SENSIBILIDAD Y ANÁLISIS DE CVA DE SWAPS LIGADOS A LA INFLACIÓN

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On the sensitivity and CVA of Inflation-Indexed Swaps

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Summary

The main contribution of this work are methodological and empirical, very useful for practitioners. On the one hand, we develop a complete procedure to calibrate the daily parameters in the Jarrow and Yildirim (2003) model, which enables us to value inflation-indexed swaps. On the other hand, we develop the corresponding algorithm in Matlab that provides the calibration and the simulation procedures in order to calculate for each derivative, the expected positive exposure, the Credit Valuation Adjustment (CVA), the 97.5% positive exposure and the expected shortfall with a confidence level of 97.5%. The results show that our methodology is very accurate and competitive with the valuation made by external consultancies.

Introduction

In recent years the trade of derivatives linked to inflation has increased through the developed economies. Inflation is defined as the increment in percentage terms, of a reference index defined as the price of a basket of goods and services. The more relevant indexes to measure inflation are the retail price index (RPI, related with a basket of goods and services that represent the total cost of a typical family) and the Consumer price index (CPI, which differs from RPI due to that the costs related to housing have been dropped out). In the European derivative markets, the CPI is the usual index, and the inflation is measured as the annual percentage change of this index. In the Eurozone, the most used index is de HICP (Harmonised Indices of Consumer Prices) or the HICP-x, similar to the previous one, but taking the tobacco out. There are other indexes, as in USA where there are the CPI-U (for all urban consumers), CPI-W (for Urban Wage Earners and Clerical Workers), CPI-E (for the elderly) and C-CPI-U (chained CPI for all urban consumers) or in France, the FCPI (Consumer Price Index for all France) and the FCPI-x (Consumer Price Index Ex-tobacco).

From an investing point of view, pension funds or any other financial institutions whose abilities are linked to an inflation index, find in those derivatives the most efficient way of hedging these flows.
The investors appetite on Inflation Linked (IL from now on) derivatives is proportional to the increasing IL government bonds issued by many countries. For example, the Spanish government issues IL bonds with maturities November 2024 and November 2033, among others. As IL bonds coupons are variables, some investors prefer to enter in a IL swap to exchange the IL coupons to fixed cash flows. Others prefer to enter in a swap to change the coupons for a fixed rate, or even for a mixed flow, with both fixed and variable flows.

This work has been proposed and developed under a fellowship program, because Laboral Kutxa was interested on reviewing the properties of inflation derivatives from both methodological and empirical point of view.

Laboral Kutxa was interested in analyse not only the present valuation, but also the future evolution of the inflation derivatives, including the evolution of the future valuation under different parameter hypothesis. This motivation comes from deep conviction on the importance of modelling all the assets in the ballance sheet under reliable metrics. Firstly, thinking from a risk management point of view, internal metrics under different macro scenarios are crucial for developing internal stress test exercises. For example, during the ICAAP/ILAAP processes of Basileas Pillar II. Furthermore, supervisory stress test exercises require for the estimation of the future market value of all the assets, including these derivatives under the parameters stated by the regulators.

The empirical analysis covers the evolution of the future values that is compulsory from a market risk management point of view. Also, estimating good future values is necessary for an appropriate estimation of the liquidity ratios as in many cases these trades are collateralized through liquid assets. From a regulatory point of view, Expected shortfall and CVA are inputs for the capital requirements.

Financial literature deals with different models to encompass the uncertainty caused by the inflation index interest rate evolution. We will use the Yarrow and Yildirim (2003) model, focusing on the sensitivity of the valuation of some standard derivatives to the change on the key parameters of the model, and computing also the simulated expected shortfall and the Credit Valuation Adjustment (CVA).

This work is organized as follows. Chapter 1 includes the basic definitions of products, the closed valuation formulas and a detailed derivation of the Jarrow and Yildirim (2003) model.

Chapter 2 focuses on the calibration techniques to obtain the parameters of the model. We present the empirical results that provide a comparison between the different calibration alternatives already studied in the literature. Finally, we propose an alternative method that improves the goodness of fit to market data.

Chapter 3 covers the valuation of some IL derivatives, calculating also the expected positive exposure, the CVA, the 97,5% positive exposure and the expected shortfall with a confidence level of 97,5%. To check the accuracy of the calibration, the results have been tested against those provided by an external consultancy firm. Moreover, we study the stability of the valuation by analysing the sensitivity of the actual and future valuation to changes of the parameters. Analysing the sensitivity, we account for the importance of calibrating the parameters adequately.

Finally, the conclusions and further ideas for possible extensions of this work are provided, suggesting different hypothesis about the parameters of the model.

Further details and the main proofs are relegated to the appendixes.

Chapter 1

Definitions and methodology

In this chapter we give the definitions of the main financial instruments related with inflation and provide the closed formulas for the derivatives we are interested in. See [1] and [2] for details.

1.1 Definitions

Let us consider the subindexes n and r for nominal and real, respectively.

- $P_n(t,T)$: price in t of a nominal zero-coupon bond with maturity T
- $P_r(t,T)$: price in t of a real zero-coupon bond with maturity T
- $I(t)$: value of the inflation index in t
- $f_k(t,T)$: instantaneous forward rate in t for any date T with $k \in \{n,r\}$. The equation of prices is as follows

$$P_k(t,T) = e^{-\int_t^T f_k(t,s)ds} \tag{1.1}$$

- $k(t) = f_k(t,t)$: instantaneous spot rate in t for $k \in \{n,r\}$, what gives us

$$P_k(t,T) = E(e^{-\int_t^T k(s)ds}|F_t) \tag{1.2}$$

- $B_k(t)$: money market account value in t for $k \in \{n,r\}$ with,

$$dB_k(t) = k(t)B_k(t)dt \tag{1.3}$$

1.1.1 Inflation Linked Bonds

An inflation-linked zero-coupon bond is a bond that pays a single cash flow at maturity T, which is the ratio of a reference index between T and $t_0 = 0$, with a nominal value of N. The value is denoted as $\mathbf{ZCILB}(t, T, I_0, N)$, in nominal basis, and the payment in T is

$$\frac{I(T)}{I_0}N \tag{1.4}$$

nominal units at maturity. The corresponding real payment is obtained by normalizing through the value of the index in T. So that, the real payment in T is

$$\frac{N}{I_0} \tag{1.5}$$

real units at maturity. We see that while the nominal payment in T is unknown the real one is known.

The real value in t of the payment of the inflation-indexed zero-coupon bond is

$$E(e^{-\int_t^T r(s)ds}\frac{N}{I_0}|F_t) = \frac{N}{I_0}P_r(t, T) \tag{1.6}$$

Taking into account that the real value in t of the ZCILB is obtained by normalizing the nominal value with the index in t, the next equation has to be fulfilled.

$$\frac{ZCILB(t, T, I_0, N)}{I(t)} = \frac{NP_r(t, T)}{I_0}. \tag{1.7}$$

Defining the bonds unit value as $P_{IL}(t, T) := ZCILB(t, T, 1, 1)$ we get

$$P_{IL}(t, T) = I(t)P_r(t, T). \tag{1.8}$$

It is seen that the inflation linked zero-coupon bond is dependent of the inflation index in t and the real zero-coupon bond. Usually, the bonds are not zero-coupon, but it is straightforward to obtain the value in t of the coupon bonds using the expression of the zero-coupon bonds. Denoting C the annual coupon rate (assuming annual coupon frequency), the nominal value in t of a inflation-linked coupon bond with payments in $T_1, T_2, ..., T_M$ (where $T_M = T$) is

$$\begin{aligned} ILB(t, T_M, I_0, N) &= \frac{N}{I_0}\left[C\sum_{i=1}^{M} P_{IL}(t, T_i) + P_{IL}(t, T_M)\right] \\ &= \frac{I(t)}{I_0}N\left[C\sum_{i=1}^{M} P_r(t, T_i) + P_r(t, T_M)\right]. \end{aligned} \tag{1.9}$$

1.1.2 Inflation-Indexed Swaps

Given a set of dates $\{T_1, ..., T_M\}$ an Inflation-Indexed Swap (IIS) is an instrument where the party A pays to B the predefined inflation rate of a period and the party B pays A a fixed rate. The inflation rate is the percentage return of the CPI, but instead of being in a year, it is calculated in the period that we are interested in. The most traded swaps are the zero coupon inflation-indexed swaps (**ZCIIS**) and the year on year inflation-indexed swaps (**YYIIS**).
The payments in the ZCIIS are:

- Party A pays to B in T_M the floating amount

$$N\Big[\frac{I(T_M) - I(T_0)}{I(T_0)}\Big], \tag{1.10}$$

  where T_0 is the reference date.

- Party B pays in T_M a fixed amount

$$N[(1+K)^{T_M} - 1], \tag{1.11}$$

  where K is the agreed fixed rate and N the nominal value.

The payments in the YYIIS are:

- In each time $T_i \in \{T_1, ..., T_M\}$, A pays the floating amount

$$N\tau_{i,A}\Big[\frac{I(T_i) - I(T_{i-1})}{I(T_{i-1})}\Big], \tag{1.12}$$

  where $\tau_{i,A}$ is the year fraction of the floating-leg in the interval $[T_{i-1}, T_i]$.

- In each time $T_i \in \{T_1, ..., T_M\}$, B pays to A the fixed amount

$$N\tau_{i,B}K, \tag{1.13}$$

  where $\tau_{i,B}$ is the year fraction of the fixed-leg in the interval $[T_{i-1}, T_i]$.

To make things easier, we fix $T_0 = 0$. It is important to remark that both swaps are quoted in the market, each of them with its corresponding fixed rate K.
The next step is to give a value to these derivatives using the no-arbitrage pricing theory.

Regarding the zero coupon inflation-indexed swaps, the value of the floating-leg in $0 \leq t \leq T_M$ is

$$ZCIIS(t, T_M, I(0), N) = NE_n\Big(e^{-\int_t^{T_M} n(u)du}\Big[\frac{I(T_M) - I(0)}{I(0)}\Big]|F_t\Big), \tag{1.14}$$

where $n(t)$ is the nominal rate. To understand how to get a equivalent expression for this value, we use the methodology proposed by Jarrow and Yildirim (2003) for modelling inflation and nominal rates, which is based on a foreign-currency equivalence. Real rates are viewed as foreign prices and nominal rates as domestic prices. The CPI is seen as the exchange rate between the nominal and real ”currencies”.

The nominal interest rate is the agreed interest between both parts of a loan, and this is equal to the sum of the real and inflation rates. Taking into account the foreign-currency analogy (Appendix A.3), the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the CPI index at the bond maturity. This implies that for $t < T_M$,

$$I(t)P_r(t,T_M) = I(t)E_r\Big(e^{-\int_t^{T_M} r(u)du}|F_t\Big) = E_n\Big(e^{-\int_t^{T_M} n(u)du}I(T_M)|F_t\Big). \tag{1.15}$$

Using this, the expresion (1.14) can be rewritten as

$$ZCIIS(t,T_M,I(0),N) = N\Big(\frac{I(t)}{I(0)}P_r(t,T_M) - P_n(t,T_M)\Big), \tag{1.16}$$

where $P_r(t,T_M)$ and $P_n(t,T_M)$ are the discount factors related to the real and nominal rates. Taking into account that the value of the market swap at the beginning is zero, one could use this to obtain an expression of $P_r(t,T_M)$. It is important to remark that we have to discount the value of the fixed-leg with the nominal rate. At $t = 0$:

$$\begin{aligned} N[(1+K)^{T_M} - 1]P_n(0,T_M) = N[P_r(0,T_M) - P_n(0,T_M)] \implies \\ P_r(0,T_M) = P_n(0,T_M)(1+K)^{T_M}. \end{aligned} \tag{1.17}$$

Similar to the ZCIIS, the value of the payment of the floating leg in T_i of the YYIIS in $0 \le t < T_i$ is

$$YYIIS(t,T_{i-1},T_i,\tau_{i,A},N) = N\tau_{i,A}E_n\Big(e^{-\int_t^{T_i} n(u)du}\Big[\frac{I(T_i) - I(T_{i-1})}{I(T_{i-1})}\Big]|F_t\Big), \tag{1.18}$$

where $n(t)$ is the nominal rate. If $t > T_{i-1}$, $I(T_{i-1})$ is known so the pricing is similar to the Zero Coupon Inflation-Indexed Swap,

$$\begin{aligned} E_n\Big(e^{-\int_t^{T_i} n(u)du}\frac{I(T_i)}{I(T_{i-1})}|F_t\Big) = \frac{P_r(t,T_i)I(t)}{I(T_{i-1})} \implies \\ YYIIS(t,T_{i-1},T_i,\tau_{i,A},N) = N\tau_{i,A}\left[\frac{P_r(t,T_i)I(t)}{I(T_{i-1})} - P_n(t,T_i)\right]. \end{aligned} \tag{1.19}$$

If $t < T_{i-1}$, using conditional expectations the expression (1.18) can be written as

$$YYIIS(t, T_{i-1}, T_i, \tau_{i,A}, N) =$$
$$= N\tau_{i,A}E_n\Big(e^{-\int_t^{T_{i-1}} n(u)du}E_n\Big(e^{-\int_{T_{i-1}}^{T_i} n(u)du}\Big[\frac{I(T_i) - I(T_{i-1})}{I(T_{i-1})}\Big]|F_{T_{i-1}}\Big)|F_t\Big). \tag{1.20}$$

Looking at the formula, it is easily seen that we can rewrite this in terms of $ZCIIS(T_{i-1}, T_i, I(T_{i-1}), N)$ as

$$\begin{aligned} &YYIIS(t, T_{i-1}, T_i, \tau_{i,A}, N) = \\ &\qquad = \tau_{i,A}E_n\Big(e^{-\int_t^{T_{i-1}} n(u)du}ZCIIS(T_{i-1}, T_i, I(T_{i-1}), N)|F_t\Big) \\ &\qquad = N\tau_{i,A}E_n\Big(e^{-\int_t^{T_{i-1}} n(u)du}[P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)]|F_t\Big) \\ &\qquad = N\tau_{i,A}E_n\Big(e^{-\int_t^{T_{i-1}} n(u)du}P_r(T_{i-1}, T_i)|F_t\Big) - N\tau_{i,A}P_n(t, T_i). \end{aligned} \tag{1.21}$$

Using the expression (1.21), the value in t of the YYIIS if we consider that we pay the fixed rate is

$$YYIIS(t, T_M, N, K) =$$
$$N\sum_{i=1}^{T_M}\Big[\tau_{i,A}E_n\Big(e^{-\int_t^{T_{i-1}} n(u)du}P_r(T_{i-1}, T_i)|F_t\Big) - N\tau_{i,A}P_n(t, T_i) - \tau_{i,B}KP_n(t, T_i)\Big]. \tag{1.22}$$

It is appreciated that the expression of the YYIIS is model dependent.

1.1.3 Inflation Linked Cap/Floor

An inflation linked cap can be seen as a series of caplets, and similarly, the floor as a series of floorlets. An inflation linked caplet (**ILCPT**) is a call option on the net increase in forward inflation index and an inflation linked floorlet (**ILFLT**) the same but instead of a call, with a put. At time T_i the payout of any of them is:

$$N\tau_i\Big[\omega\Big(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\Big)\Big]^+, \tag{1.23}$$

where κ is the strike of the caplet or floorlet, τ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N the nominal, $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

The price in t of a caplet or floorlet (**ILCFPT**) is

$$\begin{aligned} &ILCFPT(t, T_{i-1}, T_i, \tau_i, \kappa, N, \omega) \\ &= N\tau_i E_n\Big[e^{-\int_t^{T_i} n(u)du}\Big[\omega\Big(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\Big)\Big]^+ |F_t\Big] \\ &= N\tau_i P_n(t, T_i) E_n^{T_i}\Big[\Big[\omega\Big(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\Big)\Big]^+ |F_t\Big], \end{aligned} \tag{1.24}$$

where $E_n^{T_i}$ is the expectation under the nominal T_i forward measure. In the T-forward measure, instead of using the money market account as numeraire like in the risk neutral measure, it is used a zero coupon bond with maturity T, $P_n(t, T)$. The price of the cap/floor is the sum of the price of all the caplets/floorlets. Clearly this price is also model dependent.

1.2 Methodology. Closed valuation formulas

In the literature there are some alternative models that could describe the evolution of the financial variables involved in the prices of inflation-linked derivatives such us the Libor Market Model (1997) (see [3] for more details), or the Two-process Hull and White Model (for further details see [4] section 3.3), in this case we focus on the Jarrow and Yildirim model (2003), (JY from now on). Being one of the most popular models, it is also the one that provides closed formulas for some derivatives, so that the sensitive of the derivatives to the parameters gives errors limited to the evolution of the inflation rates.

The Libor Market Models (LMM) are popular due to the coherence between such models and the well-established market formulas for caps and swaptions. In this type of models, both derivatives are priced with the Black's formula, a very important fact since these derivatives become the most traded in the fixed income market. In the LMM, rather than in short rate or instantaneous forward rate models (like in Jarrow and Yidilirm (2003)), a set of forward rates are modelled, which have the advantage of being directly observed in the market. For more information see [5], chapter 6.

The Two-process Hull and White Model ignores the existence of a real economy, only assuming dynamics for the nominal instantaneous short rate and the inflation index. It is assumed a Hull and White model (see [6] for more details about the model) for both the nominal instantaneous short rate (n) and the inflation rate (i), where the inflation level is defined by

$$I(t) = I(T_0)e^{\int_{T_0}^{T} i(s)ds}$$

being $I(T_0)$ the reference inflation index.

In this chapter we develop the JY model (see [7]) to evaluate inflation-linked derivatives. We derive detailed analytical expressions for the valuation of those derivatives whose behaviours will be analysed in Chapter 2 of empirical applications. As it will be seen later, the JY model assumes a Hull and White model for nominal and real interest rates, and a Geometric Brownian Motion for the Inflation Index.

All the development made in this chapter is based in [2], some calculus are based in the ones made in this article, and others have been developed by us. The detailed explanation is in the Appendix B.

It is important to emphasize that JY (2003) use a foreign currency analogy, where real prices are seen as foreign prices, nominal prices correspond to domestic prices and the inflation index is viewed as the spot exchange rate from foreign to domestic currency.

Under the real world probability space (Ω, F, P) with associated filtration $\{F_t : t \in [0, T]\}$, the model proposed by JY for the nominal and real instantaneous forward rates and for the CPI is

$$\begin{aligned} df_n(t,T) &= \alpha_n(t,T)dt + \sigma_n(t,T)dW_n^P(t) \\ df_r(t,T) &= \alpha_r(t,T)dt + \sigma_r(t,T)dW_r^P(t) \\ dI(t) &= I(t)\mu(t)dt + \sigma_I I(t)dW_I^P(t), \end{aligned} \tag{1.25}$$

where,

- $I(0) = I_0 > 0$
- $f_k(0,T) = f_k^M(0,T)$, $k \in \{n, r\}$, being $f_n^M(0,T)$ and $f_r^M(0,T)$ the nominal and real instantaneous forward rates observed in the market in 0 for the date T
- (W_n^P, W_r^P, W_I^P) is a Brownian motion with correlations

$$\begin{aligned} dW_n^P(t)dW_r^P(t) &= \rho_{nr}dt \\ dW_n^P(t)dW_I^P(t) &= \rho_{nI}dt \\ dW_r^P(t)dW_I^P(t) &= \rho_{rI}dt \end{aligned}$$

- α_n, α_r and μ are adapted processes[*]

[*] An adapted process is a process in which the possible events until a time t only depend on past events and the process can not anticipate the future.

- σ_n and σ_n are deterministic functions
- σ_I is a positive constant

The dynamics of the nominal and real instantaneous forward rates follow the model **Heath, Jarrow and Morton** [8].

JY show that the evolutions introduced are arbitrage free and that the market is complete using the existence and uniqueness of an equivalent martingale probability measure Q such that

$$\frac{P_n(t,T)}{B_n(t)}, \frac{I(t)P_r(t,T)}{B_n(t)} \quad \text{and} \quad \frac{I(t)B_r(t)}{B_n(t)} \quad \text{are Q martingales,} \tag{1.26}$$

where $B_n(t)$ and $B_r(t)$ are the nominal and real money market account.
In fact, using Girsanov's theorem, being $\{dW_n^P(t), dW_r^P(t), dW_I^P(t)\}$ a P-Brownian motion and given that Q is a equivalent probability measure, then exists market prices of risk $\{\lambda_n(t), \lambda_r(t), \lambda_I(t)\}$ such that

$$W_l^Q(t) = W_l^P(t) - \int_0^t \lambda_l(s)ds, \quad l \in \{n, r, I\} \tag{1.27}$$

are Q-Brownian motions. We check that the instantaneous nominal and real rate dynamics follow the model of Hull and White, under the martingale measure Q, which is an arbitrage-free model (see the Appendix B). The formulas of the dynamics are

$$\begin{aligned} dn(t) &= [\nu_n(t) - \kappa_n n(t)]dt + \sigma_n dW_n^Q(t) \\ dr(t) &= [\nu_r(t) - \rho_{rI}\sigma_I\sigma_r - \kappa_r r(t)]dt + \sigma_r dW_r^Q(t), \end{aligned} \tag{1.28}$$

with

$$\nu_l(t) = \frac{\partial f_l(0,t)}{\partial T} + \kappa_l f_l(0,t) + \frac{\sigma_l^2}{2\kappa_l}(1 - e^{-2\kappa_l t}), \quad l \in \{n, r\}, \tag{1.29}$$

where $\frac{\partial f_l}{\partial T}$ denotes partial derivative of f_l with respect to its second argument.

This type of dynamics are built so that they replicate exactly the prices of the zero-coupon bonds and the plain vanilla options on interest rates (caps, floors and swaptions). Then, if there are no arbitrage opportunities, any two actives must have the same market price of risk ($\lambda(t)$) in any time t.

Next proposition provides us the necessary and sufficient conditions needed on the bond prices evolution so that the economy is arbitrage free.

Proposition 1.2.1. *(Proposition 2.2.1 in [2])* $\frac{P_n(t,T)}{B_n(t)}$, $\frac{I(t)P_r(t,T)}{B_n(t)}$ *and* $\frac{I(t)B_r(t)}{B_n(t)}$ *are Q martingales if and only if*

$$\alpha_n(t,T) = \sigma_n(t,T)\left(\int_t^T \sigma_n(t,s)ds - \lambda_n(t)\right), \tag{1.30}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_r(t,s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t)\right) \tag{1.31}$$

$$\mu_I(t) = n(t) - r(t) - \sigma_I(t)\lambda_I(t). \tag{1.32}$$

The proof is in Appendix (B.1).

Using the dynamics of the nominal and real instantaneous forward rates and the notation

$$a_l(t,T) = -\int_t^T \sigma_l(t,u)du, \tag{1.33}$$

$$b_l(t,T) = -\int_t^T \alpha_l(t,u)du + \frac{1}{2}a_l^2(t,T), \quad l \in \{n,r\} \tag{1.34}$$

we state the next proposition.

Proposition 1.2.2. *(Proposition 2.3.1 in [2]) Under the martingale measure the price processes are*

$$df_n(t,T) = -\sigma_n(t,T)a_n(t,T)dt + \sigma_n(t,T)dW_n^Q(t) \tag{1.35}$$

$$df_r(t,T) = -\sigma_r(t,T)[a_r(t,T) + \sigma_I(t)\rho_{rI}]dt + \sigma_r(t,T)dW_r^Q(t) \tag{1.36}$$

$$\frac{dI(t)}{I(t)} = [n(t) - r(t)]dt + \sigma_I(t)dW_I^Q(t) \tag{1.37}$$

$$\frac{dP_n(t,T)}{P_n(t,T)} = n(t)dt + a_n(t,T)dW_n^Q(t) \tag{1.38}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = \Big[r(t) - \sigma_I(t)\rho_{rI}a_r(t,T)\Big]dt + a_r(t,T)dW_r^Q(t) \tag{1.39}$$

$$\frac{dP_{IL}(t,T)}{P_{IL}(t,T)} := \frac{d(I(t)P_r(t,T))}{I(t)P_r(t,T)} = n(t)dt + \sigma_I(t)dW_I^Q(t) + a_r(t,T)dW_r^Q(t) \tag{1.40}$$

For more details of how to obtain these expressions see Appendix (B.1). Note that the nominal and real forward rates are normally distributed and that the inflation index is log-normally distributed.

1.2.1 Nominal and Real bonds

For the volatility functions $\sigma_n(t,T)$ and $\sigma_r(t,T)$ Jarrow and Yildirim chose an exponentially declining volatility with the expression

$$\sigma_l(t,T) = \sigma_l e^{-\kappa_l(T-t)}, \quad l \in \{n,r\} \tag{1.41}$$

where σ_l and κ_l are positive constants.
This yields,

$$a_l(t,T) = -\sigma_l \int_t^T e^{-\kappa_l(u-t)} du = -\sigma_l \beta_l(t,T), \tag{1.42}$$

where

$$\beta_l(t,T) = \frac{1}{\kappa_l}[1 - e^{-\kappa_l(T-t)}]. \tag{1.43}$$

Under the condition (1.41), it can be proofed that the dynamics of the nominal and real instantaneous short rates follow a Hull and White structure, the ones that we have defined in (1.28) with the mean reversion level (1.29). It can be also proofed that the expressions of the nominal and real bonds in terms of the nominal and real short and forward rates are

$$P_n(t,T) = \frac{P_n(0,T)}{P_n(0,t)} \exp\left(\beta_n(t,T)[f_n(0,t) - n(t)] - \frac{\sigma_n^2}{4\kappa_n} \beta_n(t,T)^2 (1 - e^{-2\kappa_n t}) \right), \tag{1.44}$$

and

$$P_r(t,T) = \frac{P_r(0,T)}{P_r(0,t)} \exp\left(\beta_r(t,T)[f_r(0,t) - r(t)] - \frac{\sigma_r^2}{4\kappa_r} \beta_r(t,T)^2 [1 - e^{-2\kappa_r t}] \right). \tag{1.45}$$

For more details of how to obtain see B.

1.2.2 Year On Year Inflation Swap

We proceed deriving the closed formula for the floating leg of the YYIIS using the JY model, since this derivatives are ones of the most traded ones in the market. Using the forward measure is obtained that

$$\begin{aligned} YYIIS(t,T,\tau,N) &= N\tau_{i(t),A} \left[\frac{I(t)}{I(T_{i(t)-1})} P_r(t,T_{i(t)}) - P_n(t,T_{i(t)}) \right] \\ &+ N \sum_{i=i(t)+1}^{M} \tau_{i,A} \left[P_n(t,T_{i-1}) \frac{P_r(t,T_i)}{P_r(t,T_{i-1})} e^{b(t,T_{i-1},T_i)} - P_n(t,T_i) \right], \end{aligned} \tag{1.46}$$

with

$$\begin{aligned} b(t,T_{i-1},T_i) = \sigma_r \beta_r(T_{i-1},T_i) \Bigg[\beta_r(t,T_{i-1}) \Bigg(\rho_{rI}\sigma_I - \frac{1}{2}\sigma_r\beta_r(t,T_{i-1}) \\ + \frac{\rho_{nr}\sigma_n}{\kappa_n + \kappa_r}(1 + \kappa_r\beta_n(t,T_{i-1})) \Bigg) - \frac{\rho_{nr}\sigma_n}{\kappa_n + \kappa_r}\beta_n(t,T_{i-1}) \Bigg], \end{aligned} \tag{1.47}$$

where $\Gamma = \{T_1, ..., T_M\}$ are the payment dates, $\tau_A = \{\tau_{1,A}, ..., \tau_{M,A}\}$ the year fraction of the floating leg and $i(t) = \min\{i : T_i > t\}$. The first cash flow has the structure of a zero-coupon inflation leg with the formula (1.16), since the inflation in $T_{i(t)-1}$ is known in t. Specifically, at $t = 0$,

$$\begin{aligned} YYIIS(0, T, \tau, N) &= N\tau_{1,A}\Big[P_r(0, T_1) - P_n(0, T_1)\Big] \\ +N\sum_{i=2}^{M}\tau_{i,A}\Big[P_n(0, T_{i-1})&\frac{P_r(0, T_i)}{P_r(0, T_{i-1})}e^{b(0,T_{i-1},T_i)} - P_n(0, T_i)\Big]. \end{aligned} \tag{1.48}$$

1.2.3 Inflation Linked Cap/Floor

In Proposition (1.2.2) is seen the dynamics of the inflation index $I(t)$, what tell us that under the risk-neutral measure Q is log-normally distributed. Then, the ratio $\frac{I(T_i)}{I(T_{i-1})}$ has also log-normal distribution. The formula (1.24) for a caplet/floorlet can be calculated using the formulas of generalized Black-Scholes,

$$\begin{aligned} &ILCFLT(t, T_{i-1}, T_i, \tau_i, \kappa, N, \omega) = \\ &wN\tau_i P_n(t, T_i)\left[\frac{P_n(t, T_{i-1})}{P_n(t, T_i)}\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}e^{b(t,T_{i-1},T_i)}\Phi(wd_1^i(t)) - (1+\kappa)\Phi(wd_2^i(t))\right], \end{aligned} \tag{1.49}$$

where

$$d_1^i(t) = \frac{ln\frac{P_n(t,T_{i-1})}{(1+\kappa)P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})} + b(t, T_{i-1}, T_i) + \frac{1}{2}V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \tag{1.50}$$

$$d_2^i(t) = d_1^i(0) - V(t, T_{i-1}, T_i), \tag{1.51}$$

being the expression of the variance

$$
\begin{aligned}
V^2(t,T_{i-1},T_i) = {} & \frac{\sigma_n^2}{2\kappa_n}\beta_n(T_{i-1},T_i)^2[1-e^{-2\kappa_n(T_{i-1}-t)}] + \sigma_I^2(T_i - T_{i-1}) \\
& \frac{\sigma_r^2}{2\kappa_r}\beta_r(T_{i-1},T_i)^2[1-e^{-2\kappa_r(T_{i-1}-t)}] \\
& -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n+\kappa_r}\beta_n(T_{i-1},T_i)\beta_r(T_{i-1},T_i)[1-e^{-(\kappa_n+\kappa_r)(T_{i-1}-t)}] \\
& +\frac{\sigma_n^2}{\kappa_n^2}\left[T_i - T_{i-1} + \frac{2}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-T_{i-1})} - \frac{3}{2\kappa_n}\right] \\
& +\frac{\sigma_r^2}{\kappa_r^2}\left[T_i - T_{i-1} + \frac{2}{\kappa_r}e^{-\kappa_r(T_i-T_{i-1})} - \frac{1}{2\kappa_r}e^{-2\kappa_r(T_i-T_{i-1})} - \frac{3}{2\kappa_r}\right] \\
& -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\left[T_i - T_{i-1} - \beta_n(T_{i-1},T_i) - \beta_r(T_{i-1},T_i) + \frac{1-e^{-(\kappa_n+\kappa_r)(T_i-T_{i-1})}}{\kappa_n+\kappa_r}\right] \\
& +2\rho_{nI}\frac{\sigma_n\sigma_I}{\kappa_n}[T_i - T_{i-1} - \beta_n(T_{i-1},T_i)] - 2\rho_{rI}\frac{\sigma_r\sigma_I}{\kappa_r}[T_i - T_{i-1} - \beta_r(T_{i-1},T_i)].
\end{aligned}
\tag{1.52}
$$

Finally, the value of the cap would be the sum of all the caplets.

Year on year inflation indexed swaps or inflation linked caps' market prices are available. In the next chapter, we will use the formulas derived in this one to calibrate the real parameters κ_r and σ_r.

Chapter 2

Calibration

The main objective of this chapter is to calibrate the JY model, to obtain the parameters that minimize the difference between today's market prices and the prices computed with the selected model. These calibrated parameters will be used to aim the final goal of estimating the daily CVA, through the calculation of expected values of the derivatives in future dates.

Although it could be considered that calibration is nothing else than a minimization problem, in practice this part is a very tough challenge for practitioners. There is a wide range of empirical studies in the literature analysing the goodness of fit for different alternatives. Unfortunately, neither of them is valid for any range of derivatives used as inputs, or any time level of interest rates or maturities. In the case of inflation-linked derivatives, with a smaller market than the one for interest rate derivatives, the process of finding accurate results becomes much more complicated. See Damr Tewolde Berhan (2012) [2], Sébastien Gurrieri, Masaki Nakabayashi and Tony Wong [9], Hongyung Li (2007) [10], Elena Sacardovi [11], for an overview on some calibration processes.

The JY has eight parameters to be calibrated, $\{\kappa_n, \sigma_n, \sigma_I, \rho_{nr}, \rho_{nI}, \rho_{rI}, \kappa_r, \sigma_r\}$. One possibility is to calibrate all of them at once, using an inflation linked instrument, as the YYIIS or the inflation caps. Due to the lack of liquidity and the high nonlinearity and complexity of the formulas, this method will not lead to reliable results.

Therefore, we will proceed to the calibration process in several stages. In particular, we propose to do it using three, each of them using different instruments according with the parameters to be calibrated.

First, we use interest caps and swaptions to calibrate the daily nominal parameters κ_n, σ_n; second, floors of zero-coupon inflation swaps for the daily inflation volatility, σ_I, and third, year on year inflation-indexed swaps or inflation caps to calibrate the daily real parameters, κ_r, σ_r. The correlations

$\rho_{nr}, \rho_{nI}, \rho_{rI}$, instead, will be estimated using historical data.

All the instruments mentioned above and those that will be mentioned later have been download from Bloomberg. When historical data are needed we have considered the sample from 03/16/2012 to 03/28/2019.

Since the nominal and real zero-coupon curves are needed for calibration, we first obtain these curves, known as discount factors. Although obtaining a nominal discount curve is very usual, the real discount curve is uncommon.

Zero-coupon nominal curves

We obtain the zero-coupon nominal curve for each maturity of the swaps, through the Interest Rate Swaps (IRS) quotes. In the last years, it is common to use different maturities for the underlying interest rate of the swaps, as Euribor 3 months, 6 months or 12 months, obtaining different values depending on the selected underlying. Also, we have to decide which curve to use to discount the payments. In this case, we have taken the 6 months Euribor to discount the payments, and the same for the underlying interest rate of the swaps. The fixed leg of the swap is collected from Bloomberg, with ticker EUSA, and the maturities selected are, in years, 0.5, 1, 1.5, 2, 2.5, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 25 and 30 years. Historical descriptive information of the fixed leg is provided in Figure 2.1. Median, mean, 95% percentile and 5% percentile are shown throughout the entire sample. As expected, when the maturity increases the values increase, tending to be stable in the long run.

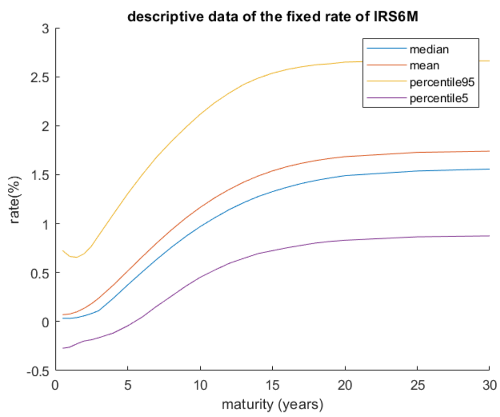


Figure 2.1: Fixed leg of 6M Eur swap from 03/16/2012 to 03/28/2019.

The value of an IRS in t is:

$$Swap(t) = N\Big[S(T_i)\sum_{j=1}^{n}\tau_{j,B}P_n(t,T_j) - \sum_{k=1}^{m}\tau_{k,A}\tilde{F}(t,\tilde{T}_{k-1},\tilde{T}_k)P_n(t,\tilde{T}_k)\Big], \tag{2.1}$$

where $S(T_i)$ is the market fixed rate of 6 months Euribor swap, N is the notional, $\tau_{j,B}$ the year fraction of the fixed-leg, $\tau_{k,A}$ the year fraction of the floating-leg and being $\tilde{F}(t,\tilde{T}_{k-1},\tilde{T}_k) = E_n^{\tilde{T}_k}(F(\tilde{T}_{k-1},\tilde{T}_{k-1},\tilde{T}_k)|F_t)$. Remembering that $E_n^{\tilde{T}_k}$ is the conditional expectation in t under the nominal $\tilde{T}_k$ forward measure. The standard market swap, the one we are using, has an annual payment of the fixed leg, while the floating leg pays semi annually, and that is why the subscripts of the summation are different. It is important to remark that $T_n = \tilde{T}_m = T_i$ for each maturity of the swaps.

Taking into account the year fraction of the fixed-leg and that the derivatives are market swaps, by definition the value in $t = 0$ has to be 0, that is:

$$S(T_i)\sum_{j=1}^{n}P_n(0,T_j) = \sum_{k=1}^{m}\tau_{k,A}\tilde{F}(0,\tilde{T}_{k-1},\tilde{T}_k)P_n(0,\tilde{T}_k), \tag{2.2}$$

In this case where the tenor of the swap and the underlying asset of the discount factor are the same, $F(t,\tilde{T}_1,\tilde{T}_2)$ is martingale under the T_2 forward measure,

$$\tilde{F}(t,\tilde{T}_1,\tilde{T}_2) = E_n^{\tilde{T}_2}(F(\tilde{T}_1,\tilde{T}_1,\tilde{T}_2)|F_t) = F(t,\tilde{T}_1,\tilde{T}_2), \tag{2.3}$$

fulfilling

$$\begin{aligned} P_n(t,\tilde{T}_2) &= P_n(t,\tilde{T}_1,\tilde{T}_2)P_n(t,\tilde{T}_1), \quad \text{where} \\ P_n(t,\tilde{T}_1,\tilde{T}_2) &= \frac{1}{1+\tau_{k,A}F(t,\tilde{T}_1,\tilde{T}_2)}. \end{aligned} \tag{2.4}$$

Solving $F(t,\tilde{T}_1,\tilde{T}_2)$ from the formula is obtained that

$$S(T_i)\sum_{j=1}^{n}P_n(0,T_j) = \sum_{k=1}^{m}\left[P_n(0,\tilde{T}_{k-1}) - P_n(0,\tilde{T}_k)\right] = 1 - P_n(0,T_i). \tag{2.5}$$

Solving the formula we get

$$P_n(0,T_i) = \frac{1 - S(T_i)\sum_{j=1}^{n-1}P_n(0,T_j)}{1+S(T_i)} \tag{2.6}$$

$$P_n(0,\tilde{T}_0) = P_n(0,0) := 1. \tag{2.7}$$

Looking at the formula, it is seen that it is necessary to obtain the nominal discount factor for the previous time to get the next one.

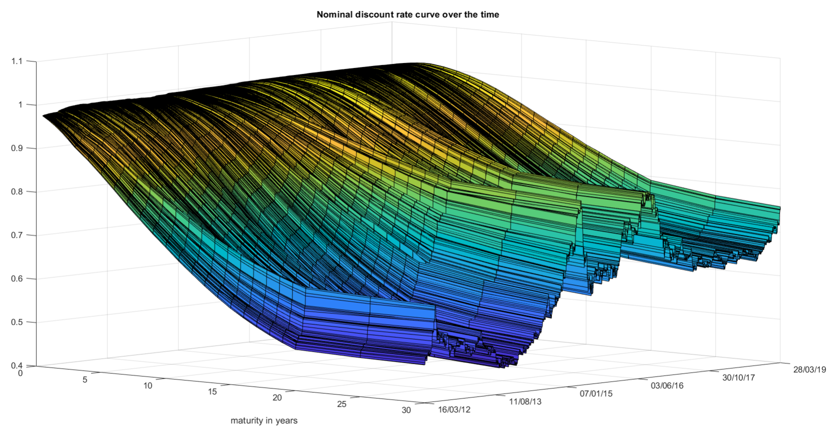


Figure 2.2: Zero coupon nominal discount factor.

The resulting zero-coupon nominal curves are shown in Figure 2.2, for different maturities and dates. It is appreciated that for long maturities the values are clearly time varying. Around the beginning of 2014 a sharp declining of the discount factors is observed, providing the lowest values, with the highest interest rates. At the end of 2016 a flattened pattern is shown. This changing behaviour reinforce the argument for a daily calibration of the model.

Zero-coupon real curves

Once the zero-coupon nominal curve is obtained, it is straightforward to obtain the zero-coupon real curve. The previous chapter shows the relationship between them,

$$P_r(0, T_i) = P_n(0, T_i)(1 + K(T_i))^{T_i}, \tag{2.8}$$

where $K(T_i)$ is the fixed-rate of the zero-coupon inflation-indexed swaps. The Bloomberg ticker of these values is EUSWI. If the real discount factors are needed for other maturities, there have to be used interpolation techniques. A briefly description of the data is shown in Figure 2.3.

As for the nominal curve, Figure 2.4 shows the real zero-coupon curve around different dates with the maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15. It is seen that at the beginning of the period, the real zero-coupon curve decreases when the maturity becomes longer. Nevertheless, at the beginning of the year 2015 the curve increases notoriously. In the last years the curve rises with the maturity, but not so quickly as in the year 2015.

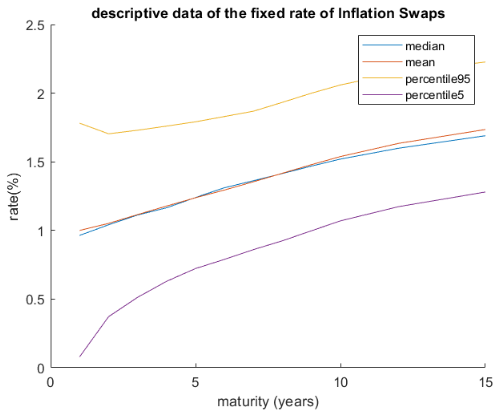


Figure 2.3: Fixed rate of ZC inflation swap from 03/16/2012 to 03/28/2019.

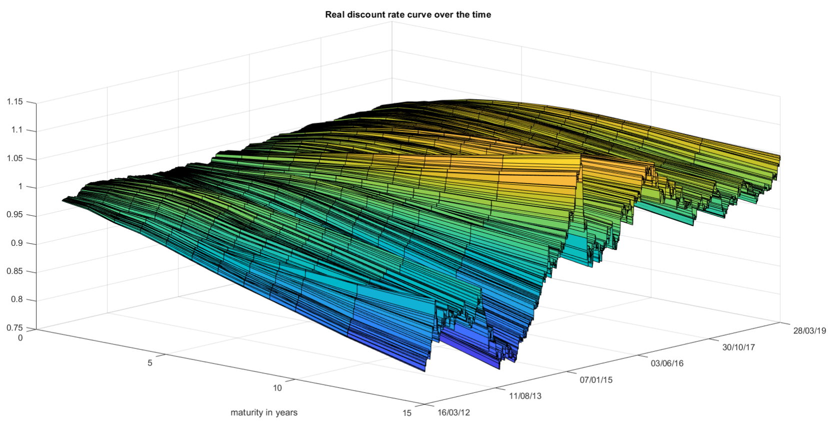


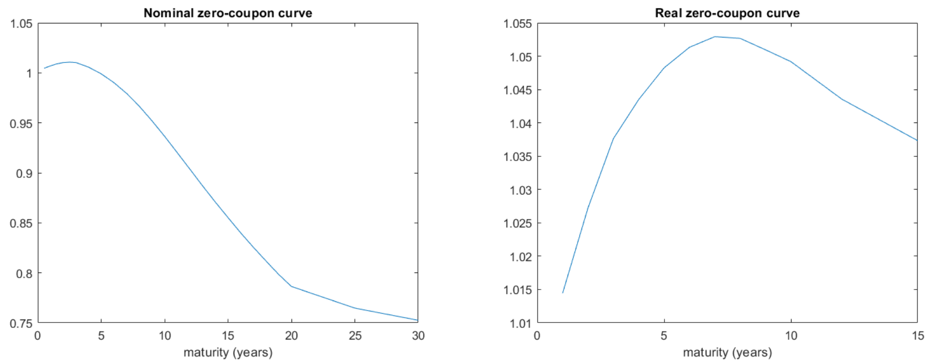
Figure 2.4: Zero coupon real discount factor.

Mat.	EUR 6M swap	ZC inf. swap	Mat.	EUR 6M swap	ZC inf. swap
0.5	-0.228	-	11	0.5397	-
1	-0.2307	0.74	12	0.6139	1.2138
1.5	-0.2228	-	13	0.6817	-
2	-0.204	0.8375	14	0.7401	-
2.5	-0.177	-	15	0.7902	1.3025
3	-0.1465	0.895	16	0.8353	-
4	-0.071	0.935	17	0.8729	-
5	0.011	0.9725	18	0.905	-
6	0.097	1.0088	19	0.934	-
7	0.1863	1.0462	20	0.9545	-
8	0.2779	1.0786	25	1.0165	-
9	0.3701	1.1113	30	1.0354	-
10	0.4569	1.1487	-	-	-

Table 2.1: 6M market IRS and ZCIIS price.

We will develop a daily calibration method and, therefore, the calculator will be able to do it for any selected day. As an illustration, we will focus on the date 03/28/2019. Table 2.1 summarizes the data linked to the fixed rate of EUR 6M swaps and zero-coupon inflation swaps, both in percentages, that have been used to obtain the nominal and real zero-coupon curves. As already mentioned, the last column of the table evidence the absence of market quotes for inflation derivatives in the long run.

Figures 2.5a and 2.5b show the zero-coupon nominal and real discount factors for our reference day 03/28/2019.



(a) Nominal curve (b) Real curve

Figure 2.5: Zero coupon nominal and real discount factors for 03-28-2019.

Using the real and nominal zero-coupon curves, is straightforward to obtain the spot rates

$$P_k(0,T_i) = \frac{1}{1+\tau_i k(T_i)} \Longrightarrow k(T_i) = \frac{1-P_n(0,T_i)}{\tau_i P_n(0,T_i)}, \quad k \in \{n, r\}. \tag{2.9}$$

The obtained nominal and real curves for the date 03/28/2019 are shown in Figure 2.6. Is seen that while the nominal curve is negative for short maturities but later becomes positive, the real one is negative for all the maturities.

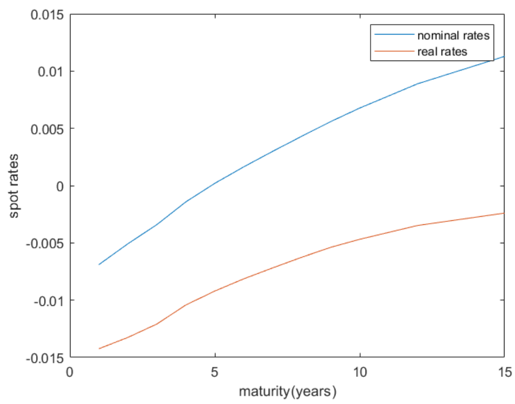


Figure 2.6: Nominal and Real rates for 03-28-2019.

2.1 Stage 1. Calibration of $\{\kappa_n, \sigma_n\}$

Recall the stochastic process of the instantaneous nominal rate,

$$dn(t) = (\nu_n(t) - \kappa_n n(t))dt + \sigma_n dW_n(t),$$

where,

$$\nu_n(t) = \frac{\partial f_n(0,t)}{\partial T} + \kappa_n f_n(0,t) + \frac{\sigma_n^2}{2\kappa_n}(1 - e^{-2\kappa_n t}),$$

The first option is to calibrate constant κ_n and σ_n simultaneously. The used instruments are Caps on Euribor 6 months. The data is taken from Bloomberg with the ticker EUCPAM and are available in Table 2.2. The prices of the caps are in units per 1, the strikes in percentages and the implied volatility in basic points.

Maturity	Market Quote	Strike	Volatility
3	0.0034	-0.1277	26.7
4	0.0066	-0.0475	32.19
5	0.0109	0.0374	36.35
6	0.0161	0.125	39.86
7	0.0223	0.2153	42.67
8	0.0292	0.3073	44.96
9	0.0367	0.3995	46.74
10	0.0447	0.4858	48.25
12	0.0617	0.6411	50.54
15	0.0889	0.9706	52.11

Table 2.2: 6M Interest Rate market Cap prices and strikes.

The calibration of both parameters simultaneously is not accurate enough, since the method is not able to obtain good estimations and the parameters are very dependant on the initial values. Therefore, we propose a method to estimate them separately. The methodology is explained in S.Gurrieri, M. Nakabayashi, T. Wong, Calibration Methods of Hull-White model, with reference [9]. Using implied volatilities of swaptions, it is possible to calculate an approximate constant κ_n and with the prices of interest rate caps (Table 2.2) calibrate σ_n.

2.1.1 Constant calibration of κ_n and σ_n

Calibration of κ_n

To obtain a constant κ_n are needed instruments that does not include σ_n in their theoretical expression. As it is proposed in [9], we use the ratio of implied volatilities of swaptions with the same maturity but different tenor. This method is only an approximation to calibrate κ_n. Once σ_n is calibrated using market interest rate caps, we recalibrate κ_n.

We summarize the steps to be followed to obtain the calibration of the mean reversion speed (κ_n), for further details of the procedure see Appendix C.1. To explain them, we denote the variance of the instantaneous short rate as $V_n(0,t) = V(n(t)) = (\sigma_n^2/2\kappa_n)e^{-2\kappa_n t})$, and $\beta_n(t,T) = (1/\kappa_n)(1 - e^{-\kappa_n(T-t)})$.

The procedure is as follows:

(i) Considering the dynamics of the nominal bond (1.38), obtain $d(P_n(t,T_1)/P_n(t,T_2))$ being $T_1 < T_2$,

$$\begin{aligned} d\frac{P_n(t,T_1)}{P_n(t,T_2)} &= \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n^2[\beta_n^2(t,T_2) - \beta_n(t,T_2)\beta_n(t,T_1)]dt \\ &+ \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n[\beta_n(t,T_2) - \beta_n(t,T_1)]dW_n^Q(t). \end{aligned} \tag{2.10}$$

(ii) Using Proposition A.1, obtain the drift under the forward measure

$$\mu^{T_2}(t) = \mu^Q(t) - \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n[\beta_n(t,T_2) - \beta_n(t,T_1)](-a_n(t,T_2)). \tag{2.11}$$

(iii) Obtain the dynamic of the bond ratio

$$d\frac{P_n(t,T_1)}{P_n(t,T_2)} = \frac{P_n(t,T_1)}{P_n^2(t,T_2)}\sigma_n[\beta_n(t,T_2) - \beta_n(t,T_1)]dW_n^{T_2}(t). \tag{2.12}$$

(iv) Obtain the integrated variance of the bond ratio

$$V_p(0,T_1,T_2) = \int_0^{T_1} \sigma_n^2[\beta_n(u,T_2) - \beta_n(u,T_1)]^2 du = V_n(0,T_1)\beta_n(T_1,T_2)^2. \tag{2.13}$$

(v) Consider

$$\tilde{S}(t,T_0,T_n) = \frac{P_n(0,T_n)}{\sum_{i=1}^n \tau_{i,B} P_n(0,T_i)}\Big[\frac{P_n(t,T_0)}{P_n(t,T_n)} - 1\Big], \tag{2.14}$$

as the approximation of the market IRS fixed rate

$$S(t,T_0,T_n) = \frac{P_n(t,T_0) - P(t,T_n)}{\sum_{i=1}^{n} \tau_{i,B} P_n(t,T_i)}, \tag{2.15}$$

where T_0 is the initial reference day and T_n the tenor of the swap. Assume that the approximated swap fixed rate has log-normal distribution under the annuity measure A (Proposition A.2.1).

(vi) Consider the dynamics of the approximated swap fixed rate under the forward measure

$$\frac{d\tilde{S}(t,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} = \frac{S(0,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} \frac{P_n(0,T_n)P_n(t,T_0)}{[P(0,T_0) - P_n(0,T_n)]P_n(t,T_n)} \cdot \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{T_n}(t). \tag{2.16}$$

(vii) Substitute in (2.16), the approximated swap fixed rate and the bond prices by their initial values and change to the annuity measure A,

$$\frac{d\tilde{S}(t,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} \simeq drift + \frac{P_n(0,T_0)}{P(0,T_0) - P_n(0,T_n)} \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{A}(t). \tag{2.17}$$

(viii) Obtain the integrated volatility of the previous expression using (2.13), as

$$IV_{swap}(T_0,T_n) = \left| \frac{P_n(0,T_0)}{P(0,T_0) - P_n(0,T_n)} \right| \sqrt{V_p(0,T_0,T_n)}, \tag{2.18}$$

where $V_p(0,T_0,T_n) = V_n(0,T_0)\beta_n(T_0,T_n)^2$.

(ix) Since the only term depending on σ_n is $V_n(0,T_0)$, take the ratio of different implied volatilities with the same maturity (M_i) for the swaptions, but different tenors (T_j and T_k) to estimate κ_n. That is,

$$\frac{IV_{swap}(M_i,T_j)}{IV_{swap}(M_i,T_k)} = \left| \frac{[P_n(0,M_i) - P_n(0,T_k)]\beta(M_i,T_j)}{[P_n(0,M_i) - P_n(0,T_j)]\beta(M_i,T_k)} \right|. \tag{2.19}$$

To do the calibration, we denote $IV_{i,j}$ as the market implied volatility of a swaption with maturity M_i and tenor T_j and minimize the sum of the different ratios for all the tenors and maturities that are available.

$$min_{\{\kappa_n\}} \qquad \sum_{i=1}^{n_m} \sum_{j=1}^{n_l - 1} \frac{w_{i,j+1}}{w_{i,j}} \left(\frac{IV_{swap}(M_i,T_{j+1})}{IV_{swap}(M_i,T_j)} - \frac{IV_{i,j+1}}{IV_{i,j}} \right)^2, \tag{2.20}$$

where $w_{i,j}$ are the weights given to the variances, n_m the number of maturities and n_l the number of tenors.

There is a problem in using this method. Since the distribution of (2.14) under the annuity measure is log-normal, the implied volatilities have as well log-normal distribution. Because of this, there are only available market implied volatilities for those swaptions where the underlying interest rate is positive, what it is a problem in the current situation with negative interest rates. As it is being implemented an automatic calculator to calibrate every day, it is necessary to chose a group of maturities and tenors that are going to have always data for the implied volatility available. The chosen maturities are 3, 4, 5, 7, 10, 15 and the tenors for the swap 7, 8, 9, 12, 15.

	Tenors				
Mat.	7	8	9	10	15
3	64.16	57.55	52.68	49.04	39.35
4	54.13	50.45	47.45	45.18	38.08
5	47.7	45.49	43.67	42.25	36.89
7	41.16	40.27	39.5	38.88	35.98
10	37.91	37.84	37.87	37.91	36.97
15	39.8	40.39	40.99	41.55	41.22

Table 2.3: EUR 6M linked market Swaptions implied volatility

The ticker for the swaptions implied volatility is EUSV and the values are given in percentages. The market values are shown in Table 2.3. To calibrate the parameter κ_n we have used the command **fminsearch** in Matlab, using different initial points for the calibration, the resultant κ_n that gives the minimum objective function for the date 03/28/2019 is: $\kappa_n = 0.1077$.

Calibration of σ_n

The first thing to note is that Bloomberg prices of caps are calculated with Black's formula, and this assumes log-normality for the interest rate EUR 6M. Some of the obtained strikes, normally for short maturities, are negative, what is not compatible with the Black model. These prices could be consistent with a Black shifted model, that consists on adding a constant value to the interest rates and strikes to become them positive (for further information see [12]). This assumption is widely used between market practitioners.

The calibration of σ_n is made using market cap prices and theoretical cap formulas when the short rate is modelled with Hull and White. Under the Hull and White model the short rate is normally distributed, then, to obtain closed formulas it is used the price of a bond, since this is log-normally distributed. The expressions for the interest rate caps are from the book *Interest Rate Models-Theory and Practice*, with reference [5].

It is important to remark that the caps can be seen as a portfolio of European zero-coupon calls on EUR 6M. We denote by $D = \{d_1, d_2, ..., d_n\}$ the payment dates of the cap, $\tau = \{\tau_1, ..., \tau_n\}$ the year-fraction between d_{i-1} and d_i and $T = \{t_1, t_2, ..., t_n\}$ the difference between d_i and the valuation day t with t_0 as reference day (in this case $t_0 = 0$). N is the nominal value of the contract and X is the strike. The cap is composed of n caplets and the value of the i-th caplet in t is

$$Cpl(t, t_{i-1}, t_i, \tau_i, N, X) = N_i' ZBP(t, t_{i-1}, t_i, X_i'), \tag{2.21}$$

where,

$$\begin{aligned} N_i' &= \frac{1}{1 + X\tau_i}, \\ N_i' &= N(1 + X\tau_i), \end{aligned} \tag{2.22}$$

being the formula of the European put option with maturity T and strike X on a unit-principal zero-coupon bond with maturity $S > T$ (ZBP),

$$ZBP(t, T, S, X) = XP_n(t, T)\Phi(-h + \sigma_p) - P_n(t, S)\Phi(-h), \tag{2.23}$$

where

$$\begin{aligned} \sigma_p &= \sigma_n \sqrt{\frac{1 - e^{-2\kappa_n(T-t)}}{2\kappa_n}} \beta_n(T, S), \\ h &= \frac{1}{\sigma_p} ln \frac{P_n(t, S)}{P_n(t, T)X} + \frac{\sigma_p}{2}. \end{aligned} \tag{2.24}$$

Using the previous information, the valuation formula for the cap in t is

$$\begin{aligned} Cap(t, T, N, X) &= N \sum_{i=1}^{n} (1 + X\tau_i) ZBP(t, t_{i-1}, t_i, \frac{1}{1 + X\tau_i}), \\ &= N \sum_{i=1}^{n} [P_n(t, t_{i-1})\Phi(-h_i + \sigma_p^i) - (1 + X\tau_i)P_n(t, t_i)\Phi(-h_i)], \end{aligned} \tag{2.25}$$

where,

$$\begin{aligned} \sigma_p^i &= \sigma_n \sqrt{\frac{1 - e^{-2\kappa_n (t_{i-1} - t)}}{2\kappa_n}} \beta_n(t_{i-1}, t_i), \\ h_i &= \frac{1}{\sigma_p^i} ln \frac{P_n(t, t_i)(1 + X\tau_i)}{P_n(t, t_{i-1})} + \frac{\sigma_p^i}{2}. \end{aligned} \tag{2.26}$$

For further details see Appendix C.2.

The calibration of constant σ_n is made using market prices of 6M interest rate caps and the expressions just obtained. The used Matlab command is **lsqnonlin** and as we have done for κ_n we have used different initial points and taken as σ_n the one that has given the minimum objective function (the minimum error). The optimum σ_n is 0.0038.

The method proposed by [9] to calibrate κ_n is just an approximation. Once σ_n is calibrated, κ_n is recalibrated using market interest rate caps obtaining $\kappa_n = 0.078$.
In order to check the accuracy of the calibration, we have iterated the calibration procedure, obtaining as final κ_n and σ_n the ones that appear in Table 2.4.

κ_n	σ_n
0.078	0.00356

Table 2.4: Calibrated constant κ_n and σ_n.

Table 2.5 shows the market and estimated quotes, together with the calibration error.

The most accurate prices are those for short and long maturities, except for the last one. The worst approximated are for the medium ones.

Since the purpose is calibration of the parameters to value inflation linked instrument in future dates, it is important to improve the accuracy. It seems clear that the prices obtained with constant σ_n are not good, so we propose a piece-wise approach for σ_n. The implied volatility is a key parameter to give value to the different options as a measure of the uncertainty involved in the market, so it makes sense to maintain κ_n constant and use a σ_n piece-wise, trying to collect this uncertainty.

Maturity	Market Quote	Model Quote	Error
3	0.0034	0.00338	0.000016
4	0.0066	0.00838	-0.00179
5	0.0109	0.0139	-0.003
6	0.0161	0.02	-0.0039
7	0.0223	0.0265	-0.0042
8	0.0292	0.0334	-0.0042
9	0.0367	0.0406	-0.0039
10	0.0447	0.0478	-0.0031
12	0.0617	0.0615	0.00016
15	0.0889	0.079	0.0099

Table 2.5: 6M Interest Rate Caps price.

2.1.2 Piece-wise calibration of σ_n

To obtain a piece-wise $\sigma_n(t)$, we use today's prices of the caps with maturities $3, 4, 5, 6, 7, 8, 9, 10, 12, 15$ shown in Table 2.2.
As in Chapter 3 we will proceed to the valuation of instruments in future dates, the expression of the nominal instantaneous short rate in those dates is needed. For this, is necessary an expression of σ_n in that dates. This does not mean that is a time-dependent function changing our calibrating date (03/28/2019), but using today's data for different maturities, is given the desired temporality to σ_n. As there are obtained more than one value for σ_n, we expect to collect better market data since we give more flexibility to this parameter.

Using the method of bootstrapping, an expression of $\sigma_n(t)$ is obtained recursively. With the price of each cap is derived a value, obtaining a piece-wise $\sigma_n(t)$ with intervals $0-3-4-5-6-7-8-9-10-12-15$.

The procedure goes as follows.

(i) Using the theoretical formulas of cap prices and today's data for the 3 years cap, calibrate a constant value of $\sigma_n(t)$ for the period [0,3].

(ii) Using the value in (i) and the strike of the cap with maturity 4 years, compute the value of the caplets up to 3 years of this second cap (maturity 4 years).

(iii) Using the values at (ii), calibrate the value of $\sigma_n(t)$ for the period (3,4].

(iv) Using the values at (i) and (iii), obtain the values of the caplets of the next cap (maturity 5 years) until 4 years, using the first σ_n for the caplets of the period [0,3] and the second value for the caplets of the period (3,4]. Using the market price of the cap with maturity 5 years, calibrate the expression for (4,5].

(v) Obtain the entire expression recursively.

In each step of the process there is one equation with one parameter to be calibrated, so the produced errors are due to the complexity and non-linearity of the formulas. The resultant values are shown in Figure 2.7. It is appreciated that the implied volatility of the nominal short rate does not follow any specific dynamics. The new prices and the error are shown in Table 2.6.

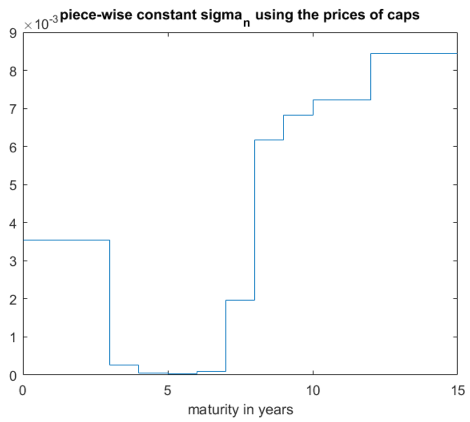


Figure 2.7: Piece-wise σ_n in the date 03/28/2019.

We realize that the values for σ_n in the period $(3, 7]$ are very small, then, when we make different simulations of the nominal instantaneous short rate along the time, there will not be significant difference between them. To avoid this, we decide to take the values for the periods $[0, 3]$ and $(8, 15]$ and obtain new values for the leftover periods using cubic interpolation. The new σ_n is shown in Figure 2.8 and the numeric values and cap prices in Table 2.7.

Maturity	$\sigma_n(t)$	Market Quote	Model Quote	Error
3	0.0036	0.0034	0.0034	$-1.07 \cdot 10^-8$
4	$1.4607 \cdot 10^-4$	0.0066	0.0078	-0.0012
5	$1.9113 \cdot 10^-5$	0.0109	0.0127	-0.0018
6	$6.2027 \cdot 10^-6$	0.0161	0.018	-0.0019
7	$2.1306 \cdot 10^-5$	0.0223	0.0236	-0.0012
8	0.0012	0.0292	0.0294	-$2.3 \cdot 10^-4$
9	0.0062	0.0367	0.0367	$-6.43 \cdot 10^-8$
10	0.0071	0.0447	0.0447	$-2.91 \cdot 10^-8$
12	0.0074	0.0617	0.0617	$-4.49 \cdot 10^-9$
15	0.0087	0.0889	0.0889	$-5.168 \cdot 10^-7$

Table 2.6: 6M Interest Rate Caps price.

With these new σ_n the errors are bigger than with the previous piece-wise function but smaller than with the constant value, except for the penultimate cap price. We have decided to use this new σ_n from now on to give more uncertainty to the dynamics of $n(t)$.

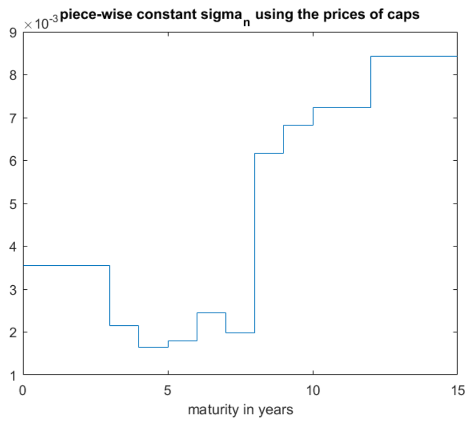


Figure 2.8: Transformed piece-wise σ_n in the date 03/28/2019.

Maturity	$\sigma_n(t)$	Market Quote	Model Quote	Error
3	0.0036	0.0034	0.0034	$-1.07 \cdot 10^{-}8$
4	0.0017	0.0066	0.0079	-0.0013
5	0.001	0.0109	0.0128	-0.0019
6	0.0011	0.0161	0.0181	-0.002
7	0.0018	0.0223	0.0238	-0.0015
8	0.0012	0.0292	0.0299	-0.0007
9	0.0062	0.0367	0.0376	-0.00089
10	0.0071	0.0447	0.0459	-0.0012
12	0.0074	0.0617	0.063	-0.0013
15	0.0087	0.0889	0.0901	-0.0013

Table 2.7: 6M Interest Rate Caps new volatility and price.

2.2 Stage 2. Calibration of σ_I

The dynamics of the inflation-index HCIP is,

$$\frac{dI(t)}{I(t)} = [n(t) - r(t)]dt + \sigma_I dW_I^Q(t).$$

Our first option to do the calibration is to use floors of zero-coupon inflation indexed swaps to calibrate a constant σ_I and if we do not get good results, a piece-wise function as it has been done with σ_n.
As mentioned before, some days is not possible to obtain market data of the inflation instruments to do the market calibration. For those cases, we propose two methods to calibrate σ_I:

- Use the data of the index to obtain a constant historical value for σ_I.
- Use an e-garch model to obtain a time-dependent expression for σ_I.

Using floors

Fortunately, data for the date 03/28/2019 is available in Bloomberg for the floors of zero coupon IL swaps. The maturities are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15$ and all of them have the strike equal to 0, the ticker is EUIZF0. The used data is in Table 2.8, the values are in basic points.

Maturity	Market Quote
1	5.6
2	6.1
3	5.8
4	6.4
5	6.6
6	7.4
7	8.7
8	9.7
9	12.6
10	13.6
12	13.2
15	13.3

Table 2.8: Market zero-coupon inflation indexed swaps floors.

Taking into account that the pay-off in T_i of the annual zero-coupon inflation swap is

$$N\omega\left[\Big(\frac{I(T_i)}{I_0}-1\Big)-\Big((1+K(i))^{T_i}-1\Big)\right], \tag{2.27}$$

where $\omega = 1$ if you pay the fixed leg and $\omega = -1$ if you pay the variable leg. Assuming that $N = 1$, the actual value off each pay-off of the floor is

$$E\left[e^{-\int_t^{T_i} n(s)ds}\left[\omega\Big(\frac{I(T_i)}{I_0}-1\Big)-\omega\Big((1+K)^{T_i}-1\Big)\right]^+ |F_t\right]. \tag{2.28}$$

Introducing the definition of the forward HCIP at time t to time T as

$$F_I(t,T)=\frac{I(t)P_r(t,T)}{P_n(t,T)},$$

it is easily seen that $F_I(t,T)$ is log-normally distributed, since the inflation index and the nominal and real bonds are log-normally distributed as well. Using Blacks formula the price of each floor in t is

$$P_n(t,T_i)\left[(1+K(i))^{T_i}N(-d_2)-\frac{F_I(t,T)}{I_0}N(-d_1)\right], \tag{2.29}$$

where

$$d_1^i = \frac{ln\left(\frac{F_I(t,T)}{I_0(1+K(i))^{T_i}}\right) + \frac{\sigma_I^2 T_i}{2}}{\sigma_I\sqrt{T_i}}, \tag{2.30}$$

$$d_2^i = d_1^i - \sigma_I\sqrt{T_i}. \tag{2.31}$$

To calibrate σ_I the formula in $t = 0$ is needed,

$$P_n(0,T_i)\left[(1+K(i))^{T_i}N(-d_2) - \frac{P_r(0,T_i)}{P_n(0,T_i)}N(-d_1)\right], \tag{2.32}$$

where

$$d_1^i = \frac{ln\left(\frac{P_r(0,T_i)}{P_n(0,T_i)(1+K(i))^{T_i}}\right) + \frac{\sigma_I^2 T_i}{2}}{\sigma_I\sqrt{T_i}}, \tag{2.33}$$

$$d_2^i = d_1^i - \sigma_I\sqrt{T_i}. \tag{2.34}$$

Trying with different initial points the calibrated parameter is $\sigma_I = 0.0073$. The accuracy of the calibration is not appropriate, since the difference between the market data and the estimated prices is very big. As explained before, calibration is very important because these are the parameters used for actual and future valuation of the derivatives. Then, we have decided to calculate a piece-wise function in a similar way as described in the case of nominal σ_n. Remark that there are available floor prices for 1 and 2 years so the intervals will be 0-1-2-3-4-5-6-7-8-9-10-12-15 (for the interest rate caps the first maturity available was 3 years, so the first interval was [0,3]). The resultant function is shown in Figure 2.9.

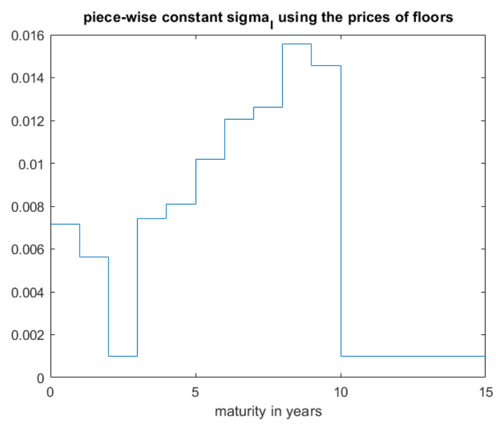


Figure 2.9: Piece-wise σ_I in the date 03-28-2019.

Table 2.9 shows the prices of market floors for different maturities for the date 03/28/2019, the obtained values with the formula (2.32) with constant and piece-wise σ_I and the corresponding error. For all the floors it is seen that the error is smaller when using the piece wise σ_I, so whenever possible we will use this expression instead of the constant one.

Maturity	Market Quote	Quotes ctant σ_I	Error	Quotes p-w σ_I	Error
1	$5.6 \cdot 10^{-}4$	$6.0096 \cdot 10^{-}4$	$4.0956 \cdot 10^{-}5$	$5.6005 \cdot 10^{-}4$	$5.2099 \cdot 10^{-}8$
2	$6.1 \cdot 10^{-}4$	$8.4143 \cdot 10^{-}4$	$2.3143 \cdot 10^{-}4$	$6.1462 \cdot 10^{-}4$	$4.6193 \cdot 10^{-}6$
3	$5.8 \cdot 10^{-}4$	$9.2509 \cdot 10^{-}4$	$3.4509 \cdot 10^{-}4$	$6.1462 \cdot 10^{-}4$	$3.4619 \cdot 10^{-}5$
4	$6.4 \cdot 10^{-}4$	$9.5355 \cdot 10^{-}4$	$3.1355 \cdot 10^{-}4$	$6.471 \cdot 10^{-}4$	$7.0996 \cdot 10^{-}6$
5	$6.6 \cdot 10^{-}4$	$9.615 \cdot 10^{-}4$	$3.015 \cdot 10^{-}4$	$6.6994 \cdot 10^{-}4$	$9.9393 \cdot 10^{-}6$
6	$7.4 \cdot 10^{-}4$	$9.6352 \cdot 10^{-}4$	$2.2352 \cdot 10^{-}4$	$7.4022 \cdot 10^{-}4$	$2.1838 \cdot 10^{-}7$
7	$8.7 \cdot 10^{-}4$	$9.6397 \cdot 10^{-}4$	$9.3967 \cdot 10^{-}5$	$8.7126 \cdot 10^{-}4$	$1.2617 \cdot 10^{-}6$
8	$9.7 \cdot 10^{-}4$	$9.6406 \cdot 10^{-}4$	$-5.944 \cdot 10^{-}6$	$9.717 \cdot 10^{-}4$	$1.703 \cdot 10^{-}6$
9	0.0013	$9.6407 \cdot 10^{-}4$	$-2.9593 \cdot 10^{-}4$	0.0013	$2.0523 \cdot 10^{-}7$
10	0.0014	$9.6407 \cdot 10^{-}4$	$-3.9593 \cdot 10^{-}4$	0.0014	$1.815 \cdot 10^{-}6$
12	0.0013	$9.6407 \cdot 10^{-}4$	$-3.5593 \cdot 10^{-}4$	0.0014	$4.1815 \cdot 10^{-}5$
15	0.0013	$9.6407 \cdot 10^{-}4$	$-3.6593 \cdot 10^{-}4$	0.0014	$3.1815 \cdot 10^{-}5$

Table 2.9: Inflation indexed swaps floors price.

In case of not available market data, as an alternative we shortly explain two methods that can be used to calibrate σ_I.

One could compute the historical volatility of the Eurostat Eurozone HICP (Harmonised Indices of Consumer Prices) Ex Tobacco (ticker CPTFEMU) with monthly frequency and a lag of 3 months from the data series available on Bloomberg. The value is given the last working day of the month and as our reference date is 03/28/2019, the March data is not available. The used data is from 03/31/2012 to 02/28/2019.

Taking into account the dynamics of the inflation index (1.37), is obtained that

$$V_t\Big(\frac{dI(t)}{I(t)}\Big) = \sigma_I^2 dt.$$

As the frequency of the data is monthly, to get the historical estimation of

σ_I is used the formula

$$\sigma_I = \left[\frac{1}{\Delta} V\Big(\frac{\Delta I(t)}{I(t)}\Big)\right]^{\frac{1}{2}}, \quad \text{where} \quad \Delta = \frac{1}{12}.$$

The resultant parameter is $\sigma_I = 0.0016$, much more smaller than the constant and piece-wise calibrated values using floors of zero-coupon inflation swaps, showing that market valuations of the derivatives also includes higher degree of uncertainty for the future.

Another way of modelling the volatility of the index is through a time series model.
Between the garch family models, we have decided to use an egarch(1,1) (for more details of the egarch model see [13]) model because its flexibility to capture and model asymmetric effects in the volatility, that is, if the effect of positive and negative yields is the same or not. The parameter that measures that effect is γ; since negative yields use to have a bigger impact, we expect γ to be negative.

The estimated parameters and their corresponding significance are shown in Table 2.10. The only parameters that are significant with a confidence level of 95% are the last two ones, since it is necessary to be the second parameter bigger than 1.96 in absolute terms (normal distribution).

Even if the parameter α_1 is not significant, since γ it is, we maintain α_1 in the equation. With the obtained expression, is possible to get an estimation of the future implied volatility of the inflation index using this formula recursively.

Parameters	Estimation	Significance
μ	0.00057	0.94
α_0	-1.799	-0.701
α_1	$5.78 \cdot 10^{-}6$	0.057
β	0.827	3.359
γ	-0.39	-6.02

Table 2.10: Egarch(1,1) parameters for the inflation index HCIP.

2.3 Stage 3. Calibration of $\{\kappa_r, \sigma_r\}$

Once the nominal and inflation parameters are calibrated, is time to obtain the real ones, κ_r and σ_r. For this, we have tried with two different type of instruments, and chosen the ones that approach better the corresponding market quotes.

- The first instruments are inflation indexed caps. The prices are from an inflation linked cap calculator (in Bloomberg). Remark that all the chosen caps are at the money. Both, the prices and strikes, are in percentages and the values are in the first and second columns of Table (2.11).

Maturity	Caps Market Quote	Caps Strike	YYIIS MQ
1	0.2277	0.941	0.74
2	0.6061	0.985	0.83692
3	1.0238	1	0.89403
4	1.4976	1.015	0.93372
5	2.0319	1.035	0.97059
6	2.5957	1.064	1.0061
7	3.1954	1.096	1.0428
8	3.7473	1.13	1.0743
9	4.3157	1.165	1.1062
10	4.9058	1.197	1.1421
12	6.0373	1.2529	1.204
15	7.8172	1.3217	1.2875

Table 2.11: market inflation indexed ATM Caps and YYIIS.

- The second instruments are year on year inflation-indexed swaps and the market fixed rates are available in the last column of Table 2.11, in percentages. The prices have been taken from Bloomberg.

In the formulas obtained in the previous chapter for the YYIIS and inflation caps also appear the correlations between nominal and real spot rates ρ_{nr}, and the correlation of both of them with the Inflation index CPI (Consumer Price Index), ρ_{nI} and ρ_{rI}. We have decided to obtain the estimation of all of them historically, to not stress more the model.

2.3.1 Correlations $\{\rho_{nr}, \rho_{nI}, \rho_{rI}\}$

We have decided to calculate all the correlations as piece-wise functions, to take into account the data available for different maturities in the case of the nominal and real zero-coupon curves. The data range is from 03/16/2012 to 03/28/2019. Using the values for the 12 maturities in common for the nominal and real discount factors 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, the implied discount factors for each date of the range are calculated. The piece-wise function for ρ_{nr} is easily obtained calculating the correlation factor between the arithmetic yields of those implied discount factors. The values are shown in the second column of the Table 2.12. It is easily seen that the correlation is very high, and gets bigger as the maturity increase, staying around 0.91 in the long run.

To obtain ρ_{nI} and ρ_{rI}, the data related to the inflation index used to calculate the historical estimation of σ_I is needed. Since the frequency of the HCIP is monthly, is necessary to obtain the nominal and real implied discount factors with monthly periodicity. Later, calculate the correlation between the arithmetic yields of each implied value and the arithmetic yields of the HICP obtaining a piece wise function. The results are shown in the last two columns of Table 2.12.

Period	ρ_{nr}	ρ_{nI}	ρ_{rI}
[0,1]	0.457	-0.05	0.175
(1,2]	0.887	-0.01	-0.031
(2,3]	0.928	-0.058	-0.055
(3,4]	0.932	-0.018	0.002
(4,5]	0.932	-0.098	-0.048
(5,6]	0.899	-0.1	-0.069
(6,7]	0.921	-0.093	-0.062
(7,8]	0.904	-0.137	-0.081
(8,9]	0.912	-0.119	-0.068
(9,10]	0.9	-0.186	-0.16
(10,12]	0.914	-0.19	-0.155
(12,15]	0.929	-0.2	-0.187

Table 2.12: Historical correlations.

2.3.2 Calibration with Caps

Using the market quotes of Table 2.11 and the formulas developed in the second chapter the parameters κ_r and σ_r are calibrated. For each caplet is used the valuation formula (1.49) in $t = 0$ and with the notional $N = 1$. Supposing that there is a caplet each $\{T_1, ..., T_M\}$, the theoretical formula of a cap is

$$\sum_{i=1}^{M} \tau_i P_n(0,T_i) \left[\frac{P_n(0,T_{i-1})}{P_n(0,T_i)} \frac{P_r(0,T_i)}{P_r(0,T_{i-1})} e^{b(0,T_{i-1},T_i)} \Phi(d_1^i(0)) - (1+k)\Phi(d_2^i(0)) \right], \tag{2.35}$$

where

$$d_1^i(0) = \frac{ln \frac{P_n(0,T_{i-1})}{(1+\kappa)P_n(0,T_i)} \frac{P_r(0,T_i)}{P_r(0,T_{i-1})} + b(0,T_{i-1},T_i) + \frac{1}{2}V^2(0,T_{i-1},T_i)}{V(0,T_{i-1},T_i)} \tag{2.36}$$

$$d_2^i(0) = d_1^i(0) - V(0,T_{i-1},T_i), \tag{2.37}$$

being

$$b(0,T_{i-1},T_i) = \sigma_r \beta_r(T_{i-1},T_i) \Bigg[\beta_r(0,T_{i-1}) \Bigg(\rho_{rI}\sigma_I - \frac{1}{2}\sigma_r\beta_r(0,T_{i-1}) \tag{2.38}$$

$$+ \frac{\rho_{nr}\sigma_n}{\kappa_n + \kappa_r}(1 + \kappa_r\beta_n(0,T_{i-1})) \Bigg) - \frac{\rho_{nr}\sigma_n}{\kappa_n + \kappa_r}\beta_n(0,T_{i-1}) \Bigg], \tag{2.39}$$

and $V(0,T_{i-1},T_i)$ the variance of the logarithm of the ratio $\frac{I(T_i)}{I(T_{i-1})}$ defined in (1.52).

κ_r	σ_r
0.9202	0.0036

Table 2.13: Calibrated κ_r and σ_r.

The obtained estimated parameters are in Table 2.13. Trying with different initial points is seen that the calibration is unstable since depending on the initial point the optimum parameters are different. In addition, the errors between the market quotes and the estimated cap prices are very big. The problem may be that the used formulas are very complex. Another option could be the little precision of Bloomberg prices due to lack of liquidity. To avoid both problems, we have decided to calibrate with year on year inflation-indexed swaps, since the formulas are easier and the liquidity is bigger.

2.3.3 Calibration with YYIIS

For the calibration with YYIIS, is minimized the difference between the formula (1.48) for the floating leg and the expression $\sum_{j=1}^{i} P_n(0,T_j)K(T_i)$ for the fixed leg. The values of $K(T_i)$ are in Table 2.11 in percentages. We have tried to calibrate both parameters at the same time using different initial points, but the estimation is very unstable. Trying to obtain a stable calibration, we have thought to use historical data of the real bonds to calibrate the parameters κ_r, σ_r historically, and use them to calibrate separately the definitive parameters.

The first step is to calculate the variance of the dynamic (1.39) of the real bond

$$\begin{aligned}&\frac{dP_r(t,T)}{P_r(t,T)} = [r(t) - \sigma_I \rho_{rI} a_r(t,T))]dt + a_r(t,T)dW_r^Q(t)\\ \Longrightarrow\ &V(\frac{dP_r(t,T)}{P_r(t,T)}) = a_r^2(t,T)dt = \frac{\sigma_r^2}{\kappa_r^2}(1 - e^{-\kappa_r(T-t)})^2 dt.\end{aligned} \tag{2.40}$$

Setting $dt = (1/365)$ since the periodicity of the discount factors is daily, is possible to calibrate the parameters κ_r and σ_r. For each maturity of the real bond 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, the left part of the equation is obtained as the historical variance of the arithmetic yields of the real discount factors. Calculating the difference between this value and the right part, is obtained the vector to be minimized to calibrate historically the parameters. The estimated values are shown in Table 2.14.

κ_r	σ_r
0.0012	0.0089

Table 2.14: Historically calibrated κ_r and σ_r.

These historical values are going to be used to calibrate separately κ_r and σ_r using market prices of year on year inflation-indexed swaps. Assuming $\sigma_r = 0.0089$, can be calibrated κ_r using market quotes of the YYIIS, obtaining $\kappa_r = 0.766$. Using this value for κ_r, σ_r is recalibrated using also the market prices of the YYIIS getting $\sigma_r = 0.0091$. To check the accuracy of the calibration, we recalibrate both parameters obtaining the definite ones shown in table 2.15. The obtained values for the fixed and floating legs and the errors are shown in Table 2.16. It is important to remark that the error is the difference between the value of the floating leg and the fixed leg, since in this case is minimized the difference between them because being market swaps, the value in $t = 0$ has to be equal to 0.

κ_r	σ_r
0.79	0.0093

Table 2.15: Calibrated κ_r and σ_r using YYIIS.

Maturity	Floating leg	Fixed leg	Error
1	0.0075	0.0075	$1.8301 \cdot 10^{-16}$
2	0.0169	0.0169	$6.825 \cdot 10^{-6}$
3	0.0271	0.0271	$1.223 \cdot 10^{-5}$
4	0.0377	0.0377	$2.7 \cdot 10^{-6}$
5	0.0488	0.0488	$-4.919 \cdot 10^{-6}$
6	0.0606	0.0606	$-7.487 \cdot 10^{-}$
7	0.073	0.073	$-2.139 \cdot 10^{-5}$
8	0.0856	0.0856	$-3.793 \cdot 10^{-5}$
9	0.099	0.099	$-3.545 \cdot 10^{-5}$
10	0.113	0.113	$-3.106 \cdot 10^{-5}$
12	0.141	0.141	$2.5434 \cdot 10^{-5}$
15	0.184	0.184	$7.0383 \cdot 10^{-5}$

Table 2.16: Calibration of YYIIS.

As it can be seen in Table 2.16, with the constant calibrated parameters the errors are more or less small, so we have decided not to calibrate a piece-wise σ_r. If someone wanted to calculate the piece-wise function, the technique is the one used previously. It has the particularity that if there is only one payment, the expression is $P_r(0, T_1) - P_n(0, T_1)$, that does not include the parameter σ_r, so the first value of the piece-wise function would be for the period $[0, 2]$. We have done this analysis to see what is the result and if there is a big difference in the precision comparing with the constant value. The sum of the squared errors is very similar so we have decided to use the constant σ_r, since in this way we do not stress the model much and the calibration is still good.

In summary, it has been shown that calibration is not trivial. In Stage 1, calibration of $\{\kappa_n, \sigma_n\}$, the first option has been calibration of both parameters together using interest rate caps, as proposed in [1],[2] and [11]. As the results were not good, we have decided to calibrate κ_n as proposed in [9]. To calibrate σ_n, instead of a constant value, we have decided to obtain a piece-wise value, greatly improving the accuracy of the calibration. To calibrate the rest of parameters, in [1],[2] and [11] is proposed to use inflation-indexed cap/floors, but we have decided to calibrate them separately. To obtain values for the correlations, we have done historically as proposed in [10]. To calibrate κ_r and σ_r, we have obtained a first approach historically as proposed in [10], and obtained the definitive parameters using YYIIS.

Chapter 3

Valuation

In Chapter 3 the main objective of this work is attained, the valuation of different inflation indexed swaps, the expected positive exposure and the expected shortfall with a confidence level of 97.5%. Also, sensitivities of those values with respect to the model parameters are given. Finally, the CVA of each swap separately and the CVA of the whole portfolio are calculated.
In Section 3.1 the different type of inflation swaps are presented, together with the method to implement the simulation and the valuation procedure.
In Section 3.2 the obtained valuations for the different swaps and the sensitivity analysis are provided.
In Section 3.3 the CVA is explained and calculated, for each swap and for the whole portfolio.

3.1 Present and future valuation

Two type of swaps will be valued. The first ones are the YYIIS, explained in Chapter 1. The second ones are particular swaps, of interest for Laboral Kutxa, the provider of data and supporter of this work. These derivatives are a kind of mixture of the derivatives described in Chapter 1.

In this derivative, one part pays the ratio of the inflation index of the moment between an initial reference inflation index multiplied by a coupon on certain dates, and the other part pays a fixed rate, the Euribor 12 months plus a spread, or a mixture between both of them, at the same dates.

Using the expression of the zero-coupon inflation-indexed swaps, the value in t of one payment made in T of the inflation leg of these swaps is

$$N_{infL} P_r(t,T) q \frac{I(t)}{I_0}, \tag{3.1}$$

where N_{infL} is the notional of the inflation leg, $P_r(t,T)$ the price in t of a zero-coupon real bond with maturity T, q the coupon of the inflation leg, $I(t)$ the inflation of the valuation moment t and I_0 the inflation taken as

reference.
The value in t of a payment made in T of the second leg depends on the type of swap:

- If the other party pays a fixed rate, then the value is

$$N_{nomL} P_n(t,T) d,$$

where N_{nomL} is the notional of this second leg, $P_n(t,T)$ the price in t of a zero-coupon nominal bond with maturity T and d the fixed rate.

- If the other party pays Euribor 12 months rate + spread agreed one year before T, the value is

$$N_{nomL} P_n(t,T)[eur(T-1,T)+d],$$

where d is the spread added to the value of the Euribor 12 months $eur(T-1,T)$, fixed in $T-1$. $eur(T-1,T)$ is obtained from the simulated value $P_{n(12)}(T-1,T)$ as

$$eur(T-1,T) = \frac{1-P_{n(12)}(T-1,T)}{P_{n(12)}(T-1,T)}.$$

It is important to remark that as Euribor 12 months is needed, $P_{n(12)}(T-1,T)$ has to be built with 12 month Interest Rate Swaps instead of with 6 months IRS (the ones used until now). So, as done previously with the 6M IRS, use the fixed rate of 12M IRS to obtain the zero coupon discount factors $P_{n(12)}(0,T)$ and using these ones obtain the implied factors $P_{n(12)}(T-1,T)$.

- If it is a mixed swap, the other party pays a fixed rate until a date previously set, called *type change date (mixed swap)*. From that date until maturity, pays Euribor 12 months + spread.

Usually, the inflation swaps also have a final exchange, where the leg that pays the inflation index pays a final amount of

$$N_{infL} \max\Big(\frac{I(T_{final})}{I_0}, 1\Big), \tag{3.2}$$

where T_{final} denotes the time to maturity, and the other leg a final amount of N_{nomL}.

As an example, for an inflation swap where we pay a fixed rate and receive the inflation leg, the value in any date t is:

$$\begin{aligned} &\sum_{i=1}^{M} \left[N_{infL} P_r(t,T_i) q \frac{I(t)}{I_0} - N_{nomL} P_n(t,T_i) d \right] \\ &+ \left[N_{infL} \max\Big(\frac{I(T_{final})}{I_0}, 1\Big) - N_{nomL} \right] P_n(t,T_{final}), \end{aligned} \tag{3.3}$$

where $T_M = T_{final}$ and there is a exchange of flows in each time T_i.
All the elements of this expression are known in $t = 0$, except $I(T_{final})$, that can be obatined by simulation. For $t > 0$, it is also necessary to obtain the expressions of $P_r(t, T_i), P_n(t, T_i)$ and $I(t)$, that will be computed by simulation.

It is interesting to remark that to get the present value, (not for future ones) we could alternatively use the closed formula and, similarly to Black-Scholes formula for a call option, we could obtain the value of the final amount in (3.2), without using the simulated values.

Before explaining the simulation method, Table 3.1 summarizes all the calibrated parameters from Chapter 2.

Constant parameters					
κ_n		κ_r		σ_r	
0.078		0.79		0.0093	
Piece-wise parameters					
period	σ_n	σ_I	ρ_{nr}	ρ_{nI}	ρ_{rI}
[0,1]	0.0036	0.0071	0.457	-0.05	0.175
(1,2]	0.0036	0.0056	0.887	-0.01	-0.031
(2,3]	0.0036	0.001	0.928	-0.058	-0.0548
(3,4]	0.0017	0.0075	0.932	-0.018	0.0024
(4,5]	0.001	0.0081	0.932	-0.098	-0.048
(5,6]	0.0011	0.0103	0.899	-0.1	-0.069
(6,7]	0.0018	0.012	0.921	-0.0925	-0.062
(7,8]	0.0012	0.0127	0.904	-0.137	-0.081
(8,9]	0.0062	0.0156	0.912	-0.119	-0.068
(9,10]	0.0071	0.0146	0.9	-0.186	-0.16
(10,12]	0.0074	0.001	0.914	-0.192	-0.155
(12,15]	0.0087	0.001	0.929	-0.2	-0.187

Table 3.1: Calibrated parameters.

For the future valuation, nominal and real discount factors in any time t are required, recall the expressions

$$P_l(t,T) = \frac{P_l(0,T)}{P_l(0,t)} \exp\left(\beta_l(t,T)[f_l(0,t) - l(t)] - \frac{\sigma_l^2}{4\kappa_l} \beta_l(t,T)^2 (1 - e^{-2\kappa_l t}) \right),$$

with

$$\beta_l(t,T) = \frac{1}{\kappa_l}(1 - e^{-\kappa_l(T-t)}), \quad \text{where} \quad l \in \{n, r\}.$$

There have to be calculated for any time t, $f_l(0,t)$ and $l(t)$.

To obtain $l(t)$ in any time t, it must be discretized the corresponding dynamics. For the instantaneous nominal rate,

$$dn(t) = [\nu_n(t) - \kappa_n n(t)]dt + \sigma_n dW_n^Q(t)$$
$$\Longrightarrow n(t+dt) = \nu_n(t)dt + n(t)[1 - \kappa_n dt] + \sigma_n \sqrt{dt}\epsilon_t,$$

where ϵ_t are *i.i.d.* $N(0,1)$ and

$$\nu_n(t) = \frac{\partial f_n(0,t)}{\partial t} + \kappa_n f_n(0,t) + \frac{\sigma_n^2}{2\kappa_n}(1 - e^{-2\kappa_k t}).$$

Since the Eonia is the market interest rate that is more similar to the instantaneous one, we have decided to take the Eonia interest rate of our valuation date as $n(0)$. For $f_n(0,t)$, we have used the Eonia zero-coupon curve (from Bloomberg). We use the approximation,

$$\frac{P_n(0,t+\Delta_t)}{P_n(0,t)} = P_n(0,t,t+\Delta_t) \approx \frac{1}{1 + \Delta_t f_n(0,t)}$$
$$\Longrightarrow f_n(0,t) = \frac{1}{\Delta_t}(\frac{P_n(0,t)}{P_n(0,t+\Delta_t)} - 1)$$

To compute the partial derivative, we use backward finite numerical differences, considering $(f_n(0,t) - f_n(0,t-\Delta_t))/\Delta_t$.

Using the Eonia zero-coupon discount factors ($P_{n(eon)}(0,t)$) and the equality (2.8), $P_r(0,t)$ is obtained. Using the same technique as for the nominal case, get $f_r(0,t)$ and the partial derivative. To obtain the dynamics of the instantaneous real rate along the time, do the same discretization as for the nominal rate. To obtain $r(0)$, it must be used the Inflation index HCIP of the previous chapter and the expression $1+n(0) = (1+r(0))(1+\pi)$, where π is the inflation rate calculated as the division between the last two inflation values.

Finally, to obtain the value of the Inflation index along the time it may be used the same discretization getting

$$I(t+dt) = [n(t) - r(t)]I(t)dt + I(t) + \sigma_I I(t)\sqrt{dt}\epsilon_t,$$

where ϵ_t are *i.i.d.* $N(0,1)$ and $I(0)$ is the last available value of the Inflation index.

To do the simulation of the nominal and real instantaneous short rates and the inflation index, take into account that the random normal variables are correlated with each other as follows

$$\begin{aligned} dW_n^Q(t)dW_r^Q(t) &= \rho_{nr}dt \\ dW_n^Q(t)dW_I^Q(t) &= \rho_{nI}dt \\ dW_r^Q(t)dW_I^Q(t) &= \rho_{rI}dt \end{aligned} \tag{3.4}$$

To generate correlated normal random variables, we use the Cholesky decomposition.

3.2 Numerical valuation and sensitivity analysis

As explained in the previous section, there are different types of inflation swaps, in this case, it is done the valuation and sensitivity analysis of 2 that the second leg is a fixed rate, 2 with a Euribor12 months+ spread, a mixed one and finally the valuation of a year on year inflation-indexed swap. More explicitly, inflation swaps that cover different IL- Spanish treasury bonds have been valued, those swaps terminate at Nov 2024 and Nov 2033.

In addition to the valuation, for each swap it has be done the calculation of the expected positive exposure on each future valuation date, that is, the mean of all the simulations that gave a positive value. This values are necessary to calculate the CVA of the derivatives later.
We also calculate the percentile 97.5 of those positive exposures and the mean of all the values that are above that percentile, thus obtaining the expected shortfall. The expected shortfall is calculated since is the calculation base according to the capital requirements of the internal model of the FRTB (Fundamental Review of Trading Book). When doing the sensitivity analysis with respect to the expected shortfall, it is possible to see how changes in the model parameters affect the capital requirements.
These values are shown for each swap in different tables one year after the reference day 03/28/2019, and at times to maturity $T_{final}/5$ and $T_{final}/3$.

In order to understand the risk involved in some inflation derivatives, it have been done a sensitivity analysis focusing on the influence of the calibrated parameters on the estimated values. That is, we have changed all of them to see how this affects the values achieved.

To check the accuracy of the model, that is, if the calibrated parameters are accurate enough and the chosen model is appropriate, after valuating the swaps we have compared some of the obtained values with an external consultancy.

We have decided to fix $\Delta_t = 1/12$, in this way the simulations of the dynamics will be done every month, and consequently we have the value of the swaps the day 28 of each month until maturity. To obtain the instantaneous forward rates, we have fixed $\Delta_t = 1/365$. Regarding the number of simulations, we have set N=1000. In this way, the Montecarlo error of calculating the valuation by simulation is $1/\sqrt{N} = 1/\sqrt{1000}$. Therefore, the higher the number of simulations, the more accurate will be the value of the derivative we are calculating.

3.2.1 The cases of Inflation vs fixed and Inflation vs variable

Inflation vs variable 2033 counterparty CaixaBank

Suppose an inflation-indexed swap where the first party pays the ratio the inflation index of the moment between a reference value and the second one the Euribor 12 months + spread with counterparty CaixaBank. The details of the swap are in Table 3.2.

SWAP INFLATION/VARIABLE 2033											
T_{final}	TT	Type of swap	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
14.688	14.667	2	1.2%	0	0	0.7%	-1	102.02533	50,000,000	50,193,149.18	CaixaBank

Table 3.2: Information of the variable swap with Counterparty CaixaBank.

Where T_{final} is the time to maturity; TT is the last valuation time, the last day 28 before maturity; Type of swap is 1 if the second leg is fixed, 2 if is variable, 3 if is mixed and 5 if is YYIIS; when type swap is 1 or 5, Swap rate 1 is the fixed rate and Swap rate 2 is 0, when is 2 Swap rate 1 is the spread of the Euribor 12 months and Swap rate 2 is 0, when is 3, Swap rate 1 is the fixed rate until the Type change date and Swap rate 2 the spread on the Euribor after the Type change date; Type change date is 0 except when Type of swap is 3, that is the time when the second party starts to pay Euribor 12 months plus spread instead of a fixed rate; Payer is 1 if we receive the inflation leg and -1 if we give it; Inf_0 is the Inflation reference value; N_{infL} is the notional of the inflation leg; N_{nomL} the notional of the second (nominal) leg and Counterparty indicates who is the second part of the swap.

Table 3.3 is organized as follows: the first row is the original value, the second row is per million, that is, the value divided by N_{nomL} and multiplied by a million. The rest of the rows are the percentage increment on the value of the considered swap (fair value, expected positive exposure, 97.5% positive exposure and expected shortfall) with respect to the original value, obtained when each parameter is changed the indicated quantity. The last rows are the percentage increment when instead of using the piece-wise for the indicated parameter, the calibrated constant values are used. We have done that change in the parameters values with the intention of comparing the sensitivities of different swaps with respect to the same parameter.

SWAP INFLATION/VARIABLE 2033													
	Valuation				Expected positive exposure			97.5% positive exposure			Expected shortfall		
	t=0	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$
original	- 1,448,372.6	- 1,557,095.6	- 1,982,127.4	- 2,876,630.9	1,709,077.4	1,619,246.7	1,559,575.2	4,883,854.8	4,740,355.1	4,663,529.5	5,826,006.6	5,780,727.4	5,723,672.9
per million	- 28,967.5	- 31,141.9	- 39,642.5	- 57,532.6	34,181.5	32,384.9	31,191.5	97,677.1	94,807.1	93,270.6	116,520.1	115,614.5	114,473.5
σ_n*(1+0.5)	23.02%	19.38%	12.65%	9.28%	0.15%	-0.52%	0.90%	1.90%	2.97%	1.45%	-0.44%	1.39%	3.22%
σ_n*(1+0.1)	3.41%	2.82%	1.76%	1.31%	0.28%	0.18%	0.87%	1.41%	0.69%	1.45%	-0.51%	0.25%	0.60%
σ_r*(1+0.5)	4.54%	4.22%	3.64%	2.52%	34.45%	32.30%	26.35%	37.04%	38.71%	35.71%	29.10%	24.06%	20.01%
σ_r*(1+0.1)	0.88%	0.81%	0.70%	0.49%	6.00%	7.21%	3.53%	7.05%	6.26%	4.82%	3.83%	5.75%	0.97%
σ_I*(1+0.5)	0.051%	0.003%	-0.08%	-0.05%	16.90%	18.46%	15.38%	27.05%	28.41%	23.83%	19.66%	19.01%	13.25%
σ_I*(1+0.1)	-0.04%	-0.04%	-0.05%	-0.03%	3.72%	2.96%	1.16%	4.93%	3.18%	0.11%	2.91%	4.76%	0.17%
$\sigma_n, \sigma_r, \sigma_I$*(1+0.1)	4.29%	3.63%	2.45%	1.79%	9.75%	7.72%	9.86%	12.50%	13.13%	9.45%	7.91%	7.63%	6.08%
κ_n*(1+0.5)	-3.91%	-3.03%	-1.63%	-1.15%	1.55%	1.76%	1.80%	3.41%	3.21%	2.55%	1.38%	1.48%	0.68%
κ_r*(1+0.5)	4.84%	5.09%	3.72%	2.55%	-21.89%	-18.36%	-17.07%	-13.58%	-10.36%	-13.65%	-13.07%	-12.75%	-11.42%
κ_n, κ_r*(1+0.5)	1.17%	2.28%	2.30%	1.54%	-19.35%	-17.93%	-14.82%	-12.77%	-10.52%	-14.65%	-12.97%	-13.05%	-12.73%
ρ_{nr}*(1-0.3)	0.23%	0.20%	0.13%	0.09%	0.09%	-0.11%	-0.75%	-0.09%	-0.65%	-0.79%	-0.19%	0.02%	0.000%
ρ_{nI}*(1-0.3)	0.013%	0.011%	0.009%	0.006%	-0.64%	0.03%	0.013%	-0.09%	-0.13%	0.00%	-0.04%	-0.007%	-0.015%
ρ_{rI}*(1-0.3)	-0.09%	-0.08%	-0.07%	-0.05%	-0.40%	-0.06%	-0.08%	-0.09%	-0.05%	-0.03%	-0.02%	-0.006%	0.006%
constant σ_n	-1.09%	-1.03%	-0.83%	-1.61%	-1.51%	-1.84%	0.59%	3.63%	5.58%	4.78%	-1.44%	-2.75%	-4.86%
constant σ_I	-0.42%	-0.46%	-0.34%	-0.20%	-4.71%	-3.89%	-5.67%	-8.54%	-7.84%	-14.45%	-15.52%	-16.03%	-16.50%
constant ρ_{nr}	-0.10%	-0.05%	-0.04%	-0.03%	0.621%	-0.65%	-1.65%	-0.50%	-0.53%	-1.22%	-0.17%	-0.22%	-0.13%
constant ρ_{nI}	-0.03%	-0.02%	-0.01%	-0.01%	-0.06%	-0.55%	-0.77%	-0.30%	-0.01%	-0.83%	0.00%	-0.12%	0.01%
constant ρ_{rI}	0.65%	0.61%	0.48%	0.34%	-0.13%	0.29%	-0.12%	0.35%	0.35%	0.14%	0.34%	0.35%	0.37%
all constants	-1.02%	-0.97%	-0.73%	-1.47%	-6.81%	-2.15%	-3.57%	-6.00%	-11.60%	-10.87%	-15.40%	-18.37%	-17.84%

Table 3.3: Obtained values for the variable swap with counterparty CaixaBank.

SWAP INFLATION/FIXED 2024													
	Valuation				**Expected positive exposure**			**97.5% positive exposure**			**Expected shortfall**		
	t=0	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$
original	2,769,669.0	3,168,282.1	3,168,619.5	3,543,418.0	3,360,682.3	3,361,185.9	3,675,269.7	6,864,583.1	6,856,699.8	7,215,336.3	7,316,078.5	7,330,270.6	7,747,101.0
per million	55,393.4	63,365.6	63,372.4	70,868.4	67,213.6	67,223.7	73,505.4	137,291.7	137,134.0	144,306.7	146,321.6	146,605.4	154,942.0
σ_n*(1+0.5)	-1.80%	-1.18%	-1.13%	-0.92%	5.27%	5.32%	3.66%	8.63%	9.31%	9.70%	5.22%	5.38%	5.38%
σ_n*(1+0.1)	-0.29%	-0.19%	-0.18%	-0.14%	0.72%	0.73%	0.65%	3.00%	3.16%	2.29%	1.51%	1.53%	1.50%
σ_r*(1+0.5)	-1.08%	-0.93%	-0.94%	-0.87%	6.21%	5.97%	3.78%	8.67%	8.71%	9.04%	3.76%	3.78%	3.62%
σ_r*(1+0.1)	-0.17%	-0.15%	-0.15%	-0.14%	0.73%	0.73%	0.33%	2.09%	2.23%	2.54%	1.06%	1.04%	1.00%
σ_I*(1+0.5)	0.39%	0.34%	0.34%	0.32%	2.07%	2.29%	2.26%	7.40%	7.13%	8.40%	3.13%	3.09%	3.09%
σ_I*(1+0.1)	0.10%	0.09%	0.09%	0.08%	0.41%	0.41%	0.67%	0.85%	0.73%	1.75%	0.72%	0.72%	0.73%
$\sigma_n, \sigma_r, \sigma_I$*(1+0.1)	-0.38%	-0.27%	-0.26%	-0.22%	1.77%	1.78%	1.79%	5.97%	5.86%	5.61%	2.92%	2.93%	2.85%
$\kappa_n * (1+0.5)$	0.09%	0.02%	0.02%	0.002%	-0.78%	-0.68%	-0.62%	-2.37%	-1.92%	-1.39%	-1.30%	-1.31%	-1.24%
$\kappa_r * (1+0.5)$	-3.50%	-3.30%	-3.29%	-2.89%	-5.20%	-5.19%	-3.78%	-6.83%	-6.86%	-5.59%	-3.99%	-3.96%	-3.65%
$\kappa_n, \kappa_r * (1+0.5)$	-3.43%	-3.29%	-3.29%	-2.90%	-5.52%	-5.53%	-4.29%	-8.10%	-8.12%	-7.12%	-5.58%	-5.57%	-5.57%
ρ_{nr}*(1-0.3)	-0.04%	-0.03%	-0.03%	-0.02%	0.12%	0.12%	0.107%	0.06%	0.47%	0.21%	0.26%	0.26%	0.25%
ρ_{nI}*(1-0.3)	0.0003%	0.0004%	0.0004%	0.0003%	0.0019%	0.0019%	0.0012%	-0.006%	0.015%	-0.03%	0.005%	0.005%	0.005%
ρ_{rI}*(1-0.3)	-0.0044%	-0.0044%	-0.0044%	-0.0038%	-0.0032%	-0.0032%	-0.003%	0.0002%	-0.012%	0.02%	0.004%	0.004%	0.003%
constant σ_n	-0.19%	-0.17%	-0.16%	-0.15%	0.28%	0.07%	0.17%	0.32%	0.01%	0.45%	0.71%	0.68%	0.65%
constant σ_r	0.24%	0.21%	0.21%	0.20%	0.78%	0.79%	0.70%	-1.44%	-1.08%	-0.53%	-0.04%	0.00%	0.15%
constant ρ_{nr}	0.024%	0.011%	0.011%	0.01%	-0.35%	-0.24%	-0.02%	-0.82%	-0.35%	0.09%	-0.27%	-0.27%	-0.25%
constant ρ_{nI}	0.010%	0.006%	0.006%	0.005%	-0.011%	-0.012%	-0.007%	-0.28%	-0.13%	0.02%	-0.10%	-0.10%	-0.10%
constant ρ_{rI}	-0.16%	-0.14%	-0.14%	-0.13%	0.10%	0.10%	-0.11%	0.14%	0.29%	0.015%	0.11%	0.11%	0.11%
all constants	-0.09%	-0.09%	-0.09%	-0.08%	0.69%	0.69%	0.74%	-1.27%	-1.05%	0.18%	0.36%	0.44%	0.61%

Table 3.4: Obtained values for the fixed swap with counterparty JP Morgan.

Inflation vs fixed 2024 counterparty JP Morgan

Now, suppose a swap where the first part pays as well the division of the inflation index of the moment between a reference value and the second one a fixed rate with counterparty JP Morgan. The details are in Table 3.5. The valuation and the rest of calculated values are in Table 3.4.

SWAP INFLATION/FIXED 2024											
T_{final}	TT	Type of swap	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
5.6822	5.667	1	0.993%	0	0	1.8%	-1	100.05803	50,000,000	59,463,157.1	JP Morgan

Table 3.5: Information of the fixed swap with Counterparty JP Morgan.

Inflation vs fixed 2033 counterparty Morgan Stanley and inflation vs variable counterparty BBVA

There is available the valuation and sensitivity of another swap where the second party pays a fixed leg with maturity Nov 2033 and counterparty Morgan Stanley. Also, another one that the second party pays Euribor 12 months + spread with maturity Nov 2024 and counterparty BBVA. The descriptive data and sensitivity information of these swaps are in Appendix E.

The conclusions of the sensitivity analysis of the 4 explained swaps, the two fixed and the two variables, are presented below.

Valuation sensitivity

Is clearly seen that the parameters that affects more the actual and future value when the maturity is Nov 2033 for both cases, fixed and Euribor 12 months + spread, is σ_n. The impact of increasing σ_r by 50% is more or less similar to the impact of raising κ_r by 50%. For the case of inflation versus variable swap, the impact of κ_n is similar but in the opposite sense, for example, when κ_n is increased by 50% the actual value is decreased by 3.91%. For the inflation versus fixed swap, changes in κ_n affect more the actual value and the value one year later than changes in σ_r or κ_r. For future valuations, the impact of κ_n decreases a little bit.

For short maturities, Nov 2024, the parameter having a bigger impact is κ_r, for example, in the case of the fixed swap, when κ_r is increased by 50% the valuation one year later is decreased by 3.3%. The impact of σ_n and σ_r is more os less similar. For both cases, variable and fixed, when both parameters increase the actual and future valuation decreases.

For the four swaps, the sensitivity with respect to the parameter σ_I and the correlations $\rho_{nr}, \rho_{nI}, \rho_{rI}$ is small. In addition, when all the volatilities are increased by 10%, the actual and future valuations are not very affected.

In general, comparing the swaps with maturity 2024 and 2033, the impact of all the parameters is bigger for long maturities. Except for κ_r, that affect more in short maturities.

Expected positive exposure, 97,5% positive exposure, expected shortfall sensitivity

For the inflation versus variable swaps, the impact of increasing σ_n and κ_n is not very big. However, the sensitivity with respect to σ_r, σ_I and κ_r is bigger than when analysing the impact on the actual and future valuation. For both maturities, σ_r is the parameter with a bigger effect. For example, for the maturity Nov 2033 when σ_r is increased by 50%, the expected shortfall within a year is increased by 29.1%.

Regarding the inflation versus fixed swaps, for long maturities the parameter having a bigger impact is σ_n. Secondly, there are the parameters σ_r, κ_n and κ_r. For the maturity Nov 2024, the biggest sensitivity is with respect to the parameters σ_n and κ_r. Note that while the effect of the volatilities is positive, the impact of the mean reversion speeds is negative. In general, the values are not strongly influenced when changing the inflation volatility σ_I.

Is important to remark that the effect of changing κ_n is always small, except for the expected positive exposure in the inflation versus fixed swap with maturity 2033.

The expected positive exposure, 97.5% positive exposure and expected shortfall are not strongly influenced when changing the correlations. Note that the correlation between the nominal and real bonds ρ_{nr} is the one having a bigger impact for all the swaps.

Sensitivity to using the constant calibrated parameters

Instead of using the piece-wise parameters to do the valuation, we could use the calibrated constant ones. In the previous chapter have been explained how hard is to calibrate the parameters accurately. Here, we have calculated the percentage increment in the valuation when using the constant parameters instead of the piece-wise ones to see if it is worth the piece-wise calibration or not.

For the maturity Nov 2033 the sensitivity in the actual and future valuation is bigger with respect to σ_n. In addition, the impact of having all constant is similar to the impact of having constant only σ_n. For the maturity Nov 2024, the sensitivity is not big with respect to any parameter.

When changing the piece-wise functions to constant ones, is appreciated that depending on the swap the impact in the expected positive exposure, 97,5% positive exposure and expected shortfall is bigger with respect to some parameters than others. Emphasising that when all the parameters are constant, the effect of some of them is counteract with the effect of others for the swaps with maturity Nov 2024, reducing in general the sensitivity to low levels. For the maturity Nov 2033, the impact of having all the parameters constant is greater than having any of them constant. Notting that for the inflation versus variable with maturity 2033 the impact of having σ_I constant is notoriously significant.

3.2.2 The case of Inflation vs mixed counterparty BBVA

The information below corresponds to the inflation versus mixed swap with maturity 2024 and counterparty BBVA. The second leg pays a fixed rate up to three years before maturity, and after pays Euribor 12 months + spread. The information about the swap is in the Table 3.6.

SWAP INFLATION/MIXED 2024											
T_{final}	TT	Type of swap	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
5.6822	5.667	3	1.30%	1.10%	2.6822	0.70%	-1	100.05803	50,000,000	51,463,157.1	BBVA

Table 3.6: Information of the mixed swap with Counterparty BBVA.

The valuation and sensitivity analysis for the mixed swap are in Table 3.7. Note that the sensitivity is really big with respect to the parameter κ_r. While in the actual and future valuation the impact is positive, in the expected positive exposure, 97.5% positive exposure and expected shortfall the effect is negative, when κ_r increases all the values decrease. With respect to the sensitivity in future dates, changes in σ_r and σ_I increase strongly the values. For example, when σ_r is raised by 50%, the expected positive exposure in $T_{final}/3$ is increased by 37.47%.

SWAP INFLATION/MIXED 2024													
	Valuation				Expected positive exposure			97.5% positive exposure			Expected shortfall		
	t=0	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$
original	-530,917.88	- 815,414.61	- 815,065.17	- 1,116,555.29	929,932.18	930,486.56	854,370.59	2,517,794.56	2,520,912.66	2,440,525.39	2,893,590.30	2,889,218.41	2,648,440.56
per million	- 10,618.36	- 16,308.29	- 16,301.30	- 22,331.11	18,598.64	18,609.73	17,087.41	50,355.89	50,418.25	48,810.51	57,871.81	57,784.37	52,968.81
σ_n*(1+0.5)	5.978%	3.009%	2.961%	1.992%	-4.905%	-5.177%	-7.089%	3.926%	3.090%	-0.385%	-0.887%	-0.881%	-0.004%
σ_n*(1+0.1)	0.904%	0.438%	0.430%	0.286%	-2.202%	-1.880%	-2.066%	0.546%	1.005%	-1.021%	-1.518%	-1.459%	-0.041%
σ_r*(1+0.5)	5.608%	3.615%	3.639%	2.712%	35.753%	35.736%	37.468%	28.708%	28.304%	23.290%	23.943%	23.918%	26.364%
σ_r*(1+0.1)	0.912%	0.587%	0.592%	0.443%	5.693%	6.353%	5.035%	7.075%	7.284%	2.817%	4.715%	4.731%	6.343%
σ_I*(1+0.5)	-2.037%	-1.291%	-1.291%	-0.959%	21.264%	21.210%	23.429%	24.114%	23.535%	15.782%	15.287%	15.418%	15.931%
σ_I*(1+0.1)	-0.524%	-0.334%	-0.334%	-0.247%	4.463%	4.109%	3.946%	5.884%	5.106%	5.359%	2.665%	2.737%	4.502%
$\sigma_n, \sigma_r, \sigma_I$*(1+0.1)	1.399%	0.761%	0.758%	0.534%	10.109%	9.394%	9.680%	12.634%	12.795%	6.455%	7.506%	7.530%	9.255%
κ_n*(1+0.5)	-0.381%	-0.138%	-0.134%	-0.078%	0.408%	-0.277%	0.283%	0.494%	0.253%	0.198%	0.011%	0.029%	0.046%
κ_r*(1+0.5)	18.276%	12.202%	12.193%	8.808%	-17.974%	-17.987%	-20.144%	-13.810%	-13.815%	-20.899%	-16.199%	-16.042%	-14.703%
κ_n, κ_r*(1+0.5)	17.983%	12.122%	12.119%	8.776%	-17.697%	-17.716%	-18.680%	-13.926%	-13.929%	-19.479%	-14.774%	-14.569%	-13.282%
$\rho_{nr} * (1-0.3)$	0.196%	0.116%	0.114%	0.075%	-0.942%	-0.621%	-1.000%	-0.685%	-0.785%	-1.373%	-1.728%	-1.714%	-0.051%
$\rho_{nI} * (1-0.3)$	-0.00139%	-0.00158%	-0.00160%	-0.00111%	0.01696%	0.01620%	-0.4152%	0.033%	0.027%	-0.207%	0.010%	0.010%	0.009%
$\rho_{rI} * (1-0.3)$	0.02303%	0.016%	0.016%	0.011%	0.046%	0.046%	0.050%	0.041%	-0.005%	0.036%	0.052%	0.052%	0.053%
constant σ_n	0.043%	0.040%	0.038%	0.030%	1.022%	1.355%	-0.515%	0.972%	0.711%	-6.155%	0.227%	0.310%	0.518%
constant σ_I	-1.235%	-0.815%	-0.819%	-0.597%	5.046%	4.679%	8.566%	1.952%	1.558%	-4.774%	-2.709%	-2.651%	-1.528%
constant ρ_{nr}	-0.124%	-0.043%	-0.043%	-0.035%	-0.221%	-0.567%	-0.488%	0.426%	0.232%	0.157%	-0.291%	-0.289%	-0.185%
constant ρ_{nI}	-0.050%	-0.025%	-0.024%	-0.015%	-0.207%	0.126%	-0.612%	0.114%	0.000%	-0.481%	-0.041%	-0.040%	-0.060%
constant ρ_{rI}	0.827%	0.540%	0.540%	0.394%	0.775%	0.432%	0.553%	0.914%	0.704%	0.457%	0.704%	0.703%	0.800%
all constants	-0.450%	-0.246%	-0.250%	-0.180%	4.151%	4.467%	9.913%	-0.797%	-0.956%	-5.070%	-2.295%	-2.233%	-0.478%

Table 3.7: Obtained values for the mixed swap with counterparty BBVA.

3.2.3 The case of YYIIS counterparty Goldman Sachs

Once carried out the sensitivity analysis of the fixed, variable and mixed swaps, is time to analyse the YYIIS with maturity Nov 2024 and counterparty Goldman Sachs. The descriptive data is in Table 3.8.

SWAP YYIIS 2024											
T_{final}	TT	Type of swap	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
5.6822	5.667	5	0.993%	0	0	100%	-1	102.02533	332,000,000	345,495,687.98	Goldman Sachs

Table 3.8: Information of the swap with Counterparty Goldman Sachs.

The obtained percentage increments due to changes in the model parameters can be seen in Table 3.9.

Valuation Sensitivity

As happened with the fixed and variable swaps with maturity 2024, the sensitivity is great with respect to the parameter κ_r. For example, the future valuation within a year increases 25.2% when κ_r is raised by 50%. The valuation is not strongly influenced by changes in κ_n.
The impact of both, changes in σ_n and σ_r have the same behaviour. The impact becomes smaller when the valuation is done at a further future date.

When changing the piece-wise function to the calibrated constant parameters, the valuation is more affected with σ_I, but all the sensitivities are small in general. If all the parameters are constant, the effect of ones is counteract with the effect of others, since there is not a big change.

Expected positive exposure, 97,5% positive exposure, expected shortfall sensitivity

In these cases, there is a bigger sensitivity to increments in all the volatilities, specifically to changes in σ_n and σ_r. However, the impact of changing them together is not so remarkable. The sensitivity with respect to the reversion speeds is bigger when both of them are increased. For example, when both of them are increased by 50% the expected shortfall in $T_{final}/3$ is decreased 11.4%. The impact of changing the correlations is not very big.

Regarding to use the constant calibrated parameters instead of the piecewise functions, the biggest impact is seen when σ_I is constant. As happened before, the impact of having all of them constant is smaller than when only some of them are constant. When calculating the expected shortfall, the effect of having only σ_I constant or all of them constant is similar.

SWAP YYIIS 2024													
	Valuation				**Expected positive exposure**			**97.5% positive exposure**			**Expected shortfall**		
	t=0	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$
original	4,336,947.0	5,105,580.1	5,105,580.1	6,296,030.0	14,589,686.8	14,589,686.8	15,268,339.3	35,979,973.5	35,979,973.5	37,769,796.8	41,087,923.6	41,087,923.6	44,264,859.3
per million	13,063.1	15,378.3	15,378.3	18,963.9	43,944.8	43,944.8	45,989.0	108,373.4	108,373.4	113,764.4	123,758.8	123,758.8	133,327.9
σ_n*(1+0.5)	10.4%	7.0%	7.0%	4.6%	19.5%	19.5%	19.9%	28.7%	28.7%	23.7%	27.7%	27.7%	22.8%
σ_n*(1+0.1)	1.6%	1.0%	1.0%	0.6%	3.3%	3.3%	2.9%	3.0%	3.0%	4.1%	5.0%	5.0%	4.4%
σ_r*(1+0.5)	7.1%	5.1%	5.1%	4.5%	20.4%	20.4%	19.0%	22.8%	22.8%	25.0%	19.6%	19.6%	19.0%
σ_r*(1+0.1)	1.1%	0.7%	0.7%	0.7%	5.4%	5.4%	3.6%	3.2%	3.2%	3.6%	3.9%	3.9%	3.0%
σ_I*(1+0.5)	1.30%	-0.06%	-0.06%	0.84%	9.0%	9.0%	6.1%	5.5%	5.5%	4.99%	11.6%	11.6%	7.4%
σ_I*(1+0.1)	0.1%	-0.1%	-0.1%	0.1%	1.6%	1.6%	0.9%	-0.5%	-0.5%	0.2%	1.7%	1.7%	1.3%
$\sigma_n, \sigma_r, \sigma_I$*(1+0.1)	3.0%	1.8%	1.8%	1.5%	9.0%	9.0%	7.5%	8.8%	8.8%	8.8%	9.7%	9.7%	8.6%
κ_n*(1+0.5)	-0.6%	-0.2%	-0.2%	0.04%	-1.9%	-1.9%	-2.3%	-3.4%	-3.4%	-3.7%	-3.4%	-3.4%	-3.2%
κ_r*(1+0.5)	13.9%	25.2%	25.2%	16.4%	-5.0%	-5.0%	-5.4%	-5.9%	-5.9%	-6.3%	-2.8%	-2.8%	-7.8%
κ_n, κ_r*(1+0.5)	13.6%	25.3%	25.3%	16.6%	-7.5%	-7.5%	-8.3%	-10.7%	-10.7%	-10.3%	-6.9%	-6.9%	-11.4%
ρ_{nr}*(1-0.3)	0.2%	0.2%	0.2%	0.1%	0.5%	0.5%	0.5%	0.8%	0.8%	0.8%	0.6%	0.6%	0.6%
ρ_{nI}*(1-0.3)	0.006%	0.002%	0.002%	0.004%	0.013%	0.013%	0.008%	0.034%	0.034%	-0.002%	0.023%	0.023%	0.004%
ρ_{rI}*(1-0.3)	-0.067%	-0.010%	-0.010%	0.028%	0.031%	0.031%	0.021%	0.027%	0.027%	0.029%	0.048%	0.048%	0.019%
constant σ_n	0.9%	0.7%	0.7%	0.6%	0.8%	0.8%	0.1%	-0.5%	-0.5%	-0.6%	0.9%	0.9%	0.2%
constant σ_I	-1.4%	-1.3%	-1.3%	-0.9%	4.4%	4.4%	1.0%	5.8%	5.8%	2.5%	5.1%	5.1%	3.3%
constant ρ_{nr}	0.2%	0.1%	0.1%	0.0%	-0.6%	-0.6%	-0.6%	-1.3%	-1.3%	-0.5%	-1.3%	-1.3%	-0.8%
constant ρ_{nI}	-0.1%	-0.02%	-0.02%	-0.03%	0.01%	0.01%	-0.1%	-0.4%	-0.4%	-0.2%	-0.3%	-0.3%	-0.2%
constant ρ_{rI}	0.1%	0.4%	0.4%	0.5%	0.8%	0.8%	0.4%	0.5%	0.5%	0.4%	0.4%	0.4%	0.2%
all constants	-0.4%	-0.04%	-0.04%	0.08%	2.4%	2.4%	0.9%	2.6%	2.6%	2.9%	5.3%	5.3%	2.7%

Table 3.9: Obtained values for the YYIIS swap with counterparty Goldman Sachs.

3.2.4 Model check

As mentioned before, we have compared some obtained values with the ones obtained by an external consultancy. The compared swaps are the inflation versus variable with maturity Nov 2033 and counterparty CaixaBank and the inflation versus fixed with maturity Nov 2024 and counterparty JP Morgan. This external consultancy gives the cash flows in each payment date for both, inflation and nominal leg. Also gives the discount factors to calculate the present value of each leg. Calculating the implied discount factors, we are able to obtain also an approximate future valuation according to the external consultancy. The obtained values are shown in Table 3.10.

		SWAP INF/VARIABLE 2033		**SWAP INF/FIXED 2024**	
		Consultancy	Our model	Consultancy	Our model
03/28/2019	nominal leg	$1.53 \cdot 10^7$	$1.346 \cdot 10^7$	$3.334 \cdot 10^6$	$3.34 \cdot 10^6$
	variable leg	$5.72 \cdot 10^6$	$5.53 \cdot 10^6$	$5.803 \cdot 10^6$	$5.76 \cdot 10^6$
11/30/2019	nominal leg	$1.47 \cdot 10^7$	$1.29 \cdot 10^7$	$2.77 \cdot 10^6$	$2.77 \cdot 10^6$
	variable leg	$5.35 \cdot 10^6$	$5.12 \cdot 10^6$	$4.85 \cdot 10^6$	$4.77 \cdot 10^6$
11/30/2020	nominal leg	$1.42 \cdot 10^7$	$1.23 \cdot 10^7$	$2.21 \cdot 10^6$	$2.2 \cdot 10^6$
	variable leg	$4.97 \cdot 10^6$	$4.74 \cdot 10^6$	$3.89 \cdot 10^6$	$3.82 \cdot 10^6$

Table 3.10: Valuation according to the external consultancy and our model.

Since the external consultancy only gives the discount factors from the payment dates to the actual valuation date (03/28/2019), using the application it is possible only to obtain an approximated future value in the payment dates, assuming that the exchange of flows has already happened.

Is seen that for the inflation versus fixed swap with maturity 2024, the obtained values are really similar, both in the actual valuation and in the future one. For the inflation versus variable swap with maturity 2033, the obtained values are approximate each other, but the difference is bigger than with the previous swap. This may be because in our model we have simulated the value of the future Euribor 12 months, and the external consultancy uses simply the implicit value.

Despite the differences, the obtained values are similar to the values of the external consultancy, meaning that the used model and calibrated parameters are reliable.

3.3 Credit Valuation Adjustment (CVA)

CVA is one of the consequences of the changes in the market after the credit crisis. Before the crisis, very few people include in the valuation the corresponding part to the CVA. CVA puts market price to the risk that has an entity with it's counterparty. That is, CVA is the difference between the risk-free value of a derivative and the real value, which takes into account the possibility of the counterparty's default. In fact, not only banks are taking into account the counterparty credit risk, also regulators are asking to include a quantity in the bank's results which corresponds to that risk. For further information see [14].

CVA is calculated every day taking into account the recovery of the counterparty in case of default, the default probability of the counterparty, and the expected positive exposure each day until maturity. The recovery of each counterparty is calculated with historical information. In the previous section there have been obtained the monthly values for the expected positive exposure for each swap. The default probability is calculated using credit defaults swaps (CDS) of the counterparty.

3.3.1 Default probability

The CDS are instruments between two parties, one buys coverage on a default and the other sells it. This contract has a notional N. Is the natural instrument to cover the counterparty risk of a bond and the counterparty of the CDS must not be correlated with the issuer of the bond. The party that sells protection pays $(1-R)N$ when the default occurs (since the counterparty of the bond pays RN) and the party that buys protection pays and spread of the notional in each predetermined date. Usually, the spreads are paid the 20 of March, June, September and December. The equilibrium spread is the one that makes the value of the CDS 0 when entering on it.

Defining τ as the default time, in the short rate models the default intensity $\lambda(t)$ is defined as

$$P(\tau \leq t+dt|\tau > t|F_{t_0}) = \lambda(t)dt, \tag{3.5}$$

where t_0 is the reference day. Using the definition of the default intensity, the survival function is defined as

$$P(\tau > t|F_{t_0}) = 1_{\tau > t_0} e^{-\int_{t_0}^{t} \lambda(s)ds}. \tag{3.6}$$

Using this, the distribution and density functions are easily calculated

- Distribution function

$$F_\tau(t) = P(\tau \leq t|F_{t_0}) = 1 - 1_{\tau > t_0} e^{-\int_{t_0}^{t} \lambda(s)ds}. \tag{3.7}$$

- Density function

$$\eta_\tau(t) = 1_{\tau > t_0} \lambda(t) e^{-\int_{t_0}^{t} \lambda(s)ds}. \tag{3.8}$$

To see how to obtain the survival function see Appendix D. Using this definitions, the price in t of a CDS with maturity T is

$$CDS(t,T) = S_{CDS} N \sum_{i=1}^{M} \gamma_i P_n(t,T_i) P(\tau > T_i|F_t)$$
$$+ S_{CDS} N \int_t^T P_n(t,s)(s - T_{\beta(s)})\eta_\tau(s)ds - (1-R)N \int_t^T P_n(t,s)\eta_\tau(s)ds, \tag{3.9}$$

where $T_{\beta(s)} = max\{T_i|T_i \leq \tau\}$ and T_M is the last payment day. The first summary corresponds to the payments of the spread until the default day, the second part corresponds to the accrued interest from the last date where the spread has been paid until the default date. The last part corresponds to the payment of the counterparty of the CDS in case of default.

For the 6 inflation swaps that have been done the sensitivity analysis, there were five different counterparties, CaixaBank, Morgan Stanley, JP Morgan, BBVA and Goldman Sachs. In our portfolio, apart of those 6 swaps, there are 6 other inflation swaps, therefore, in total the portfolio consists of 12 swaps. The details of the 12 swaps are in Table 3.11.

There are 2 new columns, in Start is indicated the effective date of the swap. In the column initial period is expressed the day count fraction for those swaps in which the first payment has not yet occurred . In that cases, in the first exchange date the nominal leg pays only the corresponding part of the coupon. For the rest of them the value is 1.

The valuation, expected positive exposure, 97.5% positive exposure and expected shortfall for all the swaps of the portfolio are shown in Table 3.12.

Descriptive data																
Swap number	Swap code	Maturity	Start	T_{final}	TT	Type ofswap	Initial period	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
1	CBEUR101833	30/11/2033	23/10/2018	14.688	14.667	2	1	1.2000%	0	0	0.7%	-1	102.02533	50,000,000.00	50,193,149.18	CaixaBank
2	MSFIX091833	30/11/2033	21/09/2018	14.688	14.667	1	1	1.5%	0	0	0.9%	-1	102.02533	332,000,000.00	345,495,687.98	Morgan Stanley
3	JPFIX011824	30/11/2024	31/01/2018	5.682	5.667	1	1	0.9330%	0	0	1.80%	-1	100.05803	50,000,000.00	59,463,157.10	JP Morgan
4	BBVAMIX031824	30/11/2024	14/03/2018	5.682	5.667	3	1	1.30%	1.10%	2.682	0.70%	-1	100.05803	50,000,000.00	51,463,157.10	BBVA
5	GSYYIIS031824	30/11/2024	11/03/2018	5.682	5.667	5	1	0.9330%	0	0	100%	1	102.02533	332,000,000.00	345,495,687.98	Goldman Sachs
6	BBVAEUR08182024	30/11/2024	04/08/2018	5.682	5.667	2	1	0.50%	0	0	1.80%	-1	100.05803	50,000,000.00	59,463,157.10	BBVA
7	JPEUR021930	30/11/2030	11/02/2019	11.6849	11.667	2	0.80	0.7%	0	0	1%	-1	100.333	250,000,000.00	282,122,013.99	JP Morgan
8	CBMIX03192030	30/11/2030	12/03/2019	11.6849	11.667	3	0.72	0.8%	0.760%	5.685	1%	-1	100.083092	50,000,000.00	57,055,875.72	CaixaBank
9	JPMIX031930	30/11/2030	12/03/2019	11.6849	11.667	3	0.72	0.8%	0.7970%	5.685	1%	-1	100.3319	50,000,000.00	57,055,876.00	JP Morgan
10	STDEUR031830	30/11/2030	06/03/2018	11.6849	11.667	2	1	0.6120%	0%	0	1%	-1	100.083092	100,000,000.00	111,116,135.58	Santander
11	JPMIX091830	30/11/2030	30/09/2018	11.6849	11.667	3	1	0.9000%	1.130%	5.685	1%	-1	100.33319	25,000,000.00	28,355,250.00	JP Morgan
12	STDMIX061830	30/11/2030	08/06/2018	11.6849	11.667	3	1	0.900%	1.150%	5.685	1%	-1	100.083092	25,000,000.00	28,156,341.18	Santander

Table 3.11: Descriptive information of the portfolio swaps.

		Valuation				Expected positive exposure			97.5% Positive exposure			Expected shortfall		
Swap number	Swap code	t=0	t=1	t=T_final/5	t=T_final/3	t=1	t=T_final/5	t=T_final/3	t=1	t=T_final/5	t=T_final/3	t=1	t=T_final/5	t=T_final/3
1	CBEUR101833	- 1,448,372.61	- 1,557,095.59	- 1,982,127.38	- 2,876,630.92	1,709,077.41	1,619,246.73	1,559,575.21	4,883,854.85	4,740,355.12	4,663,529.52	5,826,006.57	5,780,727.43	5,723,672.90
2	MSFIX091833	- 24,573,724.38	- 25,988,552.45	- 29,711,082.67	- 34,283,155.36	16,809,845.55	16,210,595.10	14,958,754.83	38,482,248.18	37,470,406.01	33,939,294.31	39,523,113.11	38,674,367.07	34,690,180.58
3	JPFIX011824	2,769,668.96	3,168,282.09	3,168,619.47	3,543,417.98	3,360,682.34	3,361,185.92	3,675,269.66	6,864,583.10	6,856,699.75	7,215,336.35	7,316,078.52	7,330,270.62	7,747,101.04
4	BBVAMIX031824	- 530,917.88	- 815,414.61	- 815,065.17	- 1,116,555.29	929,932.18	930,486.56	854,370.59	2,517,794.56	2,520,912.66	2,440,525.39	2,893,590.30	2,889,218.41	2,648,440.56
5	GSYYIIS031824	7,226,010.59	7,562,943.83	7,536,680.47	8,538,708.53	15,983,899.58	16,368,462.49	16,458,804.27	40,007,919.23	40,911,953.08	42,500,393.68	48,520,334.98	49,782,468.95	50,421,737.70
6	BBVAEUR08182024	1,975,282.54	2,716,688.91	2,716,722.84	3,408,699.40	2,819,657.94	2,819,719.93	3,453,176.00	5,528,840.67	5,534,974.45	6,255,065.74	6,079,377.54	6,073,635.38	6,751,960.21
7	JPEUR021930	- 1,613,713.96	65,268.04	1,194,969.61	1,714,487.92	10,912,667.04	11,476,138.44	12,093,435.76	29,104,466.39	30,147,684.61	31,024,568.47	33,960,756.57	34,465,032.96	35,573,351.16
8	CBMIX03192030	163,807.94	413,906.97	492,915.90	640,870.69	2,582,073.69	2,646,559.88	2,746,089.99	6,520,505.11	6,724,049.06	6,931,826.02	7,376,805.87	7,559,740.20	7,738,959.00
9	JPMIX031930	431,182.85	678,317.25	754,814.71	900,992.21	2,693,797.46	2,759,529.20	2,848,043.84	6,731,510.86	6,948,546.56	7,113,219.06	7,540,890.41	7,717,020.52	7,950,248.27
10	STDEUR031830	- 3,445,755.77	- 2,797,008.49	- 2,252,916.49	- 1,883,634.50	3,442,982.94	3,607,896.11	3,886,100.90	9,923,801.82	10,193,069.71	10,976,993.49	11,595,123.72	11,822,013.69	12,349,595.69
11	JPMIX091830	804,934.82	834,401.55	843,705.76	864,330.59	1,579,260.26	1,593,585.73	1,616,124.38	3,748,299.47	3,847,314.78	3,865,906.33	4,189,924.49	4,258,996.83	4,317,112.85
12	STDMIX061830	547,314.29	580,753.25	593,663.70	618,392.08	1,470,340.12	1,487,047.70	1,505,513.76	3,565,184.72	3,699,884.11	3,701,134.33	4,007,635.37	4,079,130.00	4,143,957.28

Table 3.12: Valuation of the portfolio swaps.

For the six counterparties, the maturities of the market CDS are the 20 of June of the years 2020, 2021, 2022, 2024, 2026 and 2029 (more or less 1, 2, 3, 5, 7, 10). The information about the recovery rate, in units per 1, and spread, in basic points, of the CDS is in Table 3.13. Using market prices of CDS, is possible to obtain $\lambda(t)$ using a bootstrapping technique.

	CaixaBank	Morgan Stanley	JP Morgan	BBVA	Goldman Sachs	Santander
			Recovery			
	0.4	0.4	0.4	0.2	0.4	0.4
			Spread CDS			
1 Year	50,9434	32,0273	26,0273	37,5022	36,9744	12.521
2 Years	63,3187	40,4445	32,1418	56,2841	47,3862	22.365
3 Years	75.1612	51,3707	39,2668	77,1471	60,8137	32.197
5 Years	104,1856	73,0448	56,7288	114,7156	89,0949	59.359
7 Years	123,6635	98,2416	79,3366	144,376	114,6748	81.12
10 Years	133,47	116,08	95,65	162,39	132,43	100.99

Table 3.13: Recovery and spreads of market CDS.

For each counterparty, to calculate a piece-wise function for $\lambda(t)$ the procedure is similar to the one used to calculate $\sigma_n(t)$. Having market CDS, the value in the expiration day (03/28/2019) has to be zero. As in the expression of the value of the CDS appear integrals, the first thing is to do the discretization of that formula. The spread of the CDS are paid every 3 months, so is set γ_i=1/4. For the integrals, we have decided to discretize obtaining values for the day 20 of each month and summing all of them. The discretized formula in $t = 0$

$$CDS(0,T) = S_{CDS} N \sum_{i=1}^{M} \frac{1}{4} P_n(0,T_i) P(\tau > T_i | F_0)$$
$$+ S_{CDS} N \sum_{j=1}^{N} P_n(0,T_j)(T_j - T_{\beta(T_j)}) \eta_\tau(T_j) \frac{1}{12} - (1-R) N \sum_{j=1}^{N} P_n(0,T_j) \eta_\tau(T_j) \frac{1}{12}, \tag{3.10}$$

where T_N is the last day 20 before maturity. Using a Matlab function that gives the value of λ making the value of the CDS with maturity 1 year zero, is possible to obtain the value for the period [0,1]. Using the obtained expression for $\lambda(t)$ until t=1 and the spread of the market CDS with maturity 2 years, is possible to obtain $\lambda(t)$ for the period (1,2]. Using the same procedure, is obtained the piece-wise function for $\lambda(t)$ for each counterparty for the period [0,10]. As the expression until the maturity of the swaps is needed, being the longer one 14,688, is supposed that the value for the last period (7,10] is valid until maturity. The obtained values are in units per 1 in Table 3.14.

Default intensity						
	CaixaBank	Morgan Stanley	JP Morgan	BBVA	Goldman Sachs	Santander
1 Year	0,0085	0,00534	0,00446	0,0047	0.0062	0.0021
2 Years	0,013	0,0085	0,0065	0.01	0.01	0.0058
3 Years	0,017	0,013	0.0092	0.0157	0.0153	0.0091
5 Years	0,026	0,0184	0.0144	0.0226	0.0232	0.018
7 Years	0,031	0,0294	0.0246	0.0294	0.0324	0.024
10 Years	0,027	0,028	0.0239	0.027	0.0311	0.027

Table 3.14: Default intensity.

3.3.2 Stand alone CVA and CVA of the whole portfolio

The CVA of each derivative, is calculated taking into account the quantity that can be lost when the counterparty makes default. Therefore, only the positive expected values of the derivative are taken into account. The numeric formula to calculate the CVA is

$$CVA(t) = (1-R)E\left[e^{-\int_t^\tau n(s)ds} V_\tau^+ 1_{\tau \leq T} | F_t\right], \tag{3.11}$$

where T is the maturity of the derivative and V_τ^+ is the positive value of the derivative in the default time for us. The formula is discretizated to obtain todays CVA, what yields that in $t = 0$

$$CVA = \sum_{i=1}^{M} (1-R)P_n(0,T_i)E(V_{T_i}^+)\Delta P_i \tag{3.12}$$

$$= \sum_{i=1}^{M} (1-R)P_n(0,T_i)E(V_{T_i}^+)[P(\tau > T_{i-1}|F_0) - P(\tau > T_i|F_0)]. \tag{3.13}$$

For our portfolio, the expected positive exposure has been computed for each month until maturity . Thus, each addend represents a month. Then, using the default intensities, the survival function is calculated for each month using (3.6). Using the expected positive exposure at the different future times and the survival function, the CVA for each swap is obtained. The CVA of the day 03/20/2019 for each swap of the portfolio taking into account its corresponding counterparty is shown in Table 3.15.

Swap number	Swap code	CVA	CVA per million	Percentage of the value
1	CBEUR101833	221,174.22	4,423.48	15%
2	MSFIX091833	2,181,798.59	6,571.68	9%
3	JPFIX011824	149,352.34	2,987.05	5%
4	BBVAMIX031824	54,252.09	1,085.04	10%
5	GSYYIIS031824	859,639.03	2,589.27	12%
6	BBVAEUR08182024	286,170.09	5,723.40	14%
7	JPEUR021930	1,326,312.29	5,305.25	82%
8	CBMIX03192030	386,294.26	7,725.89	236%
9	JPMIX031930	308,190.36	6,163.81	71%
10	STDEUR031830	465,935.27	4,659.35	14%
11	JPMIX091830	166,599.43	6,663.98	21%
12	STDMIX061830	164,821.97	6,592.88	30%

Table 3.15: CVA.

The Table 3.15 also includes the CVA value per million of notional for comparison reasons, taking into account that except for the case of BBVA as counterparty, the recovery rate is 0.4. It is seen that the smallest CVA per million is for the mixed swap 2024 with counterparty BBVA, and the biggest one for the mixed swap 2030 with counterparty CaixaBank (Swap number 8). If the maturity of the swap is bigger, it makes sense that the CVA is grater, since there are more payment dates. It is seen that the lowest values for the CVA are for those swaps with shorter maturity. Another important aspect is if there is a big probability of having a positive value in the derivative for us or not. The more paths have positive value for us in the simulation, the higher is the CVA.

It could have be done a sensitivity analysis of the CVA with respect to the model parameters, but as the CVA is proportional to the expected positive exposure, the conclusions are analogous.

To see the importance of taking into account the CVA at the time of giving a value to a derivative, in the last column of Table 3.15 is seen the percentage in absolute terms that represents the CVA from the theoretical value.
For example, for the mixed swap 2030 with counterparty CaixaBank, the absolute value of the CVA is more than twice the actual value of the swap. This indicates that taking into account the CVA at the time of valuing the derivatives is something really important. In the rest of the cases, the value of the CVA is always smaller than the actual value in absolute terms. For both fixed swaps, maturity 2024 and 2033 the CVA is a small percentage of the actual value.

To calculate the CVA of the whole portfolio, an option could be to sum all the stand alone CVA-s, obtaining that the CVA is EUR 6,501,401.37.

Generally, there are netting agreements and collateralization. The netting agreement consists on considering all the derivatives related with the same counterparty together. Thus, in each valuation date is not considered the individual value of each swap, but the sum of all the values together. By this way, the resultant CVA is always smaller since

$$\Big(\sum_{i=1}^{N} \text{value}_i(\tau)\Big)^+ \leq \sum_{i=1}^{N} \text{value}_i(\tau)^+, \tag{3.14}$$

where N is the number of swaps with that counterparty. Then, in case of default, the losses are always smaller or equal if there is a netting agreement.

Once done the netting agreement, usually a collateral agreement is also considered when trading the derivatives . If the present value of the derivative is bigger than a fixed value, the party for which the value is positive receives a guarantee, this procedure is usually checked in daily, weekly, or monthly basis. By these way, the counterparty risk is reduced, since we are exposed to the change in valuation during only the period considered for collateralization.

In our case the collateral agreement is done as follows: with today's calibrated parameters, are calculated the values of the swaps previously netted in each future calculation date (for each date there are 1000 values since there are 1000 simulations). As well, is calculated the value one day earlier. For each path, is made the difference between the value of that date and the value one day before. With the 1000 obtained values for each valuation future date t, the expected positive exposure is done and whit this the CVA is calculated. To obtain the CVA of the whole portfolio, the CVA-s obtained for each counterparty are added.

Then, the CVA with one counterparty is

$$\sum_{i=1}^{M} P_n(0,T_i) E\left[(1-R)\left(\sum_{j=1}^{N} V_{T_i} - \sum_{j=1}^{N} V_{T_i-1}\right)^+ \Delta P_i\right], \tag{3.15}$$

where M represents the number of dates where the valuation is made in the future and N the number of swaps with that counterparty. $\sum_{j=1}^{N} V_{T_i-1}$ is the value of the netted swaps one day earlier.

The obtained CVA for the date 03/28/2019 with netting and collateral agreement is EUR 3,106,056.35. The value is much more smaller than the sum of the stand alone CVA-s, but still big, what means that taking into account the CVA when valuing inflation derivatives is important.

Conclusions

In this work we have developed a complete procedure to value inflation indexed swaps and their daily Credit Valuation Adjustment. This is a challenging task, since the practical calibration of the parameters involved becomes a very complicated issue. For this reason, Laboral Kutxa proposed this objective for this work, since it is a big, and expensive task for practitioners, that needs a very deep knowledge of the main financial instruments, their dynamics, the related formula for valuation and the development of computing algorithms.

To obtain the daily calibration of the parameters, we have used different financial instruments. For the nominal parameters, $\{\kappa_n, \sigma_n\}$, we have used interest rate swaptions and caps. For the inflation volatility, $\{\sigma_I\}$, we have used floors of zero-coupon inflation indexed-swaps. For the real parameters, $\{\kappa_r, \sigma_r\}$, we have tried with inflation caps and year on year inflation-indexed swaps. For the correlations, we have assumed that they are piece-wise constant and they have been estimated using historical data of the nominal and real bonds, and the inflation index.

The nominal and real instantaneous rates have been modeled through the Hull and White model and the inflation index with a Geometric Brownian Motion, as in Jarrow and Yildirim (2003).

Our calibration proposal is based in several techniques, proposed by some authors as [9] or [10].

As remarked in all related works, the calibration becomes a cumbersome procedure and we must adjust the methodology in order to obtain reasonable results. In our case, it becomes clear that the best results are attained when the nominal and inflation volatilities, $\{\sigma_n, \sigma_I\}$, are considered to be piece-wise functions instead of a constant, accounting for different risks depending on time to maturity. The results obtained in Chapter 2 show that the procedure is able to calibrate in an appropriate manner.

Once the parameters are calibrated, in Chapter 3, a simulation procedure has been developed to value the derivatives we are interested on. In total, we have valued four types of inflation indexed swaps. In the first 3 swaps one party pays the division of the inflation index between a fixed value multiplied by a factor and the other one a fixed, variable or mixed value. In the last type, one party pays the division of the inflation index between the inflation index one year before and the other party pays a fixed rate.

For all the derivatives, we have derived several measures of risk exposure, expected positive exposure, 97,5% positive exposure and expected shortfall for future dates. Moreover, we have made an analysis of the sensitivity of this risk measures to changes in the calibrated parameters.

Finally, we approach the main goal of calculating the daily CVA. To do that, we still need the intensity of default, which has been obtained using market CDS and a bootstrapping technique.
The results indicate that the CVA is a very important measure, representing a big percentage of the theoretical value. In fact, for the mixed swap 2030 with counterparty CaixaBank is more than twice the obtained value with the theoretical formula.

Of course, all the results highly depend on the accuracy of our estimations. Therefore, it is important to remark that we have compared the results of this work, when we had the same product, with the results given by an external firm that provides consultancy to Laboral Kutxa and, therefore, the risk measures used in practice. The results were very similar.

As the main conclusion we can say that we have developed a complete procedure to calculate the main risk measures associated to inflation index derivatives. This is an initial work that opens future challenges for research. One is to use different products for calibration, as interest rate swaptions to calibrate the parameters κ_n and σ_n, or fixing the mean reversion speed (κ_n) with historical values and calibrate alone the parameter σ_n. Also, we could calibrate daily correlations using market prices instead of doing it historically. In case of using another models, an option could be to use the Libor Market Models to value the inflation linked instruments, or the Two-process Hull and White model. In the current situation with negative interest rates, is quite common to obtain negative values for the mean reversion speed. For those cases, we propose to analyse a model without a drift.

Another possible future lines of research is to study the evolution of parameters along the time and search for an econometric model that relates a high percentage of the variations of the swaps value with respect to the evolution of market prices of inflation derivatives, or the evolution of the nominal interest rates.

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Appendix A

Concepts and definitions

A.1 Change of Numeraire

This proposition has been taken from [5], page 33.

Proposition A.1.1. *Consider a diffusion process whose dynamics under Q^S is given by*

$$dX(t) = \mu_X^S(t) + \sigma_X(t)dW^S(t), \tag{A.1}$$

where W^S is a standard Brownian motion and $\mu_X^S(t)$,$\sigma_X(t)$ are scalars. Let us assume that the two numeraires S and U evolve under Q^U according to

$$\begin{aligned} dS(t) &= (\cdots)dt + \sigma^S(t)dW^U(t) \\ dU(t) &= (\cdots)dt + \sigma^U(t)dW^U(t), \end{aligned} \tag{A.2}$$

where W^U is a standard Brownian motion and $\sigma^S(t)$, $\sigma^U(t)$ are scalars. Then, the drift of the process X under the numeraire U is

$$\mu_X^U(t) = \mu_X^S(t) - \sigma_X(t)\rho\Big(\frac{\sigma^S(t)}{S(t)} - \frac{\sigma^U(t)}{U(t)}\Big), \tag{A.3}$$

where ρ is the correlation factor between $dW^S(t)$ and $dW^U(t)$.

Usually, this proposition is used to change from the risk neutral measure Q to the nominal forward T_1 measure. The numeraire of the measure Q is the money market account, with the dynamic $dB_k(t) = k(t)B_k(t)dt$, $k \in \{n, r\}$, so in this case $\sigma^S(t) = 0$. With the T_1 forward measure, the numeraire is $P_n(t, T_1)$ with the dynamic $dP_n(t, T_1) = (\cdots)dt + a_n(t, T_1)P_n(t, T_1)dW_n^{T_1}(t)$, then, $\frac{\sigma^U(t)}{U(t)} = a_n(t, T_1)$.

A.2 Annuity measure

Proposition A.2.1. *The annuity measure A is the measure in which*

$$N_t := \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k), \tag{A.4}$$

is taken as numeraire, where $P(t, T_1), ..., P(t, T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$.

A.3 Foreign currency analogy

Consider a foreign market where the price of an asset is X_f under the associated martingale measure Q_f. The foreign and domestic money market accounts are B_f and B_d respectively. The process of exchange is modelled by H, where 1 unit of the foreign currency are $H(t)$ units of the domestic currency in t. If we think in X_f as a derivative which pays $X_f(T_M)$ in T_M, the price in t should be:

$$V_f(t) = B_f(t) E_f\Big(\frac{X_f(T_M)}{B_f(T_M)} | F_t\Big). \tag{A.5}$$

If is wanted the price at t in the domestic currency, it has to be used the relation $V_d(t) = V_f(t)H(t) \longrightarrow V_f(t) = \frac{V_d(t)}{H(t)}$. Then,

$$V_d(t) = H(t) B_f(t) E_f\Big(\frac{X_f(T_M)}{B_f(T_M)} | F_t\Big). \tag{A.6}$$

If it is a domestic investor who buys the derivative, the payout in T_M is $H(T_M)X_f(T_M)$. Let consider now a domestic derivative that pays $H(T_M)X_f(T_M)$ at t_M, the price in t is

$$V_d(t) = B_d(t) E_d\Big(\frac{X_f(T_M)H(T_M)}{B_d(T_M)} | F_t\Big). \tag{A.7}$$

To avoid arbitrage, the price (in the domestic currency) of both terms in t has to be the same:

$$V_d(t) = H(t) B_f(t) E_f\Big(\frac{X_f(T_M)}{B_f(T_M)} | F_t\Big) = B_d(t) E_d\Big(\frac{X_f(T_M)H(T_M)}{B_d(T_M)} | F_t\Big). \tag{A.8}$$

Appendix B

Development of the formulas of JY model

In this chapter are developed the formulas to obtain Propositions and expressions of the section 1.2 of Chapter 1.

B.1 Propositions (1.2.1) and (1.2.2)

In this section are developed the processes to obtain the expressions and dynamics of the Propositions 1.2.1 and 1.2.2. To obtain the conditions of the first proposition the technique is the same for all of them, so the proof is shown only for the first one.

Using the general expressions

$$\begin{aligned} df_k(t,T) = \alpha_k(t,T)dt + \sigma_k(t,T)dW_k^P(t) \Longrightarrow \\ f_k(t,T) = f_k(0,T) + \int_0^t \alpha_k(s,T)ds + \int_0^t \sigma_k(s,T)dW_k^P(s) \end{aligned} \tag{B.1}$$

and

$$P_k(t,T) = e^{-\int_t^T f_k(t,s)ds}, \quad k \in \{n,s\}, \tag{B.2}$$

is obtained that the log-bond price process is

$$
\begin{aligned}
lnP_k(t,T) &= -\int_t^T f_k(t,u)du \\
= -\int_t^T f_k(0,u)du - \int_0^t \left[\int_s^T \alpha_k(s,u)du\right] ds &- \int_0^t \left[\int_s^T \sigma_k(s,u)du\right] dW_k^P(s) \\
&= lnP_k(0,T) + \int_0^t f_k(0,u)du - \int_0^t \left[\int_s^T \alpha_k(s,u)du\right] ds \\
&- \int_0^t \left[\int_s^T \sigma_k(s,u)du\right] dW_k^P(s).
\end{aligned}
\tag{B.3}
$$

Using the notation (1.33) expression (B.3) can be rewritten as

$$
\begin{aligned}
lnP_k(t,T) = lnP_k(0,T) + \int_0^t [k(s) + b_k(s,T)]ds - \frac{1}{2}\int_0^t a_k^2(s,T)ds + \\
\int_0^t a_k(s,T)dW_k^P(s).
\end{aligned}
\tag{B.4}
$$

So,

$$
dlnP_k(t,T) = \left[k(t) + b_k(t,T) - \frac{1}{2}a_k^2(t,T)\right]dt + a_k(t,T)dW_k^P(t). \tag{B.5}
$$

To get the process of $dP_k(t,T)$ apply the Ito's lemma with the function $f(lnP_k(t,T)) = f(x) = e^x = P_k(t,T)$ and this yields

$$
dP_k(t,T) = P_k(t,T)[k(t) + b_k(t,T)]dt + P_k(t,T)a_k(t,T)dW_k^P(t). \tag{B.6}
$$

What it can be expressed as

$$
dP_k(t,T) = P_k(t,T)\left[k(t) - \int_t^T \alpha_k(t,u)du + \frac{1}{2}a_k^2(t,T)\right]dt + P_k(t,T)a_k(t,T)dW_k^P(t), \tag{B.7}
$$

The equation (B.7) is used to proof Proposition (1.2.1). The first thing to do is to express the process of $f_n(t,T)$ under the risk-neutral measure Q,

$$
\begin{aligned}
df_n(t,T) &= \alpha_n(t,T)dt + \sigma_n(t,T)dW_n^P(t) \\
&= [\alpha_n(t,T) + \lambda_n(t)\sigma_n(t,T)]dt + \sigma_n(t,T)dW_n^Q(t)
\end{aligned}
\tag{B.8}
$$

We need to proof that $d(P_n(t,T)/B_n(t))$ it can be expressed as a stochastic integral, i.e., the drift is equal to 0.

$$\begin{aligned} d\frac{P_n(t,T)}{B_n(t)} &= \frac{1}{B_n(t)}dP_n(t,T) - \frac{P_n(t,T)}{B_n(t)^2}dB_n(t) \\ &-\frac{1}{B_n^2(t)}dP_n(t,T)dB_n(t) + \frac{P_n(t,T)}{B_n(t)^3}dB_n^2(t) \\ &= \frac{1}{B_n(t)}dP_n(t,T) - \frac{P_n(t,T)}{B_n(t)^2}n(t)B_n(t)dt. \end{aligned} \tag{B.9}$$

Using (B.7), (B.9) can be rewritten as

$$\begin{aligned} d\frac{P_n(t,T)}{B_n(t)} = \frac{P_n(t,T)}{B_n(t)}\left[n(t) - \int_t^T \alpha_n(t,u)du + \frac{1}{2}a_n^2(t,T)\right] + \\ \frac{P_n(t,T)}{B_n(t)}\left[a_n(t,T)dW_n^P(t) - n(t)dt\right]. \end{aligned} \tag{B.10}$$

Next step is to express this process under the risk-neutral measure Q,

$$\begin{aligned} d\frac{P_n(t,T)}{B_n(t)} = \frac{P_n(t,T)}{B_n(t)}\left[-\int_t^T \alpha_n(t,u)du + \frac{1}{2}a_n^2(t,T) + \lambda_n(t)a_n(t,T)\right] \\ +\frac{P_n(t,T)}{B_n(t)}a_n(t,T)dW_n^Q(t). \end{aligned} \tag{B.11}$$

As said before, $(P_n(t,T)/B_n(t))$ is Q martingale if and only if the drift of $d(P_n(t,T)/B_n(t))$ is equal to 0. This yields,

$$\begin{aligned} \int_t^T \alpha_n(t,u)du = \frac{1}{2}a_n^2(t,T) + \lambda_n(t)a_n(t,T) \Longrightarrow \\ \alpha_n(t,T) = \frac{1}{2}\frac{\partial a_n^2(t,T)}{\partial T} + \frac{\partial a_n(t,T)}{\partial T}\lambda_n(t) \\ = \sigma_n(t,T)\int_t^T \sigma_n(t,s)ds - \sigma_n(t,T)\lambda_n(t), \end{aligned} \tag{B.12}$$

what gives us the aspect that has to have the drift under the real world probability measure so that the evolutions are arbitrage free and the market is complete.

Using the expression (B.7) and the proposition just proofed, the dynamics of the nominal and real bonds of the Proposition (1.2.2) are obtained. Using the nominal drift condition (B.12), under P the dynamics of the nominal zero-coupon bond is

$$\frac{dP_n(t,T)}{P_n(t,T)} = \left[n(t) - \frac{1}{2}a_n^2(t,T) - \lambda_n(t)a_n(t,T) + \frac{1}{2}a_n^2(t,T)\right]dt + a_n(t,T)dW_n^P(t), \tag{B.13}$$

and under Q

$$\frac{dP_n(t,T)}{P_n(t,T)} = n(t)dt + a_n(t,T)dW_n^Q(t). \tag{B.14}$$

As done with the nominal drift condition, using the expression of the real drift,

$$\int_t^T \alpha_r(t,u)du = \frac{1}{2}a_r^2(t,T) + [\lambda_r(t) + \sigma_I(t)\rho_{rI}]a_r(t,T),$$

the dynamics of the real zero-coupon bond under P is

$$\frac{dP_r(t,T)}{P_r(t,T)} = \Big[r(t) - \frac{1}{2}a_r^2(t,T) - [\lambda_r(t) + \sigma_I(t)\rho_{rI}]a_r(t,T) + \frac{1}{2}a_r^2(t,T)\Big]dt + a_r(t,T)dW_r^P(t), \tag{B.15}$$

and under Q

$$\frac{dP_r(t,T)}{P_r(t,T)} = \Big[r(t) - \sigma_I(t)\rho_{rI}a_r(t,T)\Big]dt + a_r(t,T)dW_r^Q(t). \tag{B.16}$$

The dynamics of the instantaneous forward rates and the inflation index under the risk neutral measure Q are easily obtained using the drift conditions of the proposition (1.2.1). To obtain the dynamic of the inflation bond apply multivariate Ito's lemma to the function $f(t, P_r(t,T), I(t)) = I(t)P_r(t,T)$,

$$\begin{aligned} d(I(t)P_r(t,T)) &= P_r(t,T)dI(t) + I(t)dP_r(t,T) + dI(t)dP_r(t,T) \\ &= [n(t) - r(t)]I(t)P_r(t,T)dt + \sigma_I(t)I(t)P_r(t,T)dW_I^Q(t) \\ &+[r(t) - \sigma_I(t)\rho_{rI}a_r(t,T)]I(t)P_r(t,T)dt + a_r(t,T)I(t)P_r(t,T)dW_r^Q(t) \\ &+P_r(t,T)I(t)a_r(t,T)\sigma_I(t)\rho_{rI}dt \\ &= n(t)I(t)P_r(t,T)dt + \sigma_I(t)I(t)P_r(t,T)dW_I^Q(t) + a_r(t,T)I(t)P_r(t,T)dW_r^Q(t). \end{aligned}$$

B.2 Nominal instantaneous rate and bond

Using the expression (1.43) and the dynamics (1.35), the forward rate under Q evolves as

$$f_n(t,T) = f_n(0,T) + \sigma_n^2 \int_0^t \beta_n(s,T)e^{-\kappa_n(T-s)}ds + \sigma_n \int_0^t e^{-\kappa_n(T-s)}dW_n^Q(s). \tag{B.17}$$

Then, the instantaneous spot rate evolves as

$$\begin{aligned} n(t) &= f_n(t,t) \\ &= f_n(0,t) + \sigma_n^2 \int_0^t \beta_n(s,t) e^{-\kappa_n(t-s)} ds + \sigma_n \int_0^t e^{-\kappa_n(t-s)} dW_n^Q(s) \\ &= f_n(0,t) + \frac{\sigma_n^2}{2}\beta_n(0,t)^2 + \sigma_n \int_0^t e^{-\kappa_n(t-s)} dW_n^Q(s). \end{aligned} \tag{B.18}$$

To obtain the dynamics of $dn(t)$, take differences in the previous expression,

$$dn(t) = \left[\frac{\partial f_n(0,t)}{\partial t} + \frac{\sigma_n^2}{2} 2\beta_n(0,t)\frac{\partial \beta_n(0,t)}{\partial t}\right] dt$$
$$+\sigma_n dW_n^Q(t) + \sigma_n \int_0^t e^{-\kappa_n(t-s)}(-\kappa_n) dW_n^Q(s) dt.$$

Then, substitute

$$-\kappa_n \left[\sigma_n \int_0^t e^{-\kappa_n(T-s)} dW_n^Q(s)\right]$$

by

$$-\kappa_n \left[n(t) - f_n(0,t) - \frac{\sigma_n^2}{2}\beta_n(0,t)^2\right]$$

and using (B.18) obtain the dynamics of the nominal instantaneous rate.

$$\begin{aligned} dn(t) = &\left[\frac{\partial f_n(0,t)}{\partial t} + \frac{\sigma_n^2}{2} 2\beta_n(0,t)\frac{\partial \beta_n(0,t)}{\partial t} - \kappa_n n(t) + \kappa_n f_n(0,t)\right. \\ &\left. + \kappa_n \frac{\sigma_n^2}{2}\beta_n(0,t)^2\right] dt + \sigma_n dW_n^Q(t) \\ = &\left[\frac{\partial f_n(0,t)}{\partial t} + \kappa_n f_n(0,t) - \kappa_n n(t) + \frac{\sigma_n^2}{2\kappa_n}[1 - e^{-2\kappa_n t}]\right] dt + \sigma_n dW_n^Q(t). \end{aligned} \tag{B.19}$$

Therefore, the dynamics of the nominal instantaneous short rate follows a Hull and White model with κ_n as the mean reversion speed and

$$\nu_n(t) = \frac{\partial f_n(0,t)}{\partial t} + \kappa_n f_n(0,t) + \frac{\sigma_n^2}{2\kappa_n}[1 - e^{-2\kappa_n t}]$$

as the mean reversion level.

To obtain the expression of the nominal zero-coupon bond, it has to be developed the integral of (B.18),

$$\int_0^t n(u)du = -lnP_n(0,t)+\frac{\sigma_n^2}{2}\int_0^t \beta_n(0,s)^2 ds+\sigma_n \int_0^t \left[\int_0^u e^{-\kappa_n(u-s)}dW_n^Q(s)\right]du. \tag{B.20}$$

To solve the double integral define a new variable,

$$Y(t) = \int_0^t e^{\kappa_n s}dW_n^Q(s),$$

and using Ito's lemma

$$d(e^{-\kappa_n t}Y(t)) = -\kappa_n e^{-\kappa_n t}Y(t)dt + e^{-\kappa_n t}dY(t) = -\kappa_n e^{-\kappa_n t}Y(t)dt + dW_n^Q(t).$$

Integrating the obtained expression,

$$e^{-\kappa_n t}Y(t) = -\kappa_n \int_0^t e^{-\kappa_n u}Y(u)du + W_n^Q(t).$$

If we take the definition of $Y(t)$ the expression above yields

$$e^{-\kappa_n t}\int_0^t e^{\kappa_n s}dW_n^Q(s) = W_n^Q(t) - \kappa_n \int_0^t \left[e^{-\kappa_n u}\int_0^u e^{\kappa_n s}dW_n^Q(s)\right]du.$$

So, the double integral is

$$\begin{aligned}\kappa_n \int_0^t \left[e^{-\kappa_n u}\int_0^u e^{\kappa_n s}dW_n^Q(s)\right]du &= \int_0^t (1-e^{-\kappa_n(t-s)})dW_n^Q(s)\\ &= \kappa_n \int_0^t \beta_n(s,t)dW_n^Q(s).\end{aligned}$$

Taking into account the expression of the double integral, the equation (B.20) can be rewritten as

$$\int_0^t n(s)ds = -lnP_n(0,t) + \frac{\sigma_n^2}{2}\int_0^t \beta_n(0,s)^2 ds + \sigma_n \int_0^t \beta_n(s,t)dW_n^Q(s). \tag{B.21}$$

Using the dynamics (1.38) of the nominal zero-coupon bond that the nominal forward rate has a normal distribution, is easy to demonstrate that the bond price has log-normal distribution. Using the Black-Scholes solution for log-normal distributions,

$$\begin{aligned}P_n(t,T) &= P_n(0,T)\exp\left(\int_0^t \left(n(s) - \frac{a_n(s,T)^2}{2}\right)ds + \int_0^t a_n(s,T)dW_n^Q(s)\right) \\ &= P_n(0,T)\exp\left(\int_0^t \left(n(s) - \frac{\sigma_n^2\beta_n(s,T)^2}{2}\right)ds - \int_0^t \sigma_n\beta_n(s,T)dW_n^Q(s)\right) \\ &= \frac{P_n(0,T)}{P_n(0,t)}\exp\left(\frac{\sigma_n^2}{2}\int_0^t \left(\beta_n(0,s)^2 - \beta_n(s,T)^2\right)ds + \sigma_n\int_0^t (\beta_n(s,t) - \beta_n(s,T))dW_n^Q(s)\right).\end{aligned} \tag{B.22}$$

Is necessary to do some calculus to represent the expression above without stochastic integrals. Use the representation (B.18) of $n(t)$ to solve the integral. Being

$$\beta_n(s,t) - \beta_n(s,T) = \frac{1}{\kappa_n}(e^{-\kappa_n(T-s)} - e^{-\kappa_n(t-s)})$$

and

$$\begin{aligned}\beta_n(t,T)n(t) &= \beta_n(t,T)f_n(0,t) + \frac{\sigma_n^2}{2}\beta_n(t,T)\beta_n(0,t)^2 \\ &+\frac{\sigma_n}{\kappa_n}\int_0^t (e^{-\kappa_n(t-s)} - e^{-\kappa_n(T-s)})dW_n^Q(s),\end{aligned}$$

the part of the stochastic integral of the equation (B.22) can be represented as

$$\begin{aligned}\sigma_n\int_0^t (\beta_n(s,t) - \beta_n(s,T))dW_n^Q(s) &= \beta_n(t,T)[f_n(0,t) - n(t)] \\ +\beta_n(t,T)\beta_n(0,t)^2\frac{\sigma_n^2}{2} &= \beta_n(t,T)[f_n(0,t) - n(t)] \\ +\frac{\sigma_n^2}{2\kappa_n^3}\Big[1 - 2e^{-\kappa_n t} + e^{-2\kappa_n t} &- e^{-\kappa_n(T-t)} + 2e^{-\kappa_n T} - e^{-\kappa_n(T+t)}\Big].\end{aligned} \tag{B.23}$$

Next step is to develop the first part of the expression (B.22)

$$\begin{aligned}\frac{\sigma_n^2}{2}\int_0^t \left(\beta_n(0,s)^2 - \beta_n(s,T)^2\right)ds &= \frac{\sigma_n^2}{2\kappa_n^3}\Big[2e^{-\kappa_n t} - 2 - \frac{1}{2}e^{-2\kappa_n t} + \frac{1}{2} \\ & 2e^{-\kappa_n(T-t)} - 2e^{-\kappa_n T} - \frac{1}{2}e^{-2\kappa_n(T-t)} + \frac{1}{2}e^{-2\kappa_n T}\Big].\end{aligned} \tag{B.24}$$

Taking into account the obtained expressions for the integrals of (B.22) and simplifying some terms,

$$P_n(t,T) = \frac{P_n(0,T)}{P_n(0,t)}\exp\Bigg(\beta_n(t,T)[f_n(0,t)-n(t)]+$$

$$\frac{\sigma_n^2}{4\kappa_n^3}\Big[-1+2e^{\kappa_n(T-t)}-e^{-2\kappa_n(T-t)}+e^{-2\kappa_n t}-2e^{-2\kappa_n(T+t)}+e^{-2\kappa_n T}\Big]\Bigg)$$

$$= \frac{P_n(0,T)}{P_n(0,t)}\exp\Bigg(\beta_n(t,T)[f_n(0,t)-n(t)]$$

$$-\frac{\sigma_n^2}{4\kappa_n^3}\Big[(1-2e^{\kappa_n(T-t)}+e^{-2\kappa_n(T-t)})(1-e^{-2\kappa_n t})\Big]\Bigg)$$

$$= \frac{P_n(0,T)}{P_n(0,t)}\exp\Bigg(\beta_n(t,T)[f_n(0,t)-n(t)]-\frac{\sigma_n^2}{4\kappa_n}\beta_n(t,T)^2(1-e^{-2\kappa_n t})\Bigg). \tag{B.25}$$

B.3 Real instantaneous rate and bond

In this case the dynamics of $f_r(t,T)$ has one more term, but the procedure is going to be equal to the nominal case. Using the equation (1.36) and assuming a constant volatility for the inflation index, σ_I, the instantaneous real spot rate evolves as

$$r(t) = f_r(t,t) = f_n(0,t) + \sigma_n^2\int_0^t \beta_n(s,t)e^{-\kappa_n(t-s)}ds$$

$$-\sigma_r\sigma_I\rho_{rI}\int_0^t e^{-\kappa_r(t-s)}ds + \sigma_n\int_0^t e^{-\kappa_n(t-s)}dW_n^Q(s)$$

$$= f_n(0,t) + \frac{\sigma_n^2}{2}\beta_n(0,t)^2 - \sigma_r\sigma_I\rho_{rI}\beta_r(0,t) + \sigma_n\int_0^t e^{-\kappa_n(t-s)}dW_n^Q(s). \tag{B.26}$$

Using this and the dynamics of the nominal instantaneous short rate, is easily obtained the dynamic for the real one

$$dr(t) = \Bigg[\frac{\partial f_r(0,t)}{\partial t} + \frac{\sigma_r^2}{2}2\beta_r(0,t)\frac{\partial \beta_r(0,t)}{\partial t} - \sigma_r\sigma_I\rho_{rI}\frac{\partial \beta_r(0,t)}{\partial t} - \kappa_r r(t) + \kappa_r f_r(0,t)$$

$$+\kappa_r\frac{\sigma_r^2}{2}\beta_r(0,t)^2 - \kappa_r\sigma_r\sigma_I\rho_{rI}\beta_r(0,t)\Bigg]dt + \sigma_r dW_r^Q(t)$$

$$= \Bigg[\frac{\partial f_r(0,t)}{\partial t} + \kappa_r f_r(0,t) - \kappa_r r(t) + \frac{\sigma_r^2}{2\kappa_r}[1-e^{-2\kappa_r t}] - \sigma_r\sigma_I\rho_{rI}\Bigg]dt + \sigma_r dW_r^Q(t).$$

Therefore, the dynamics of the real instantaneous short rate follows a Hull and White model as

$$dr(t) = [\nu_r(t) - \sigma_r\sigma_I\rho_{rI} - \kappa_r r(t)]dt + \sigma_r dW_r^Q(t), \tag{B.27}$$

where the mean reversion level is

$$\nu_r(t) = \frac{\partial f_r(0,t)}{\partial t} + \kappa_r f_r(0,t) + \frac{\sigma_r^2}{2\kappa_r}[1 - e^{-2\kappa_r t}].$$

To obtain a closed formula for the real bond is used that

$$\begin{aligned} \int_0^t r(s)ds = -lnP_r(0,t) + \frac{\sigma_r^2}{2}\int_0^t \beta_r(0,s)^2 ds \\ -\sigma_r\sigma_I\rho_{rI}\int_0^t \beta_r(0,s)ds + \sigma_r\int_0^t \beta_r(s,t)dW_r^Q(s). \end{aligned} \tag{B.28}$$

Knowing that the real forward rate has a normal distribution, it can be easily demonstrated that the bond price has log-normal distribution. Using the Black-Scholes solution for log-normal distributions is obtained that

$$\begin{aligned} P_r(t,T) = P_r(0,T)\text{exp}\Bigg(\int_0^t \Big(r(s) - a_r(s,T)\sigma_I\rho_{rI} - \frac{a_r(s,T)^2}{2}\Big)ds \\ + \int_0^t a_r(s,T)dW_r^Q(s)\Bigg) \\ = P_r(0,T)\text{exp}\Bigg(\int_0^t \Big(r(s) + \beta_r(s,T)\sigma_r\sigma_I\rho_{rI} - \frac{\sigma_r^2\beta_r(s,T)^2}{2}\Big)ds \\ - \int_0^t \sigma_r\beta_r(s,T)dW_r^Q(r)\Bigg) \\ = \frac{P_r(0,T)}{P_r(0,t)}\text{exp}\Bigg(\frac{\sigma_r^2}{2}\int_0^t \Big(\beta_r(0,s)^2 - \beta_r(s,T)^2\Big)ds \\ +\sigma_r\sigma_I\rho_{rI}\int_0^t (\beta_r(s,T) - \beta_r(0,s))ds + \sigma_r\int_0^t (\beta_r(s,t) - \beta_r(s,T))dW_r^Q(s)\Bigg). \end{aligned} \tag{B.29}$$

Using the solution obtained for the nominal bond with the real expressions

$$\begin{aligned} P_r(t,T) = \frac{P_r(0,T)}{P_r(0,t)}\text{exp}\Bigg(\frac{\sigma_r^2}{2}\int_0^t \Big(\beta_r(0,s)^2 - \beta_r(s,T)^2\Big)ds \\ +\sigma_r\sigma_I\rho_{rI}\int_0^t (\beta_r(s,T) - \beta_r(0,s))ds + \beta_r(t,T)[f_r(0,t) - r(t)] \\ +\frac{\sigma_r^2}{2}\beta_r(t,T)\beta_r(0,t)^2 - \sigma_r\sigma_I\rho_{rI}\beta_r(t,T)\beta_r(0,t)\Bigg). \end{aligned} \tag{B.30}$$

Developing the second integral of (B.30) (the first one is known since is equal to the expression of the nominal bond),

$$\sigma_r\sigma_I\rho_{rI}\int_0^t(\beta_r(s,T)-\beta_r(0,s))ds = \sigma_r\sigma_I\rho_{rI}\int_0^t\frac{1}{\kappa_r}(e^{-\kappa_r s}-e^{-\kappa_r(T-s)})ds$$
$$=\sigma_r\sigma_I\rho_{rI}\frac{1}{\kappa_r^2}[-e^{-\kappa_r t}+1-e^{-\kappa_r(T-t)}+e^{-\kappa_r T}] = \sigma_r\sigma_I\rho_{rI}\beta_r(0,t)\beta_r(t,T). \tag{B.31}$$

Using the expression (B.31) and the ones obtained when calculating the nominal bond,

$$P_r(t,T)=\frac{P_r(0,T)}{P_r(0,t)}\exp\left(\beta_r(t,T)[f_r(0,t)-r(t)]-\frac{\sigma_r^2}{4\kappa_r}\beta_r(t,T)^2[1-e^{-2\kappa_r t}]\right). \tag{B.32}$$

B.4 Year on Year Inflation Swap

In this section is explained how to obtain the closed formulas for the YYIIS using the T-forward measure. In this way, one could express (1.21) as

$$YYIIS(t,T_{i-1},T_i,\tau_{i,A},N)=N\tau_{i,A}P_n(t,T_{i-1})E_n^{T_{i-1}}(P_r(T_{i-1},T_i)|F_t)-N\tau_{i,A}P_n(t,T_i). \tag{B.33}$$

To use these expression is necessary to put the dynamics of the real bond under the T_1 forward measure. Using Proposition (A.1.1) with $S(t)$ as the real market account $B_r(t)$, and $U(t)$ as the nominal bond $P_n(t,T_1)$,

$$\frac{dP_r(t,T_2)}{P_r(t,T_2)}=\Big[r(t)-\sigma_I(t)\rho_{rI}a_r(t,T_2)+\rho_{nr}a_r(t,T_2)a_n(t,T_1)\Big]dt+a_r(t,T_2)dW_r^{T_1}(t) \tag{B.34}$$

As said previously, the real bond is log-normally distributed. Then, the solution is

$$P_r(t,T_2)=P_r(0,T_2)\exp\Bigg(\int_0^t(r(s)-\sigma_I\rho_{rI}a_r(s,T_2)+\rho_{nr}a_r(s,T_2)a_n(s,T_1))ds$$
$$-\int_0^t\frac{a_r^2(s,T_2)}{2})ds+\int_0^t a_r(s,T_2)dW_r^{T_1}(s)\Bigg) \tag{B.35}$$

And the ratio of two real bonds with different maturities

$$\frac{P_r(t,T_2)}{P_r(t,T_1)} = \frac{P_r(0,T_2)}{P_r(0,T_1)}\exp\Bigg(\int_0^t \sigma_I\rho_{rI}(a_r(s,T_1)-a_r(s,T_2)) + \rho_{nr}a_n(s,T_1)(a_r(s,T_2)-a_r(t,T_1))ds - \int_0^t (\frac{a_r^2(s,T_2)}{2}) - \frac{a_r^2(s,T_1)}{2}))ds + \int_0^t (a_r(s,T_2)-a_r(s,T_1))dW_r^{T_1}(s)\Bigg). \tag{B.36}$$

Next step is to evaluate the expression in T_1 but under the real world filtration F_t, since our intention is to calculate $P_r(T_1,T_2)$ for later evaluate the forward measure expectation

$$P_r(T_1,T_2)|F_t = \frac{P_r(t,T_2)}{P_r(t,T_1)}\exp\Bigg(\int_t^{T_1} \sigma_I\rho_{rI}(a_r(s,T_1)-a_r(s,T_2)) + \rho_{nr}a_n(s,T_1)(a_r(s,T_2)-a_r(t,T_1))ds - \int_t^{T_1} (\frac{a_r^2(s,T_2)}{2}) - \frac{a_r^2(s,T_1)}{2}))ds + \int_t^{T_1} (a_r(s,T_2)-a_r(s,T_1))dW_r^{T_1}(s)\Bigg). \tag{B.37}$$

Finally, the nominal expectation under the T_1 forward measure is calculated,

$$E_n^{T_1}[P_r(T_1,T_2)|F_t] = \frac{P_r(t,T_2)}{P_r(t,T_1)}\exp\Bigg(\int_t^{T_1} \sigma_I\rho_{rI}(a_r(s,T_1)-a_r(s,T_2)) + \rho_{nr}a_n(s,T_1)(a_r(s,T_2)-a_r(t,T_1))ds - \int_t^{T_1} (\frac{a_r^2(s,T_2)}{2}) - \frac{a_r^2(s,T_1)}{2})ds\Bigg) \cdot E_n^{T_1}\Bigg(\exp\Bigg[\int_t^{T_1} (a_r(s,T_2)-a_r(s,T_1))dW_r^{T_1}(s)\Bigg]|F_t\Bigg). \tag{B.38}$$

Solving the expectation with the moment generating function,

$$E_n^{T_1}\Bigg(\exp\Bigg[\int_t^{T_1} (a_r(s,T_2)-a_r(s,T_1))dW_r^{T_1}(s)\Bigg]|F_t\Bigg) = \exp\Bigg[\frac{1}{2}\int_t^{T_1} (a_r(s,T_2)-a_r(s,T_1))^2 ds\Bigg]. \tag{B.39}$$

So, the expectation under the T_1 forward measure is

$$E_n^{T_1}(P_r(T_1,T_2)|F_t) = \frac{P_r(t,T_2)}{P_r(t,T_1)}\exp\left(\int_t^{T_1} \sigma_I\rho_{rI}(a_r(s,T_1) - a_r(s,T_2)) \right.$$

$$\left. +\rho_{nr}a_n(s,T_1)(a_r(s,T_2) - a_r(t,T_1))ds + \int_t^{T_1}(a_r^2(s,T_1) - a_r(s,T_1)a_r(s,T_2))ds\right)$$

$$= \frac{P_r(t,T_2)}{P_r(t,T_1)}\exp\left(\int_t^{T_1}[a_r(s,T_2) - a_r(s,T_1)][a_n(s,T_1)\rho_{nr} - \sigma_I\rho_{rI} - a_r(s,T_1)]ds\right) \tag{B.40}$$

What in a reduced way can be expressed as

$$E_n^{T_1}(P_r(T_1,T_2)|F_t) = \frac{P_r(t,T_2)}{P_r(t,T_1)}e^{b(t,T_1,T_2)}, \tag{B.41}$$

where

$$b(t,T_1,T_2) = \int_t^{T_1}[a_r(s,T_2) - a_r(s,T_1)][a_n(s,T_1)\rho_{nr} - \sigma_I\rho_{rI} - a_r(s,T_1)]ds. \tag{B.42}$$

Is seen that the expectation of the future real zero-coupon bond under the nominal T_1 forward measure is equal to the current forward price of the real bond, multiplied by a correction factor ($e^{b(t,T_1,T_2)}$).
Using (B.41) the inflation leg of the year on year swap can be represented as

$$YYIIS(t,T_{i-1},T_i,\tau_{i,A},N) = N\tau_{i,A}\left[P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{b(t,T_{i-1},T_i)} - P_n(t,T_i)\right], \tag{B.43}$$

where

$$b(t,T_{i-1},T_i) = \sigma_r\beta_r(T_{i-1},T_i)\left[\beta_r(t,T_{i-1})\left(\rho_{rI}\sigma_I - \frac{1}{2}\sigma_r\beta_r(t,T_{i-1}) \right.\right.$$

$$\left.\left. + \frac{\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}(1+\kappa_r\beta_n(t,T_{i-1}))\right) - \frac{\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}\beta_n(t,T_{i-1})\right]. \tag{B.44}$$

For further details of how to get to this formula look at B.4.1. The expression (B.44) is called the convexity adjustment for a year on year inflation indexed swap.
The value of the inflation leg is obtained by summing up the values of all the payments.

B.4.1 Convexity adjustment

In this part is shown how to obtain (B.44).

$$b(t,T_1,T_2) = \int_t^{T_1} [a_r(s,T_2) - a_r(s,T_1)][a_n(s,T_1)\rho_{nr} - \sigma_I \rho_{rI} - a_r(s,T_1)]ds, \tag{B.45}$$

where

$$a_l(t,T) = \frac{-\sigma_l}{\kappa_l}(1 - e^{-\kappa_l(T-t)}) \quad l \in \{n,r\}.$$

Developing the formula is obtained

$$\begin{aligned}
b(t,T_1,T_2) = \int_t^{T_1} \Bigg[& \rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}(1-e^{-\kappa_r(T_2-s)})(1-e^{-\kappa_n(T_1-s)}) + \rho_{rI}\frac{\sigma_r\sigma_I}{\kappa_r}(1-e^{-\kappa_r(T_2-s)}) \\
& -\frac{\sigma_r^2}{\kappa_r^2}(1-e^{-\kappa_r(T_2-s)})(1-e^{-\kappa_r(T_1-s)}) - \rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}(1-e^{-\kappa_r(T_1-s)})(1-e^{-\kappa_n(T_1-s)}) \\
& -\rho_{rI}\frac{\sigma_r\sigma_I}{\kappa_r}(1-e^{-\kappa_r(T_1-s)}) + \frac{\sigma_r^2}{\kappa_r^2}(1-e^{-\kappa_r(T_1-s)})^2 \Bigg] ds \\
= \int_t^{T_1} \Bigg[& \rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}\Big(1-e^{-\kappa_r(T_2-s)} - e^{-\kappa_n(T_1-s)} + e^{-\kappa_r(T_2-s)-\kappa_n(T_1-s)}\Big) \\
& +\rho_{rI}\frac{\sigma_r\sigma_I}{\kappa_r}(1-e^{-\kappa_r(T_2-s)} - 1 + e^{-\kappa_r(T_1-s)}) \\
& -\frac{\sigma_r^2}{\kappa_r^2}\Big(1-e^{-\kappa_r(T_2-s)} - e^{-\kappa_r(T_1-s)} e^{-\kappa_r(T_1+T_2-2s)}\Big) \\
& -\rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}\Big(1-e^{-\kappa_r(T_1-s)} - e^{-\kappa_n(T_1-s)} + e^{-(\kappa_r+\kappa_n)(T_1-s)}\Big) \\
& +\frac{\sigma_r^2}{\kappa_r^2}\Big(1-2e^{-\kappa_r(T_1-s)} + e^{-2\kappa_r(T_1-s)}\Big)\Bigg] ds.
\end{aligned} \tag{B.46}$$

Solving the integral and grouping terms,

$$\begin{aligned}
b(t,T_1,T_2) &= \rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}\Bigg[-\frac{1}{\kappa_r}(e^{-\kappa_r(T_2-T_1)}-e^{-\kappa_r(T_2-t)})+\beta_r(t,T_1) \\
&+\frac{1}{\kappa_n+\kappa_r}(-1+e^{-\kappa_r(T_2-T_1)}+e^{-(\kappa_n+\kappa_r)(T_1-t)}-e^{-\kappa_r(T_2-t)-\kappa_n(T_1-t)})\Bigg] \\
&\rho_{rI}\frac{\sigma_r\sigma_I}{\kappa_r}\Big[-\frac{1}{\kappa_r}(e^{-\kappa_r(T_2-T_1)}-e^{-\kappa_r(T_2-t)})+\beta_r(t,T_1)\Big] \\
&+\frac{\sigma_r^2}{\kappa_r^2}\Big[-\beta_r(t,T_1)+\frac{1}{\kappa_r}(e^{-\kappa_r(T_2-T_1)}-e^{-\kappa_r(T_2-t)}) \\
&+\frac{1}{2\kappa_r}(1-e^{-2\kappa_r(T_1-t)}-e^{-\kappa_r(T_2-T_1)}+e^{-\kappa_r(T_1+T_2-2t)})\Big] \\
&= \rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}\Bigg[-\frac{1}{\kappa_r}(e^{-\kappa_r(T_2-T_1)}-e^{-\kappa_r(T_2-t)})+\beta_r(t,T_1) \\
&+\frac{1}{\kappa_n+\kappa_r}(-1+e^{-\kappa_r(T_2-T_1)}+e^{-(\kappa_n+\kappa_r)(T_1-t)}-e^{-\kappa_r(T_2-t)-\kappa_n(T_1-t)})\Bigg] \\
&+\sigma_r\beta_r(T_1,T_2)\Bigg[\beta_r(t,T_1)\Big(\rho_{rI}\sigma_I-\frac{1}{2}\beta_r(t,T_1)\Big)\Bigg].
\end{aligned} \tag{B.47}$$

For the fist part is necessary to do long but straightforward calculations. We have to obtain

$$\frac{\sigma_r\beta_r(T_1,T_2)\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}\Bigg[\beta_r(t,T_1)(1+\kappa_r\beta_n(t,T_1))-\beta_n(t,T_1)\Bigg],$$

what it can be expressed as

$$\frac{\sigma_r\beta_r(T_1,T_2)\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}\Big[\beta_r(t,T_1)-e^{-\kappa_r(T_1-t)}\beta_n(t,T_1)\Big]. \tag{B.48}$$

Is easy to see that

$$\begin{aligned}
&\beta_r(T_1,T_2)e^{-\kappa_r(T_1-t)}\beta_n(t,T_1)= \\
&=\frac{1}{\kappa_n\kappa_r}\Big(e^{-\kappa_r(T_1-t)}-e^{-\kappa_r(T_2-t)}-e^{-(\kappa_n+\kappa_r)(T_1-t)}+e^{-\kappa_r(T_2-t)-\kappa_n(T_1-t)}\Big).
\end{aligned} \tag{B.49}$$

The last two terms appear in the formula (B.47), but the first two not, so summing this first terms (with opposite sign) to the rest of the formula (B.47) we should get the part of (B.48) not used yet, that is

$$\frac{\sigma_r\beta_r(T_1,T_2)\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}\beta_r(t,T_1).$$

The first part of (B.47) yields

$$\rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_r\kappa_n}\left[-\frac{1}{\kappa_r}(e^{-\kappa_r(T_2-T_1)}-e^{-\kappa_r(T_2-t)})+\beta_r(t,T_1)\right]=\rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_n}\beta_r(T_1,T_2)\beta_r(t,T_1). \tag{B.50}$$

Putting together the terms of (B.47) not used yet and the two that we have summed to obtain (B.49),

$$\begin{aligned}\frac{\sigma_r\rho_{nr}\sigma_n}{(\kappa_n+\kappa_r)\kappa_r\kappa_n}&\left[1-e^{-\kappa_r(T_2-T_2)}+e^{-\kappa_r(T_1-t)}-e^{-\kappa_r(T_2-t)}\right]\\&=-\frac{\sigma_r\rho_{nr}\sigma_n\kappa_r}{(\kappa_n+\kappa_r)\kappa_n}\beta_r(T_1,T_2)\beta_r(t,T_1).\end{aligned} \tag{B.51}$$

Adding the two terms (B.50) and (B.51)

$$\begin{aligned}\rho_{nr}\frac{\sigma_r\sigma_n}{\kappa_n}\beta_r(T_1,T_2)\beta_r(t,T_1)&-\frac{\sigma_r\rho_{nr}\sigma_n\kappa_r}{(\kappa_n+\kappa_r)\kappa_n}\beta_r(T_1,T_2)\beta_r(t,T_1)\\&=\frac{\sigma_r\beta_r(T_1,T_2)\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}\beta_r(t,T_1).\end{aligned} \tag{B.52}$$

Once obtained the last terms, taking out the things that are in common in all the expressions is obtained the expression (B.44)

$$\begin{aligned}b(t,T_1,T_2)=\sigma_r\beta_r(T_1,T_2)&\Bigg[\beta_r(t,T_1)\Bigg(\rho_{rI}\sigma_I-\frac{1}{2}\sigma_r\beta_r(t,T_1)\\&+\frac{\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}(1+\kappa_r\beta_n(t,T_1))\Bigg)-\frac{\rho_{nr}\sigma_n}{\kappa_n+\kappa_r}\beta_n(t,T_1)\Bigg].\end{aligned} \tag{B.53}$$

B.5 Inflation Linked Cap/Floor

In this section are obtained the expressions for the inflation cap and floors. Calling $\frac{I(t)}{I(T_{i-1})}=X$, $E(X)=m$ and $Std[ln(X)]=\nu$ and using the Black-Scholes generalized formula we get that

$$E([\omega(X-K)]^+)=wm\Phi\Big(w\frac{ln\frac{m}{(1+k)}+\frac{1}{2}\nu^2}{\nu}\Big)-\omega K\Phi\Big(w\frac{ln\frac{m}{(1+k)}-\frac{1}{2}\nu^2}{\nu}\Big). \tag{B.54}$$

The conditional expectation of $\frac{I(t)}{I(t_{i-1})}$ is easily obtained using (B.43). Taking into account that

$$\begin{aligned}YYIIS(0,T,\tau,N)&=N\tau_{1,A}\left[P_n(t,T_i)E_n^{T_i}\Big(\frac{I(t)}{I(t_{i-1})}|F_t\Big)-P_n(t,T_i)\right]\\&=N\tau_{i,A}\left[P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{b(t,T_{i-1},T_i)}-P_n(t,T_i)\right],\end{aligned} \tag{B.55}$$

is obtained that

$$E_n^{T_i}\left(\frac{I(T_i)}{I(T_{i-1})}|F_t\right) = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{b(t,T_{i-1},T_i)}. \tag{B.56}$$

As a change of measure does not affect the variance, it can be calculated under the risk-neutral measure Q,

$$Var^{T_i}\left(ln\frac{I(T_i)}{I(T_{i-1})}|F_t\right) = V^2(t,T_{i-1},T_i). \tag{B.57}$$

B.5.1 Variance of Cap formula

Here is explained how to obtain the expression $V^2(t,T_{i-1},T_i)$, it is not difficult but it is necessary to do some calculus. In order to do things easier, the final result is obtained part by part. It has to be obtained an analytical formula for

$$Var^{T_i}\left(ln\frac{I(T_i)}{I(T_{i-1})}|F_t\right) = V^2(t,T_{i-1},T_i). \tag{B.58}$$

The first step is to get $\frac{I(T_i)}{I(T_{i-1})}$. Remaining that under the risk-neutral measure Q the dymanics of the inflation index is

$$dI(t) = [n(t)-r(t)]dt + \sigma_I dW_I^Q(t),$$

then,

$$I(T_i) = I(t)\exp\left(\int_t^{T_i}[n(s)-r(s)-\frac{1}{2}\sigma_I^2]ds + \sigma_I\int_t^{T_i}dW_I^Q(s)\right),$$

what yields that

$$ln\frac{I(T_i)}{I(T_{i-1})} = \int_{T_{i-1}}^{T_i}[n(s)-r(s)-\frac{1}{2}\sigma_I^2]ds + \sigma_I\int_{T_{i-1}}^{T_i}dW_I^Q(s). \tag{B.59}$$

In the section B.2 have been obtained that

$$\int_0^{T_i}n(s)ds = -lnP_n(0,T_i) + \frac{\sigma_n^2}{2}\int_0^{T_i}\beta_n(0,s)^2ds + \sigma_n\int_0^{T_i}\beta_n(s,T_i)dW_n^Q(s). \tag{B.60}$$

Using this expression is obtained the integral between T_{i-1} and T_i as

$$\int_{T_{i-1}}^{T_i}n(s)ds = \int_0^{T_i}n(s)ds - \int_0^{T_{i-1}}n(s)ds.$$

For the integral of the real rate it has to be made the same, but in this case the formula is

$$\int_0^{T_i} r(s)ds = -lnP_r(0,T_i) + \frac{\sigma_r^2}{2}\int_0^{T_i} \beta_r(0,s)^2 ds$$
$$-\sigma_r\sigma_I\rho_{rI}\int_0^{T_i} \beta_r(0,s)ds + \sigma_r\int_0^{T_i} \beta_r(s,T_i)dW_r^Q(s). \tag{B.61}$$

Since it has to been calculated the variance of (B.59), we are only interested in the random parts (the integral between 0 and t is known). In fact, what we have to obtain is

$$Var\Bigg[\sigma_n\int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n\int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)$$
$$-\sigma_r\int_t^{T_i} \beta_r(s,T_i)dW_r^Q(s) + \sigma_r\int_t^{T_{i-1}} \beta_r(s,T_{i-1})dW_r^Q(s) \tag{B.62}$$
$$+\sigma_I W_I^Q(T_i - T_{i-1})|F_t\Bigg],$$

What can be represented as

$$Var\Bigg[\sigma_n\int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n\int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)|F_t\Bigg]$$
$$+Var\Bigg[\sigma_r\int_t^{T_i} \beta_r(s,T_i)dW_r^Q(s) + \sigma_r\int_t^{T_{i-1}} \beta_r(s,T_{i-1})dW_r^Q(s)|F_t\Bigg]$$
$$+Var(\sigma_I W_I^Q(T_i - T_{i-1})|F_t)$$
$$+2Cov\Bigg[\sigma_n\int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n\int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s), \sigma_I\int_{T_{i-1}}^{T_1} dW_I^Q(s)|F_t\Bigg]$$
$$-2Cov\Bigg[\sigma_r\int_t^{T_i} \beta_r(s,T_i)dW_r^Q(s) + \sigma_r\int_t^{T_{i-1}} \beta_r(s,T_{i-1})dW_r^Q(s), \sigma_I\int_{T_{i-1}}^{T_1} dW_I^Q(s)|F_t\Bigg]$$
$$-2Cov\Bigg[\sigma_n\int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n\int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s),$$
$$\sigma_r\int_t^{T_i} \beta_r(s,T_i)dW_r^Q(s) + \sigma_r\int_t^{T_{i-1}} \beta_r(s,T_{i-1})dW_r^Q(s)|F_t\Bigg]. \tag{B.63}$$

The first and the second part are equal, so only the first one is calculated,

$$
\begin{aligned}
Var\Big[\sigma_n \int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)|F_t\Big] \\
= \sigma_n^2 E\Big[\int_t^{T_i} \beta_n(s,T_i)^2 ds + \int_t^{T_{i-1}} \beta_n(s,T_{i-1})^2 ds \qquad \text{(B.64)} \\
-2\int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)|F_t\Big].
\end{aligned}
$$

Solving part by part,

$$
\begin{aligned}
\sigma_n^2 E\Big(\int_t^{T_i} \beta_n(s,T_i)^2 ds|F_t\Big) &= \frac{\sigma_n^2}{\kappa_n^2}\int_t^{T_i}(1-2e^{-\kappa_n(T_i-s)} + e^{-2\kappa_n(T_i-s)})ds \\
&= \frac{\sigma_n^2}{\kappa_n^2}[T_i - t - \frac{2}{\kappa_n} + \frac{2}{\kappa_n}e^{-\kappa_n(T_i-t)} + \frac{1}{2\kappa_n} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-t)}].
\end{aligned}
\qquad \text{(B.65)}
$$

Using the expression above is easy to obtain the second part

$$
\begin{aligned}
\sigma_n^2 E\Big(\int_t^{T_{i-1}} \beta_n(s,T_{i-1})^2 ds|F_t\Big) = \\
= \frac{\sigma_n^2}{\kappa_n^2}[T_{i-1} - t - \frac{2}{\kappa_n} + \frac{2}{\kappa_n}e^{-\kappa_n(T_{i-1}-t)} + \frac{1}{2\kappa_n} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_{i-1}-t)}].
\end{aligned}
\qquad \text{(B.66)}
$$

To solve the last part is necessary to take into account that the Brownian motion is linearly independent when the limits of the integral are disjointed, this yields

$$
\begin{aligned}
-2\sigma_n^2 E\Big(\int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)|F_t\Big) \\
= -2\sigma_n^2 E\Big(\Big[\int_t^{T_{i-1}} \beta_n(s,T_i)dW_n^Q(s) + \int_{T_{i-1}}^{T_i} \beta_n(s,T_i)dW_n^Q(s)\Big] \\
\cdot \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)|F_t\Big) \\
= -2\sigma_n^2 \int_t^{T_{i-1}} \beta_n(s,T_i)dW_n^Q(s) \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s) \\
= -2\sigma_n^2 \int_t^{T_{i-1}} \beta_n(s,T_i)\beta_n(s,T_{i-1})ds - 2\frac{\sigma_n^2}{\kappa_n^2}\Big[T_{i-1} - t - \frac{1}{\kappa_n} + \frac{1}{\kappa_n}e^{-\kappa_n(T_{i-1}-t)} \\
-\frac{1}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} + \frac{1}{\kappa_n}e^{-\kappa_n(T_i-t)} + \frac{1}{2\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} - \frac{1}{2\kappa_n}e^{-\kappa_n(T_i+T_{i-1}-2t)}\Big].
\end{aligned}
$$

Joining all the parts and simplifying is obtained

$$
\begin{aligned}
&Var\left[\sigma_n \int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)|F_t\right] \\
&= \frac{\sigma_n^2}{\kappa_n^2}\left[T_i - T_{i-1} - 2t - 2T_{i-1} + 2t - \frac{4}{\kappa_n} + \frac{2}{\kappa_n} + \frac{1}{\kappa_n} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-t)}\right. \\
&\left. - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_{i-1}-t)} + \frac{2}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} - \frac{1}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} + \frac{1}{\kappa_n}e^{-\kappa_n(T_i+T_{i-1}-2t)}\right] \\
&= \frac{\sigma_n^2}{\kappa_n^2}\left[T_i - T_{i-1} + \frac{2}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-T_{i-1})} - \frac{3}{2\kappa_n}\right] \\
&+ \frac{\sigma_n^2}{\kappa_n^2}\left[\frac{1}{2\kappa_n} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_{i-1}-t)} - \frac{1}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})}\right. \\
&\left. + \frac{1}{\kappa_n}e^{-\kappa_n(T_i+T_{i-1}-2t)} + \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-T_{i-1})} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-t)}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{1}{2\kappa_n} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_{i-1}-t)} - \frac{1}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} \\
&+ \frac{1}{\kappa_n}e^{-\kappa_n(T_i+T_{i-1}-2t)} + \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-T_{i-1})} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-t)} \\
&= \frac{1}{2\kappa_n}(1 - e^{-\kappa_n(T_i-T_{i-1})})^2(1 - e^{-2\kappa_n(T_{i-1}-t)}) \\
&= \frac{\kappa_n}{2}\beta_n(T_{i-1},T_i)^2(1 - e^{-2\kappa_n(T_{i-1}-t)}).
\end{aligned}
$$

So, the first three parts of (B.63) can be represented as

$$
\begin{aligned}
&\frac{\sigma_n^2}{2\kappa_n}\beta_n(T_{i-1},T_i)^2(1 - e^{-2\kappa_n(T_{i-1}-t)}) + \sigma_i^2(T_i - T_{i-1}) \\
&+ \frac{\sigma_n^2}{2\kappa_r}\beta_r(T_{i-1},T_i)^2(1 - e^{-2\kappa_r(T_{i-1}-t)}) \\
&+ \frac{\sigma_n^2}{\kappa_n^2}\left[T_i - T_{i-1} + \frac{2}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} - \frac{1}{2\kappa_n}e^{-2\kappa_n(T_i-T_{i-1})} - \frac{3}{2\kappa_n}\right] \\
&\frac{\sigma_r^2}{\kappa_r^2}\left[T_i - T_{i-1} + \frac{2}{\kappa_r}e^{-\kappa_r(T_i-T_{i-1})} - \frac{1}{2\kappa_r}e^{-2\kappa_r(T_i-T_{i-1})} - \frac{3}{2\kappa_r}\right].
\end{aligned}
\tag{B.67}
$$

Next step is to give a value to the first and second covariances, the structure of both of them is the same so only the first one is calculated.

$$\begin{aligned}
&+2Cov\left[\sigma_n \int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s), \sigma_I \int_{T_{i-1}}^{T_1} dW_I^Q(s)|F_t\right] \\
&= 2E\left[\left(\sigma_n \int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s)\right) \cdot \sigma_I \int_{T_{i-1}}^{T_1} dW_I^Q(s)|F_t\right] \\
&= 2E\left[\sigma_n\sigma_I \int_{T_{i-1}}^{T_i} \beta_n(s,T_i)dW_n^Q(s) \int_{T_{i-1}}^{T_1} dW_I^Q(s)|F_t\right] \\
&= 2\sigma_n\sigma_I\rho_{nI} \int_{T_{i-1}}^{T_i} \beta_n(s,T_i)ds = \frac{2\sigma_n\sigma_I\rho_{nI}}{\kappa_n}\left[T_i - T_{i-1} - \beta_n(T_{i-1},T_i)\right].
\end{aligned} \tag{B.68}$$

Finally, with long but standard calculations the last one is obtained.

$$\begin{aligned}
&-2Cov\Bigg[\sigma_n \int_t^{T_i} \beta_n(s,T_i)dW_n^Q(s) - \sigma_n \int_t^{T_{i-1}} \beta_n(s,T_{i-1})dW_n^Q(s), \\
&\qquad \sigma_r \int_t^{T_i} \beta_r(s,T_i)dW_r^Q(s) + \sigma_r \int_t^{T_{i-1}} \beta_r(s,T_{i-1})dW_r^Q(s)|F_t\Bigg] \\
&= -2\sigma_n\sigma_r \int_t^{T_i} \beta_n(s,T_i)\beta_r(s,T_i)\rho_{nr}ds - 2\sigma_n\sigma_r \int_t^{T_{i-1}} \beta_n(s,T_{i-1})\beta_r(s,T_{i-1})\rho_{nr}ds \\
&+2\sigma_n\sigma_r \int_t^{T_{i-1}} \beta_n(s,T_i)\beta_r(s,T_{i-1})\rho_{nr}ds + 2\sigma_n\sigma_r \int_t^{T_{i-1}} \beta_n(s,T_{i-1})\beta_r(s,T_i)\rho_{nr}ds.
\end{aligned} \tag{B.69}$$

The first two expressions are obtained in the same way and the third and fourth similarly.

$$\begin{aligned}
&-2\sigma_n\sigma_r\rho_{nr} \int_t^{T_i} \beta_n(s,T_i)\beta_r(s,T_i)\rho_{nr}ds \\
&= -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r} \int_t^{T_i} 1 - e^{-\kappa_r(T_i-s)} - e^{-\kappa_n(T_i-s)} + e^{-(\kappa_r+\kappa_n)(T_i-s)}ds \\
&= -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\Bigg[T_i - t - \frac{1}{\kappa_r} + \frac{1}{\kappa_r}e^{-\kappa_r(T_i-t)} - \frac{1}{\kappa_n} + \frac{1}{\kappa_n}e^{-\kappa_n(T_i-t)} \\
&\qquad + \frac{1}{\kappa_r+\kappa_n} - \frac{1}{\kappa_r+\kappa_n}e^{-(\kappa_r+\kappa_n)(T_i-t)}\Bigg].
\end{aligned} \tag{B.70}$$

So the second part is

$$-2\sigma_n\sigma_r\int_t^{T_{i-1}}\beta_n(s,T_{i-1})\beta_r(s,T_{i-1})\rho_{nr}ds = -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\Bigg[T_{i-1}-t-\frac{1}{\kappa_r}$$
$$+\frac{1}{\kappa_r}e^{-\kappa_r(T_{i-1}-t)}-\frac{1}{\kappa_n}+\frac{1}{\kappa_n}e^{-\kappa_n(T_{i-1}-t)}+\frac{1}{\kappa_r+\kappa_n}-\frac{1}{\kappa_r+\kappa_n}e^{-(\kappa_r+\kappa_n)(T_{i-1}-t)}\Bigg]. \tag{B.71}$$

Solving the third part is obtained

$$2\sigma_n\sigma_r\int_t^{T_{i-1}}\beta_n(s,T_{i-1})\beta_r(s,T_i)\rho_{nr}ds$$
$$=2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\int_t^{T_{i-1}}1-e^{-\kappa_r(T_{i-1}-s)}-e^{-\kappa_n(T_i-s)}+e^{-\kappa_nT_i-\kappa_rT_{i-1}+(\kappa_r+\kappa_n)s}ds$$
$$=2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\Bigg[T_{i-1}-t-\frac{1}{\kappa_r}+\frac{1}{\kappa_r}e^{-\kappa_r(T_{i-1}-t)}-\frac{1}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})}+\frac{1}{\kappa_n}e^{-\kappa_n(T_i-t)}$$
$$+\frac{1}{\kappa_r+\kappa_n}e^{-\kappa_n(T_i-T_{i-1})}-\frac{1}{\kappa_r+\kappa_n}e^{-\kappa_n(T_i-t)-\kappa_r(T_{i-1}-t)}\Bigg]. \tag{B.72}$$

Using the previous one is straightforward to obtain the fourth one

$$2\sigma_n\sigma_r\int_t^{T_{i-1}}\beta_n(s,T_{i-1})\beta_r(s,T_i)\rho_{nr}ds = 2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\Bigg[T_{i-1}-t-\frac{1}{\kappa_n}+\frac{1}{\kappa_n}e^{-\kappa_n(T_{i-1}-t)}$$
$$-\frac{1}{\kappa_r}e^{-\kappa_r(T_i-T_{i-1})}+\frac{1}{\kappa_r}e^{-\kappa_r(T_i-t)}+\frac{1}{\kappa_r+\kappa_n}e^{-\kappa_k(T_i-T_{i-1})}-\frac{1}{\kappa_r+\kappa_n}e^{-\kappa_r(T_i-t)-\kappa_n(T_{i-1}-t)}\Bigg]. \tag{B.73}$$

Summing the four parts the expression for the last covariance is obtained

$$
\begin{aligned}
&-2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\Bigg[T_i - T_{i-1} - \frac{1}{\kappa_r} - \frac{1}{\kappa_n} + \frac{2}{\kappa_r+\kappa_r} + \frac{1}{\kappa_n}e^{-\kappa_n(T_i-T_{i-1})} + \frac{1}{\kappa_r}e^{-\kappa_r(T_i-T_{i-1})} \\
&+\frac{1}{\kappa_r+\kappa_n}\Bigg(- e^{-(\kappa_r+\kappa_n)(T_i-t)} - e^{-(\kappa_r+\kappa_n)(T_{i-1}-t)} - e^{-\kappa_n(T_i-T_{i-1})} - e^{-\kappa_r(T_i-T_{i-1})} \\
&\qquad +e^{-\kappa_n(T_i-t)-\kappa_r(T_{i-1}-t)} + e^{-\kappa_r(T_i-t)-\kappa_n(T_{i-1}-t)}\Bigg)\Bigg] \\
&= -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\left[T_i - T_{i-1} - \beta_n(T_{i-1},T_i) - \beta_r(T_{i-1},T_i) + \frac{1-e^{-(\kappa_n+\kappa_r)(T_i-T_{i-1})}}{\kappa_n+\kappa_r}\right] \\
&-2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r(\kappa_n+\kappa_r)}\Bigg[e^{-(\kappa_n+\kappa_r)(T_i-T_{i-1})} + 1 - e^{-(\kappa_r+\kappa_n)(T_i-t)} - e^{-(\kappa_r+\kappa_n)(T_{i-1}-t)} \\
&\quad -e^{-\kappa_n(T_i-T_{i-1})} - e^{-\kappa_r(T_i-T_{i-1})} + e^{-\kappa_n(T_i-t)-\kappa_r(T_{i-1}-t)} + e^{-\kappa_r(T_i-t)-\kappa_n(T_{i-1}-t)}\Bigg] \\
&= -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\left[T_i - T_{i-1} - \beta_n(T_{i-1},T_i) - \beta_r(T_{i-1},T_i) + \frac{1-e^{-(\kappa_n+\kappa_r)(T_i-T_{i-1})}}{\kappa_n+\kappa_r}\right] \\
&\qquad -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\frac{(1-e^{-\kappa_n(T_i-T_{i-1})})}{\kappa_n}\frac{(1-e^{-\kappa_r(T_i-T_{i-1})})}{\kappa_r}(1-e^{-(\kappa_r+\kappa_n)(T_{i-1}-t)}) \\
&= -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\left[T_i - T_{i-1} - \beta_n(T_{i-1},T_i) - \beta_r(T_{i-1},T_i) + \frac{1-e^{-(\kappa_n+\kappa_r)(T_i-T_{i-1})}}{\kappa_n+\kappa_r}\right] \\
&\qquad -2\rho_{nr}\frac{\sigma_n\sigma_r}{\kappa_n\kappa_r}\beta_n(T_{i-1},T_i)\beta_r(T_{i-1},T_i)(1-e^{-(\kappa_r+\kappa_n)(T_{i-1}-t)}).
\end{aligned}
\tag{B.74}
$$

Putting together all the variances and covariances the expression (1.52) of the second chapter is obtained.

Appendix C

Development of the formulas of the second chapter

C.1 Development of the formulas of the estimation of κ_n

In this appendix are developed the formulas proposed in [9] to estimate independently κ_n.

The first thing to do is defining on a general form the next expressions:

- Variance of the instantaneous short rate:

  $$V_n(s,t) = V(n(t)|F_s),$$

  where F_s is the σ-field that captures the information generated by $n(t)$ until s.

- $E(t) = e^{\int_0^t \kappa_n(u)du}$.

With the second expression one can define the variance of the short rate as

$$V_n(0,t) = \frac{1}{E^2(t)} \int_0^t E^2(u)\sigma^2(u)du, \tag{C.1}$$

and the expression $\beta_n(t,T)$ as

$$\beta_n(t,T) = E(t) \int_t^T \frac{du}{E(u)}. \tag{C.2}$$

If we take both parameters constant as done in the second chapter, next expressions are obtianed:

$$V_n(0,t) = \frac{\sigma_n^2}{2\kappa_n}(1 - e^{-2\kappa_n t}), \tag{C.3}$$

$$\beta_n(t,T) = \frac{1}{\kappa_n}(1 - e^{-\kappa_n (T-t)}). \tag{C.4}$$

Note that the formula of $\beta_n(t,T)$ is the one defined in Chapter 1. Recall the expression nominal bond in time t and maturity T

$$\begin{aligned} P_n(t,T) &= \frac{P_n(0,T)}{P_n(0,t)}\exp\left(\beta_n(t,T)[f_n(0,t)-n(t)] - \frac{\sigma_n^2}{4\kappa_n}\beta_n^2(t,T)[1-e^{-2\kappa_n t}]\right) \\ &= \frac{P_n(0,T)}{P_n(0,t)}\exp\left(\beta_n(t,T)[f_n(0,t)-n(t)] - \frac{1}{2}V_n(0,t)\beta_n^2(t,T]\right), \end{aligned}$$

and the process of the bond price

$$\frac{dP_n(t,T)}{P_n(t,T)} = n(t)dt + a_n(t,T)dW_n^Q(t), \tag{C.5}$$

where

$$a_n(t,T) = -\int_t^T \sigma_n(t,u)du = -\sigma_n\beta_n(t,T). \tag{C.6}$$

The stochastic process of $d(P_n(t,T_1)/P_n(t,T_2))$ under the risk-neutral measure with $T_1 < T_2$ is needed to obtain the expression of the implied volatility of the swaptions. Using the Ito's Lemma,

$$\begin{aligned} d\frac{P_n(t,T_1)}{P_n(t,T_2)} &= 0\cdot dt + \frac{1}{P_n(t,T_2)}dP_n(t,T_1) - \frac{P_n(t,T_1)}{P_n^2(t,T_2)}dP_n(t,T_2) \\ &\quad + \frac{P_n(t,T_1)}{P_n^3(t,T_2)}(dP_n(t,T_2))^2 - \frac{1}{P_n^2(t,T_2)}dP_n(t,T_1)dP_n(t,T_2) \\ &= \frac{P_n(t,T_1)}{P_n(t,T_2)}[n(t)dt + a_n(t,T_1)dW_n^Q(t)] - \frac{P_n(t,T_1)}{P_n(t,T_2)}[n(t)dt + a_n(t,T_2)dW_n^Q(t)] \\ &\quad + \frac{P_n(t,T_1)}{P_n(t,T_2)}a_n^2(t,T_2)dt - \frac{P_n(t,T_1)}{P_n(t,T_2)}a_n(t,T_1)a_n(t,T_2)dt \\ &= \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n[\beta_n(t,T_2)-\beta_n(t,T_1)]dW_n^Q(t) \\ &\quad + \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n^2[\beta_n^2(t,T_2) - \beta_n(t,T_2)\beta_n(t,T_1)]. \end{aligned}$$

Next step is to obtain this expression under the forward measure. For this purpose the Proposition of numeraire change is used, which gives us the new drift, since the diffusion is the same. The drift under the forward measure is (Proposition A.1):

$$\mu^{T_2}(t) = \mu^Q(t) - \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n[\beta_n(t,T_2)-\beta_n(t,T_1)](-a_n(t,T_2)). \tag{C.7}$$

Developing a little bit, the drift is cancelled and the expression above yields

$$d\frac{P_n(t,T_1)}{P_n(t,T_2)} = \frac{P_n(t,T_1)}{P_n(t,T_2)}\sigma_n[\beta_n(t,T_2)-\beta_n(t,T_1)]dW_n^{T_2}(t), \tag{C.8}$$

with integrated variance

$$
\begin{aligned}
V_p(0,T_1,T_2) &= \int_0^{T_1} \sigma_n^2[\beta_n(u,T_2)-\beta_n(u,T_1)]^2 du \\
&= \frac{\sigma_n^2}{\kappa_n^2}\int_0^{T_1} \left[e^{-\kappa_n(T_2-u)} - e^{-\kappa_n(T_1-u)}\right]^2 du \\
&= \frac{\sigma_n^2}{\kappa_n^2}\int_0^{T_1} \left[e^{-2\kappa_n(T_1-u)} - 2e^{-\kappa_n(T_1+T_2-2u)} + e^{-2\kappa_n(T_2-u)}\right] du \\
&= \frac{\sigma_n^2}{\kappa_n^2}\left(\frac{1}{2\kappa_n}(e^{-2\kappa_n T_1}) - \frac{1}{\kappa_n}(1-e^{-\kappa_n(T_2-T_1)} - e^{-\kappa_n(T_1+T_2)})\right) \\
&\quad + \frac{\sigma_n^2}{\kappa_n^2}\left(\frac{1}{2\kappa_n}(e^{-2\kappa_n(T_2-T_1)} - e^{-2\kappa_n T_2})\right) \\
&= \frac{\sigma_n^2}{2\kappa_n^3}(1-e^{-2\kappa_n T_1})(1-e^{-\kappa_n(T_2-T_1)})^2.
\end{aligned}
$$

So, the integrated variance of the bond ratio is:

$$V_p(0,T_1,T_2) = V_n(0,T_1)\beta_n(T_1,T_2)^2. \tag{C.9}$$

Using this the swaptions implied volatility is obtained. The idea is to do an approximation to the fixed rate swap with the intention of obtaining the stochastic process of it, to finally calculate the implied volatility. Recall the expression (2.5) but with a general $\tilde{T_0} = T_0 \neq 0$, taking into account that we are working with a market swap, what means that the fixed rate swap is agreed so that the value of the swap is 0, we have:

$$S(t,T_0,T_n) = \frac{P_n(t,T_0) - P(t,T_n)}{\sum_{i=1}^n \tau_{i,B} P_n(t,T_i)}, \tag{C.10}$$

where T_0 is the maturity of the swaption and T_n the tenor of the swap.

We could have written the expression as we did before but we have decided to put on a more general form without using the assumptions that we did. To get a formula of the ratio of implied volatilities that do not contain the parameter σ_n the approximated fixed swap rate $\tilde{S}(t,T_0,T_n)$ proposed in [9] is used. Is also assumed that this rate has log-normal distribution under the annuity measure A (Proposition A.2.1)

$$\tilde{S}(t,T_0,T_n) = \frac{P_n(0,T_n)}{\sum_{i=1}^n \tau_{i,B} P_n(0,T_i)}\left[\frac{P_n(t,T_0)}{P_n(t,T_n)} - 1\right]. \tag{C.11}$$

Knowing the process of $d(P_n(t,T_0)/P_n(t,T_n))$ in (C.8) and using Ito's Lemma it is easy to get:

$$\frac{d\tilde{S}(t,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} = \frac{1}{\tilde{S}(t,T_0,T_n)} \frac{P_n(0,T_n)}{\sum_{i=1}^{n} \tau_{i,B} P_n(0,T_i)} d\frac{P_n(t,T_0)}{P_n(t,T_n)}$$

$$= \frac{1}{\tilde{S}(t,T_0,T_n)} \frac{P_n(0,T_n)P_n(t,T_0)}{P_n(t,T_n)\sum_{i=1}^{n} \tau_{i,B} P_n(0,T_i)} \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{T_n}(t)$$

$$= \frac{S(0,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} \frac{P_n(0,T_n)P_n(t,T_0)}{[P(0,T_0) - P_n(0,T_n)]P_n(t,T_n)} \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{T_n}(t).$$

Our intention is to get an approximation of the implied volatility of the swaptions. For this, our proposal is to substitute the approximated fixed rate and bond prices by their initial values so that we get a known expression. The resultant is:

$$\frac{d\tilde{S}(t,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} \simeq \frac{S(0,T_0,T_n)}{\tilde{S}(0,T_0,T_n)} \frac{P_n(0,T_n)P_n(0,T_0)}{[P(0,T_0) - P_n(0,T_n)]P_n(0,T_n)} \cdot \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{T_n}(t)$$

$$= \frac{P_n(0,T_0)}{P(0,T_0) - P_n(0,T_n)} \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{T_n}(t).$$

Finally, change to the annuity measure A, since it has been assumed that under A $\tilde{S}(t,T_0,T_n)$ has log-normal distribution. There is a drift, but we do not care about it because we want to calculate the integrated variance and for this only the diffusion is needed.

$$\frac{d\tilde{S}(t,T_0,T_n)}{\tilde{S}(t,T_0,T_n)} \simeq drift + \frac{P_n(0,T_0)}{P(0,T_0) - P_n(0,T_n)} \sigma_n[\beta_n(t,T_n) - \beta_n(t,T_0)]dW_n^{A}(t). \tag{C.12}$$

The expression (C.12) is similar to (C.8), and knowing the resultant integrated variance gotten before, it is easy to find that

$$V_{swap}(T_0,T_n) = \int_0^{T_0} \left(\frac{P_n(0,T_0)}{P(0,T_0) - P_n(0,T_n)} \sigma_n[\beta_n(u,T_n) - \beta_n(u,T_0)] \right)^2 du$$

$$= \left(\frac{P_n(0,T_0)}{P(0,T_0) - P_n(0,T_n)} \right)^2 V_p(0,T_0,T_n), \tag{C.13}$$

where $V_p(0,T_0,T_n) = V_n(0,T_0)\beta_n(T_0,T_n)^2$ is the variance of the bond ratio. It is important to remark that this is only an approximation to do the calibration of the nominal mean reversion speed κ_n and so that it can not be used as real expressions for the implied volatility.

We realize that the Hull and White implied volatility depends on the parameter σ_n only in the term $V_n(0,T_0)$. So, if we take the ratio of different implied volatilities with the same maturity for the swaptions but different tenors, the result does not depend on σ_n and it will be possible to estimate κ_n.

Taking the ratio of two different implied variances with the characteristics mentioned before,

$$\frac{V_{swap}(M_i,T_j)}{V_{swap}(M_i,T_k)} = \left(\frac{[P_n(0,M_i)-P_n(0,T_k)]\beta(M_i,T_j)}{[P_n(0,M_i)-P_n(0,T_j)]\beta(M_i,T_k)}\right)^2. \tag{C.14}$$

C.2 Development of the cap formula under the Hull and White model

Since under the Hull and White model the nominal interest rate is log-normally distributed, the value of each caplet has to be represented in terms of a European put option on a zero-coupon bond.

$$\begin{aligned}
Cpl(t,t_{i-1},t_i,\tau_i,N,X) &= E\left(e^{-\int_t^{t_i} n(s)ds}N\tau_i(R(t_{i-1},t_i)-X)^+|F_t\right)\\
&= E\left(E\left(e^{-\int_t^{t_i} n(s)ds}N\tau_i(R(t_{i-1},t_i)-X)^+|F_{t-1}\right)|F_t\right)\\
&= E\left(e^{-\int_t^{t_{i-1}} n(s)ds}N\tau_i(R(t_{i-1},t_i)-X)^+E\left(e^{-\int_{t_{i-1}}^{t_i} n(s)ds}|F_{t-1}\right)|F_t\right)\\
&= E\left(e^{-\int_t^{t_{i-1}} n(s)ds}P_n(t_{i-1},t_i)N\tau_i(R(t_{i-1},t_i)-X)^+|F_t\right).
\end{aligned}$$

Taking into account that

$$P_n(t_{i-1},t_i) = \frac{1}{1+\tau_i R(t_{i-1},t_i)}, \tag{C.15}$$

The value of the caplet in t is

$$\begin{aligned}
Cpl(t,t_{i-1},t_i,\tau_i,N,X) &=\\
&= NE\left(e^{-\int_t^{t_{i-1}} n(s)ds}P_n(t_{i-1},t_i)\Big[\frac{1}{P_n(t_{i-1},t_i)}-(1+X\tau_i)\Big]^+|F_t\right)\\
&= N(1+X\tau_i)E\left(e^{-\int_t^{t_{i-1}} n(s)ds}\Big[\frac{1}{(1+X\tau_i)}-P_n(t_{i-1},t_i)\Big]^+|F_t\right).
\end{aligned} \tag{C.16}$$

To obtain closed formulas for the cap are needed the ones for the European put option with maturity T and strike X on a unit-principal zero-coupon bond with maturity $S > T$. The value of this derivative in t is

$$ZBP(t,T,S,X) = E\Bigg(e^{\int_t^T n(s)ds}(X - P_n(T,S))^+ | F_t\Bigg). \tag{C.17}$$

The change of numeraire to the forward measure Q^T allows us to express the above as

$$ZBP(t,T,S,X) = P_n(t,T)E^T\Bigg((X - P_n(T,S))^+ | F_t\Bigg). \tag{C.18}$$

What it can be represented as

$$ZBP(t,T,S,X) = XP_n(t,T)\Phi(-h+\sigma_p) - P_n(t,S)\Phi(-h), \tag{C.19}$$

where

$$\begin{aligned}\sigma_p &= \sigma_n\sqrt{\frac{1-e^{-2\kappa_n(T-t)}}{2\kappa_n}}\beta_n(T,S),\\ h &= \frac{1}{\sigma_p}ln\frac{P_n(t,S)}{P_n(t,T)X} + \frac{\sigma_p}{2}.\end{aligned} \tag{C.20}$$

The valuation formula (C.16) of the caplet is very similar to the one of the ZBP (C.17). Using the closed formula just obtained for the ZBP is easy to get the one for each caplet

$$Cpl(t,t_{i-1},t_i,\tau_i,N,X) = N_i'ZBP(t,t_{i-1},t_i,X_i'), \tag{C.21}$$

where,

$$\begin{aligned}N_i' &= \frac{1}{1+X\tau_i},\\ N_i' &= N(1+X\tau_i).\end{aligned} \tag{C.22}$$

So, the valuation formula for the cap in t is

$$\begin{aligned}Cap(t,T,N,X) &= N\sum_{i=1}^{n}(1+X\tau_i)ZBP(t,t_{i-1},t_i,\frac{1}{1+X\tau_i}),\\ &= N\sum_{i=1}^{n}[P_(t,t_{i-1})\Phi(-h_i+\sigma_p^i) - (1+X\tau_i)P_n(t,t_i)\Phi(-h_i)],\end{aligned} \tag{C.23}$$

where,

$$\begin{aligned}\sigma_p^i &= \sigma_n\sqrt{\frac{1-e^{-2\kappa_n(t_{i-1}-t)}}{2\kappa_n}}\beta_n(t_{i-1},t_i),\\ h_i &= \frac{1}{\sigma_p^i}ln\frac{P_n(t,t_i)(1+X\tau_i)}{P_n(t,t_{i-1})} + \frac{\sigma_p^i}{2}.\end{aligned} \tag{C.24}$$

Appendix D

CVA formulas

Survival function

Remembering the definition of the default intensity

$$P(\tau \leq t + dt | \tau > t | F_{t_0}) = \lambda(t) dt, \tag{D.1}$$

the procedure to obtain the survival function is the next one. On the one hand is the equality

$$P(\tau > t + dt | F_{t_0}) = P(\tau > t + dt \cap \tau > t | F_{t_0}). \tag{D.2}$$

On the other hand there is the definition of the conditional probability

$$\begin{aligned} P(\tau > t + dt \quad \cap \quad \tau > t | F_{t_0}) &= P(\tau > t + dt | \tau > t | F_{t_0}) P(\tau > t | F_{t_0}) \\ &= (1 - \lambda(t) dt) P(\tau > t | F_{t_0}). \end{aligned}$$

So,

$$P(\tau > t + dt | F_{t_0}) = P(\tau > t | F_{t_0}) - \lambda(t) P(\tau > t | F_{t_0}) dt. \tag{D.3}$$

Using the notation $P(t) = P(\tau > t | F_{t_0})$, the equation can be expressed as

$$P(t + dt) = P(t) - \lambda(t) P(t) dt \Longrightarrow dP(t) = -\lambda(t) P(t) dt \tag{D.4}$$

$$\Longrightarrow \frac{dP(t)}{P(t)} = -\lambda(t) dt \Longrightarrow \frac{P(t)}{P(t_0)} = e^{-\int_{t_0}^{t} \lambda(s) ds}. \tag{D.5}$$

Using the notation defined at the beginning

$$P(\tau > t | F_{t_0}) = P(\tau > t_0 | F_{t_0}) e^{-\int_{t_0}^{t} \lambda(s) ds} = 1_{\tau > t_0} e^{-\int_{t_0}^{t} \lambda(s) ds}. \tag{D.6}$$

Price of CDS

The first part of the formula is easily obtained taking into account that the spread is paid every 3 months as long as default does not occur. The part that corresponds to the received cash-flow in case of default is obtained using the iterative law of expectations.

$$\begin{aligned}(1-R)E\Big[e^{-\int_t^{\tau} n(s)ds}1_{\tau\leq T}|F_t\Big] &= (1-R)E\Big[E\Big(e^{-\int_t^{\tau} n(s)ds}1_{\tau\leq T}|\tau=y|F_t\Big)|F_t\Big] \\ &= (1-R)\int_t^{\infty} E\Big(e^{-\int_t^{\tau} n(s)ds}1_{\tau\leq T}|\tau=y|F_t\Big)\eta_\tau(y)dy \\ &= (1-R)\int_t^{\infty} E\Big(e^{-\int_t^{y} n(s)ds}1_{y\leq T}|F_t\Big)\eta_\tau(y)dy \\ = (1-R)\int_t^{T} E\Big(e^{-\int_t^{y} n(s)ds}|F_t\Big)\eta_\tau(y)dy &= (1-R)\int_t^{T} P_n(t,y)\eta_\tau(y)dy. \end{aligned} \tag{D.7}$$

Finally, is explained how to obtain the part corresponding to the accrued interest. If the default occurs before the maturity of the derivative and it is been a while since the last spread was paid, it has to be taking into account the accrued interest. The value in the default time is

$$V(\tau) = S_{CDS}(\tau - T_{\beta(\tau)})1_{\tau\leq T}, \tag{D.8}$$

where $T_{\beta(\tau)}$ is the last day where spread was paid. Then, the value in t is

$$V(t) = S_{CDS}E\Big(e^{-\int_t^{\tau} n(s)ds}(\tau - T_{\beta(\tau)})1_{\tau\leq T}\Big). \tag{D.9}$$

Using the iterative law of expectations as done previously is obtained

$$V(t) = S_{CDS}\int_t^{T} P_n(t,s)(s - T_{\beta(s)})\eta_\tau(s)ds \tag{D.10}$$

CVA

To calculate the expression of the CVA is necessary to calculate the value of the derivative if there is a default or if there is not. $V(t,T)$ is defined as the cash-flows between t and T, V_τ^+ is the value in τ if it is positive for us and V_τ^- if it is negative for us, $V(\tau) = V_\tau^+ + V_\tau^-$. Then,

$$X(t,T) = 1_{\tau>T}V(t,T) + 1_{\tau\leq T}[V(t,\tau) + RV_\tau^+ + V_\tau^-] \tag{D.11}$$

The value in t of the derivative is

$$V(t) = E\Big(e^{-\int_t^{T} n(s)ds}1_{\tau>T}V(t,T) + e^{-\int_t^{\tau} n(s)ds}1_{\tau\leq T}[V(t,\tau) + RV_\tau^+ + V_\tau^-]|F_t\Big) \tag{D.12}$$

Taking into account $1_{\tau>T} + 1_{\tau\leq T} = 1$, is obtained $V(t,T)1_{\tau>T} = V(t,T) - V(t,T)1_{\tau\leq T}$. Then,

$$\begin{aligned} X(t,T) &= V(t,T) + 1_{\tau\leq T}[V(t,\tau) - V(t,T)] + 1_{\tau\leq T}[RV_\tau^+ + V_\tau^- + V_\tau^+ - V_\tau^+] \\ &= V(t,T) + 1_{\tau\leq T}[V(t,\tau) - V(t,T)] + 1_{\tau\leq T}[-(1-R)V_\tau^+ + V(\tau)] \end{aligned} \tag{D.13}$$

Using the definition of $V(\tau)$ and the iterative law of expectations

$$E\Big(V(\tau)|F_t\Big) = E\Big(E[V(\tau,T)|F_\tau]|F_t\Big) = E\Big(V(\tau,T)|F_t\Big). \tag{D.14}$$

Then, calling to $E\Big(e^{-\int_t^T n(s)ds}V(t,T)|F_t\Big) = V(t)^{rf}$

$$V(t) = V(t)^{rf} + E\left(e^{-\int_t^\tau n(s)ds}1_{\tau\leq T}[-V(t,T)+V(t,\tau)+V(\tau,T)-(1-R)V_\tau^+]|F_t\right)$$
$$= V(t)^{rf} - E\left(e^{-\int_t^\tau n(s)ds}1_{\tau\leq T}(1-R)V_\tau^+|F_t\right)$$

Then, the CVA if there is independence between the default probability and the positive exposure is

$$CVA(t) = E\left(e^{-\int_t^\tau n(s)ds}1_{\tau\leq T}(1-R)V_\tau^+|F_t\right) = (1-R)\sum_{i=1}^{M} P_n(t,T_i)E(V_{T_i}^+|F_t)\Delta P_t \tag{D.15}$$

Appendix E

Extra Tables

Information about the swap inflation versus fixed with maturity Nov 2033 and counterparty Morgan Stanley is in Table E.1. The valuation and sensitivities in Table E.3.

SWAP INFLATION/FIXED 2033											
T_{final}	TT	Type of swap	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
14.688	14.667	1	1.5%	0	0	0.9%	-1	102.02533	332,000,000	345,495,688	Morgan Stanley

Table E.1: Information of the fixed swap with Counterparty Morgan Stanley.

Information about the swap inflation versus Euribor 12 months with maturity Nov 2024 and counterparty BBVA is in Table E.2. The valuation and sensitivities in Table E.4.

SWAP INFLATION/VARIABLE 2024											
T_{final}	TT	Type of swap	Swap rate 1	Swap rate 2	Type change date (Mixed swap)	Inflation coupon	Payer	Inf_0	N_{infL}	N_{nomL}	Counterparty
5.6822	5.667	2	0.5%	0	0	1.8%	-1	100.05803	50,000,000	59,463,157.1	BBVA

Table E.2: Information of the variable swap with Counterparty BBVA.

SWAP INFLATION/FIXED 2033													
	Valuation				Expected positive exposure			97.5% positive exposure			Expected shortfall		
	t=0	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$
original	-24,573,724.4	- 25,988,552.4	- 29,711,082.7	- 34,283,155.4	16,809,845.6	16,210,595.1	14,958,754.8	38,482,248.2	37,470,406.0	33,939,294.3	39,523,113.1	38,674,367.1	34,690,180.6
per million	- 74,017.2	- 78,278.8	- 89,491.2	- 103,262.5	50,632.1	48,827.1	45,056.5	115,910.4	112,862.7	102,226.8	119,045.5	116,489.1	104,488.5
σ_n*(1+0.5)	16.25%	13.83%	10.21%	9.43%	27.42%	26.26%	29.95%	7.36%	8.74%	7.18%	6.54%	8.16%	6.85%
σ_n*(1+0.1)	2.50%	2.10%	1.52%	1.41%	5.54%	4.81%	9.00%	0.58%	1.27%	1.34%	1.23%	0.91%	1.28%
σ_r*(1+0.5)	1.78%	1.68%	1.62%	1.42%	12.43%	11.71%	12.85%	1.67%	0.55%	1.35%	1.97%	0.08%	2.17%
σ_r*(1+0.1)	0.34%	0.32%	0.31%	0.27%	1.64%	0.75%	3.37%	0.72%	0.09%	0.17%	0.61%	-0.50%	0.41%
σ_I*(1+0.5)	0.020%	0.00%	-0.040%	-0.03%	3.49%	2.24%	7.81%	0.67%	1.16%	1.201%	0.740%	0.662%	1.10%
σ_I*(1+0.1)	-0.01%	-0.02%	-0.02%	-0.02%	0.53%	1.39%	2.75%	0.46%	0.03%	0.07%	0.25%	-0.60%	0.12%
$\sigma_n, \sigma_r, \sigma_I$*(1+0.1)	2.84%	2.42%	1.82%	1.68%	5.90%	9.25%	12.84%	0.69%	1.79%	1.46%	1.35%	1.66%	1.80%
κ_n*(1+0.5)	-2.92%	-2.32%	-1.47%	-1.38%	-9.59%	-9.06%	-6.49%	-1.10%	-2.16%	-1.63%	-1.15%	-2.74%	-1.52%
κ_r*(1+0.5)	1.89%	2.10%	1.70%	1.45%	-8.87%	-5.96%	-4.19%	-1.38%	0.16%	-0.89%	-1.01%	-0.57%	-1.30%
κ_n, κ_r*(1+0.5)	-0.94%	-0.13%	0.32%	0.15%	-15.92%	-17.20%	-16.98%	-2.77%	-3.04%	-3.26%	-2.02%	-3.69%	-3.85%
ρ_{nr}*(1-0.3)	0.09%	0.08%	0.06%	0.05%	0.15%	0.63%	1.34%	0.45%	0.05%	0.09%	0.08%	-0.54%	0.09%
ρ_{nI}*(1-0.3)	0.0049%	0.0043%	0.0039%	0.0034%	-0.35%	0.12%	0.14%	-0.0119%	0.0006%	0.0012%	0.0015%	0.0016%	0.0020%
ρ_{rI}*(1-0.3)	-0.04%	-0.03%	-0.03%	-0.03%	0.0008%	-0.0050%	-0.0076%	0.0053%	0.0018%	0.0011%	0.0033%	0.0029%	0.0017%
constant σ_n	-3.37%	-3.20%	-2.82%	-3.38%	1.20%	-0.35%	2.44%	-0.01%	0.45%	2.76%	0.52%	0.10%	2.13%
constant σ_I	-0.16%	-0.18%	-0.15%	-0.11%	0.43%	1.73%	3.49%	0.12%	0.07%	-0.07%	-0.003%	-0.52%	-0.16%
constant ρ_{nr}	-0.04%	-0.02%	-0.02%	-0.02%	0.49%	-0.53%	0.08%	-0.01%	-0.02%	-0.03%	-0.08%	-0.07%	-0.05%
constant ρ_{nI}	-0.01%	-0.01%	-0.005%	-0.003%	0.14%	-0.38%	0.20%	0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.03%
constant ρ_{rI}	0.26%	0.25%	0.22%	0.19%	-0.47%	-0.52%	0.73%	0.02%	0.001%	0.04%	0.02%	0.02%	0.04%
all constants	-3.34%	-3.18%	-2.77%	-3.31%	-4.46%	-7.61%	-6.57%	1.16%	-1.63%	4.04%	1.11%	-0.72%	2.22%

Table E.3: Obtained values for the fixed swap with counterparty Morgan Stanley.

SWAP INFLATION/VARIABLE 2024													
	Valuation				Expected positive exposure			97.5% positive exposure			Expected shortfall		
	t=0	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$	t=1	t=$T_{final}/5$	t=$T_{final}/3$
original	1,975,282.5	2,716,688.9	2,716,722.8	3,408,699.4	2,819,657.9	2,819,719.9	3,453,176.0	5,528,840.7	5,534,974.4	6,255,065.7	6,079,377.5	6,073,635.4	6,751,960.2
per million	39,505.7	54,333.8	54,334.5	68,174.0	56,393.2	56,394.4	69,063.5	110,576.8	110,699.5	125,101.3	121,587.6	121,472.7	135,039.2
σ_n*(1+0.5)	-1.27%	-0.76%	-0.73%	-0.62%	-0.41%	-0.17%	-0.47%	-0.258%	-0.78%	0.35%	-0.211%	-0.25%	-0.17%
σ_n*(1+0.1)	-0.18%	-0.11%	-0.10%	-0.09%	0.11%	-0.09%	-0.08%	0.09%	0.22%	0.07%	-0.03%	-0.04%	0.00%
σ_r*(1+0.5)	-1.51%	-1.08%	-1.09%	-0.90%	8.13%	8.36%	4.90%	16.95%	16.55%	14.16%	14.56%	14.51%	12.76%
σ_r*(1+0.1)	-0.25%	-0.18%	-0.18%	-0.15%	2.29%	2.28%	0.36%	3.54%	3.30%	2.63%	3.18%	3.12%	2.80%
σ_I*(1+0.5)	0.55%	0.40%	0.40%	0.33%	3.74%	3.62%	2.15%	12.48%	11.89%	9.42%	9.79%	9.75%	8.96%
σ_I*(1+0.1)	0.14%	0.10%	0.10%	0.08%	0.97%	1.07%	0.33%	2.64%	2.20%	1.73%	2.01%	2.03%	1.83%
$\sigma_n, \sigma_r, \sigma_I$*(1+0.5)	-0.32%	-0.20%	-0.20%	-0.17%	2.52%	2.41%	0.81%	5.02%	4.92%	4.56%	4.89%	4.88%	4.37%
κ_n*(1+0.5)	0.10%	0.05%	0.04%	0.03%	0.05%	0.15%	0.03%	-0.17%	-0.15%	0.34%	-0.06%	-0.06%	0.04%
κ_r*(1+0.5)	-4.91%	-3.85%	-3.84%	-3.01%	-5.40%	-5.39%	-3.72%	-7.78%	-8.07%	-7.15%	-10.24%	-10.23%	-8.21%
κ_n, κ_r*(1+0.5)	-4.84%	-3.82%	-3.81%	-2.99%	-5.37%	-5.37%	-3.70%	-7.38%	-7.61%	-6.92%	-9.94%	-9.92%	-7.88%
ρ_{nr}*(1-0.3)	-0.05%	-0.03%	-0.03%	-0.02%	-0.03%	0.18%	-0.02%	-0.54%	-0.15%	-0.15%	-0.22%	-0.19%	-0.15%
ρ_{nI}*(1-0.3)	0.0004%	0.0005%	0.0005%	0.0004%	0.0011%	0.0010%	0.0006%	-0.04%	-0.03%	-0.02%	-0.00006%	-0.00010%	0.00205%
ρ_{rI}*(1-0.3)	-0.0062%	-0.0051%	-0.0051%	-0.0039%	0.0998%	-0.0032%	-0.0036%	0.00%	-0.02%	0.03%	0.02%	0.02%	0.02%
constant σ_n	0.03%	0.02%	0.02%	0.01%	0.78%	0.88%	-0.08%	-0.72%	-0.37%	-0.34%	0.14%	0.09%	0.09%
constant σ_I	0.33%	0.25%	0.25%	0.20%	1.37%	1.37%	0.92%	1.60%	1.47%	0.50%	-0.80%	-0.76%	-0.72%
constant ρ_{nr}	0.03%	0.01%	0.01%	0.01%	0.10%	0.11%	0.00%	0.04%	0.18%	-0.11%	-0.02%	-0.03%	-0.07%
constant ρ_{nI}	0.01%	0.01%	0.01%	0.01%	0.00%	0.11%	0.00%	0.32%	0.23%	0.03%	0.05%	0.05%	0.02%
constant ρ_{rI}	-0.22%	-0.16%	-0.17%	-0.13%	0.28%	0.18%	-0.12%	0.16%	0.18%	0.37%	0.31%	0.31%	0.28%
all constants	0.16%	0.10%	0.10%	0.09%	1.50%	1.49%	0.83%	1.83%	2.01%	1.00%	-0.27%	-0.24%	-0.35%

Table E.4: Obtained values for the variable swap with counterparty BBVA.

Appendix F

Used Matlab programs

In this Appendix are explained all the used programs. All the functions are called from excel using macros, and it is possible to obtain all the calculated values using only excel buttons. With excel macros, excel and Matlab are connected so that the inputs for Matlab are inserted from excel, the functions are executed in Matlab and the results are reflected again in excel.

Figure F.1: Excel tab.

In the tab market data is obtained the market data used to calculate the zero coupon curves and to implement the calibration. In these tab are the first and second buttons. In the tab calibrated parameters, the nominal, inflation and real parameters are calibrated. The corresponding buttons are the third and the fourth. In the third tab the valuation and sensitivities for the chosen 6 swaps are calculated using the fifth button. In the tab CVA are calculated the default intensities and the survival functions, using the sixth and seventh buttons. In the last tab, cartera, the stand alone CVA for each derivative of the portfolio and the CVA with netting and collateral agreement are calculated. The corresponding buttons are the eighth and the ninth.

The first button is used to take from Bloomberg the data for the selected date.

The second button corresponds to the calculation of the zero coupon nominal and real curves, and the Matlab function is

```
curvas_cupon_cero
```

The third button is used to visualise the calibrated constant and piece-wise nominal parameters. The Matlab function is

`calibracion_nominal`

Inside this program different programs are used to do the calibration. The function to be optimized to calibrate κ_n is

`sacarkappa`

The function to be optimized to calibrate constant σ_n is called

`cali`

The function to be optimized to calibrate the piece-wise function for σ_n is

`sacarcap_cambiante`

For this function, is needed another one that calculates the value of the caplets until the previous maturity to obtain the new value for σ_n. That function is called

`valorar_cap_cambiante`

The fourth button is to calibrate the inflation and real parameters and the Matlab function is

`calibracion_real`

To calibrate the constant inflation volatility σ_I the function to be optimized is

`cali_sigmaI`

To calibrate the piece-wise function

`sacarfloor_cambianteI`

and to obtain the value of the floorlets until the previous maturity

`valorarfloor_cambianteI`

To calibrate the real parameters historically the function to be optimized is

`cali_real_historico`

To calibrate the real parameters using market prices of inflation caps the function to be optimized is

`cali_real`

To calibrate the real parameters using market prices of year on year inflation indexed swaps separately the used function is

`yyiis_calcular`

there is a dichotomous variable that indicates if the wanted parameter to be calibrated is κ_r or σ_r. To calibrate the piece-wise function for σ_r the used functions are

`sacar_r_cambiante, anteriores_yyiis`

The fifth button is to calculate the actual and future valuation, expected positive exposure, 97,5% positive exposure and expected shortfall of the different swaps and the Matlab function is

`simulacion`

When sensitivity about different parameters are calculated, is programmed in excel macros to call the function simulation to obtain the corresponding values. That is, every time that any parameter is changed, the function simulacion is called to obtain the values mentioned before.

The sixth button is to calculate the piece-function for the default intensities for each counterparty and the Matlab function is

`intesidad_default`

Inside this function, the one to be optimized to calibrate the piece-wise function for $\lambda(t)$ is

`sacar_lambda`

The one to calculate the value of the CDS until the previous maturity is

`sacar_parte_lambda_anterior`

The seventh button is to calculate the survival function using the intensity of default in some determined dates and the function is

`probabilidad`

The eighth button is to calculate the CVA of each inflation swap taking into account the counterparty, and the function is

`CVA`

The ninth button is to calculate the CVA with respect to each counterparty, with netting and collateral agreement. The Matlab function is

`CVA_net_col`