitu nahi dogu, hala nola, indize global baten bidez adierazitako merkatu faktore bat. Gainera, korrelazio lineala baino haratago doazen menpekotasun neurriak interesatzen jakuz.

Lan honetan Europar herrialdeetako indizeen arteko erlazioetan zentratuko gara. Ezaguna danez, indizeen artean dagoen menpekotasuna nabaria dala eta menpekotasun horren zati bat Euro Stoxx indizearen bitartez azalduta egon beharko leuke.

Beraz, indize global batekin herrialdeetako indize desberdinen analisiak interes esanguratsua dauka Ekonomia eta Finantza arloetako aspektu batzuetan. Ohituta gagoz korrelazio lineala neurtzen dauan Pearsonen korrelazio koefizientea kalkulatzen, baina zer gertatzen da indizeen arteko menpekotasuna erlazio lineal baten bidez ez badator? Pearsonen korrelazio koefizientea kalkulatzeko ez ezik interpretatzeko ere erraza da, baina baliteke nahikoa ez izatea. Menpekotasun neurri aproposa da banaketa normala jarraitzen daben aldagaietarako baina ez normala izatetik urruti dagozan aldagaietarako. Izan ere, finantza aldagaien arteko erlazioak konplikatuagoak izan daitekez eta hori dala eta orokorragoa dan beste zerbait beharrezkoa da. Sarritan, zorroen aktiboetako errentagarritasunen banaketa funtzio marjinalak desberdinak izaten dira, baita asimetrikoak edo mutur handietakoak ere eta egoera honeetan korrelaziokoefiziente lineala ez da neurri egokia. Orduan, menpekotasun egitura *kopula* deituriko funtzio batzuen bitartez modelizatu daiteke. Kopulak baterako banaketa funtzioak dira eta banaketa funtzio marginalak bateratzea ahalbidetzen dabe. Beraz, kopula eta banaketa funtzio marginalen bitartzez aldagaien baterako banaketa funtzioa definitu daiteke. Banaketa funtzio marjinalak baterako banaketa

*UPV/EHU University of Basque Country

funtzioa baino modelizatzeko errazagoak izateak abantaila bat suposatzen dau. Kopulen teoria interes handikoa da aktiboen arteko baterako banaketa funtzioa aztertzerako orduan, bereziki VaR (Value at Risk) (John C. Hull (2012), Jorion (2003) and Parra and Koodi (2006) among others) bezalako neurriak kalkulatzerako orduan.

Arrazoi honeek direla eta, kopulen bitartez herrialde desberdinetako indizeen arteko erlazioa aztertuko dogu eta baita, kopula baldintzatuak erabiliz, indize globalari (Euro Stoxx) baldintzatutako erlazioak ere. Gainera, menpekotasuna eta baldintzatutako menpekotasuna neurtuko doguz korrelazio koefiziente linealaren ezberdinak diren Kendall-en tau koefizientea bezalako beste neurri batzuekin.

Aipatu bezala, analisi enpirikoan Europar indizeetako erlazioan zentratuko gara. Zehatz-mehatz hurrengo herrialdeetako indizeen eguneroko datuak erabiliko doguz: Espainia (IBEX 35), Frantzia (CAC 40), Alemania (DAX), Erresuma batua (FTSE 100) eta suitza (SMI). Indize global moduan Euro Stoxx (EUROSTOXX 50) indizea erabiliko dogu.

Artikulua bost zatitan egituratuta dago. Lehenengo atalean baldintztutako kopulak estimatzeko prozedura deskribatzen da. Bigarren atalean Euro Stoxx-arekiko menpekotasuna aztertzen da. Kopulen teorian azaltzen diran menpekotasun neurriak azaltzeaz gain, neurri desberdinen arteko desberdintasunak eztabaidatuko dira. Hirugarren atalean aukeratutako neurriarekin (Kendall-en tau koefizientea) menpekotasunaren testa burutuko dogu. Behin jakinda indizeen artean menepekotasuna dagoela, laugarren atalean erregresio eredu ez-parametriko bat proposatzen dogu eta hurrengo galdera erantzuten saiatuko gara: Ba al dago inolako menpekotasun gehigarririk jatorrizko aldagaietatik Euro Stoxx-arekiko batazbesteko menpekotasuna kenduz gero? Azkenik, lortutako emaitzak errazagoa dan erregresio linealaren bidez lortutako emaitzekin konparatuko doguz. Bosgarren atala ondorioen atalari dagokio.

Introduction

The analysis of relationships between financial assets is a topic of great interest in Economics and Finance. Every day, portfolios, assets or indexes undergo changes and the relationship between them might also be affected. It is well known that the comovements between portfolios are time-varying. This fact is supported by works as, among others, Krishnan, Petkova and Ritchken (2009), Ferreira, Gil and Orbe (2011) and Ferreira and Orbe (2015). Our interest is to detect whether these comovements variation can be explained by some global risk factors as a market factor represented by a global index. Moreover, we are interested in measures of dependece beyond linear correlation.

We are going to focus on the relation between domestic European indexes. It's known that there is a great dependence between domestic indexes and that part of this dependence should be explained by Euro Stoxx index.

So, the analysis of domestic indexes with global indexes has a significant interest for some aspects in Economics and Finance. We are used to calculate the Pearson's correlation coefficient, which measures linear correlation, but what if the dependence between indexes is not given by a linear relationship? Pearson's correlation coefficient is very intuitive and easy to calculate but this might not be enough. It is a perfect measure of dependence for normal variables but not for variables are far from being normal distributed. In fact, relationships between financial variables can be more complicated and that's why something more general is needed. In addition, many times marginal distributions of assets returns of a portfolio are different, asymmetric or have big tails and in those situations the linear correlation coefficient is not the adequate measure. Then, one way to model the dependence structure is through functions called *copulas*, which allow constructing a joint distribution function to represent de returns dependence better than an elliptic distribution. Copulas allow for obtaining, together with marginal distribution function, the joint distribution function. An advantage comes from the fact that marginal distribution functions are always easier modelling than the joint distribution. Copula theory has a high interest while studying the joint probability distribution function between assets, particularly to obtain measures such as VaR (Value at Risk) (John C. Hull (2012), Jorion (2003) and Parra and Koodi (2006) among others).

For these reasons, we will study the relation between domestic indexes using copulas, and the relation conditioned to the global index (Euro Stoxx), using conditional copulas. Moreover, we will measure the dependence and the conditional dependence using other measures than the linear correlation, as the Kendall's tau.

As we have mentioned before, for the empirical analysis we will focus on the relationships between European indexes. Specifically, we will use daily stock indexes (1994-2016) for Spain (IBEX 35), France (CAC 40), Germany (DAX), UK (FTSE 100) and Switzerland (SMI). The global index we are going to use is the Euro Stoxx index (EUROSTOXX 50).

This work is structured as follows. In Section 1 we describe the procedure to estimate conditional copulas. In Section 2 we test for Eurostoxx dependence. We present different measures in copula theory and then we discuss about differences between them. In Section 3 we test for dependence with the chosen measure, Kendall's tau. Once we know that there exists dependence between index returns, Section 4 considers a nonparametric regression model and try to answer the following question: Is there any dependence left if we remove the effect of the conditional expectation? Finally we compare the results with the ones obtained if we would consider the easier linear regression model. The last section, Section 5, corresponds to the conclusions.

1. Nonparametric conditional copulas

1.1. Definition of copula

As mentioned above, when we have two variables we are interested in studying the relationship between them and the point is how to model this relationship. One way to model the dependence structure is through *copulas*, joint distribution functions whose marginals are uniform $U \sim (0,1)$ variables. So, let Y_1 and Y_2 be two random variables and $F_1(y_1)$ and $F_2(y_2)$ their marginal distribution functions respectively, where $y_1 \in Y_1$ and $y_2 \in Y_2$. Denote $u_1 = F_1(y_1)$ and $u_2 = F_2(y_2)$.

A *Copula* is a $C(u_1, u_2) = P[U_1 \leq u_1, U_2 \leq u_2]$ function defined in the domain $C : [0,1] \times [0,1] \rightarrow [0,1]$ with the following properties:

(i) $C : [0,1] \times [0,1] \rightarrow [0,1]$.
(ii) $C(u_1, 0) = C(0, u_2) = 0$.
(iii) $C(u_1, 1) = u_1$ and $C(1, u_2) = u_2$.
(iv) For all u_1, u_2, v_1, v_2 in$[0,1]$, where $u_1 \leq u_2$ and $v_1 \leq v_2$: $C(v_1, v_2) - C(u_1, v_2) - C(v_1, u_2) + C(u_1, u_2) \geq 0$.

An important theorem in copula theory is the Sklar's theorem:

(**Sklar's theorem**)Let F be a joint distribution function and F_1, F_2 marginals of F. Then, there exist a copula C where for all $x, y \in \overline{\mathbf{R}}$, such that

$$F(x, y) = C(F_1(x), F_2(y)) \tag{1}$$

If F_1 and F_2 marginal functions are continuous, then C is unique; otherwise, C is uniquely determined on $Ran(F_1) \times Ran(F_2)$ domain. Conversely, if C ia a copula and F_1 and F_2 are distribution functions, then the function F defined by (1) is a joint distribution function with margins F_1 and F_2.

In copula theory there are some basic copulas. A copula which is consistent with variables independence is $\Pi(u_1, u_2) = u_1 \cdot u_2$ the *product copula*. In this case, the joint distribution function would be the product of the marginals. An other basic copula is $M(u_1, u_2) = min(u_1, u_2)$ and is called *Fréchet-Hoeffding upper bound*. When variables are related with this copula we say that there is a perfect positive dependence between the variables. The third copula is the *Fréchet-Hoeffding lower bound* and it is defined as $W(u_1, u_2) = max(u_1 + u_2 - 1, 0)$. Variables related with this copula have a perfect negative dependence. Moreover, *Fréchet-Hoeffding bounds inequality* is defined as

$$W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2),$$

for all $(u_1, u_2) \in \mathbf{I}^2 = [0,1]^2$ and every copula C.

In many applications, the variables represent the lifetimes of individuals in some population. The probability of an individual surviving is given by *the survival function* $\overline{F}_1(y_1) = P[Y_1 > y_1] = 1 - F_1(y_1)$, where F_1 is the distibution function of Y_1. For variables Y_1 and Y_2 with joint distribution function F, *the joint survival function* is $\overline{F}(y_1, y_2) = P[Y_1 > y_1, Y_2 > y_2]$. The marginals of $\overline{F}$ are the univariate survival functions $\overline{F}_1$ and $\overline{F}_2$. Furthermore, *the survival copula* $\hat{C}$ is a function from $\mathbf{I}^2$ to $\mathbf{I}$ defined as

$$\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2).$$

Then, the joint survival function can be written as

$$\overline{F}(u_1, u_2) = \hat{C}(\overline{F}_1(y_1), \overline{F}_2(y_2)).$$

1.2. Conditional copulas

The dependence structure between two variables can be highly influenced by a covariate, and it is also interesting to know how the dependence structure changes with the value of the covariate. So, we want to know what is the relationship between the variables Y_1 and Y_2 conditionally upon the given value of the covariate $X = x$ and whether this relationship changes with the values of x. A function that describes the dependence structure between the variables Y_1 and Y_2 given $X = x$ is called *conditional copula* and represents the conditional joint distribution function of the pair (Y_1, Y_2).

Denote the marginal distribution functions of Y_1 and Y_2, conditionally upon $X = x$ as

$$\begin{array}{ll} F_{1x}(y_1) = P(Y_1 \leq y_1 \mid X = x), & y_1 \in Y_1 \\ F_{2x}(y_2) = P(Y_2 \leq y_2 \mid X = x), & y_2 \in Y_2 \end{array}$$

and let F_x be the conditional joint distribution function

$$F_x(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2 \mid X = x).$$

The Sklar's theorem can be adapted to conditional copulas in the following way. Let F_{1x} and F_{2x} be the marginal distribution functions of F_x. Then there is a unique copula such that

$$F_x(y_1, y_2) = C_x(F_{1x}(y_1), F_{2x}(y_2)).$$

As has been mentioned before, what we want is to analyze the relationship between different variables. To do this, we are going to estimate the dependence structure they have, that is, we are going to estimate the copula. There are two methods of estimation: parametric estimation (Craiu (2009)) or nonparametric estimation. The estimates in this paper will be obtained with nonparametric estimators, so unlike parametric estimation techniques, we will make no assumptions about the probability distributions of the variables.

Thus, we use the inverted Sklar's theorem, which enables to express the conditional copula C_x in terms of the joint and marginal distribution functions:

$$C_x(u_1, u_2) = F_x(F_{1x}^{-1}(u_1), F_{2x}^{-1}(u_2)), \, (u_1, u_2) \in [0, 1]^2,$$

where $F_{1x}^{-1}(u) = inf\{y : F_{1x}(y) \geq u\}$ and $F_{2x}^{-1}(u)$ are the conditional quantile functions of Y_1 and Y_2 given $X = x$ respectively.

1.3. Nonparametric estimation of conditional copulas

The estimation of conditional copulas will be done using nonparametric estimation, which implies that we won't make any assumption about the distribution functions. Suppose that we observe independent identically distributed three-dimensional vectors $(Y_{11}, Y_{21}, X_1), \ldots, (Y_{1n}, Y_{2n}, X_n)$ from the cumulative distribution function $F(y_1, y_2, x)$. Based on the sample of observations we have the following conditional estimator for $F_x(y_1, y_2)$:

$$F_x(y_1, y_2) = \textstyle\sum_{i=1}^{n} w_{ni}(x, h)\mathbf{I}\{Y_{1i} \leq y_1, Y_{2i} \leq y_2\},$$

where $\{w_{ni}(x, h)\}$ is a sequence of weights depending on how close is X_i to x, $h > 0$ is the bandwidth tending to zero as the sample size increases and $\mathbf{I}\{A\}$ is an indicator of an event A. Taking into account this expression for the distribution function estimator, we can define the estimator of the copula function as

$$\begin{aligned} C_{xh}(u_1, u_2) &= F_{xh}(F_{1xh}^{-1}(u_1), F_{2xh}^{-1}(u_2)) \\ &= \sum_{i=1}^{n} w_{ni}(x, h)\mathbf{I}\{Y_{1i} \leq F_{1xh}^{-1}(u_1), Y_{2i} \leq F_{2xh}^{-1}(u_1)\}, \end{aligned} \tag{2}$$

where $0 \leq u_1, u_2 \leq 1$ and F_{1xh} and F_{1xh} are the marginal distribution functions of F_{xh}. Although the copula estimator given in (1) seem very natural because it mimics the structure of the true copula C_x, it can lead to wrong conclusions: suppose that Y_1 and Y_2 are conditionally independent given $X = z$, but that their conditional distributions are increasing with z. Then, larger values of Y_1 will occur with larger values of Y_2 only because they have the same trend respect to the covariate z, creating an artificial dependence. Gijbels, Veraverbeke and Omelka (2011) confirmed this intuition by Monte Carlo experiments in which they observed that this way the estimator C_{xh} may be biased if any of the conditional distributions change with the value of the covariate $X = x$. They also observed that this bias could be reduced to almost at all by removing the effect of the covariates on the marginal distribution functions. Given that the invariance to increasing transformations is a copula property, they suggested that if one knew the marginals F_{1X} and F_{2X}, one could base the estimator C_{xh} on transformed observations $\{(U_{1i}, U_{2i}), i = 1, ...n\}$ where

$$(U_{1i}, U_{2i}) = (F_{1X_i}(Y_{1i}), F_{2X_i}(Y_{2i})),$$

whose marginal distributions are uniform. However, we usually don't know which are the theoretical marginal distribution functions F_{1X_i} and F_{2X_i}, but we can estimate them in the following way

$$F_{1X_ig_1}(y) = \sum_{j=1}^{n} w_{nj}(X_i, g_1)\mathbf{I}\{Y_{1j} \leq y\},$$
$$F_{2X_ig_2}(y) = \sum_{j=1}^{n} w_{nj}(X_i, g_2)\mathbf{I}\{Y_{2j} \leq y\},$$

where g_1 and g_2 tend to zero as n tends to infinity.

Once we have estimated the marginal distribution functions, first we transform the original observed variables to reduce the effect of the covariate by

$$(\tilde{U}_{1i}, \tilde{U}_{2i}) = (F_{1X_ig_1}(Y_{1i}), F_{2X_ig_2}(Y_{2i})),\ i = 1, \ldots, n.$$

Then, we use the transformed variables $(\tilde{U}_{1i}, \tilde{U}_{2i})$ as they were the original ones and we construct

$$\tilde{C}_{xh}(u_1, u_2) = \tilde{G}_{xh}(\tilde{G}_{1xh}^{-1}(u_1), \tilde{G}_{2xh}^{-1}(u_2)),$$

where

$$\tilde{G}_{xh}(u_1, u_2) = \textstyle\sum_{i=1}^{n} w_{ni}(x, h)\mathbf{I}\{\tilde{U}_{1i} \leq u_1, \tilde{U}_{2i} \leq u_2\}$$

and $\tilde{G}_{1xh}$ and $\tilde{G}_{2xh}$ are the corresponding marginals.

While estimating conditional copulas, it is very important the choice of the weight function. An approach to represent the weight sequence $\{w_{ni}(x)\}_{i=1}^{n}$ in each point x, is to describe the shape of the weight function w_{ni} using a shape function with a scale parameter that adjusts the size and the form of the weights near the point x. The shape function is known as *kernel* K and satisfies the condition

$$\textstyle\int_{-\infty}^{\infty} K(x)dx = 1.$$

There are many common choices for weights, but we are going to use the classic kernel estimator proposed by Nadaraya-Watson (1964). This leads to define the weight function as

$$w_{ni}(x,h) = \frac{K(\frac{X_i - x}{h})}{\sum_{j=1}^{n} K(\frac{X_j - x}{h})},$$

where h is the bandwith and K is the kernel function. There are many choices to the kernel function too. We are going to use the Epanechnikov kernel function because is the most efficient kernel. This kernel is defined as follows

$$K(u) = \frac{3}{4}(1 - u^2)\mathbf{1}_{\{|u|<1\}}.$$

An other thing to take care with is the bandwidth parameter's choice. It's even much more important than the kernel's choice. It would be easy if it would exist an automatic method to chose the bandwidth value, but the area of the bandwidth selection in the context of copulas has not been investigated yet. So, we are going to take a starting point. An easy approach is to take $h = 1.06\sigma n^{-1/5}$, where σ is the standard deviation of the covariate. A quick way of choosing the parameter would be to estimate σ from the data and then to substitute the estimate into the formula. This expression could work well if the data is normally distributed, but it may oversmooth if the variable is multimodal. If we write the expression in terms of interquartile range R, we have $h = 0.79Rn^{-1/5}$ (Silverman (1986)). This expression used for bimodal distributions makes things worse because it oversmooths even more. We can obtain the best of both by

$$h = 0.9An^{-1/5}, \text{ using the adaptative estimate of spread } A = min\{\sigma, \frac{R}{1.34}\}.$$

From now on we will use the expression above as bandwidth value for copula estimates; which although is not an optimal value, is a good starting point. For our study this expression takes $h = 0.0018$ value.

Example of estimated conditional copulas will be given in empirical analysis.

2. Measures of concordance

In previous section we have described the method to estimate nonparametric conditional copulas, that is, the copulas subject to some fixed value of the covariate. Now, we want to quantify that dependence. As Jodgeo (1982) notes, "dependence relations between random variables is one of the most widely studied subjects in probability and statistics. The nature of the dependence can take a variety of forms and unless some specific assumptions are made about the dependence, no meaningful statistical model can be contemplated". To do so, it is necessary to examine which role copulas play in the study of dependence. As mentioned before, copulas are invariant under strictly increasing transformations of the variables and many of these measures preserve the "scale-invariant" property. Moreover, we know that dependence properties and measures of association are interrelated. In the literature, we can find different measures of dependence. The association measures we are going to describe measure a form of dependence known as concordance. Let (x_i, y_i) and (x_j, y_j) be two observations from a vector (X, Y). (x_i, y_i) and (x_j, y_j) are said to be *concordant* if $(x_i - x_j)(y_i - y_j) > 0$. In the same way, (x_i, y_i) and (x_j, y_j) are said to be *discordant* if $(x_i - x_j)(y_i - y_j) < 0$. Now we define a concordance function Q. Let (Y_1, Y_2) and (Y_1', Y_2') be two random variables with joint distribution functions H_1 and H_2 respectively, where Y_1' and Y_2' are independent copies of Y_1 and Y_2 and F_1 and F_2 common margins of Y_1 and Y_2 and Y_1' and Y_2' respectively. Let C_1 and C_2 be copulas of (Y_1, Y_2) and (Y_1', Y_2'), so that $H_1(x, y) = C_1(F_1(x), F_2(x))$ and $H_2(x, y) = C_2(F_1(x), F_2(x))$. Thus, Q is defined as the difference between the probabilities of concordance and discordance:

$$Q = P[(Y_1 - Y_2)(Y_1' - Y_2') > 0] - P[(Y_1 - Y_2)(Y_1' - Y_2') < 0].$$

Then,

$$Q = Q(C_1, C_2) = 4\int\int_{\mathbf{I}^2} C_2(u_1, u_2)dC_1(u_1, u_2) - 1.$$

The concordance function plays an important role while defining measures of dependence. So, we summarize some useful properties of the concordance function Q. Let C_1, C_2 be two copulas and Q a concordance function. Then,

- Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$.
- Q is nondecreasing in each argument: if $C_1 \prec C_1'$ and $C_2 \prec C_2'$ for all $(u_1, u_2) \in \mathbf{I}^2$, then $Q(C_1, C_2) \leq Q(C_1', C_2')$.
- $Q(C_1, C_2) = Q(\hat{C}_1, \hat{C}_2)$, where $\hat{C}$ is a survival copula.

Moreover, any measure of association κ between two continuous variables with copula C is a measure of concordance if it satisfies those properties:

1. κ is defined for every pair Y_1, Y_2 of continuous random variables.
2. $-1 \leq \kappa_{Y_1,Y_2} \leq 1$, $\kappa_{Y_1,Y_1} = 1$ and $\kappa_{Y_1,-Y_1} = -1$.
3. $\kappa_{Y_1,Y_2} = \kappa_{Y_2,Y_1}$.
4. If Y_1 and Y_2 are independent, then $\kappa_{Y_1,Y_2} = \kappa_\Pi = 0$.
5. $\kappa_{-Y_1,Y_2} = \kappa_{Y_1,-Y_2} = -\kappa_{Y_1,Y_2}$.
6. If C_1 and C_2 are copulas such that $C_1 \prec C_2$, then $\kappa_{C_1} \leq \kappa_{C_2}$.
7. If $\{(Y_{1n}, Y_{2n})\}$ is a sequence of continuous random variables with copulas C_n, and if $\{C_n\}$ converges to C, then $lim_{n\to\infty}\kappa_{C_n} = \kappa_C$.

2.1. Kendall's tau

One of the most used measure of dependence in nonparametric estimation is *Kendall's tau*. While Pearson's correlation coefficient is a measure of linear correlation between two variables, Kendall's tau correlation coefficient measures the relationship of two variables in a much more general way; measures the ordinal association between two measured quantiles, that is, measures the probability that large (small) values of one variable go with large (small) values of the second variable. Kendall's tau takes values between -1 and 1. Let Y_1 and Y_2 be two random variables. Then, the population version of Kendall's tau is defined as

$$\tau_{Y_1,Y_2} = P[(Y_1 - Y_1')(Y_2 - Y_2') > 0] - P[(Y_1 - Y_1')(Y_2 - Y_2') < 0] = 2P((Y_1 - Y_1')(Y_2 - Y_2') > 0) - 1.$$

where Y_1' and Y_2' are independent copies of Y_1 and Y_2. Looking at Nelsen (2006)(Theorem 5.1.1), τ can be also expressed in terms of copulas, so

$$\tau_{Y_1,Y_2} = \tau_C = Q(C, C) = 4\int\int_{\mathbf{I}^2} C(u_1, u_2)dC(u_1, u_2) - 1$$

Since the integral that appears in the previous expression could be interpreted as the expected value of the function $C(u_1, u_2)$ of $U(0, 1)$ uniform variables U_1 and U_2 whose joint distribution function is C, Kendall's tau can be written as

$$\tau_C = 4E(C(U_1, U_2)) - 1,$$

where C is one of the parametric family of copulas.

2.2. Spearman's rho

As Kendall's tau, the population version of Spearman's rho coefficient is also based on concordance and discordance. Let (Y_1, Y_2), (Y_1', Y_2') and (Y_1'', Y_2'') be three random vectors with common joint distribution function F, margins F_1 and F_2 and copula C. Then, the population version of Spearman's rho is defined to be proportional to the difference between the probability of concordance and discordance for the two pairs (Y_1, Y_1') and (Y_2, Y_2''), that is,

$$\rho = 3(P[(Y_1 - Y_1')(Y_2 - Y_2'') > 0) - P((Y_1 - Y_1')(Y_2 - Y_2'') < 0]),$$

where (Y_1', Y_2') and (Y_1'', Y_2'') are independent copies of (Y_1, Y_2). This leads to the following expression in terms of de concordance function. Then, the population version of Spearman's rho is defined as

$$\begin{aligned}\rho_{Y_1,Y_2} = \rho_C = 3Q(C,\Pi) &= 12\int\int_{\mathbf{I}^2} u_1 u_2 dC(u_1,u_2) - 3\\ &= 12\int\int_{\mathbf{I}^2} C(u_1,u_2)du_1du_2 - 3.\end{aligned}$$

The constant 3, which appears in the previous formula, is a normalization constant since $Q(C,\Pi) \in [-\frac{1}{3}, \frac{1}{3}]$. Moreover, the last equality comes because of the symmetry property of concordance measure Q ($Q(C_1, C_2) = Q(C_2, C_1)$).

As well as for Kendall's tau, a population version for the conditional Spearman's rho can be obtained for the conditional copula C_x:

$$\rho(x) = 12\int\int C_x(u_1,u_2)du_1du_2 - 3.$$

According to Gijbels, Veraverbeke and Omelka (2011) the first estimator of a nonparametric conditional Spearman's rho is

$$\hat{\rho}_n(x) = 12\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ni}(x,h_n)(1-\hat{U}_{1i})(1-\hat{U}_{2i}) - 3,$$

where $\hat{U}_{1i} = F_{1xh}(Y_{1i})$ and $\hat{U}_{2i} = F_{2xh}(Y_{2i})$.

The relationship between Spearman's rho and Pearson's correlation coefficient has been studied for different authors. Spearman's rho is often called "grade" correlation coefficient. Grades are the population analogs of ranks (if y_1 and y_2 are observations from Y_1 and Y_2 with distribution functions F_1 and F_2 respectively, then the grades of y_1 and y_2 are given by $u_1 = F_1(y_1)$ and $u_2 = F_2(y_2)$). It's known that, since grades are observations from uniform variables $U_1 = F_1(Y_1)$ and $U_2 = F_2(Y_2)$ whose joint distribution function is C copula and U_1 and U_2 both have mean $\frac{1}{2}$ and variance $\frac{1}{12}$, the expression for Spearman's rho can be written as

$$\begin{aligned}\rho_{Y_1,Y_2} = \rho_C &= 12\int\int_{\mathbf{I}^2} u_1 u_2 dC(u_1,u_2) - 3\\ &= 12E(U_1U_2) - 3 = \frac{E(U_1U_2) - \frac{1}{4}}{\frac{1}{12}} = \frac{E(U_1U_2) - E(u_1)E(u_2)}{\sqrt{Var(U_1)}\sqrt{Var(U_1)}}\end{aligned}$$

So, looking at this last expression, Spearman's rho for Y_1 and Y_2 is identical to Pearson's correlation coefficient for $U_1 = F_1(Y_1)$ and $U_2 = F_2(Y_2)$.

We have seen that both Kendall's tau and Spearman's rho are measures defined dependent on concordance between two variables with a given copula, but the values of τ and ρ are usually different. However, Daniels (1950) found and proved that those two coefficients are related somehow. Let Y_1 and Y_2 be two variables and τ and ρ Kendall's tau and Spearman's rho respectively. Then, the relationship between τ and ρ is given by

$$-1 \leq 3\tau - 2\rho \leq 1.$$

Kendall's tau and Spearman's rho are both measures of concordance (for proof see Nelsen (2006) Theorem 5.1.9).

2.3. Gini's coefficient

Although Kendall's tau and Spearman's rho are the most common and used measures, besides those two measures of depencence there are other measures too. One of them is the Gini's coefficient. In the 1910's Corrado Gini introduced a measure of association g

$$g = \frac{1}{\lfloor n^2/2 \rfloor}[\textstyle\sum_{i=1}^{n} \mid p_i + q_i - n - 1 \mid - \sum_{i=1}^{n} \mid p_i - q_i \mid],$$

where $\lfloor x \rfloor$ is the integer part of x and p_i and q_i are the ranks in a sample of size n of two continuous variables Y_1 and Y_2 respectively. Let $U_1 = F_1(Y_1)$ and $U_2 = F_2(Y_2)$ the marginal distribution functions of the variables Y_1 and Y_2 with joint distribution function or copula C. The population parameter γ estimated by this statistic is defined by

$$\gamma = 2 \textstyle\int\int_{\mathbf{I}^2} (|u_1 + u_2 - 1| - |u_1 - u_2|) dC(u_1, u_2).$$

Gini's measure of association can be written also in terms of the measure of concordance Q,

$$\gamma_{Y_1,Y_2} = \gamma_C = Q(C, M) + Q(C, W),$$

where M and W are the *Fréchet-Hoeffding upper bound* and the *Fréchet-Hoeffding lower bound* respectively. Spearman's rho ($\rho = 3Q(C, \Pi)$) measures a concordance relationship between the joint distribution function of Y_1 and Y_2 represented by the copula C and the indepent copula Π. However, Gini's γ coefficient ($\gamma_C = Q(C, M) + Q(C, W)$) measures a concordance relationship between the joint distribution function between Y_1 and Y_2, represented by the C copula, and the monotone dependence, represented by the Frchet-Hoeffding bounds copulas M and W.

2.4. Blomqvist coefficient

The last measure of association we are going to explain is the *Blomqvist coefficient*. Consider again the expression for the measure of concordance Q:

$$Q = P[(Y_1 - Y_2)(Y_1' - Y_2') > 0] - P[(Y_1 - Y_2)(Y_1' - Y_2') < 0].$$

Instead of taking two independent copies of Y_1 and Y_2, we take a fixed point for each variable. Thus,

$$Q = P[(Y_1 - y_{10})(Y_2 - y_{20}) > 0] - P[(Y_1 - y_{10})(Y_2 - y_{20}) < 0],$$

for some choice of (y_{10}, y_{20}) in $\mathbf{R}^2$. Blomqvist (1950) studied this measure taking the respective population medians for the values of y_{10} and y_{20}. Making this choice for the point (y_{10}, y_{20}), he called this measure *medial correlation coefficient*, which was denoted β and given by

$$\beta = \beta_{Y_1,Y_2} = P[(Y_1 - \tilde{y_1})(Y_2 - \tilde{y_2}) > 0] - P[(Y_1 - \tilde{y_1})(Y_2 - \tilde{y_2}) < 0],$$

where $\tilde{y_1}$ and $\tilde{y_2}$ are the medians of Y_1 and Y_2 respectively. In particular, if Y_1 and Y_2 are continuous with F joint distribution function and margins F_1 and F_2 and copula C, then $F_1(\tilde{y_1}) = F_2(\tilde{y_2}) = 1/2$. Hence,

$$\begin{aligned}
\beta &= P[(Y_1 - \tilde{y_1})(Y_2 - \tilde{y_2}) > 0] - P[(Y_1 - \tilde{y_1})(Y_2 - \tilde{y_2}) < 0] \\
&= 2P[(Y_1 - \tilde{y_1})(Y_2 - \tilde{y_2}) > 0] - 1 \\
&= 2\{P[(Y_1 < \tilde{y_1}, Y_2 < \tilde{y_2}] + P[(Y_1 > \tilde{y_1}, Y_2 > \tilde{y_2})]\} - 1 \\
&= 2\{F(\tilde{y_1}, \tilde{y_2}) + [1 - F_1(\tilde{y_1}) - F_2(\tilde{y_2}) + F(\tilde{y_1}, \tilde{y_2})]\} - 1 = 4F(\tilde{y_1}, \tilde{y_2}) - 1
\end{aligned}$$

On the other hand, note that $F(\tilde{y_1}, \tilde{y_2}) = C(\frac{1}{2}, \frac{1}{2})$. Hence,

$$\beta = \beta_C = 4C(\tfrac{1}{2}, \tfrac{1}{2}) - 1,$$

The Blomqvist's β then depends on the copula only through the middle value of $\mathbf{I}^2$, but however it often can provide an accurate approximation to Spearman's ρ and Kendall's τ.

As well as Kendall's tau and Spearman's rho, Gini's γ coefficient and Blomqvist's β are also measures of concordance, since the properties of a concordance measure are satisfied.

2.5. Comparison between measures of concordance

The Pearson's correlation coefficient is one of the most often used statistical estimator. The problem is that its value may be affected by only one outlier. On the other hand, there are Spearman's rho and Kendall's tau, which are most used nonparametric measures. Therefore, there is no much investigation about γ and β coefficients' efficiency to compare with *tau* and *rho*. Croux and Denon (2010) confirm the general belief that Kendall's tau and Spearman's rho nonparametric measures are robust to outliers. Moreover, although they aren't as efficient as Pearson's coefficient, they provide a good proportion between robustness and efficiency. Even so, Kendall's correlation measure is more robust and slightly more efficient than Spearman's rho, making it preferable. In addition, as the sample size increases, Kendall's tau approaches a normal distribution more quickly than Spearman's rho. Thus, Kendall's tau will be the dependence measure for our testing work. In the same way that Kendall's population version has been defined, it is also posible to define a population version for the conditional Kendall's tau, which expression is

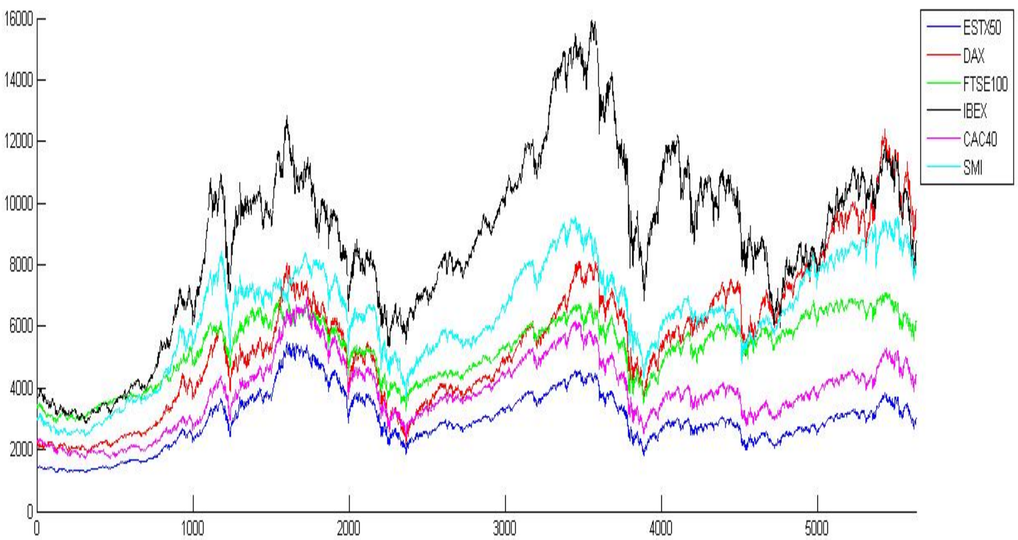
$$\tau(x) = 4 \int \int C_x(u_1, u_2) dC_x(u_1, u_2) - 1,$$

where C_x is the conditional copula.

Gijbels, Veraverbeke and Omelka (2011) checked that the best experience was obtained with the following expression for estimating the conditional Kendall's tau

$$\hat{\tau}_n(x) = \frac{4}{1 - \sum_{i=1}^{n} w_{ni}^2(x, h_n)} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ni}(x, h) w_{nj}(x, h) \mathbf{I}\{Y_{1i} < Y_{1j}, Y_{2i} < Y_{2j}\} - 1. \tag{3}$$

Figure 1: European indexes and Euro Stoxx from 1994 to 2016. Daily data.



This figure shows daily data for the five domestic indexes and Euro Stoxx. The period runs from 1994-03-01 to 2016-03-02.

3. Empirical analysis of dependence between domestic indexes and Euro Stoxx

We consider daily indexes[2] for five countries: Spain (IBEX 35), France (CAC 40), Germany (DAX), Switzerland (SMI) and the United Kingdom (FTSE 100). Moreover, we consider as global index the Euro Stoxx. The sample runs from 1994-03-01 to 2016-03-02 and contains the closing daily data for each domestic index and Euro Stoxx. The idea is to test for Euro Stoxx dependence with Kendall's tau measure of dependence. Figure 1 shows the evolution on daily data for the indexes.

We can see that all the indexes have the same trend and in fact, they follow Euro Stoxx trend. So the Euro Stoxx movements affect on domestic indexes. The domestic index that better mimics Euro Stoxx in this period is CAC 40, which correspond to France. We can see that the comovements between them are similar. Table 1 presents the main descriptive statistics of the domestic indexes and Euro Stoxx.

In the same way as it could be seen in Figure 1, note that IBEX 35 takes the biggest range of values, but it has the highest volatility also. The index with lowest value is FTSE 100. On the other hand, looking at those values we can see that effectively CAC 40 and Euro Stoxx have almost the same mean and volatility. In addition, the standard deviation for all the indexes is small and the mean values are all around zero.

As illustration we show in Table 2 the unconditional linear correlation coefficients of the returns between the five domestic indexes. The correlation between domestic indexes is over 0.7 for all domestic indexes. In addition, the correlation with Euro Stoxx is even bigger. So we can see that there is a significant relationship between domestic indexes and with Euro Stoxx.

[2]Source: http://finance.yahoo.com/.

Table 1: Descriptive statistics of daily domestic indexes and Euro Stoxx

	Min	Max	Mean	Std.
EUROSTOXX 50	-0.0821	0.1044	0.000138	0.0147
DAX	-0.0743	0.1080	0.000273	0.0152
FTSE 100	-0.0926	0.0938	0.000110	0.0119
IBEX 35	-0.0959	0.1348	0.000164	0.0151
CAC 40	-0.0947	0.1060	0.000124	0.0148
SMI	-0.0907	0.1079	0.000184	0.0122

This table presents the descriptive statistics for the following daily indexes: EUROSTOXX 50 (Euro Stoxx), DAX (Germany), FTSE 100 (United Kingdom), IBEX 35 (Spain), CAC 40 (France) and SMI(Switzerland).

Table 2: Domestic index returns and Euro Stoxx: unconditional linear correlation coefficients

	DAX	FTSE 100	IBEX 35	CAC 40	SMI	EUROSTOXX
DAX	1	0.790	0.772	0.857	0.763	0.9202
FTSE 100		1	0.767	0.860	0.780	0.8648
IBEX 35			1	0.845	0.725	0.8809
CAC 40				1	0.793	0.9552
SMI					1	0.814

This table presents the unconditional linear correlation coefficients for daily indexes: DAX (Germany), FTSE 100 (United Kingdom), IBEX 35 (Spain), CAC 40 (France) and SMI (Switzerland).

Taking into account the procedure described in Section 1 and in order to show some estimated copulas, we have calculated the conditional copula estimates for several values of Euro Stoxx. The values of Euro Stoxx to condition on will be quantiles 0.01, 0.1, 0.25, 0.50, 0.75, 0.9 and 0.99, which are represented by $q_{0.01}, q_{0.1}, q_{0.25}, q_{0.5}, q_{0.75}, q_{0.9}$ and $q_{0.99}$ respectively. Figure 2 shows the estimates for DAX and FTSE 100. It can be observed that when we condition to tail values there are not many observations to estimate each point; that's why we obtain a stepped shape. However, the closer is Euro Stoxx value to the mean the more plane the distribution function is.

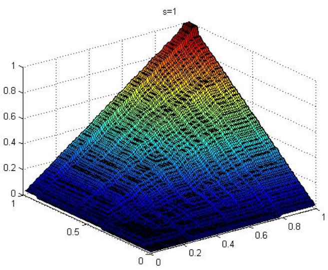
We would like to test for Euro Stoxx dependence with Kendall's tau measure of dependence. To do so, we are going to condition to some values of Euro Stoxx, such as quantile 0.01, 0.1, 0.25, 0.5, 0.75, 0.9 and 0.99 (which are represented by $q_{0.01}, q_{0.1}, q_{0.25}, q_{0.5}, q_{0.75}, q_{0.9}$ and $q_{0.99}$ respectively). Then, we will try to answer the following question: Are the measures for the conditioning values equal and the same to the uncoditional measure or not?

So, we suggest the following hypothesis test:

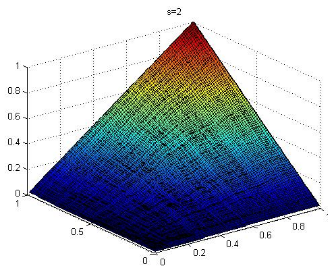
$$\begin{cases} H_0 : \tau = \tau_{q_{0.01}} = \tau_{q_{0.1}} = \tau_{q_{0.25}} = \tau_{q_{0.5}} = \tau_{q_{0.75}} = \tau_{q_{0.9}} = \tau_{q_{0.99}} \\ \\ H_a : \tau \neq \tau_i, \quad i = q_{0.01}, q_{0.1}, q_{0.25}, q_{0.5}, q_{0.75}, q_{0.9}, q_{0.99} \end{cases}$$

If H_0 is satisfied, Kendall's tau coefficient is the same for all conditioning values of Eurostoxx, that is, it must be equal to the unconditional Kendall's coefficient. There is no doubt that calculating the unconditional coefficient is easier than calculating the conditional one. Note that $\hat{\tau}_n$ coincides with the estimator defined in (3) when parameters h, g_1 and g_2 go to infinity. The nonparametric unconditional Kendall's tau version is given by

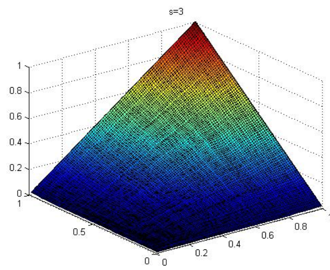
Figure 2: Conditional copula estimates for DAX and FTSE 100.



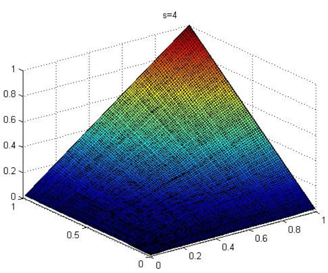
(a) quantile 0.01



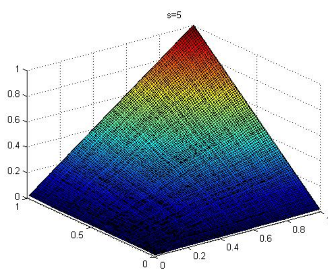
(b) quantile 0.1



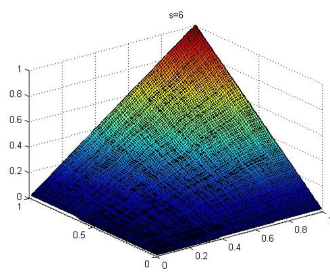
(c) quantile 0.25



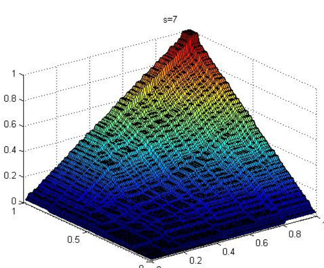
(d) quantile 0.50



(e) quantile 0.75



(f) quantile 0.9



(g) quantile 0.99

This figure shows the estimations of conditional copula for DAX and FTSE 100 of the Euro Stoxx quantiles.

$$\hat{\tau}_n = \frac{4}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{I}\{Y_{1i} < Y_{1j}, Y_{2i} < Y_{2j}\} - 1.$$

In this case, Kendall's tau could be considered the same, no matter what the conditionant value is. On the contrary, if the alternative hypothesis is satisfied, it's not necessary to do the complex procedure of calculating conditional copulas and it's enough with the general theory of unconditional copulas. Moreover, given this fact, the relationship between index returns does not depend on Euro Stoxx values.

To decide somehow when the null hypothesis is satisfied we calculate the confidence interval for uncoditional Kendall's tau,

$$[\hat{\tau} - \sigma_\tau u_{1-\frac{\alpha}{2}}, \hat{\tau} + \sigma_\tau u_{1-\frac{\alpha}{2}}],$$

where $u_{1-\frac{\alpha}{2}}$ is the corresponding quantile of the standard normal distribution and $\alpha \in (0,1)$ confidence level. σ_τ makes reference to the standard deviation of τ. We don't know the real value, so we will estimate it via *Jackknife estimator.*

3.1. Estimator of Kendall's tau variance.

The *jackknife* is a resampling technique developed by Quenouille (1949) to estimate the bias of an estimator. Tukey (1958) expanded the use of the procedure to include variance estimation. This procedure is strongly related to an other resampling method such as the bootstrap (i.e., the jackknife is often a linear approximation of the bootstrap), which is the main technique for computational estimation of population parameters. The jackknife estimation of a parameter is an iterative process.

Let θ be the parameter and T its jackknife estimate. The sample of N observations is a set denoted $\{X_1, \cdots, X_n, \cdots, X_N\}$. The sample estimate of the parameter θ is a function $T = f(X_1, \cdots, X_n, \cdots, X_N)$ of the observations in the sample. An estimation of the population parameter obtained removing the n-th observation, called the n-th *partial prediction* and denoted T_{-n}, is a function $T_{-n} = f(X_1, \cdots, X_{n-1}, \cdots, X_{n+1}, \cdots, X_N)$. Moreover, $T_\bullet$ is obtained as the mean of the partial predictions

$$T_\bullet = \sum_{i=1}^{n} T_{-i}$$

Hence, the variance of the parameter estimates is denoted

$$\hat{\sigma}_T^2 = \frac{n-1}{n} \sum_{i=1}^{n} (T_{-i} - T_\bullet)^2$$

Our parameter is Kendall's tau, so what we would like to estimate is tau's variance. Thus, the estimated variance will be

$$\hat{\sigma}_\tau^2 = \frac{n-1}{n} \sum_{i=1}^{n} (\hat{\tau}_{-i} - \hat{\tau}_\bullet)^2,$$

where $\hat{\tau}_\bullet = \frac{1}{n} \sum_{i=1}^{n} \hat{\tau}_{-i}$.

Instead of doing directly a leave-one-out procedure, we have implemented the procedure leaving out 1, 5, 10, 15 and 20 observations. The results obtained were similar and we reached to the same conclusions, so due to programming efficiency and to getting a representative number of estimations, we are going to leave out 10 observations of the sample on the calculation of each Kendall's tau. This way we have 538 estimated coefficients to calculate the estimated variance.

In order to see the sensitivity of Kendall's tau to the bandwidth choice, we consider different values for the h based on the one used to estimate the conditional copula. The value $h = 0.0018$ considered before implies taking about the nearest 9 observations to estimate each point. For the values of τ conditioned to quantiles

0.01 and 0.99 it is necessary to increase the h value (i.e. we take $3\dot{h}$ as we did before when calculating copulas) because there is a small amount of observations on the tails while estimating each point.

Table 3 shows the unconditional Kendall's tau and the confidence intervals that correspond to each pair of indexes.

Table 3: Unconditional Kendall's τ and its confidence interval for each pair of index returns with 95% and 99% of confidence level

	FTSE 100	IBEX 35	CAC 40	SMI
DAX	0.5737	0.566	0.661	0.558
	[0.540 0.607]	[0.532 0.600]	[0.631 0.690]	[0.535 0.580]
	[0.530 0.616]	[0.522 0.610]	[0.622 0.699]	[0.529 0.586]
FTSE 100		0.540	0.636	0.554
		[0.515 0.565]	[0.605 0.668]	[0.533 0.575]
		[0.509 0.571]	[0.595 0.677]	[0.519 0.590]
IBEX 35			0.633	0.505
			[0.615 0.651]	[0.482 0.528]
			[0.609 0.657]	[0.476 0.535]
CAC 40				0.574
				[0.550 0.598]
				[0.542 0.606]

This table presents the unconditional Kendall's tau coefficients for the five index daily returns: DAX (Germany), FTSE 100 (United Kingdom), IBEX 35 (Spain), CAC 40 (France) and SMI(Switzerland). The second and the third row represent the confidence intervals with 95% and 99% of confidence level respectively.

Table 4 to Table 13 show Kendall's tau coefficients for pairs of indexes. Results are obtained for several bandwidth values which are close to the one indicated by $h = 0.9An^{1/5}$. Once the procedure above has been implemented, we obtain the following results. Therefore the notation is the same for all results tables.

On the other hand, in Subsection 1.3 we have discussed the bandwidth choice's importance. That's why it would be interesting to study the sensitivity of Kendall's τ to the bandwidth choice. Thus, we calculate the conditional coefficient for different values around h such as

$$[\tfrac{3}{4}h \;\; h \;\; \tfrac{5}{4}h \;\; \tfrac{3}{2}h \;\; 2h]$$

Table 4: Kendall's τ between DAX and FTSE 100 conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.164	0.035	-0.022	-0.003	0.031	0.0008	-0.034
$\frac{3}{4}h$	0.289	0.030	-0.020	-0.017	0.025	0.023	-0.099
$\frac{5}{4}h$	0.157	0.019	-0.019	-0.013	0.028	0.014	-0.003
$\frac{3}{2}h$	0.176	0.017	-0.014	0.028	0.029	0.027	0.003
$2h$	0.196	0.020	0.0002	0.059	0.053	0.061	0.027

This table presents the estimated Kendall's tau coefficients for DAX and FTSE 100 conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 5: Kendall's τ between DAX and IBEX 35 conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.115	-0.161	-0.069	-0.134	-0.163	-0.162	-0.261
$\frac{3}{4}h$	-0.093	-0.158	-0.075	-0.152	-0.178	-0.153	-0.303
$\frac{5}{4}h$	-0.078	-0.171	-0.063	-0.121	-0.140	-0.156	-0.207
$\frac{3}{2}h$	-0.069	-0.158	-0.058	-0.102	-0.124	-0.145	-0.178
$2h$	-0.045	-0.133	-0.037	-0.059	-0.082	-0.118	-0.109

This table presents the estimated Kendall's tau coefficients for DAX and IBEX 35 conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 6: Kendall's τ between DAX and CAC 40 conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.246	-0.137	-0.041	-0.099	-0.064	-0.004	-0.004
$\frac{3}{4}h$	-0.230	-0.124	-0.041	-0.106	-0.058	-0.010	-0.100
$\frac{5}{4}h$	-0.139	-0.147	-0.029	-0.078	-0.050	-0.0004	0.044
$\frac{3}{2}h$	-0.020	-0.137	-0.011	-0.052	-0.030	-0.006	-0.063
$2h$	0.146	-0.106	0.023	-0.003	0.006	-0.004	0.140

This table presents the estimated Kendall's tau coefficients for DAX and CAC 40 conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 4 presents the Kendall's tau coefficients between DAX and FTSE 100. The results show that there is only some dependence for the smallest values of Euro Stoxx returns. Although the values conditional in the rest of the Euro Stoxx values change their sign, we realize that is almost negligible. Attending to the sensitivity of Kendall's tau to the bandwidth choice, we can see that in this case the coefficient is not too much sensitive. Table 5 presents the coefficients' value for DAX and IBEX 35. In this case, we can see that the dependence is always negative, so DAX and IBEX 35 indexes' returns are inversely proportional. The dependence between them is in general insignificant for low values of Euro Stoxx returns and takes more importance while Euro Stoxx values are higher, mostly from the mean zone on. Then, the bigger are the returns of Euro Stoxx, the bigger is the dependence between DAX and IBEX 35. Hence, the relationship between them is stronger for periods where the Euro Stoxx yields well and there is almost no dependence for Euro Stoxx low returns. The values for τ are bigger when the bandwidth is far from h (look at the coefficients for $2h$). Even though, there is no an appreciable sensitivity of tau to the bandwidth for the different values of the bandwidth.

Table 6 shows the results for DAX and CAC 40. In this case the dependence between DAX and CAC 40

Table 7: Kendall's τ between DAX and SMI conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.087	0.107	0.062	0.078	0.103	0.0043	−0.167
$\frac{3}{4}h$	0.046	0.116	0.058	0.067	0.104	0.052	−0.199
$\frac{5}{4}h$	0.131	0.087	0.069	0.095	0.104	0.051	−0.125
$\frac{3}{2}h$	0.174	0.084	0.082	0.102	0.103	0.056	−0.105
$2h$	0.215	0.087	0.099	0.119	0.110	0.078	−0.048

This table presents the estimated Kendall's tau coefficients for DAX and SMI conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 8: Kendall's τ between FTSE 100 and IBEX 35 conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	−0.027	−0.100	0.109	−0.005	−0.002	0.010	0.011
$\frac{3}{4}h$	−0.072	−0.116	0.123	−0.014	0.007	0.004	0.056
$\frac{5}{4}h$	0.007	−0.082	0.097	0.010	0.006	0.025	0.020
$\frac{3}{2}h$	0.021	−0.070	0.086	0.025	0.017	0.029	0.051
$2h$	0.065	−0.049	0.077	0.051	0.036	0.054	0.094

This table presents the estimated Kendall's tau coefficients for FTSE 100 and IBEX 35 conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 9: Kendall's τ between FTSE 100 and CAC 40 conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	−0.027	0.188	0.223	0.113	0.134	0.187	0.197
$\frac{3}{4}h$	−0.185	0.178	0.228	0.125	0.143	0.231	0.103
$\frac{5}{4}h$	0.034	0.210	0.213	0.120	0.149	0.167	0.220
$\frac{3}{2}h$	0.100	0.221	0.203	0.129	0.154	0.157	0.239
$2h$	0.209	0.241	0.197	0.160	0.165	0.164	0.264

This table presents the estimated Kendall's tau coefficients for FTSE 100 and CAC 40 conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

is also negative but it is only appreciable when we condition to percentile 1 and decile 1, that is, when we condition to small values of Euro Stoxx returns. So, from the bigger 75

Table 7 shows that the dependence between DAX and SMI is negative only when Euro Stoxx takes big values. On the other hand, we can say that there is some little positive dependence when we condition to decile 1 and quantile 3 of Euro Stoxx. Even so, in general, the dependence between DAX and SMI is small. If we analyze the sensitivity to the bandwidth, we conclude that when we increase the bandwidth value, the coefficient becomes bigger.

In Table 8 we can see Kendall's coefficients for FTSE 100 and IBEX 35. This time, the dependence between the indexes is almost insignificant: values are negative only for the 10

Table 11 and Table 12 show the results for IBEX 35 with CAC 40 and SMI. In the first case, the dependence is negative when we condition to the percentile 1, quantile 3 and decile 9 of Euro Stoxx; that is, when we condition to very small and relatively big (but not the biggest) values. However, the dependence is only significant for very small values of Euro Stoxx and almost negligible for the rest of the conditionants. In the second case, tau

Table 10: Kendall's τ between FTSE 100 and SMI conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.179	0.319	0.252	0.127	0.115	0.187	0.176
$\frac{3}{4}h$	0.137	0.348	0.262	0.123	0.109	0.173	0.185
$\frac{5}{4}h$	0.228	0.304	0.243	0.132	0.120	0.201	0.165
$\frac{3}{2}h$	0.261	0.289	0.228	0.141	0.122	0.213	0.175
$2h$	0.303	0.265	0.219	0.158	0.138	0.217	0.221

This table presents the estimated Kendall's tau coefficients for FTSE 100 and SMI conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 11: Kendall's τ between IBEX 35 and CAC 40 conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.203	0.047	0.023	0.002	-0.027	-0.042	0.079
$\frac{3}{4}h$	-0.317	0.054	0.023	-0.005	-0.035	-0.026	0.118
$\frac{5}{4}h$	-0.097	0.044	0.025	0.020	-0.022	-0.039	0.086
$\frac{3}{2}h$	-0.038	0.043	0.030	0.041	-0.001	-0.014	0.113
$2h$	0.050	0.055	0.058	0.082	0.033	0.023	0.165

This table presents the estimated Kendall's tau coefficients for IBEX 35 and CAC 40 conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 12: Kendall's τ between IBEX 35 and SMI conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.162	-0.036	0.044	0.019	0.033	0.057	-0.042
$\frac{3}{4}h$	0.170	-0.062	0.060	-0.004	0.045	0.029	-0.012
$\frac{5}{4}h$	0.175	-0.024	0.038	0.034	0.021	0.067	-0.058
$\frac{3}{2}h$	0.156	-0.018	0.028	0.045	0.018	0.083	-0.041
$2h$	0.167	-0.015	0.018	0.068	0.023	0.091	-0.008

This table presents the estimated Kendall's tau coefficients for IBEX 35 and SMI conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

takes negative values only for decile 1 and percentile 99 of Euro Stoxx. However, there is some appreciable dependence only when Euro Stoxx yields very little. In this second case, there is not sensitivity to bandwidth choice. In Table 13 we can see that Kendall's tau is always positive, but the values are so small that we couldn't say there is a significant dependence between CAC 40 and SMI. Nevertheless, when we double the bandwith value, there is some little dependence for the tails' values (the biggest and smallest values) and the middle values of Euro Stoxx.

Attending to the confidence intervals, we check that the null hypothesis is not satisfied. Thus, the seven conditional tau coefficients are different to the unconditional one in all cases. That is, it's not enough with estimating the unconditional copula. On the other hand, Kendall's coefficients are also different each other, so it is important and affects which value of Euro Stoxx we condition to. As we have seen, in some cases (i.e. when analyzing the relationship between CAC 40 and SMI) there is no appreciable dependence but in other cases (i.e. if we take FTSE 100 and SMI), the value of Euro Stoxx we condition to affects to the dependence. Although sometimes Kendall's coefficient changes with bandwidth value, the conclusions we obtain are the same

Table 13: Kendall's τ between CAC 40 and SMI conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.005	0.049	0.088	0.088	0.044	0.046	0.036
$\frac{3}{4}h$	0.015	0.044	0.089	0.090	0.038	0.039	0.080
$\frac{5}{4}h$	0.022	0.064	0.088	0.092	0.057	0.055	0.041
$\frac{3}{2}h$	0.067	0.072	0.094	0.100	0.063	0.072	0.069
$2h$	0.147	0.083	0.105	0.127	0.078	0.098	0.133

This table presents the estimated Kendall's tau coefficients for CAC 40 and SMI conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

whatever the value of h is. So, we could conclude that, in this sense, there is no sensitivity to bandwith choice in Kendall's tau estimation.

4. Testing for dependence when the conditional expectation is removed

4.1. Nonparametric conditional mean

In previous sections we have seen that there exists dependence on Eurostoxx for the different indexes. In this section we want to try to answer the question: Is the dependence totally explained by the conditional mean?

We propose to estimate the following nonparametric regression model using kernel estimation

$$Y_i = m(X_i) + \varepsilon_i,\ i = 1, ..., n.$$

where $m(X_i)$ is a smooth unknown function of X and ε_i is the error term.

If we remove the $m(X_i)$ effect from the original observations, then we are working with residuals. The main advantage of nonparametric estimation is that we do not need to make any assumption about the functional form of the regression function. The kernel estimator proposed by Nadaraya (1964) and Watson (1964) for $i = 1, ..., n$ is

$$\hat{m}(X_i) = \frac{\frac{1}{ng}\sum_{j=1}^{n} K(\frac{X_j - X_i}{g})Y_j}{\frac{1}{ng}\sum_{j=1}^{n} K(\frac{X_j - X_i}{g})},$$

where g is the bandwidth. Due to the sensitivity the bandwidth's value causes, we need to use an appropiate value for g to estimate $m(X_i)$, $i = 1, ..., n$. After checking with several values

Hence, the residual is defined as the difference between the original observations and the estimated systematic part

$$\hat{\varepsilon}_i = Y_i - \hat{m}(X_i),\ \text{for all } i = 1, ..., n,$$

where $\hat{m}(X_i)$ is the nonparametric estimate of $m(X_i)$.

So now we use the estimated residuals as if they were the original observations. In the same way we did in Section 1, we take the bandwidth value $h = 0.0018$ to estimate the copula dependent on Euro Stoxx. The goal of considering a regression model was to see whether there is something of Euro Stoxx left in the residuals or not. To test it, in a similar way as before, the suggested hypothesis test is defined as

$$\begin{cases} H_0 : \tau = \tau_{q_{0.01}} = \tau_{q_{0.1}} = \tau_{q_{0.25}} = \tau_{q_{0.5}} = \tau_{q_{0.75}} = \tau_{q_{0.9}} = \tau_{q_{0.99}} \\ H_a : \tau \neq \tau_i, \quad i = q_{0.01}, q_{0.1}, q_{0.25}, q_{0.5}, q_{0.75}, q_{0.9}, q_{0.99} \end{cases}$$

First of all, we show in the following table the unconditional Kendall's τ and the confidence intervals for each pair of residuals with 95% and 99% confidence levels.

Table 14: Unconditional Kendall's τ and its confidence interval for each pair of residuals with 95% and 99% of confidence level

	FTSE 100	IBEX 35	CAC 40	SMI
DAX	0.195	0.126	0.268	0.220
	[0.164 0.227]	[0.088 0.163]	[0.238 0.298]	[0.173 0.267]
	[0.154 0.237]	[0.077 0.174]	[0.218 0.318]	[0.150 0.290]
FTSE 100		0.470	0.314	0.264
		[0.121 0.210]	[0.268 0.359]	[0.221 0.307]
		[0.112 0.219]	[0.257 0.370]	[0.213 0.316]
IBEX 35			0.254	0.141
			[0.205 0.303]	[0.107 0.175]
			[0.190 0.317]	[0.083 0.198]
CAC 40				0.232
				[0.200 0.265]
				[0.190 0.275]

This table presents the unconditional Kendall's tau coefficients for the residuals of the index returns. The second and the third row represent the confidence intervals with 95% and 99% of confidence level respectively

Table 15: Kendall's τ between DAX and FTSE 100 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.110	0.027	-0.026	-0.009	0.023	-0.0002	-0.054
$\frac{3}{4}h$	0.277	0.026	-0.031	-0.018	0.019	0.023	-0.097
$\frac{5}{4}h$	0.086	0.007	-0.040	-0.0009	0.015	0.007	-0.054
$\frac{3}{2}h$	0.087	0.002	-0.042	0.004	0.012	0.019	-0.065
$2h$	0.042	-0.004	-0.043	0.014	0.021	0.031	-0.080

This table presents the estimated Kendall's tau coefficients for DAX and FTSE 100 residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 16: Kendall's τ between DAX and IBEX 35 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.045	-0.186	-0.092	-0.148	-0.187	-0.175	-0.269
$\frac{3}{4}h$	-0.168	-0.181	-0.093	-0.157	-0.195	-0.160	-0.305
$\frac{5}{4}h$	-0.140	-0.196	-0.094	-0.147	-0.171	-0.178	-0.241
$\frac{3}{2}h$	-0.165	-0.188	-0.098	-0.141	-0.165	-0.174	-0.240
$2h$	-0.201	-0.176	-0.102	-0.125	-0.147	-0.169	-0.245

This table presents the estimated Kendall's tau coefficients for DAX and IBEX 35 residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 17: Kendall's τ between DAX and CAC 40 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.219	-0.173	-0.074	-0.129	-0.089	-0.022	-0.053
$\frac{3}{4}h$	-0.190	-0.154	-0.070	-0.125	-0.073	-0.006	-0.116
$\frac{5}{4}h$	-0.223	-0.190	-0.070	-0.123	-0.093	-0.037	0.077
$\frac{3}{2}h$	-0.215	-0.187	-0.066	-0.112	-0.085	-0.053	-0.115
$2h$	-0.183	-0.178	0.066	-0.095	-0.081	-0.080	-0.126

This table presents the estimated Kendall's tau coefficients for DAX and CAC 40 residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 15 presents the Kendall's tau coefficients between DAX and FTSE 100. The results show that there is only some dependence for the smallest values of Euro Stoxx returns. The values conditional in the rest of the Euro Stoxx values show that there is no dependence between FTSE 100 and DAX. That is, all the dependence between those indexes is due to Euro Stoxx. Table 16 presents the coefficients for DAX and IBEX 35. The results show that the dependence is negative again whatever the value we condition to. There is some dependence for small values of Euro Stoxx returns and also is bigger from the mean zone on. In addition, the gratest dependence between the indexes is on the right tail. Hence, the relationship between them is stronger as the Euro Stoxx returns are higher. Something similar happens with CAC40 (Table 17). In this case the dependence between DAX and CAC40 is also negative but it is still appreciable when we condition to percentile 1 and decile 1. So, we can say that there is some dependence for small values of Euro Stoxx returns and less dependence as Euro Stoxx returns are bigger. Table 18 shows that the dependence between DAX and SMI continues being negative only when Euro Stoxx takes big values. Working with returns has decreased the depencence, so now the dependence between DAX and SMI is negative and significant only on the right tail and positive but almost negligible.

In Table 19 we can see Kendall's coefficients for FTSE 100 and IBEX 35. This time, we would say that there is very litlte dependence between the indexes and only for the decile 1 of Euro Stoxx. Things change when we discuss about FTSE 100 and CAC40 or FTSE and SMI. If we analyze the results in Table 20, we can see that dependence is negative (but is almost negligible) for very small values of Euro Stoxx and that there is more

Table 18: Kendall's τ between DAX and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.054	0.092	0.055	0.065	0.095	0.046	−0.213
$\frac{3}{4}h$	−0.029	0.104	0.054	0.059	0.96	0.061	−0.221
$\frac{5}{4}h$	0.103	0.075	0.056	0.076	0.091	0.047	−0.204
$\frac{3}{2}h$	0.121	0.071	0.064	0.78	0.091	0.045	−0.204
$2h$	0.110	0.064	0.070	0.080	0.087	0.055	−0.164

This table presents the estimated Kendall's tau coefficients for DAX and SMI residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 19: Kendall's τ between FTSE 100 and IBEX 35 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	−0.011	−0.118	0.102	−0.011	−0.003	−0.013	0.054
$\frac{3}{4}h$	−0.048	−0.130	0.120	−0.016	0.003	−0.026	0.098
$\frac{5}{4}h$	0.003	−0.103	0.086	−0.0003	0.003	0.004	0.054
$\frac{3}{2}h$	−0.001	−0.0978	0.073	0.008	0.007	0.009	0.069
$2h$	0.025	−0.086	0.049	0.014	0.014	0.024	0.068

This table presents the estimated Kendall's tau coefficients for FTSE 100 and IBEX 35 residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 20: Kendall's τ between FTSE 100 and CAC 40 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	−0.043	0.185	0.212	0.101	0.140	0.173	0.221
$\frac{3}{4}h$	−0.202	0.175	0.218	0.117	0.148	0.220	0.135
$\frac{5}{4}h$	−0.020	0.204	0.201	0.101	0.149	0.146	0.229
$\frac{3}{2}h$	−0.003	0.209	0.191	0.103	0.146	0.133	0.235
$2h$	0.047	0.217	0.172	0.115	0.134	0.124	0.229

This table presents the estimated Kendall's tau coefficients for FTSE 100 and CAC 40 residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

dependence between FTSE 100 and CAC40 when we condition to the decile 1 and percentile 99 values of Euro Stoxx. Looking at results in Table 21, the dependence of FTSE 100 with SMI is positive whatever the Euro Stoxx value we condition to. Even so, the dependence is bigger for small values of Euro Stoxx (for decile and quantile 1). In addition, it seems that the dependence on the tails is similar and it is lower around the mean.

Table 22 and Table 23 show the results for IBEX 35 with CAC40 and SMI. In the first case, the dependence is negative almost always and if not, the values are very small. However, as we had before, the dependence is only significant for very small values of Euro Stoxx and almost negligible for the rest of the values. In the second case, tau takes negative values only for decile 1 and percentile 99 of Euro Stoxx, but those values are also small. However, the dependence is only significant for the left tail of Euro Stoxx. In Table 24 we can see that Kendall's tau is positive, but the values are so small again that we can't say there is a significant dependence between CAC40 and SMI.

Attending to the confidence intervals, the results change when we use the residuals of the indexes. In the most of the cases, the null hypothesis is rejected again. So in those cases the seven conditional tau coefficients

Table 21: Kendall's τ between FTSE 100 and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.151	0.313	0.245	0.120	0.115	0.182	0.160
$\frac{3}{4}h$	0.122	0.346	0.257	0.119	0.112	0.169	0.162
$\frac{5}{4}h$	0.195	0.295	0.233	0.120	0.118	0.194	0.152
$\frac{3}{2}h$	0.230	0.277	0.218	0.127	0.118	0.203	0.161
$2h$	0.237	0.252	0.204	0.132	0.121	0.195	0.192

This table presents the estimated Kendall's tau coefficients for FTSE 100 and SMI residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 22: Kendall's τ between IBEX 35 and CAC 40 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	−0.182	0.024	0.004	−0.014	−0.046	−0.078	−0.028
$\frac{3}{4}h$	−0.231	0.039	0.014	−0.014	−0.045	−0.050	−0.002
$\frac{5}{4}h$	−0.139	0.014	−0.001	−0.005	−0.050	−0.087	−0.032
$\frac{3}{2}h$	−0.116	0.003	−0.006	0.001	−0.042	−0.073	−0.014
$2h$	−0.078	−0.004	−0.006	0.010	−0.032	−0.052	0.002

This table presents the estimated Kendall's tau coefficients for IBEX 35 and CAC 40 residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 23: Kendall's τ between IBEX 35 and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.186	−0.045	0.031	0.003	0.027	0.037	−0.017
$\frac{3}{4}h$	0.175	−0.068	0.049	−0.018	0.038	0.011	0.026
$\frac{5}{4}h$	0.196	−0.033	0.022	0.015	0.016	0.046	−0.053
$\frac{3}{2}h$	0.187	−0.030	0.008	0.021	0.011	0.061	−0.054
$2h$	0.171	−0.035	−0.012	0.032	0.006	0.059	−0.040

This table presents the estimated Kendall's tau coefficients for IBEX 35 and SMI residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

are different to the unconditional one and this means that the dependence between indexes is dependent on Euro Stoxx. Moreover, Kendall's coefficients are also different each other, so it affects which value of Euro Stoxx we condition to. Hence, we conclude that there is still some dependence in the residuals after removing the conditional mean and so, the dependence is not totally explained by the conditional mean. However, there are two exceptions. If we analyze the dependence of SMI with FTSE 100 and IBEX 35, we can realize that for values according to decile and quantile 1 of Euro Stoxx in the first case and for percentile 1 of Euro Stoxx in the second case, the null hypothesis is satisfied. In those cases, the conditional Kendall's tau is the unconditional one and then, this dependence between the indexes doesn't depend on Euro Stoxx (so in those cases the dependence is totally explained by the conditional mean). On the other hand, we wanted to analyze also the sensitivity of Kendall's tau to the bandwidth choice. Looking at the tables above, can be seen that the conclusions don't change with different bandwidth values.

Table 24: Kendall's τ between CAC 40 and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.024	0.035	0.075	0.075	0.039	0.025	0.038
$\frac{3}{4}h$	-0.030	0.033	0.080	0.083	0.036	0.031	0.110
$\frac{5}{4}h$	0.046	0.049	0.073	0.074	0.046	0.021	0.021
$\frac{3}{2}h$	0.051	0.059	0.073	0.077	0.046	0.027	0.032
$2h$	0.070	0.057	0.066	0.089	0.049	0.037	0.075

This table presents the estimated Kendall's tau coefficients for CAC 40 and SMI residuals conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

4.2. Linear conditional mean

But, why do we use a nonparametric regression model to remove the Euro Stoxx's effect instead of using a linear regression model? Estimating a nonparametric regression model implies doing no assumption about the model or any especification either, but on the other hand, it's known that the linear regression model is definitely so much easier to estimate. The linear regression model is a parametric regression model, and contrary to the nonparametric regression, we don't need to think about Kernel or optimal bandwidth choices. Thus, the linear regression model is defined as

$$Y_i = \alpha + \beta X_i + \varepsilon_i,\ i = 1, ..., n.$$

This time the residuals we are going to work with are estimated as

$$\hat{\varepsilon}_i = Y_i - \hat{\alpha} - \hat{\beta} X_i,\ i = 1, ..., n,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are estimates obtained by OLS method. As for the previous cases, we have the confidence intervals for each pair of indexes.

Tables 26 to 35 above show Kendall's tau coefficients for the residuals calculated by linear regression of different pairs of indexes. The values have the same trend if we calculate the residuals by nonparametric regression or linear regression, i.e. if there was more dependence for big values of Euro Stoxx before, now there is also more dependence for these values of Euro Stoxx. However, the conclusions obtained are different. News in conclusions come from the changes in confidence intervals, due to changes in unconditional coefficients' quantities. We are going to get conclusions based in the confidence interval with 99% of confidence level because has more financial relevance.

If we analyze the relationship between DAX and FTSE 100, the alternative hypothesis is rejected when we condition to $q_{0.5}, q_{0.75}$ and $q_{0.9}$ values (i.e. big values but not extreme values) of Euro Stoxx. So in those cases, Kendall's tau is considered to be the same as the uncoditional. That is, if Euro Stoxx takes small or very big values the relationship between the indexes is affected by Euro Stoxx significantly and depends on its value. In the relationship of DAX with IBEX 35, the alternative hypothesis is rejected for the right tail of Euro Stoxx, so when Euro Stoxx returns are high the relationship between the indexes is Euro Stoxx dependent. Something similar happens with DAX and SMI, but although the alternative hypothesis is rejected and the covariate value affects, the dependence between them is almost insignificant. However, in the relationship of DAX and CAC 40 the relevant are the extreme values of Euro Stoxx. Then, when Euro Stoxx has values around the mean there is nothing of Euro Stoxx in the dependence between DAX and CAC 40. In addition, the results show that the relationship between IBEX 35 and CAC 40 and IBEX 35 and SMI are Euro Stoxx small value dependent. We can also see that in the case of FTSE 100 and CAC 40 Kendall's tau is dependent on Euro Stoxx for the small

Table 25: Unconditional Kendall's τ and its confidence interval for each pair of index returns with 95% and 99% of confidence level

	FTSE 100	IBEX 35	CAC 40	SMI
DAX	-0.0008	-0.162	-0.112	0.065
	$[-0.036\ 0.034]$	$[-0.200\ \ -0.124]$	$[-0.147\ \ -0.076]$	[0.026 0.104]
	$[-0.060\ 0.058]$	$[-0.211\ \ -0.113]$	$[-0.172\ \ -0.051]$	[0.014 0.116]
FTSE 100		0.004	0.151	0.172
		$[-0.037\ 0.046]$	[0.096 0.207]	[0.139 0.205]
		$[-0.047\ 0.056]$	[0.080 0.223]	[0.115 0.228]
IBEX 35			-0.030	-0.001
			$[-0.065\ 0.004]$	$[-0.039\ 0.037]$
			$[-0.077\ 0.016]$	$[-0.050\ 0.048]$
CAC 40				0.067
				[0.032 0.102]
				[0.032 0.103]

This table presents the unconditional Kendall's tau coefficients for the residuals of the index returns. The second and the third row represent the confidence intervals with 95% and 99% of confidence level respectively

Table 26: Kendall's τ between DAX and FTSE 100 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.118	0.026	-0.043	-0.007	0.022	0.0008	-0.054
$\frac{3}{4}h$	0.277	0.025	-0.036	-0.014	0.018	0.018	-0.097
$\frac{5}{4}h$	0.094	0.007	-0.048	-0.002	0.016	0.010	-0.054
$\frac{3}{2}h$	0.096	0.003	-0.051	-0.0005	0.013	0.023	-0.065
$2h$	0.056	-0.003	-0.054	0.0009	0.018	0.032	-0.080

This table presents the estimated Kendall's tau coefficients for DAX and FTSE 100 residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

extreme values. Nevertheless, between FTSE 100 and SMI Kendall's tau is conditional for values which are around the mean of Euro Stoxx.

Hence, in the most of the cases when we remove Euro Stoxx dependence linearly, there is nothing of Euro Stoxx dependece left, that is, the dependence between indexes is linear.

On the other hand, we wanted to analyze also the sensitivity of Kendall's tau to the bandwidth choice. Looking at the tables above, we can see that the conclusions don't change with different bandwidth values this time either.

Once we have analyzed the results we can draw some conclusions. When we were working with residuals obtained from the nonparametric regression model the alternative hypothesis was satisfied for all values of Kendall's tau. Now, while working with residuals obtained from the parametric regression model, sometimes the alternative hypothesis is satisfied, but most of cases is rejected (which means that in those cases the conditional Kendall's tau is just the unconditional Kendall's tau). So, in this point we have opposite results with parametric and nonparametric models. This means that this time doesn't work using linear regression model, so we can't use the simple model and we have to go to more general models such as the nonparametric regression model.

Table 27: Kendall's τ between DAX and IBEX 35 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.130	-0.187	-0.100	-0.150	-0.193	-0.175	-0.268
$\frac{3}{4}h$	-0.159	-0.181	-0.100	-0.157	-0.201	-0.159	-0.306
$\frac{5}{4}h$	-0.132	-0.196	-0.102	-0.155	-0.180	-0.180	-0.240
$\frac{3}{2}h$	-0.163	-0.188	-0.108	-0.153	-0.177	-0.177	-0.239
$2h$	-0.204	-0.175	-0.119	-0.151	-0.165	-0.173	-0.247

This table presents the estimated Kendall's tau coefficients for DAX and IBEX 35 residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 28: Kendall's τ between DAX and CAC 40 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.236	-0.175	-0.080	-0.136	-0.097	-0.022	-0.058
$\frac{3}{4}h$	-0.195	-0.156	-0.176	-0.129	-0.079	-0.006	-0.116
$\frac{5}{4}h$	-0.235	-0.191	-0.077	-0.136	-0.107	-0.037	0.080
$\frac{3}{2}h$	-0.227	-0.186	-0.078	-0.133	-0.102	-0.052	-0.117
$2h$	-0.195	-0.176	-0.087	-0.128	-0.107	-0.079	-0.127

This table presents the estimated Kendall's tau coefficients for DAX and CAC 40 residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 29: Kendall's τ between DAX and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.037	0.091	0.052	0.063	0.093	0.048	-0.209
$\frac{3}{4}h$	-0.032	0.104	0.052	0.059	0.092	0.061	-0.218
$\frac{5}{4}h$	0.096	0.075	0.053	0.070	0.094	0.049	-0.200
$\frac{3}{2}h$	0.123	0.072	0.060	0.069	0.091	0.047	-0.201
$2h$	0.115	0.064	0.065	0.064	0.084	0.055	-0.166

This table presents the estimated Kendall's tau coefficients for DAX and SMI residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

5. Conclusions

Dependence is very important when we discuss about assets. In particular, we analyze the dependence between some domestic European indexes conditional on a more general index such as Euro Stoxx. One way to analyze depencende is using copulas, which are joint distribution functions that allow describing the dependence structure between variables. Moreover, the dependence structure between indexes can be influenced by the Euro Stoxx and it would be interesting to know how the dependence structure changes with its value. In this context, we work with conditional copulas. Since we want to quantify the strenght of the dependence, we use Kendall's tau correlation coefficient. To test for Euro Stoxx dependence we have designed an hypothesis test based on Kendall's tau. To decide about the hypothesis test for conditional tau, we use confidence intervals based on the unconditional Kendall's coefficient. In all cases the coefficients are different and also different to the unconditional one. The unconditional coefficient is different to zero and takes high values for all pair of indexes. This fact was hopefully since the correlation between indexes is also high. Moreover, the conditional Kendall's

Table 30: Kendall's τ between FTSE 100 and IBEX 35 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.016	-0.119	0.100	-0.012	0.001	-0.014	0.056
$\frac{3}{4}h$	-0.045	-0.132	0.119	-0.017	0.006	-0.029	0.099
$\frac{5}{4}h$	0.0003	-0.104	0.082	-0.002	0.008	0.003	0.055
$\frac{3}{2}h$	-0.003	-0.098	0.070	0.002	0.010	0.010	0.067
$2h$	0.018	-0.086	0.044	0.0005	0.013	0.023	0.066

This table presents the estimated Kendall's tau coefficients for FTSE 100 and IBEX 35 residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 31: Kendall's τ between FTSE 100 and CAC 40 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.038	0.189	0.210	0.099	0.148	0.171	0.208
$\frac{3}{4}h$	-0.191	0.178	0.217	0.116	0.155	0.215	0.129
$\frac{5}{4}h$	-0.028	0.207	0.201	0.096	0.156	0.146	0.212
$\frac{3}{2}h$	-0.012	0.211	0.192	0.096	0.150	0.133	0.216
$2h$	0.043	0.218	0.174	0.100	0.135	0.122	0.214

This table presents the estimated Kendall's tau coefficients for FTSE 100 and CAC 40 residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 32: Kendall's τ between FTSE 100 and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.164	0.312	0.241	0.118	0.118	0.178	0.159
$\frac{3}{4}h$	0.131	0.346	0.254	0.118	0.114	0.165	0.159
$\frac{5}{4}h$	0.202	0.294	0.230	0.117	0.123	0.190	0.151
$\frac{3}{2}h$	0.235	0.276	0.215	0.122	0.122	0.201	0.160
$2h$	0.247	0.250	0.201	0.122	0.123	0.192	0.192

This table presents the estimated Kendall's tau coefficients for FTSE 100 and SMI residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 33: Kendall's τ between IBEX 35 and CAC 40 residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	-0.179	0.018	0.0006	-0.020	-0.050	-0.083	-0.031
$\frac{3}{4}h$	-0.232	0.033	0.011	-0.018	-0.048	-0.055	-0.009
$\frac{5}{4}h$	-0.139	0.008	-0.006	-0.011	-0.057	-0.094	-0.032
$\frac{3}{2}h$	-0.119	-0.002	-0.012	-0.011	-0.055	-0.080	-0.012
$2h$	-0.082	-0.009	-0.020	-0.021	-0.053	-0.060	0.006

This table presents the estimated Kendall's tau coefficients for IBEX 35 and CAC 40 residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 34: Kendall's τ between IBEX 35 and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.175	-0.046	0.027	0.0003	0.026	0.033	-0.017
$\frac{3}{4}h$	0.170	-0.069	0.045	-0.019	0.037	0.006	0.026
$\frac{5}{4}h$	0.184	-0.034	0.018	0.010	0.018	0.043	-0.054
$\frac{3}{2}h$	0.167	-0.031	0.004	0.015	0.012	0.057	-0.053
$2h$	0.146	-0.036	-0.019	0.018	0.004	0.053	-0.038

This table presents the estimated Kendall's tau coefficients for IBEX 35 and SMI residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

Table 35: Kendall's τ between CAC 40 and SMI residuals conditional on Euro Stoxx

	$\tau_{q_{0.01}}$	$\tau_{q_{0.1}}$	$\tau_{q_{0.25}}$	$\tau_{q_{0.5}}$	$\tau_{q_{0.75}}$	$\tau_{q_{0.9}}$	$\tau_{q_{0.99}}$
h	0.045	0.034	0.072	0.069	0.043	0.021	0.039
$\frac{3}{4}h$	-0.004	0.032	0.076	0.079	0.039	0.028	0.113
$\frac{5}{4}h$	0.051	0.049	0.070	0.068	0.050	0.014	0.021
$\frac{3}{2}h$	0.050	0.059	0.072	0.071	0.049	0.019	0.036
$2h$	0.062	0.056	0.066	0.074	0.050	0.028	0.079

This table presents the estimated Kendall's tau coefficients for CAC 40 and SMI residuals obtained by linear regression conditional on Euro Stoxx for different values of h. The value of the bandwidth h is $h = 0.0018$.

coefficient depends on the condition, that is, depends on Euro Stoxx. In fact, when we control by Euro Stoxx the coefficients change a lot. Paying attention to the values we can see that the conditional dependence is low. So the results show that the dependence structure is effectively influenced by the Euro Stoxx and that the strenght of the dependence changes with the value of the covariate. In addition, when the dependence structure is controled by Euro Stoxx the relationship between domestic indexes is much lower.

In this point, we wanted to know if, after removing the effect of the conditional expectation of Euro Stoxx by regression techniques, there is still any dependence between the domestic indexes. Hence, we estimate a nonparametric regression model and we use the residuals to estimate conditional copulas. The residuals obtained are used in a similar way as the original observations. The unconditional coefficients are this time also different to zero but the values have been reduced. This means that when we remove the effect of the conditional mean there is less dependence between indexes. The values obtained in Table 14 to Table 23 show that the relationship between indexes decreases a lot and almost all the values are between -0.2 and 0.2. While testing for dependence, it can be concluded that the values are statistically different from each other for most of the index pairs. The results show that in those cases there is still dependence on Euro Stoxx between the indexes after removing the effect of the conditional mean. The values are not statistically signifiant only for the relationship of SMI with FTSE 100 and IBEX 35. If we analyze the relation between SMI and FTSE 100, we can see that when Euro Stoxx takes smaller values than the mean there aren't statistical differences in the dependence. Something similar happens with SMI and IBEX 35. In the relation between them Kendall's tau is not statistically significant for Euro Stoxx takes left tail values. In those two cases the dependence between the indexes is totally explained by the conditional mean for small values of Euro Stoxx. Furthermore, we wanted to see if it was necessary to remove the Euro Stoxx effect conditional mean or it was enough with removing it linearly. When a linear regression model is used, in most of the cases the dependence between the indexes is explained totally by the linear relationship and there is no Euro Stoxx dependence left in the residuals.

Hence, we obtain two main conclusions. First of all we conclude that the linear regression model it's not enough and we need a more general model such as nonparametric regression models, which have the advantage of no assumptions about the functional form of the model. In addition, we also conclude that that there is still some dependence on Euro Stoxx between domestic indexes when we remove the effect by the conditional mean. Particularly, there is still some dependence in the left tail, that is, when Euro Stoxx takes very low values, which is the zone that matters. Hence, it requires further research.

Ondorioak

Aktiboetaz hitz egiten gagozanean oso garrantzitsua da menpekotasun kontzeptua. Lan honetan indize domestiko desberdinen arteko dagoen menpekotasuna aztertzen da indize orokorrago batera, hala nola Euro Stoxx-era, baldintzatzen dogunean. Menpekotasuna aztertzeko modu bat kopulak erabiltzea da. Esan bezala, kopulak baterako banaketa funtzioak dira eta aldagai desberdinen arteko menpekotasun egitura deskribatzea ahalbidetzen dabe. Indizeen arteko menpekotasun egiturak Euro Stoxx-aren eragina jaso ahal dauanez, interesgarria litzateke menpekotasuna Euro Stoxx-aren balioekin zelan aldatzen dan ezagutzea. Testuinguru honetan, baldintzatutako kopulekin lan egiten dogu. Menpekotasuna zein gogorra izan daitekeen aztertu nahi dogunez, Kendall-en korrelazio koefizientea erabiltzen dogu. Euro Stoxx-arekiko menpekotasuna aztertu ahal izateko hipotesi kontraste bat diseinatu dogu, Kendall-en baldintzatu gabeko korrelazio koefizientean oinarritzen dana. Gainera, konfidantza tarteetaz baliatu gara emaitzetatik ondorioak atara ahal izateko. Emaitzek esaten dabenaren arabera, baldintzatutako korrelazio koefiziente guztiak desberdinak dira euren artean eta baita baldintzatu gabeko Kendall-en koefizientearekin ere. Baldintzatu gabeko koefizientea zeroren desberdina da eta balio altuak hartzen dauz kasu guztietan. Egia esan emaitza hau espero genduan, indizeen arteko korrelazio linealak ere altuak baitira. Gainera, baldintzatutako Kendall-en koefizientea baldintzaren menpekoa da, hau da, Euro Stoxx-aren menpekoa da. Izan ere, Euro Stoxx-aren balioa finkatzen dogunean koefizienteak asko aldatzen dira. Balioak arretaz begiratuz, ikus daiteke baldintzatutako menpekotasuna baxua dala. Beraz, emaitzek adierazten daben bezala Euro Stoxx-aren balioak indizeen arteko menpekotasun egituran benetan eragina dauka eta gainera eragin hori Euro Stoxx-ak hartutako balioen araberakoa da. Gainera, Euro Stoxx-aren balioa finkatzen dogunean indizeen arteko erlazioa txikitu egiten da.

Guzti hau dala eta, indizeen artean inolako menpekotasunik dagoen jakin nahi genduan Euro Stoxx-aren baldintzazko itxaropenaren efektua erregresio tekniken bitartez kendu ostean. Hortaz, erregresio eredu ez-parametriko bat estimatzen dogu eta hondarrak erabiltzen doguz baldintzatutako kopulak estimatzeko. Lortutako hondarrak jatorrizko balioak bailiran erabiltzen dira. Baldintzatu gabeko koefizienteak oraingoan ere zeroren desberdinak dira baina balioak txikiagoak dira. Beraz, baldintzatutako batazbestekoaren efektua kentzen dogunean indizeen arteko menpekotasuna gitxitu egiten da. 14. eta 23. taulen arteko balioek indizeen erlazioa asko txikitu dala erakusten dabe eta balioen gehiengoa -0.2 eta 0.2 bitartean mugitzen da. Menpekotasunaren testa egiterako orduan, ondorioztatu daiteke balioak estatistikoki desberdinak direla euren artean indize bikote gehienetarako. Emaitzek esaten dabenaren arabera, kasu honeetan oraindik indizeen artean badago Euro Stoxx-arekiko menpekotasuna baldintzatutako batazbestekoaren efektua kendu arren. SMI eta FTSE 100 arteko indizeen eta SMI eta IBEX 35 arteko indizeen kasuan soilik balioak ez dira estatistikoki esanguratsuak. SMI eta FTSE 100 indizeen arteko erlazioa aztertuz gero, Euro Stoxx-ak batazbestekoa baino balio txikiagoak hartzen dauzanean menpekotasunean ez dago desberdintasun esanguratsurik. Antzeko zerbait gertatzen da SMI eta IBEX 35 indizeen artean. Euren arteko erlazioan Euro Stoxx-ak baliorik txikienak hartzen dauzanean Kendall-en tau koefizientea ez da esanguratsua. Bi kasu honeetan indizeen arteko menpekotasuna baldintzatutako batazbestekoarekin guztiz azalduta dago. Gainera, Euro Stoxx-aren efektua baldintzatutako batazbestekoarekin kendu beharrean linealki kentzearekin nahikoa dan ikusi nahi izan dogu. Erregresio eredu lineal bat erabiltzen danean, gehienetan indizeen arteko menpekotasuna erlazio linealaren bitartez guztiz azaltzen da eta ez dago hondarretan Euro Stoxx-arekiko inolako menpekotasunik.

Hortaz, ondorio garrantzitsu bi lortzen doguz. Alde batetik, ez da nahikoa erregresio eredu lineal bat kontsideratzearekin eta erregresio eredu ez-parametrikoa bezalako eredu orokorrago bat beharrezkoa da. Eredu ez-parametrikoetan ez da forma funtzionalari buruz inolako hipotesirik egiten eta horrek abantaila bat suposatzen dau. Beste alde batetik, indizeen artean badago oraindik nolabaiteko menpekotasuna nahiz eta baldintzatutako batazbestekoaren efektua kendu. Bereziki, badago nolabaiteko menpekotasuna Euro Stoxx-ak balio oso txikiak hartzen dauzanean, hau da, zonalderik kritikoenean. Hortaz, ikerketa gehiago egin beharra dago.

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Technical report

Jone Ascorbebeitia Bilbatua
Master's Degree in Banking and Quantitative Finance

Supervisors:
*Eva Ferreira**
Susan Orbe-Mandaluniz [1]

[1] UPV/EHU University of Basque Country

Kapitulua 1

Bi aldagaietako kopulak

Definizioa 1. Izan bitez S_1 eta S_2 $\overline{\mathbf{R}}$-ren azpimultzoak eta F $Dom(F) = S_1 \times S_2$ eremuan definitutako funtzioa. Izan bedi baita erpin guztiak $Dom(F)$-n daukazan $J = [x_1, x_2] \times [y_1, y_2]$ laukizuzena. Orduan, *J-ren F-bolumena*

$$V_F(J) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$$

formularen bidez emonda dago.
Ohartu $V_F(J)$ J laukizuzenaren F-masa baita dala: J laukizuzenean F-ren lehen ordenako deribatuak $\Delta_{x_1}^{x_2} F(x, y) = F(x_2, y) - F(x_1, y)$ eta $\Delta_{y_1}^{y_2} F(x, y) = F(x, y_2) - F(x, y_1)$ eran definitu badaitekez, orduan J laukizuzenaren F-bolumena J gaineko F-ren bigarren ordenako deribatua da,

$$V_F(J) = \Delta_{x_1}^{x_2} \Delta_{y_1}^{y_2} F(x, y)$$

Definizioa 2. F aldagai biko funtzio erreala *2-gorakorra* dala esaten da $V_F(J) \geq 0$ bada erpinak $Dom(F)$-n daukazan J laukizuzen guztietarako. Bestela, F *n-beherakorra* dala esaten da.

F 2-gorakorra danean, J laukizuzenaren F-bolumenari askotan *J-ren F-neurria* esaten jako. 2-gorakorrak diren funtzioei *sasi monotonoak* ere esaten jake. Ohartu baita F 2-gorakorra dala esateak ez dauala inplikatzen, ezta inplizituki azaltzen ere ez, F argumentu bakoitzean ez-beherakorra izatea.

Adibidea 1.

- Izan bedi $\mathbf{I}^2 = [0, 1]^2$ eremuan definitutako $F(x, y) = max(x, y)$ funtzioa. F funtzio ez beherakorra da bai x-en eta baita y-n ere. Baina,
  $V_F([0, 1]^2) = \Delta_{y_1}^{y_2} \Delta_{x_1}^{x_2} max(x, y) = max(x_2, y_2) - max(x_1, y_2) - max(x_2, y_1) + max(x_1, y_1) = -1$ da eta hortaz, F ez da 2-gorakorra.

- Izan bedi orain $\mathbf{I}^2 = [0, 1]^2$ eremuan definitutako $F(x, y) = (2x - 1)(2y - 1)$ funtzioa.
  $V_F([0, 1]^2) = \Delta_{y_1}^{y_2} \Delta_{x_1}^{x_2} [(2x - 1)(2y - 1)] = [(2x_2 - 1) - (2x_1 - 1)][(2y_2 - 1) - (2y_1 - 1)] = 4 \geq 0$. F funtzio 2-gorakorra da, baina x-en funtzio

beherakorra da $y \in (0, \frac{1}{2})$ bakoitzerako eta y-n funtzio beherakorra da $x \in (0, \frac{1}{2})$ bakoitzerako.

Suposatu S_1 eta S_2-k a_1 eta a_2 elementu minimoak daukiezela, hurrenez hurren. $F : S_1 \times S_2 \to \mathbf{R}$ funtzioa *oinarrituta* dago $(x_1, x_2) \in S_1 \times S_2$ guztietarako, $F(x, a_2) = 0 = F(a_1, y)$ betetzen bada.

Lema 1.0.1. *Izan bitez S_1 eta S_2 $\overline{\mathbf{R}}$-ren azpimultzo bi eta F $S_1 \times S_2$ eremuan definitutako funtzio oinarritua eta 2-gorakorra. Orduan, F argumentu bakoitzean ez-beherakorra da.*

Orain suposatu S_1 eta S_2-k b_1 eta b_2 elementu handien bat daukela. Orduan $F : S_1 \times S_2 \to \mathbf{R}$ funtzioak F_1 eta F_2 hurrengo marjinalak daukaz:

$$F_1(x) = F(x, b_2),\ x \in S_1 \text{ guztietarako.}$$
$$F_2(y) = F(b_1, y),\ y \in S_2 \text{ guztietarako.}$$

Ohartu F_1 funtzioa $Dom(F_1) = S_1$ eremuan definituta dagoela eta F_2 funtzioa $Dom(F_2) = S_2$ eremuan.

Adibidea 2. Izan bedi $[-1, 1] \times [0, \infty]$ eremuan definitutako

$$F(x, y) = \tfrac{(x+1)(e^y-1)}{x+2e^y-1} \text{ funtzioa.}$$

$F(x, 0) = 0$ eta $F(-1, y) = 0$ betetzen diranez, F oinarrituta dago.

F-ren marjinalak hurrengoak dira:

$F_1(x) = F(x, \infty) = \frac{(x+1)(e^\infty-1)}{x+2e^\infty-1} \to \frac{x+1}{2}$

$F_2(y) = F(1, y) = \frac{e^y-1}{e^y} = 1 - e^{-y}$

Definizioa 3. *2 dimentsioko azpikopula edo 2-azpikopula C'* funtzio bat da hurrengo propietateak betetzen dauzana:

- C'-ren eremua $Dom(C') = S_1 \times S_2$ da, non S_1 eta S_2 0 eta 1 barne daukiezan $\mathbf{I} = [0, 1]$-ren azpimultzo bi diren.
- C' oinarritua eta 2-gorakorra da.
- $u_1 \in S_1$ eta $u_2 \in S_2$ guztietarako, $C'(u_1, 1) = u_1$ eta $C'(1, u_2) = u_2$.

Ohartu $(u_1, u_2) \in Dom(C')$ guztietarako, $0 \le C'(u_1, u_2) \le 1$, beraz $Ran(C')$ ere $\mathbf{I} = [0, 1]$-ren azpimultzo bat da.

Definizioa 4. *Bi dimentsioko kopula bat, 2-kopula bat edo beste barik kopula* $\mathbf{I}^2 = [0, 1]^2$ eremuan definitutako C 2-azpikopula bat da.
Baliokideki, *kopula* bat $C : [0, 1] \times [0, 1] \to [0, 1]$ eremuan definitutako $C(u_1, u_2) = P[U_1 \le u_1, U_2 \le u_2]$ funtzio bat da eta ondoko propietateak betetzen dauz:

1. $C : [0,1] \times [0,1] \to [0,1]$.
2. $C(u_1, 0) = C(0, u_2) = 0$.
3. $C(u_1, 1) = u_1$ eta $C(1, u_2) = u_2$.
4. $[0,1]$ tarteko u_1, u_2, v_1, v_2 guztietarako, $u_1 \leq u_2$ eta $v_1 \leq v_2$ izanik: $V_C([u_1,v_1] \times [u_2,v_2]) = C(v_1,v_2) - C(u_1,v_2) - C(v_1,u_2) + C(u_1,u_2) \geq 0$.

Nahiz eta kopula eta azpikopula kontzeptuen arteko desbardintasuna oso txikia dala iruditu, definizio eremuan dago bakarrik desbardintasuna, Sklar-en teoreman garrantzitsua izango da desbardintasun hori.

Nahiz eta kopularen kontzeptua definituta egon, badauka beste definizio formalago bat: Izan bedi $C : [0,1] \times [0,1] \to [0,1]$ funtzioa.
Orduan, C kopula da baldin eta soilik baldin $C(u_1, u_2) = P(U_1 \leq u_1, U_2 \leq u_2)$, non $U_1, U_2 \sim U(0,1)$.
Beraz,

$$\begin{aligned} C(F_1(x), F_2(y)) &= P(U_1 \leq F_1(x), u_2 \leq F_2(y)) = P(F_1^{-1}(u_1) \leq x, F_2^{-1}(u_2) \leq y) \\ &= P(X \leq x, Y \leq y) = F(x,y) \end{aligned}$$

baterako banaketa funtzioa lortzen da.

Kopulen artean ondoren aurkeztuko diran kopulak garrantzia handikoak dira. Aldagaien independentziarekin konsistentea dan kopula $\Pi(u_1, u_2) = u_1 \cdot u_2$ *biderkadura kopula* da. Beraz, kasu honetan, baterako banaketa funtzioa banaketa funtzio marjinalen biderkadura izango da. Beste kopula garrantzitsu bat $M(u_1, u_2) = min(u_1, u_2)$ kopula da eta *Fréchet-en goi bornea* esaten jako. Beste kopula batek ere ezin daike kopula honek baino balio handiagoa hartu. Zorizko aldagaiak kopula honen bidez erlazionatuta dagozanean menpekotasun zehatz positiboa daukela esaten da. Hirugarren kopula garrantzitsua $W(u_1, u_2) = max(u_1 + u_2 - 1, 0)$ kopula da eta *Fréchet-en behe bornea* esaten jako. Ez dago balio txikiagoa hartu daiken beste kopularik. Zorizko aldagaiak kopula honen bidez erlazionatuta dagozanean menpekotasun zehatz negatiboa daukela esaten da. Esaten da kopula batek menpekotasun positiboa edo negatiboa inplikatzen dauala Fréchet-en kopuletako batera konbergitzen badau parametroen bat aldatzen dan heinean. Gainera, $(u_1, u_2) \in \mathbf{I}^2 = [0,1]^2$ eta C kopula guztietarako,

$$W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2)$$

Fréchet-Hoeffding borneen desberdintza deitzen da.

Sklar-en teorema kopulen teoriako teorema nagusia da. Teorema honek banaketa funtzio marjinalen eta baterako banaketa funtzioen arteko erlazioan kopulek daukien eginkizuna argitzen dau. Sklar-en teorema aurkeztu aurretik zenbait kontzeptu gogoratuko doguz.

Definizioa 5. Banaketa funtzio bat $\overline{\mathbf{R}}$-n definitutako G funtzio bat da eta propietate bi betetzen dauz:

1. G ez-beherakorra da.
2. $G(-\infty) = 0$ eta $G(\infty) = 1$

Definizioa 6. *Baterako banaketa funtzio bat* $\overline{\mathbf{R}}^2$-n definitutako F funtzio bat da, non

1. F 2-gorakorra da.
2. $F(x, -\infty) = F(-\infty, y) = 0$ eta $F(\infty, \infty) = 1$

Hortaz, bigarren baldintzagatik F oinarritua da. Gainera, $Dom(F) = \overline{\mathbf{R}}$ danez, F-k $F_1(x) = F(x, \infty)$ eta $F_2(y) = F(\infty, y)$ moduan definitutako marjinalak daukaz. Funtzio marjinal honeek ere banaketa funtzioak dira era berean.

Teorema 1.0.2. ***(Sklar-en teorema)*** *Izan bedi F baterako banaketa-funtzio bat eta F_1, F_2 F-ren banaketa funtzio marjinalak. Orduan, C kopula bat existitzen da non $x, y \in \overline{\mathbf{R}}$ guztietarako,*

$$F(x, y) = C(F_1(x), F_2(y)) \tag{1}$$

F_1 eta F_2 banaketa funtzio marjinalak jarraituak badira orduan C kopula bakarra da eta hortaz modu unibokoan zehaztuta dago $ran(F_1) \times ran(F_2)$ eremuan. Alderantziz, C kopula bat bada eta F_1 eta F_2 banaketa funtzioak badira, orduan (1) *adierazpenean definitutako F funtzioa banaketa funtzio bateratua da F_1 eta F_2 marjinalekin.*

Sklar-en teoremak baterako banaketa funtziorako adierazpen bat emoten dau kopularen eta banaketa funtzio marginal biren menpean. Adierazpen horri buelta emonez gero, kopula baterako banaketa funtzioaren eta funtzio marjinalen alderantzizkoen menpe jarri daitekez. Hala ere, banaketa funtzio marjinalen bat hertsiki gorakorra ez bada, orduan alderantzizkoak ez dauka ohiko zentzua. Hori dala eta, banaketa funtzioen "sasi alderantzizkoak"definitzea beharrezkoa da.

Definizioa 7. Izan bedi F banaketa funtzioa. Orduan, *F-ren sasi alderantzizkoa* hurrengo bi propietateak betetzen dauzan $[0, 1]$ tartean definitutako edozein $F^{(-1)}$ funtzio da:

- $t \in Im(F)$ eta $x \in \mathbb{R}$ badira, $F^{(-1)}(t) = x$ y $F(x) = t$. Hortaz, $t \in Im(F)$ guztietarako,

$$F(F^{(-1)}(t)) = t$$

- $t \notin Im(F)$ bada, $F^{(-1)}(t) = inf\{x : F(x) \geq t\} = sup\{x : F(x) \leq t\}$.

F hertsiki gorakorra danean sasi alderantzizkoa ohiko alderantzizko funtzioa da.

Sklar-en alderantzizko teoremaren arabera, edozein motatako banaketa funtzio bi bateratu daitekez (ez da beharrezkoa familia bardinekoak izatea) edozein kopularekin; eta ondorioz aldagai biko banaketa funtzio baliagarri bat definituta egongo da. Zein da emaitza honen erabilgarritasuna? Ekonomia eta estatistikan aldagai bakarreko banaketa funtzio parametriko ugari dagoz, baina aldagai anitzeko banaketa funtzioen multzoa oso txikia da. Izan ere, baliteke aldagai anitzeko banaketa funtzioen konplexutasunak edo funtzio honeen iraganeko erabilera eskasak arazo hau sortarazo izana. Hala ere, Hutchinson eta Lai (1990), Joe Clyton (1997) eta Nelsen-en (1999) lanetan ikusi daitekeen moduan, kopula parametriko ugari dagoz aldagai anitzeko dentsitate funtzioen modelizaziorako. Orduan, aipatutako funtzioak bakarrik erabilita aldagai anitzeko banaketa funtzio parametriko posibleen multzoa izugarri handitzen da Sklar-en teoremari esker. Ideia bat egin ahal izateko izan bitez M, N eta P aldagai anitzeko banaketa funtzio, aldagai bakarreko banaketa funtzio eta kopula kopurua hurrenez hurren. Esandakoaren arabera, $N >> M$ da. Sklar-en teoremarekin aldagai anitzeko banaketa funtzio parametrikoen kopurua M izatetik $N^2 \cdot P >> M$ izatera pasatu da. Argi dago banaketa funtzio guzti honeek ez dirala enpirikoki erabilgarriak, baina aukera multzoa handitzeak datuekiko egokitze aproposa emoten daben ereduak aurkitzeko aukera hobetzen dau.

Korolarioa 1.0.3. *Izan bitez H bi aldagaiko banaketa funtzioa eta F eta G bere banaketa funtzio marjinal jarraituak. Izan bitez baita $F^{(-1)}$ eta $G^{(-1)}$ banaketa funtzio marjinalen sasialderantzizkoak. Orduan, $C : [0,1] \times [0,1] \to [0,1]$ kopula bakarra existitzen da non*

$$C(u_1, u_2) = H(F^{(-1)}(u_1), G^{(-1)}(u_2)), \quad (u_1, u_2) \in [0,1] \times [0,1] guztietarako.$$

Sklar-en teoremaren korolario honek hurrengoa dino: aldagai anitzeko edozein banaketa funtzioren kopula ataratzea posiblea da eta funtzio hau jatorrizko banaketa funtzioaren marjinalak edozein direlarik erabilten da. Hortaz, propietate honetan ikusi daiteke aldagai anitzeko banaketa funtzioen multzoa are gehiago handitzen dala.

Orduan, Sklar-en alderantzizko teorematik edozein banaketa funtzio marjinal bi eta kopula bat emonda banaketa funtzio bateratua lortu daiteke. Aurreko korolarioari esker, edozein baterako banaketa funtziorako kopula inplizitua eta banaketa funtzio marjinalak lortu daitez.

Oharra 1. Izan bitez X eta Y zorizko aldagaiak eta suposatu hurrengo adierazpena:

$$F(x,y) = C(F_X(x), F_Y(y))$$

Sklar-en teorema jarraituz, baterako probabilitatea kopula eta marjinalen artean banantzen da eta kopulak X eta Y-ren arteko lotura bakarrik adierazten dau. Banaketa funtzioaren bitartez adierazitako probabilitate bateratuan ez bezala, kopulek portaera marjinala multzotik banantzen dabe. Hori dala eta, kopulak menpekotasun funtzio bezala izendatzen dira. Kasu honetan, C kopulari *X eta Y zorizko aldagaien arteko kopula* esaten jako eta C_{XY} adierazten da.

Proposizioa 1.0.4.

- *Izan bedi C_{XY} X eta Y-ren kopula eta α eta β funtzio hertsiki gorakorrak. Orduan, $C_{\alpha(X)\beta(Y)} = C_{XY}$. Hortaz, C_{XY} inbariantea da X eta Y-ren edozein transformazio hertsiki gorakorrerako.*
- *Izan bitez C_1 eta C_2 bi kopula. Orduan, C_1 C_2 baino txikiagoa da (eta $C_1 \prec C_2$ idazten da) $C_1(u_1, u_2) \leq C_2(u_1, u_2)$ bada $u_1, u_2 \in \mathbf{I} = [0,1]$ guztietarako.*

Kopula baten dentsitatea definitu ahal izateko, lehenik eta behin teorema bat beharrezkoa da.

Teorema 1.0.5. *Izan bedi C kopula bat. $u_1, u_2 \in \mathbf{I} = [0,1]$ guztietarako $\frac{\partial C}{\partial u_1}$ deribatu partziala existitzen da eta u_1 eta u_2 horreetarako*

$$0 \leq \frac{\partial C(u_1,u_2)}{\partial u_1} \leq 1.$$

Era berean, $u_1, u_2 \in \mathbf{I} = [0,1]$ guztietarako $\frac{\partial C}{\partial u_1}$ deribatu partziala existitzen da eta u_1 eta u_2 horreetarako

$$0 \leq \frac{\partial C(u_1,u_2)}{\partial u_2} \leq 1.$$

Gainera, $u_1 \mapsto \frac{\partial C(u_1,u_2)}{\partial u_2}$ eta $u_2 \mapsto \frac{\partial C(u_1,u_2)}{\partial u_1}$ funtzioak ez beherakorrak dira eta definituta dagoz $\mathbf{I} = [0,1]$ eremuan.

Definizioa 8. Kopula baten dentsitatea, c adierazten dana, hurrengo adierazpenarekin definitzen da:

$$c(u_1, u_2) = \frac{\partial C(u_1, u_2)}{\partial u_1 \partial u_2} \tag{2}$$

$c(u_1, u_2)$ kopularen dentsitatea eukinda F banaketa funtzioaren f dentsitate funtzioa hurrengo eran lortu daiteke:

$$f(x,y) = c(F_X(x), F_Y(y)) \cdot f_X(x) \cdot f_Y(y) \tag{3}$$

Azken adierazpen hau Sklar-en kopulen dentsitateetarako teorema moduan ere ezagutzen da.

1.0.1 Kopula motak

Kopula funtzio mota desbardin asko dagoz baina zaila da mota guztie sailkapen zehatz bat aurkitzea, irizpide desbardinak baitagoz. Kopulak menpekotasunaren arabera, euskarriaren arabera (jarraitua edo diskretua), etab. sailkatu daitekez.

- Kopula inplizitua edo esplizitua
  - Kopula inplizituak: Kopularen forma funtzionalak bat egiten dau banaketa funtzio ezagun bategaz (Gauss-en kopula, student-en t kopula).
  - Kopula esplizituak: Kopula bakoitzak forma funtzional erraz bat dauka (Gumbel, Clayton eta Frank-en kopulak) .
- Menpekotasunaren arabera sailkatutako kopulak: Kopula eliptikoak, kopula normalak, mutur balioetako kopulak, kopula arkimedearrak eta HRT kopulak.

Orain kopula desbardinak aztertuko dira.

1.1 Kopula eliptikoak

Definizioa 9. *Kopula eliptikoak* banaketa funtzio eliptikoekin lotutako kopulak dira. Kopula eliptikoek menpekotasun erlazio simetrikoak adierazten dabez, banaketa funtzioaren ezkerreko zein eskumako aldea aztertzen dan arduratu barik. Kopula eliptiko guztiek itxura bera daukie:

$$
\begin{aligned}
C_\rho(u_1, u_2) &= \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi_{g,1}^{-1}(u_1)} \int_{-\infty}^{\Phi_{g,2}^{-1}(u_2)} g(\frac{x^2 - \rho xy + y^2}{\sqrt{1-\rho^2}}) dx \cdot dy \\
&= H_\rho(\Phi_{g,1}^{-1}(u_1), \Phi_{g,2}^{-1}(u_2))
\end{aligned}
$$

Gauss-en eta Student-en t kopulak kopula eliptikoak dira, simetrikoak eta erabiltzerako orduan errazak dira lotuta dagozan banaketak ezagunak direlako.

Aipatu bezala, kopula normala edo Gauss-en kopula eta Student-en t kopula kopula elptiko ezagunenak dira.

1.1.1 Gauss-en kopula

Gauss-en kopula banaketa funtzio normalari lotutako menpekotasun funtzioa da eta kopula mota hau kopula inplizitua da. Izan bedi ρ positiboki definitutako matrize diagonala non diagonaleko elementuak 1 diren. Orduan, Gauss-en kopula hurrengo eran definitzen da:

$$C(u_1, u_2; \rho) = \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$$
$$= \int_0^{\Phi^{-1}(u_1)} \int_0^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} exp(-\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1-\rho^2)} dx_1 dx_2 \tag{4}$$

Gainera, kopula mota honen dentsitatea

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} exp(\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1-\rho^2)} + \frac{x_1^2 + x_2^2}{2})$$

Gauss-en kopula kopula simetrikoa da, hau da, $C(u_1, u_2) = C(u_2, u_1)$. Kopula honek alboetan menpekotasun nulua edo oso baxua dauka, aldagaien korrelazioa 1 ez bada. Beraz, finantza-errentagarritasunen menpekotasuna modelizatzeko ez da kopula oso egokia. Gainera, badirudi finantza-errentagarritasunak handiagoak eta negatiboak direnean korrelazio handiagoa daukela handiak eta positiboak direnean baino, alboetako menpekotasuna asimetrikoa dala aditzera emoten dauana.

1.1.2 Student-en t kopula

Student-en t kopula Student-en t banaketa funtzioari lotutako menpekotasun funtzioa da eta hau ere kopula inplizitua da. Aldagai biren kasuan ondoko eran definitzen da:

$$C(u_1, u_2; \rho, \nu) = t_{\rho,\nu}(t^{-1}(u_1), t^{-1}(u_2)) \tag{5}$$

Lehenik eta behin Student-en t banaketaren dentsitatea ezagutzea beharrezkoa da kopula definitu ahal izateko. Gogoratu kopulak aldagaitzak direla transformazio gorakor monotonoetarako, $C(t(\nu, \mu, \Sigma)) = C(t(\nu, 0, P))$ eta P korrelazio matrizea. Beraz,

$$f(x) = \frac{1}{2\pi\sqrt{1-\rho^2}}(1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\nu(1-\rho^2)})^{-\frac{\nu+2}{2}} \text{ da,}$$

non ν askatasun graduak eta ρ korrelazio koefizientea diren. Student-en t banaketak orokorrean $X \stackrel{d}{=} \mu + \sqrt{\frac{\nu}{S}} \cdot Z$ adierazpena dauka, non $Z \sim N(0, \Sigma)$ dan. Beraz, Studen-en t kopula ondoko eran definitu daiteke:

$$C(u_1, u_2) = \mathbf{t}_\nu(t_{\nu_1}^{-1}(u_1), t_{\nu_2}^{-1}(u_2)) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} f(x)dx$$
$$= \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}}(1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\nu(1-\rho^2)})^{-\frac{\nu+2}{2}} dx \tag{6}$$

eta Student-en t kopularen dentsitatea

$$c(u_1, u_2) = K\frac{1}{\sqrt{(1-\rho^2)}}[1+\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\nu(1-\rho^2)}]^{-\frac{\nu+2}{2}}[(1+\nu^{-1}x_1^2)(1+\nu^{-1}x_2^2)]^{\frac{\nu+1}{2}} \quad (7)$$

formularen bidez emonda dago, non $K = \frac{\nu}{2}\Gamma(\frac{\nu}{2})^2\Gamma(\frac{\nu+1}{2})^{-2}$ eta ν kopularen askatasun gradu kopurua diren.

1.2 Mutur balioetako kopulak

Mota honetako kopulak oso erabilgarriak dira banaketa funtzio marginaletan muturretako gertaerei garrantzi handiagoa emoten erlazioak adierazteko. Mutur balioetako kopulak askeak eta berdinki banatutako laginen maximoei lotutako kopulen limiteak dira, beti ere limitea existitzen dan kasuetan. Izan bedi bi dimentsiotako zorizko aldagaiez sortutako $(X_1, Y_1), ..., (X_n, Y_n)$ lagina, non aldagaiak askeak eta berdinki banatuak diren F_X eta G_Y marginalekiko eta H_{XY} baterako banaketarekiko. Sklar-en teoremaren arabera H_{XY} baterako banaketa funtzioari C kopula bat lotzen jako: $H_{XY} = C(F_X(x), G_Y(y))$. Izan bitez $M_n = max\{X_1, ..., X_n\}$ eta $N_n = max\{Y_1, ..., Y_n\}$ bi aldagai, $F^n(x) = P[M_n \leq x]$ eta $G^n(y) = P[N_n \leq y]$ euren banaketa funtzioak hurrenez hurren eta $H_n(x, y) = P[M_n \leq x, N_n \leq y]$ baterako banaketa funtzioa. C (M_n, N_n) bikoteari eta bere limiteari lotutako kopula bada (n-k infinitorantz joten dauanean), orduan *C mutur balioko kopula* bat dala esaten da. Mutur balioko kopulek menpekotasun positiboa modelizatzen dabe.

Dehuelves- en teoremaren arabera mutur balioko C kopula batek hurrengo baldintza bete behar dau:

$$C^t(u_1^{\frac{1}{t}}, u_2^{\frac{1}{t}}) = C(u, v), \text{ para todo } t > 0$$

Mota honetako kopulek , *Pickands*-en (1981) hurrengo teoreman azaltzen dan bezala, adierazpen bat daukie lotuta.

Teorema 1.2.1. ***Mutur balioko kopulen adierazpena*** *C kopula bat mutur balioko kopula bat da baldin eta soilik baldin hurrengo erlazioa betetzen dauan [0, 1] tartean definitutako A funtzio erreal bat existitzen bada:*

$$C(u_1, u_2) = exp\{-log(u_1 \cdot u_2) \cdot A(\frac{log(u_2)}{log(u_1 \cdot u_2)})\} \quad (8)$$

edo baliokideki:

$$C(e^{-u_1}, e^{-u_2}) = exp\{-(u_1 + u_2) \cdot A(\frac{u_2}{log(u_1 + u_2)})\} \quad (9)$$

A funtzioa ***Pickands-en menpekotasun funtzioa*** *da eta hurrengo baldintzak betetzen dauz:*

- *Ganbila da* $[0,1]$ *tartean.*
- $max(t, 1-t) \leq A(t) \leq 1$, $t \in [0,1]$ *guztietarako.*

1.3 Kopula arkimedearrak

Kopula arkimedearrak, finantzetako eta geoestatistikako arloetan erabiliak izan aurretik, beste arlo batzuetan aplikatuak izan ziran Clayton, Oakes eta Cook eta Johnsonengatik. Arrazoi asko dira kopula arkimedearrak hainbeste erabili izana. Inplementatzeko, bornatzeko eta simulazio desberdinak egiteko erraztasuna eta baita malgutasuna edo daukiezan propietate matematikoak egiten dabez erakargarriak mota honetako kopulak. Hala ere, kopula arkimedearrek badaukie desabantailaren bat. Hain zuzen, argumentuekiko simetria, hau da, $C(u_1, u_2) = C(u_2, u_1)$.

Definizioa 10. Izan bedi Ψ $\psi : [0,1] \to [0, \infty)$ funtzioen multzoa. ψ funtzioak jarraituak, hertsiki beherakorrak eta ganbilak dira eta $\psi(0) = \infty$ eta $\psi(1) = 0$. Orduan Ψ multzoko ψ funtzio bakoitzak hurrengo adierazpena jarraitzen dauan C kopula bat sorrarazten dau:

$$C(u_1, u_2) = \psi^{-1}(\psi(u_1) + \psi(u_2)), \text{ non } 0 \leq u_1, u_2 \leq 1$$

ψ funtzioari **kopularen sortzailea** esaten jako eta ψ^{-1} ψ funtzioaren sasialderantzizkoa hurrengo eran dago definituta:

$$\psi^{-1}(t) = \begin{cases} \psi^{-1}(t) & 0 \leq t < \psi(0) \\ \\ 0 & \psi(0) \leq t < \infty \end{cases}$$

Aurrerago ikusiko dan moduan, kopula arkidemedearren adierazpenak aldagai anitzeko kopula baten ikerketa aldagai bakarreko funtzio batera murrizten dau. Esan bezala, ψ funtzioa C kopularen sortzaile arkimedearra da. $\psi(0) = \infty$ bada, C kopula hertsiki arkimedearra dala esaten da.

Orain ikusi daiguzan kopula arkimedearren familian parte hartzen daben kopula mota batzuk.

1.3.1 Kopula independentea

Kopula independentearen kasuan funtzio sortzailea $\psi(t) = -ln\ t$ da eta hortaz, bere alderantzizkoa $\psi^{-1}(x) = e^{-x}$. Orduan, eta aurrekoa kontuan hartuz, kopula independentearen adierazpena lortu daiteke:

$$\begin{aligned} C(u_1, u_2) &= \psi^{-1}(\psi(u_1) + \psi(u_2)) = \psi^{-1}(-ln\ u_1 - ln\ u_2) \\ &= \psi^{-1}(-(ln\ u_1 + ln\ u_2)) = \psi^{-1}(-ln\ (u_1 u_2)) = e^{-(-ln\ u_1 u_2)} \\ &= u_1 \cdot u_2 \end{aligned} \tag{10}$$

Aurreko adierazpena u_1-ekiko deribatzen da C_1 lortzeko:

$$C_1(u_1,u_2) = (\frac{\partial C(u_1,u_2)}{\partial u_1}) = u_2 \tag{11}$$

Orduan kopularen dentsitatea:

$$c(u_1,u_2) = (\frac{\partial C(u_1,u_2)}{\partial u_1 \partial u_2}) = 1 \tag{12}$$

Kontuan hartu, kopula independentea aurrerago definitutako $\Pi(u_1,u_2) = u_1 \cdot u_2$ biderkadura kopula dala.

1.3.2 Frank kopula

Frank kopularen kasuan funtzio sortzailea hurrengoa da:

$$\psi(t) = -ln\frac{e^{-\delta t}-1}{e^{-\delta}-1}, \quad \delta \in \mathbb{R} \tag{13}$$

Orduan, ψ^{-1} alderantzizko funtzioa $\psi^{-1}(x) = -\frac{1}{\delta}ln[1+\frac{e^{-\delta}-1}{e^x}]$ dala ondorioztatu daiteke.

Hortaz, Frank kopula hurrengo eran definitzen da:

$$\begin{aligned} C(u_1,u_2) = \psi^{-1}(\psi(u_1)+\psi(u_2)) &= \psi^{-1}(-ln\frac{e^{-\delta u_1}-1}{e^{-\delta}-1} - ln\frac{e^{-\delta u_2}-1}{e^{-\delta}-1}) \\ &= \psi^{-1}(ln\frac{(e^{-\delta}-1)^2}{(e^{-\delta u_1}-1)(e^{-\delta u_2}-1)}) \\ &= -\frac{1}{\delta}ln(1+\frac{(e^{-\delta u_1}-1)(e^{-\delta u_2}-1)}{e^{-\delta}-1}) \end{aligned} \tag{14}$$

$C(u_1,u_2)$ kopula funtzioa u_1-ekiko deribatuz gero:

$$C_1(u_1,u_2) = (\frac{\partial C(u_1,u_2)}{\partial u_1}) = e^{-\delta u_1}\frac{e^{-\delta u_2}-1}{(e^{-\delta}-1)+(e^{-\delta u_1}-1)(e^{-\delta u_2}-1)} \tag{15}$$

Orduan dentsitate funtzioa hurrengoa izango litzateke:

$$c(u_1,u_2) = (\frac{\partial C(u_1,u_2)}{\partial u_1 \partial u_2}) = \frac{-\delta e^{-\delta u_1}e^{-\delta u_2}(e^{-\delta}-1)}{[(e^{-\delta}-1)+(e^{-\delta u_1}-1)(e^{-\delta u_2}-1)]^2} \tag{16}$$

1.3.3 Gumbel kopula

Gumbel kopula kopula arkimedear mota bat da. Kasu honetan funtzio sortzailea $\psi(t) = (-ln\ t)^{\delta}$ da, non $\delta \geq 1$ dan. ψ funtzioaren alderantzizkoa $\psi^{-1}(x) = exp(-x^{\frac{1}{\delta}})$. Orduan,

$$\begin{aligned} C(u_1, u_2) &= \psi^{-1}(\psi(u_1) + \psi(u_2)) = \psi^{-1}((-ln\ u_1)^{\delta} + (-ln\ u_2)^{\delta}) \\ &= exp(-[(-ln\ u_1)^{\delta} + (-ln\ u_2)^{\delta}]^{\frac{1}{\delta}}) \end{aligned} \tag{17}$$

Aurreko adierazpen hau u_1-ekiko deribatuz gero:

$$C_1 = (\frac{\partial C(u_1, u_2)}{\partial u_1}) = C(u_1, u_2)[(-ln\ u_1)^{\delta} + (-ln\ u_2)^{\delta}]^{-1+\frac{1}{\delta}} \frac{-ln\ u_1^{\delta-1}}{u_1} \tag{18}$$

Beraz, kopularen dentsitatea hurrengoa da:

$$\begin{aligned} c(u_1, u_2) &= (\frac{\partial C(u_1, u_2)}{\partial u_1 \partial u_2}) \\ &= C(u_1, u_2)[(-ln\ u_1)^{\delta} + (-ln\ u_2)^{\delta}]^{\frac{2}{\delta}+2}[ln\ u_1 \cdot ln\ u_2]^{\delta-1} \cdot \\ &\cdot \{1 + (\delta - 1)[(-ln\ u_1)^{\delta} + (-ln\ u_2)^{\delta}]^{-\frac{1}{\delta}}\} \end{aligned}$$

Gumbel kopulak menpekotasun positiboa dauka goiko buztanean $\delta > 1$ bada. δ-k balio desberdinak hartzen dauzan heinean Gumbel kopula independentzia ($\delta = 1$) eta menpekotasun zehatz positiboaren artean mugitzen da, $\delta \to \infty$ danean Gumbel kopulak Fréchetеn goi borneko kopulara konbergitzen baitau.

1.3.4 Clayton kopula

Clayton kopularen kasuan funtzio sortzailea $\psi(t) = \frac{1}{\delta}(t^{-\delta} - 1)$ da. ψ funtzioaren alderantzizkoa $\psi^{-1}(x) = (\delta x + 1)^{-\frac{1}{\delta}}$. Orduan,

$$\begin{aligned} C(u_1, u_2) &= \psi^{-1}(\psi(u_1) + \psi(u_2)) = \psi^{-1}(\frac{1}{\delta}(u_1^{-\delta} - 1) + \frac{1}{\delta}(u_2^{-\delta} - 1)) \\ &= \psi^{-1}(\frac{1}{\delta}(u_1^{-\delta} + u_2^{-\delta} - 2)) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-\frac{1}{\delta}} \end{aligned} \tag{19}$$

Adierazpen hau u_1-ekiko deribatzen bada:

$$C_1 = (\frac{\partial C(u_1, u_2)}{\partial u_1}) = u_1^{-\delta-1}(u_1^{-\delta} + u_2^{-\delta} - 1)^{-1-\frac{1}{\delta}} \tag{20}$$

Hortaz, Clayton kopularen dentsitatea:

$$c(u_1,u_2) = (\frac{\partial C(u_1,u_2)}{\partial u_1 \partial u_2}) = (u_1u_2)^{-\delta-1}(\delta+1)(u_1^{-\delta}+u_2^{-\delta}-1)^{-2-\frac{1}{\delta}} \quad (21)$$

Clayton kopulak menpekotasun asimetrikoa dauka buztanetan. Kopula mota honek menpekotasun nulua dauka goiko buztanean eta menpekotasun positiboa beheko buztanean $\delta > 0$ danean. δ-ren balio desberdinetarako Clayton kopulak menpekotasun gradu desberdinak hartu daikez. Menpekotasuna zehatz positiboa izango da $\delta \to \infty$ doan heinean, Gumbel kopulan bezala Frécheten goi borneko kopulara konbergitzen baitau.

1.4 Kopula baldintzatuak

Orain, zorizko aldagaien erlazioa aztertzeaz gain, koaldagai baten balioa emonda erlazio hori koaldagaiaren balio desberdinekin zelan aldatzen dan aztertzea izango da helburua. Izan bitez X, Y zorizko aldagaiak eta $Z = z$ koaldagaia. Aurrerago ikusiko dan moduan, $Z = z$ emonda, X eta Y-ren menpekotasun egitura *kopula baldintzatua* izeneko funtzio baten bidez deskribatzen da. Orduan, (X, Y) aldagaien baterako banaketa funtzioa eta funtzio marjinalak, $Z = z$ baldintzapean,

$$F_z(x,y) = P(X \leq x, Y \leq y | Z = z),$$
$$F_{1z}(x) = P(X \leq x | Z = z) \text{ eta } F_{2z}(y) = P(Y \leq y | Z = z).$$

dira hurrenez hurren.

Definizioa 11. U_1 eta U_2-rekiko deribatu partzialek C kopulari loturiko kopula baldintzatuak definitzen dabez (C kopularen monotonia dala eta, u_1 eta u_2 ia guztietarako existitzen dira):

- u_1 finko baterako, u_1-rekiko kopula baldintzatua U_2-ren funtzio bat da, $u_2 \to C_1(u_1,u_2) = C(u_2|u_1) = \frac{\partial C}{\partial u_1}(u_1,u_2)$
- u_2 finko baterako, u_2-rekiko kopula baldintzatua U_1-ren funtzio bat da, $u_1 \to C_2(u_1,u_2) = C(u_1|u_2) = \frac{\partial C}{\partial u_2}(u_1,u_2)$

Definizioa 12. Aldagai biko banaketa funtzio baldintzatua eskuinetik jarraitua dan $F_z : \overline{\mathbf{R}}^2 \to [0,1]$ funtzioa da eta ondoko propietateak betetzen dauz:

- $F_z(y_1, -\infty | Z = z) = F_z(-\infty, y_2 | Z = z) = 0$ eta $F_z(\infty, \infty | Z = z) = 1$
- $x_1, x_2, y_1, y_2 \in \overline{\mathbf{R}}$ guztietarako eta $x_1 \leq x_2,\ y_1 \leq y_2$, $V_{F_z}([x_1,x_2]\times[y_1,y_2]) \equiv F_z(x_2,y_2|Z=z) - F_z(x_1,y_2|Z=z) - F_z(x_2,y_1|Z=z) + F_z(x_1,y_1|Z=z) \geq 0$.

Lehenengo baldintzak banaketa funtzioaren goi eta behe borneak emoten dauz. Bigarren baldintzan aldiz, $[x_1, x_2] \times [y_1, y_2]$ eremuko edozein punturen probabilitatea ez-negatiboa dala ziurtatzen da.

Definizioa 13. Kopula baldintzatu bat $C_z : [0,1] \times [0,1] \to [0,1]$ funtzio bat da hurrengo propietateak betetzen dauzana:

1. $C_z(u_1, 0|Z = z) = C_z(0, u_2|Z = z) = 0$, $u_1, u_2 \in [0,1]$ guztietarako.

2. $C_z(u_1, 1|Z = z) = u_1$ eta $C_z(1, u_2|Z = z) = u_2$, $u_1, u_2 \in [0,1]$ guztietarako.

3. $[0,1]$ tarteko u_1, u_2, v_1, v_2 guztietarako, $u_1 \leq u_2$ eta $v_1 \leq v_2$ izanik: $V_{C_z}([u_1, v_1] \times [u_2, v_2]|Z = z) \equiv C_z(v_1, v_2|Z = z) - C_z(u_1, v_2|Z = z) - C_z(v_1, u_2|Z = z) + C_z(u_1, u_2|Z = z) \geq 0$.

Oraingoan ere lehenengo baldintzak banaketa funtzioaren goi eta behe borneak emoten dauz. Gainera, $C_z(u_1, 1|Z = z)$ eta $C_z(1, u_2|Z = z)$ banaketa marjinalak uniformeak izatea ziurtatzen dau. Bigarren baldintzak, lehengo legez, puntu baten probabilitatearen ez negatibotasuna adierazten dau.

Aurreko baldintza biak $\overline{\mathbf{R}}^2$ eremura hedatuz, kopula baldintzatua (U_{1t}, U_{2t}) zorizko aldagaien baterako banaketa funtzio baldintzatua moduan definitu daiteke, non marjinalak $Unif(0,1)$ banaketa uniformea jarraitzen daben:

$$C_z^*(u_1, u_2|Z = z) = \begin{cases} 0 & u_1 < 0 \; edo \; u_2 < 0, \\ C_z(u_1, u_2|Z = z) & (u_1, u_2) \in [0,1] \times [0,1], \\ u_1 & u_1 \in [0,1], \; u_2 > 1, \\ u_2 & u_1 > 1, \; u_2 \in [0,1], \\ 1 & u_1 > 1, \; u_2 > 1, \end{cases}$$

2-kopula baterako kopula baldintzatu biren banaketa funtzioak definitu daitekez:

$$C(u_1|u_2) = P(U_1 < u_1|U_2 = u_2)$$
$$C(u_2|u_1) = P(U_2 < u_2|U_1 = u_1)$$

Arinago ikusitako Sklar-en bi teoremetatik aparte, badago kopula baldintzatueri egokitutako Sklar-en teorema bat ere.

Teorema 1.4.1. *Izan bitez F_z aldagai biren baterako banaketa funtzioa eta F_{1z} eta F_{2z} bere marjinal jarraituak. Orduan, $C_z : [0,1] \times [0,1] \to [0,1]$ kopula baldintzatu bakarra existitzen da non $x, y \in \overline{\mathbf{R}} = [-\infty, \infty]$ guztietarako,*

$$F_z(x, y|Z = z) = C_z(F_{1z}(x|Z = z), F_{2z}(y|Z = z)|Z = z) \tag{22}$$

Alderantziz, C_z kopula baldintzatua eta F_{1z} eta F_{2z} X eta Y zorizko aldagai biren marjinal baldintzatuak badira, orduan (22) ekuazioarekin definitutako F_z funtzioa F_{1z} eta F_{2z} marjinalak daukazan banaketa funtzio baldintzatua da.

Kapitulua 2

Aldagai anitzeko kopulak

Atal honetan aurreko ataletako emaitzak aldai anitzeko kasura luzatuko doguz. Egia esan, definizio eta teorema asko antzekoak dira aldagai anitzeko bertsioan. Hala ere, ez da kasu guztietan emoten eta hori dala eta kontu handiz jokatu behar da.
Lehenik eta behin, aurrerago baliagarriak izango diran zenbait kontzeptu definituko dira.

Definizioa 14. Izan bedi n zenbaki oso positibo bat. n-kutxa bat $\overline{\mathbb{R}}$-ko n tarte itxiren arteko edozein biderketa kartesiarra da. $J = [a_1, b_1] \times ... \times [a_n, b_n]$ edozein n-kutxa $J = [(a_1, ..., a_n), (b_1, ..., b_n)]$ ere idazten da. $a_i = a$ eta $b_i = b$ bada $i = 1, ..., n$ guztietarako, $J = [a, b]^n$ idatzi daiteke.

Gainera, esaten da *J n-kutxa endekatua* dala existitzen bada i, $1 \leq i \leq n$, non $a_i = b_i$. Bestela, J *n-kutxa ez endekatua edo endekatu gabea* dala esaten da. Beste alde batetik, J-ren *erpinak* $(c_1, ..., c_n)$ puntuak dira, non $i = 1, ..., n$ bakoitzerako $c_i = a_i$ edo $c_i = b_i$. J endekatu gabea bada, orduan 2^n erpin daukaz.

Definizioa 15. Izan bitez $S_1, S_2, ..., S_n$ $\overline{\mathbb{R}}$-ren azpimultzoak eta F n-dimentsioko funtzio errala non $Dom(F) = S_1 \times S_2 \times ... \times S_n$. Izan bedi baita erpinak $Dom(F)$-n dauzan J n-kutxa endekatu gabea. Orduan, *J n-kutxaren F-bolumena* hurrengo eran emonda dago:

$$V_F(J) = \sum sgn(\mathbf{c})F(\mathbf{c}), \tag{1}$$

non batukaria J-ren $\mathbf{c}$ erpin guztien gainean hartzen dan eta $sgn(\mathbf{c})$

$$sgn(\mathbf{c}) = \begin{cases} 1 & c_k = a_k \; bada \; k \;\; indize \; kopurua \; bikoitia \; danean \\ \\ -1 & c_k = a_k \; bada \; k \;\; indize \; kopurua \; bakoitia \; danean \end{cases}$$

moduan definituta dagoen.

*J=[***a***,***b***]-ren F-bolumena* beste modu baliokide baten ere definitu daiteke. Hain zuzen, J-ren F-bolumena F-ren J gaineko n ordenako aldakuntza da, hau da,

$$V_F(J) = \Delta_{\mathbf{a}}^{\mathbf{b}} F(\mathbf{t}) = \Delta_{a_n}^{b_n} \Delta_{a_{n-1}}^{b_{n-1}} ... \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} F(\mathbf{t}), \tag{2}$$

non lehenengo ordenako n aldakuntzak

$$\Delta_{a_k}^{b_k} F(\mathbf{t}) = F(t_1, ..., t_{k-1}, b_k, t_{k+1}, ..., t_n) - F(t_1, ..., t_{k-1}, a_k, t_{k+1}, ..., t_n) \tag{3}$$

moduan definituta dagozan.

Adibidea 3. Izan bitez F $\overline{\mathbb{R}}^3$-ko funtzioa eta J $[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ 3-kutxa. Orduan, J-ren F-bolumena:

$$\begin{aligned} V_F(J) = \Delta_{\mathbf{a}}^{\mathbf{b}} F(\mathbf{t}) &= \Delta_{a_3}^{b_3} \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} F(\mathbf{t}) \\ &= \Delta_{a_3}^{b_3} \Delta_{a_2}^{b_2} [F(b_1, t_2, t_3) - F(a_1, t_2, t_3)] \\ &= \Delta_{a_3}^{b_3} \{[F(b_1, b_2, t_3) - F(b_1, a_2, t_3)] - [F(a_1, b_2, t_3) - F(a_1, a_2, t_3)]\} \\ &= \Delta_{a_3}^{b_3} \{F(b_1, b_2, t_3) - F(b_1, a_2, t_3) - F(a_1, b_2, t_3) + F(a_1, a_2, t_3)\} \\ &= [F(b_1, b_2, b_3) - F(b_1, b_2, a_3)] - [F(b_1, a_2, b_3) - F(b_1, a_2, a_3)] \\ &- [F(a_1, b_2, b_3) - F(a_1, b_2, a_3)] + [F(a_1, a_2, b_3) - F(a_1, a_2, a_3)] \\ &= F(b_1, b_2, b_3) - F(b_1, b_2, a_3) - F(b_1, a_2, b_3) + F(b_1, a_2, a_3) \\ &- F(a_1, b_2, b_3) + F(a_1, b_2, a_3) + F(a_1, a_2, b_3) - F(a_1, a_2, a_3) \end{aligned}$$

Kontuan izanda kasu honetan $\mathbf{a} = (a_1, a_2, a_3) = (x_1, y_1, z_1)$ eta $\mathbf{b} = (b_1, b_2, b_3)$ $= (x_2, y_2, z_2)$ direla:

$$\begin{aligned} V_F(J) = &F(x_2, y_2, z_2) - F(x_2, y_2, z_1) - F(x_2, y_1, z_2) + F(x_2, y_1, z_1) \\ &- F(x_1, y_2, z_2) + F(x_1, y_2, z_1) + F(x_1, y_1, z_2) - F(x_1, y_1, z_1). \end{aligned}$$

Definizioa 16. F n aldagaiko funtzio erreala *n-gorakorra* dala esaten da $V_F(J) \geq 0$ bada erpinak $Dom(F)$-n daukazan J n-kutxa guztietarako. Bestela, F *n-beherakorra* dala esaten da.

F funtzioa n-beherakorra da baldin eta soilik baldin -F n-gorakorra bada. Beste alde batetik, J kutxa endekatu gabea bada, orduan $V_F(J) = 0$ da. Suposatu F n aldagaiko funtzio erreal baten eremua $Dom(F) = S_1 \times ... \times S_n$ dala non S_k bakoitzak a_k elementu minimo bat daukan.*F oinarrituta* dagoela esaten da gitxienez k baterako $t_k = a_k$ diren $\mathbf{t} \in Dom(F)$ guztietarako $F(\mathbf{t}) = 0$ bada. S_k bakoitza ez-hutsa bada eta b_k elementu handien bat badauka, orduan F-k marjinalak daukaz. *F-ren dimentsio bateko marjinalak* $Dom(F_k) = S_k$ eremuan definitutako F_k funtzioak dira,

$$F_k(x) = F(b_1, ..., b_{k-1}, x, b_{k+1}, ..., b_n), \ x \in S_k \ guztietarako. \tag{4}$$

Adibidea 4. Izan bedi $[-1, 1] \times [0, \infty] \times [0, \frac{\pi}{2}]$ eremuan definitutako

$$F(x,y,z) = \frac{(x+1)(e^y-1)sin\ z}{x+2e^y-1} \text{ funtzioa.}$$

$F(x,y,0)=0$, $F(x,0,z)=0$ eta $F(x,y,0)=0$ betetzen diranez, F oinarrituta dago.

F-ren dimentsio bateko marjinalak hurrengoak dira:

$F_1(x) = F(x,\infty,\frac{\pi}{2}) = \frac{(x+1)(e^\infty-1)}{x+2e^\infty-1} \to \frac{x+1}{2}$

$F_2(y) = F(1,y,\frac{\pi}{2}) = \frac{e^y-1}{e^y} = 1-e^{-y}$

$F_3(z) = F(1,\infty,z) = \frac{e^\infty-1}{e^\infty} sin\ z = sin\ z$

F-ren bi dimentsiotako marjinalak:

$F_{1,2}(x,y) = F(x,y,\frac{\pi}{2}) = \frac{(x+1)(e^y-1)}{x+2e^y-1}$

$F_{2,3}(y,z) = F(1,y,z) = \frac{e^y-1}{e^y} sin\ z = (1-e^{-y}) sin\ z$

$F_{1,3}(x,z) = F(x,\infty,z) = \frac{(x+1)(e^\infty-1)sin\ z}{x+2e^\infty-1} \to \frac{(x+1)sin\ z}{2}$

Aurrerantzean dimentsio bateko marjinalei *marjinala* bakarrik esango jake eta $k \geq 2$ danean k dimentsioko marjinalak *k-marjinalak* izango dira.

Lema 2.0.2. *Izan bitez $S_1,...,S_n$ $\overline{\mathbf{R}}$-ren azpimultzo es hutsak eta F funtzio n-gorakor oinarritua $S_1\times...\times S_n$ eremuarekin. Orduan, F ez-beherakorra da argumentu bakoitzean; hau da, $(t_1,...,t_{k-1},x,t_{k+1},...,t_n)$ eta $(t_1,...,t_{k-1},y,$ $t_{k+1},...,t_n)$ F-ren eremuan badagoz eta $x<y$ bada, orduan*

$$F(t_1,...,t_{k-1},x,t_{k+1},...,t_n) \leq F(t_1,...,t_{k-1},y,t_{k+1},...,t_n).$$

Definizioa 17. *n dimentsioko azpikopula edo n-azpikopula C'* funtzio bat da hurrengo propietateak betetzen dauzana:

- C'-ren eremua $Dom(C') = S_1 \times ... \times S_n$ da, non S_k bakoitza 0 eta 1 barne daukazan $\mathbf{I}=[0,1]$-ren azpimultzo bat dan.
- C' oinarritua eta n-gorakorra da.
- C'-k (dimentsio bateko) C'_k marjinalak daukaz, $k=1,...,n$ eta $u \in S_k$ guztietarako $C'_k(u)=u$ betetzen dabe.

Ohartu $\mathbf{u} \in Dom(C')$ guztietarako $0 \leq C'(\mathbf{u}) \leq 1$, beraz $Ran(C')$ ere $\mathbf{I}=[0,1]$-ren azpimultzo bat da.

Definizioa 18. *n dimentsioko kopula bat edo n-kopula bat* $\mathbf{I}^n$ eremuan definitutako C azpikopula bat da.
Baliokideki, *n-kopula* $C:[0,1]^n \to [0,1]$ eran definitutako funtzio bat da eta ondoko propietateak betetzen dauz:

1. $\mathbf{u} \in \mathbf{I}^n$ guztietarako, $C(\mathbf{u}) = 0$ da gitxienez $\mathbf{u}$-ren koordenatu bat 0 bada; eta $\mathbf{u}$-ren koordenatu guztiak 1 badira u_k izan ezik, orduan $C(\mathbf{u}) = u_k$.

2. $\mathbf{a}, \mathbf{b} \in \mathbf{I}^n$ guztietarako non $\mathbf{a} \leq \mathbf{b}$, $V_C([\mathbf{a}, \mathbf{b}]) \geq 0$.

Adibidea 5. Izan bedi $C(u_1, u_2, u_3) = u_3 \cdot min(u_1, u_2)$.

1. $u_3 = 0$ bada, $C(u_1, u_2, 0) = 0$
   $u_1, u_2 = 1$ bada, $C(1, 1, u_3) = u_3 \cdot min(1, 1) = u_3$

2. Izan bedi $J = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, non $a_k \leq b_k$. Orduan,

$$\begin{aligned} V_C(J) &= \Delta_{a_3}^{b_3} \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} C(u_1, u_2, u_3) = \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} (C(u_1, u_2, b_3) - C(u_1, u_2, a_3)) \\ &= \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} ((b_3 - a_3) min(u_1, u_2)) = (b_3 - a_3) \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} min(u_1, u_2) \geq 0 \end{aligned}$$

3. C-ren 2-marjinalak 2-kopulak dira lehenago ikusi bezala.

   $C_{1,2}(u_1, u_2) = C(u_1, u_2, 1) = min(u_1, u_2) = M(u_1, u_2)$

   $C_{1,3}(u_1, u_3) = C(u_1, 1, u_3) = u_3 \cdot min(u_1, 1) = \Pi(u_1, u_3)$

   $C_{2,3}(u_2, u_3) = C(1, u_2, u_3) = u_3 \cdot min(1, u_2) = \Pi(u_2, u_3)$

Definizioa 19. n dimentsioko banaketa funtzio bat $\overline{\mathbf{R}}^n$-n definitutako F funtzio bat da eta propietate bi betetzen dauz:

1. F n-gorakorra da.

2. $\mathbf{t} \in \overline{\mathbf{R}}^n$ guztietarako $F(\mathbf{t}) = 0$, non $t_k = -\infty$ gitxienez k baterako eta $F(\infty, ..., \infty) = 1$

Hortaz, bigarren baldintzagatik F oinarritua da. Gainera, $Dom(F) = \overline{\mathbf{R}}$ danez, 1.5.1 lema jarraituz, (24)-ren bitartez adierazitako n dimentsioko banaketa funtzioaren marjinalak baneketa funtzioak dira era berean. Banaketa funtzio marjinal honei $F_1, ..., F_n$ esango jake $n \geq 3$ danean.

Teorema 2.0.3. ***(Sklar-en teorema n dimentsiotan)*** *Izan bedi F n-dimentsioko banaketa-funtzio bat eta $F_1, ..., F_n$ F-ren banaketa-funtzio marjinalak. Orduan, n-dimentsioko $C : [0, 1]^n \to [0, 1]$ kopula bat existitzen da non $X = (x_1, ..., x_n) \in \overline{R}^n$ guztietarako,*

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{5}$$

$F_1, ..., F_n$ banaketa-funtzio marjinal guztiak jarraituak badira orduan C kopula bakarra da eta hortaz modu unibokoan zehaztuta dago $ran(F_1) \times ... \times ran(F_n)$ eremuan. Alderantziz, C kopula bat bada eta $F_1, ..., F_n$ banaketa funtzioak badira, orduan (25) adierazpenean definitutako F funtzioa n dimentsioko banaketa funtzio bateratua da $F_1, ..., F_n$ banaketa funtzio marjinalekin.

Korolarioa 2.0.4. *Izan bitez F banaketa funtzio bateratua, C n-kopula eta $F_1, ..., F_n$ banaketa funtzio marjinalak eta izan bitez baita $F_1^{(-1)}, ..., F_n^{(-1)}$ aurreko $F_1, ..., F_n$ banaketa funtzioen sasi-alderantzizkoak hurrenez hurren. Orduan, $\mathbf{u} \in \mathbf{I}^n = [0,1]^n$ guztietarako,*

$$C(u_1, ..., u_n) = F(F_1^{(-1)}(u_1), ..., F_n^{(-1)}(u_n)).$$

Argi dagoen moduan, zorizko aldagaietarako n dimentsioko Sklar-en teorema aurreko 1.5.2 teoremaren antzerakoa da:

Teorema 2.0.5. *Izan bitez $F_1, ..., F_n$ $X_1, ..., X_n$ zorizko aldagaien banaketa funtzioak hurrenez hurren eta F banaketa funtzio bateratua. Orduan, $F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n))$ betetzen dauan n-kopula bat existitzen da. $F_1, ..., F_n$ guztiak jarraituak badira, C n-kopula bakarra da. Bestela modu unibokoan zehaztuta dago $ran(F_1) \times ... \times ran(F_n)$ eremuan.*

M, Π eta W 2-kopulen n dimentsiotarako hedapenak M^n, Π^n eta W^n izendatzen dira (goi-indizeak dimentsioari egiten deutso erreferentzia, ez da berretzaile bat) eta

$$\begin{aligned} M^n(\mathbf{u}) &= min(u_1, ..., u_n) \\ \Pi^n(\mathbf{u}) &= u_1 \cdot ... \cdot u_n \\ W^n(\mathbf{u}) &= max(u_1 + ... + u_n - n + 1, 0) \end{aligned}$$

M^n eta Π^n funtzioak n-kopulak dira $n \geq 2$ guztietarako. W^n funtzioa aldiz, ez da n-kopula $n > 2$ danean. Holan eta guztiz ere, Frchet-Hoeffding borneen inekuazioaren n dimentsiorako bertsio bat existitzen da.

Teorema 2.0.6. *C' edozein n-azpikopularako eta $\mathbf{u} \in Dom(C')$ guztietarako,*

$$W^n(\mathbf{u}) \leq C'(\mathbf{u}) \leq M^n(\mathbf{u}).$$

Teorema 2.0.7. *Izan bedi $X_1, ..., X_n$ zorizko aldagai jarraituak, non $n \geq 2$. Orduan*

1. *$X_1, ..., X_n$ askeak edo independenteak dira baldin eta soilik baldin $X_1, ...,$ X_n-ren n-kopula Π^n bada.*

2. *$X_1, ..., X_n$ zorizko aldagai bakoitza ziur aski besteetariko edozein aldagairen funtzio hertsiki gorakorra da baldin eta soilik baldin $X_1, ..., X_n$-ren n-kopula M^n bada.*

2.0.1 Gauss-en kopula

Aldagai biko kopula n aldagaira ere hedatu daiteke, $n > 2$. Bi aldagairen kasuan Gauss-en kopula baterako banaketa funtzio normaletik eta baita marjinal normaletatik ondorioztatzen da. Aldagai anitzeko kasuan ere, aldagai anitzeko banaketa funtziotik eta bere marjinaletatik ondorioztatzen da. Beraz, ondoko eran defintzen da:

$$C(u_1, ..., u_n) = P(\Phi(X_1) \leq u_1, ..., \Phi(X_n) \leq u_n) = \Phi_n(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)) \tag{6}$$

Kopula honek ez dau formula esplizitúrik onartzen eta integral baten bitartez bakarrik adierazi daiteke. Beste alde batetik, Gauss-en kopulan parametroak Γ korrelazio matrizearen osagaiak dira. Kopula independentea ($\Gamma = I_n$ identitate matrizea danean) eta goi borne kopula ($\Gamma = 1_{n\times n}$ matrizea danean) Gauss-en kopularen kasu partikularrak dira. Gauss-en kopularen dentsitatea

$$c(u_1, ..., u_n; \mathbf{\Gamma}) = |\mathbf{\Gamma}|^{-\frac{1}{2}} exp(-\frac{1}{2}\xi'(\mathbf{\Gamma}^{-1} - I_n)\xi) \tag{7}$$

da, non $\xi = (\xi_1, ..., \xi_n)'$ eta ξ_i $N(0,1)$ banaketa normalaren u_i koantila diren:

$$u_i = P(N(0,1) < \xi_i), \ i = 1, ..., n.$$

Gauss-en kopula (27), $\xi = (\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))$-ren bitartez, $u_1, ..., u_n$-ren funtzio moduan kontsideratu behar da, ez $\xi = (\xi_1, ..., \xi_n)'$-ren funtzio bat.

2.0.2 Student-en t kopula

Student-en t kopula Student-en t banaketa funtzioari lotutako menpekotasun funtzioa da eta hau ere kopula inplizitua da. Lehenik eta behin Student-en t banaketaren dentsitatea ezagutzea beharrezkoa da kopula definitu ahal izateko. Orduan, izan bitez $X_1, ..., X_n$ zorizko aldagaiak. Beraz, $t \sim t_n(\nu, \mu, \Sigma)$ aldagai anitzeko banaketaren dentsitate funtzioa:

$$f(x) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^n|\Sigma|}}(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu})^{-\frac{\nu+n}{2}} \text{ da,}$$

non ν askatasun graduak, μ batazbestekoen bektorea eta Σ positiboki definitutako dispertsio matrizea diren. Aldagai anitzeko Student-en t banaketak $X \overset{d}{=} \mu + \sqrt{\frac{\nu}{S}} \cdot Z$ adierazpena dauka, non $Z \sim N(0, \Sigma)$ eta $S \sim \chi^2_\nu$ diren. Beraz, Studen-en t kopula ondoko eran definitu daiteke:

$$\begin{aligned} C(u_1, ..., u_n; \Gamma) &= \mathbf{t}_\nu(t^{-1}_{\nu_1}(u_1), ..., t^{-1}_{\nu_n}(u_n)) = \int_{-\infty}^{t^{-1}_\nu(u_1)} \cdots \int_{-\infty}^{t^{-1}_\nu(u_n)} f(x)dx \\ &= \int_{-\infty}^{t^{-1}_\nu(u_1)} \cdots \int_{-\infty}^{t^{-1}_\nu(u_n)} \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^n|\Sigma|}}(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu})^{-\frac{\nu+n}{2}} dx \end{aligned} \tag{8}$$

Ikusi daitekeen moduan, Student-en t baterako banaketaren marjinalen askatasun graduak ez dira zertan bardinak izan behar. Gainera, Student-en t kopularen dentsitatea

$$c(u_1, ..., u_n; \Gamma) = K \frac{1}{|\Gamma|^{1/2}} [1 + \nu^{-1} \mathbf{x}' \mathbf{\Gamma}^{-1} \mathbf{x}]^{-\frac{\nu+n}{2}} \prod_{i=1}^{n} (1 + \nu^{-1} x_i^2)^{\frac{\nu+1}{2}} \qquad (9)$$

formularen bidez emonda dago, non $\mathbf{x} = (T_{\nu_1}^{-1}(u_1), ..., T_{\nu_n}^{-1}(u_n))$, $K = \Gamma(\frac{\nu}{2})^{n-1} \Gamma(\frac{\nu+1}{2})^{-n} \Gamma(\frac{\nu+n}{2})$ eta ν kopularen askatasun gradu kopurua diren.

Kasu honetan $\Gamma = I_n$ danean ez da kopula independentea edo askea lortzen, aldagai anitzeko t banaketan korrelazio hutsak ez baitau independentzia inplikatzen. Aldiz, $\Gamma = 1_{n \times n}$ danean goi borne kopula lortzen da oraingoan ere.

2.1 kopula arkimedearrak

Aurreko atal baten hiru kopula oso garrantzitsu definitu dira. Kopula horren n dimentsiotarako hedapen modu oso naturalean egin izan da; adibidez biderkadura kopularen kasuan $\Pi(u_1, u_2) = u_1 \cdot u_2 \rightarrow \Pi^n(u_1, ..., u_n) = u_1 \ldots u_n$. Era berean orokortu daiteke kopula arkimedearren definizioa n aldagaien kasura:

$$C^n(u_1, \ldots, u_n) = \psi^{-1}(\psi(u_1) + \ldots \psi(u_n)), \ non \ 0 \leq u_1, \ldots u_n \leq 1. \qquad (10)$$

C^n funtzioak 2-kopula arkimedearrak iteratuz lortu daitekez, hau da, $C^n(u_1, \ldots, u_n) = C(C^{n-1}(u_1, \ldots, u_{n-1}), u_n)$. Baina kopulak konposatzeko teknika honek ez dau beti balio. $\psi(t) = 1 - t$ C^n kopularen formulan ordezkatuz gero W^n kopula lortzen da eta arinago esaten dan bezala, W^n ez da kopula bat $n > 3$ danean. Beraz, *psi* funtzioak propietateak zeintzuk izan behar diran aztertzea beharrezkoa da.

Definizioa 20. $g(t)$ funtzio bat *guztiz monotonoa* da J tarte batean tarte horretan jarraitua bada eta seinuz aldatzen daben ordena guztietako deribatuak badaukaz, hau da,

$$(-1)^k \frac{d^k g(t)}{dt^k} \geq 0 \text{ betetzen bada,}$$

$t \in J$ guztietarako eta $k = 0, 1, 2, \ldots$.

Ondorioz, g funtzioa guztiz monotonoa bada $[0, \infty)$ tartean eta $g(c) = 0$ $c > 0$ batzuetarako, orduan g zeroren bardina da $[0, \infty)$ tartean. Hori dala eta, ψ^{-1} ψ sortzaile arkimedearraren sasi alderantzizkoa guztiz monotonoa bada, orduan positiboa izan behar da $[0, \infty)$ tartean.
Hurrengo teoremak baldintza beharrezkoak eta nahikoak emoten dauz ψ sortzaile batek n-kopula arkimedearrak sortu ahal izateko $n \geq 2$ danean.

Teorema 2.1.1. *Izan bedi $\psi : \mathbf{I} = [0,1] \to [0,\infty]$ hertsiki beherakorra eta jarraitua dan funtzio bat, non $\psi(0) = \infty$ eta $\psi(1) = 0$ diren eta izan bedi baita ψ^{-1} ψ funtzioaren alderantzizkoa. $C^n : \mathbf{I}^n \to \mathbf{I}$ funtzioa (32) adierazpenaren bidez emondako funtzioa bada, orduan C^n n-kopula bat da $n > 2$ guztietarako baldin eta soilik baldin ψ^{-1} guztiz monotonoa bada $[0,\infty)$.*

Orokorrean kopula arkimedearren dentsitate funtzioa

$$c(u_1, \ldots, u_n) = \psi^{-1}_{(n)}(\psi(u_1) + \ldots + \psi(u_n)) \prod_{i=1}^{n} \psi'(u_i)$$

adierazpenaren bidez emonda dago.

Nahiz eta nahiko erraza izan n-kopula arkimedearrak eraikitzea, muga batzuk ere badaukiez. Lehenik eta behin, n-kopula arkimedear baten k-marjinal guztiak orokorrean bardinak dira. Gainera, normalean parametro bat edo bi egoteak menpekotasun egituraren izaera mugatzen dau familia honetan.

2.1.1 Kopula independentea

Gogoratu kopula independentearen kasuan funtzio sortzailea $\psi(t) = -ln\ t$ eta bere alderantzizkoa $\psi^{-1}(x) = e^{-x}$ direla. Orduan, n aldagaietarako hedapena holan definitzen da:

$$\begin{aligned} C(u_1, \ldots, u_n) &= \psi^{-1}(\psi(u_1) + \ldots + \psi(u_n)) = \psi^{-1}(-(ln\ u_1 + \ldots + ln\ u_n)) \\ &= \psi^{-1}(-ln\ (u_1 \cdots u_n)) = e^{-(-ln\ u_1 \cdots u_n)} \\ &= u_1 \cdots u_n \end{aligned} \tag{11}$$

Orduan kopularen dentsitatea:

$$c(u_1, \ldots, u_n) = (\frac{\partial C(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}) = 1 \tag{12}$$

Kontuan hartu, bi aldagaietako kasuan bezala, kopula independentea aurrerago definitutako Π^n biderkadura kopula dala.

2.1.2 Frank kopula

Frank kopularen kasuan gogoratu funtzio sortzailea eta bere alderantzizkoa:

$$\begin{aligned} \psi(t) &= -ln\frac{e^{-\delta t} - 1}{e^{-\delta} - 1}, \quad \delta \in \mathbb{R} \\ \psi^{-1}(x) &= -\frac{1}{\delta} ln[1 + \frac{e^{-\delta} - 1}{e^x}] \end{aligned}$$

Ordezkatuz, Frank n-kopula hurrengoa da:

$$
\begin{aligned}
C(u_1,\ldots,u_n) &= \psi^{-1}(\psi(u_1)+\ldots+\psi(u_n)) = \psi^{-1}(-ln\frac{e^{-\delta u_1}-1}{e^{-\delta}-1}-\ldots-ln\frac{e^{-\delta u_n}-1}{e^{-\delta}-1}) \\
&= \psi^{-1}(ln\frac{(e^{-\delta}-1)^n}{(e^{-\delta u_1}-1)\cdots(e^{-\delta u_n}-1)}) \\
&= -\frac{1}{\delta}ln(1+\frac{(e^{-\delta u_1}-1)\cdots(e^{-\delta u_n}-1)}{(e^{-\delta}-1)^{n-1}})
\end{aligned}
\tag{13}
$$

2.1.3 Gumbel kopula

$\psi(t)=(-ln\ t)^\delta$ $(\delta\geq 1)$ eta $\psi^{-1}(x)=exp(-x^{\frac{1}{\delta}})$ funtzioak gogoratuz, Gumbel n-kopula:

$$
\begin{aligned}
C(u_1,\ldots,u_n) &= \psi^{-1}(\psi(u_1)+\ldots+\psi(u_n)) = \psi^{-1}((-ln\ u_1)^\delta+\ldots+(-ln\ u_n)^\delta) \\
&= exp(-[(-ln\ u_1)^\delta+\ldots+(-ln\ u_2)^\delta]^{\frac{1}{\delta}})
\end{aligned}
\tag{14}
$$

2.1.4 Clayton kopula

Clayton kopularen kasuan funtzio sortzailea eta bere alderantzizkoa $\psi(t)=\frac{1}{\delta}(t^{-\delta}-1)$ eta $\psi^{-1}(x)=(\delta x+1)^{-\frac{1}{\delta}}$ ziran hurrenez hurren. Orduan, Clayton n-kopula hurrengo eran definitzen da:

$$
\begin{aligned}
C(u_1,\ldots,u_n) &= \psi^{-1}(\psi(u_1)+\ldots+\psi(u_n)) = \psi^{-1}(\frac{1}{\delta}(u_1^{-\delta}-1)+\ldots+\frac{1}{\delta}(u_n^{-\delta}-1)) \\
&= \psi^{-1}(\frac{1}{\delta}(u_1^{-\delta}+\ldots+u_n^{-\delta}-n)) = (u_1^{-\delta}+\ldots+u_n^{-\delta}-n+1)^{-\frac{1}{\delta}}
\end{aligned}
\tag{15}
$$

Kasu honetan erraza da Clayton n-kopularen dentsitatea kalkulatzea:

$$
c(u_1,\ldots,u_n) = (\frac{\partial C(u_1,\ldots,u_n)}{\partial u_1\cdots\partial u_n}) = (\sum_{i=1}^{n}u_i^{-\delta}-n+1)^{-\frac{1}{\delta}-n}\prod_{i=1}^{n}[(1+(i-1)\delta)u_i^{-\delta-1}]
\tag{16}
$$

2.2 Kopula baldintzatuak

Aurreko ataletan kopula baldintzatuak aztertu dira aldagai biren kasuan. Orain, ikusitako emaitzak aldagai anitzen kasura hedatuko dira. Izan bitez $Y_1,\ldots,Y_n$ zorizko aldagaiak eta $X=x$ koaldagaia. Orduan, lehengo gisa, $(Y_1,\ldots,Y_n)$ aldagaien baterako banaketa funtzioa eta funtzio marjinalak, $X=x$ baldintzapean,

$$F_x(y_1,\dots,y_n) = P(Y_1 \le y_1,\dots Y_n \le y_n|X = x),$$
$$F_{1x}(y_1) = P(Y_1 \le y_1|X = x),\dots,F_{nx}(y_n) = P(Y \le y|X = x).$$

dira hurrenez hurren.

Definizioa 21. $U_1,\dots,U_n$-rekiko deribatu partzialek C kopulari loturiko kopula baldintzatuak definitzen dabez (C kopularen monotonia dala eta, $u_1,\dots,u_n$ ia guztietarako existitzen dira):

- u_1 finko baterako, u_1-rekiko kopula baldintzatua $U_2,\dots,U_n$-ren funtzio bat da, $(u_2,\dots,u_n) \to C_1(u_1,\dots,u_n) = C(u_2,\dots,u_n|u_1) = \frac{\partial C}{\partial u_1}(u_1,\dots,u_n)$
- $\vdots$
- u_n finko baterako, u_n-rekiko kopula baldintzatua $U_1,\dots,U_{n-1}$-ren funtzio bat da, $(u_1,\dots,u_{n-1}) \to C_1(u_1,\dots,u_{n-1}) = C(u_1,\dots,u_{n-1}|u_n) = \frac{\partial C}{\partial u_n}(u_1,\dots,u_n)$

Definizioa 22. Kopula baldintzatu bat $C_x : [0,1]^{n-1} \times \mathcal{X} \to [0,1]$ funtzio bat da hurrengo propietateak betetzen dauzana:

1. $\mathbf{u} \in \mathbf{I}^n$ guztietarako eta $x \in \mathcal{X}$ bakoitzerako, $C(\mathbf{u}|\mathcal{X} = x) = 0$ da gitxienez $\mathbf{u}$-ren koordenatu bat 0 bada; eta $\mathbf{u}$-ren koordenatu guztiak 1 badira u_k izan ezik, orduan $C(\mathbf{u}|\mathcal{X} = x) = u_k$.
2. $\mathbf{a},\mathbf{b} \in \mathbf{I}^n$ guztietarako non $\mathbf{a} \le \mathbf{b}$ eta $x \in \mathcal{X}$ bakoitzerako, $V_C([\mathbf{a},\mathbf{b}]|\mathcal{X} = x) \ge 0$.

Arinago ikusitako Sklar-en bi teoremetatik aparte, badago kopula baldintzatueri egokitutako Sklar-en teorema bat ere.

Teorema 2.2.1. *Izan bitez F_x aldagai biren baterako banaketa funtzioa eta $F_{1x},\dots,F_{nx}$ bere marjinal jarraituak. Orduan, $C_x : [0,1]^n \times \mathcal{X} \to [0,1]$ kopula baldintzatu bakarra existitzen da non $y_1,\dots,y_n \in \overline{\mathbf{R}}^n$ guztietarako,*

$$F_x(y_1,\dots,y_n|\mathcal{X} = x) = C_x(F_{1x}(y_1|\mathcal{X} = x),\dots,F_{nx}(y_n|\mathcal{X} = x)|\mathcal{X} = x) \tag{17}$$

Alderantziz, C_x kopula baldintzatua eta $F_{1x},\dots,F_{nx}$ $Y_1,\dots,Y_n$ zorizko aldagai biren marjinal baldintzatuak badira, orduan aurreko (17) ekuazioarekin definitutako F_x funtzioa $F_{1x},\dots,F_{nx}$ marjinalak daukazan banaketa funtzio baldintzatua da.