Horace Hearne Jr.
Laboratory for Theoretical Physics
Louisiana State University

Quantum scalar field in loop quantum gravity with spherical symmetry

Jorge Pullin
Horace Hearne Institute
for Theoretical Physics
Louisiana State University

With Rodolfo Gambini
University of the Republic of Uruguay

Overview:

We consider a spherically symmetric quantum scalar field coupled to spherically symmetric quantum gravity.

Since the problem has a constraint algebra with structure functions, we will use the “uniform discretizations” approach.

We will construct the state corresponding to the vacuum and show that it results in a quantum state peaked around flat space (minus a deficit angle) for the gravitational variables and a state closely resembling the Fock vacuum for the scalar field.

Plan

- Introduction: previous work.
- Spherically symmetric gravity.
- QFT in CST limit.
- A technique for minimizing the master constraint.
- Definition of a vacuum for the matter fields.
- Minimization of the master constraint in the full theory.

Previous work:

Loop quantum gravity is being extended systematically to situations of greater and greater complexity:

-Loop quantum cosmology (lots of people).

-Spherically symmetric vacuum gravity (our previous work)

-Gowdy cosmologies (Madrid group).

In all these cases, however, one has never had to confront “the problem of dynamics”, namely, that the constraint algebra of gravity has structure functions.

In loop quantum cosmology there is only one constraint with trivial algebra.
In the spherically symmetric case, special gauge fixings were used that rendered the constraint algebra Abelian. In the Gowdy case, the issue was avoided by polymerizing only partially the variables.

Spherical symmetry with the new variables

Previous work with the new variables, Bengtsson (1988) Kastrup and Thiemann (1993) and Bojowald and Swiderski (2005, 2006). Choose connections and triads adapted to spherical symmetry,

$$A = A_x(x)\Lambda_3 dr + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2)\, d\theta + ((A_1(x)\Lambda_2 - A_2(x)\Lambda_1)\sin\theta + \Lambda_3\cos\theta)\, d\varphi,$$

$$E = E^x(x)\Lambda_3 \sin\theta \frac{\partial}{\partial x} + \left(E^1(x)\Lambda_1 + E^2(x)\Lambda_2\right)\sin\theta \frac{\partial}{\partial\theta} + \left(E^1(x)\Lambda_2 - E^2(x)\Lambda_1\right)\frac{\partial}{\partial\varphi},$$

Λ's are generators of su(2).

It simplifies the constraints if one introduces a "polar" canonical transformation in the variables A_φ, P^φ, β, P^β

$$\begin{aligned} A_1 &= A_\varphi \cos\beta, \\ A_2 &= -A_\varphi \sin\beta, \\ E^\varphi &= \sqrt{(E^1)^2 + (E^2)^2}. \end{aligned}$$

$$\begin{aligned} P^\varphi &= 2E^1\cos\beta - 2E^2\sin\beta, \\ P^\beta &= -2E^1 A_\varphi \sin\beta + 2E^2 A_\varphi \cos\beta, \end{aligned}$$

To fix asymptotic problems (Bojowald, Swiderski), one does a further canonical change,

$$\begin{aligned} A_\varphi \to \bar{A}_\varphi &= 2\cos\alpha A_\varphi, \\ \beta \to \eta &= \alpha + \beta, \end{aligned}$$

$$P^\beta = P^\eta, \quad P^\varphi = 2E^\varphi\cos\alpha,$$

Leading to the canonical pairs $A_x, E^x, \bar{A}_\varphi, E^\varphi, \eta, P^\eta$.

One can introduce gauge invariant variables (Gauss' law is then gone),

$$2\gamma K_x = A_x + \eta' \quad A_\varphi = 2\gamma K_\varphi .$$

The canonically conjugate pairs are now E^x, K_x and E^φ, K_φ.

Their relation to the traditional metric variables is,

$$g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \qquad g_{\theta\theta} = |E^x|,$$

$$K_{xx} = -\mathrm{sign}(E^x)\frac{(E^\varphi)^2}{\sqrt{|E^x|}} K_x \qquad K_{\theta\theta} = -\sqrt{|E^x|}\frac{A_\varphi}{2\gamma},$$

The constraints take a relatively simple form with the usual 1+1 diffeo/Hamiltonian algebra of constraints (with structure functions),

$$\begin{aligned} C_r &= (|E^x|)' K_x - E^\varphi (K_\varphi)' - P_\phi \phi' \\ H &= -\frac{E^\varphi}{2\sqrt{|E^x|}} - 2K_\varphi\sqrt{|E^x|}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{|E^x|}} + \frac{((|E^x|)')^2}{8\sqrt{|E^x|}E^\varphi} \\ &\quad - \frac{\sqrt{|E^x|}(|E^x|)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{|E^x|}(|E^x|)''}{2E^\varphi} + \frac{P_\phi^2}{2\sqrt{|E^x|}E^\varphi} + \frac{(|E^x|)^{3/2}(\phi')^2}{2E^\varphi} \end{aligned}$$

The quantization of this model directly is therefore a hard thing since it has the "problem of dynamics".

We will fix partially the gauge to simplify things. We choose $E^x = x^2$. We then eliminate the diffeomorphism constraint by solving for K_x

$$K_x = \frac{E^\varphi (K_\varphi)' + P_\phi \phi'}{2x},$$

And rescaling the lapse, $N_{\rm old} = N_{\rm new} (E^x)'/E^\varphi$ we get:

$$H = H_{\rm vac} + H_{\rm matt}$$

$$\begin{aligned}
H_{\rm vac} &= \left(-x - x K_\varphi^2 + \frac{x^3}{(E^\varphi)^2}\right)', \\
H_{\rm matt} &= \frac{P_\phi^2}{(E^\varphi)^2} + \frac{x^4 (\phi')^2}{(E^\varphi)^2} - 2\frac{x K_\varphi P_\phi \phi'}{E^\varphi}.
\end{aligned}$$

And the spatial integral of $H_{\rm matt}$ gives the mass of the space-time.

QFT in CST

To gain intuition on what to expect, we will rederive well known results of quantum field theory in curved space-time in the notation we are using. We fix the background space-time to flat in usual spherical coordinates where $K_\varphi=0$. One has,

$$H_{\rm matt} = \frac{P_\phi^2}{x^2} + \frac{x^2(\phi')^2}{2}.$$

With evolution equation: $$\phi'' - \ddot{\phi} + 2\frac{\phi'}{x} = 0.$$

And solution:

$$\phi(x,t) = \int_0^\infty d\omega \frac{\left(C(\omega)\exp(-i\omega t) + \bar{C}(\omega)\exp(i\omega t)\right)\sin(\omega x)}{\sqrt{\pi\omega}x},$$

And from Hamilton's equations:

$$P_\phi(x,t) = \int_0^\infty d\omega \frac{\left(-iC\omega\exp(-i\omega t) + i\bar{C}\exp(i\omega t)\right)x\sin(\omega x)}{\sqrt{\pi\omega}}.$$

To quantize, start from $[\hat{\phi}(x,t), \hat{P}_\phi(y,t)] \;=\; i\delta(x-y),$

and one gets $[\hat{C}(\omega), \hat{\bar{C}}(\omega')] \;=\; i\delta(\omega-\omega')$

One can define Fock states and the Hamiltonian has the usual singularities, which we regularize by working on a lattice, with usual conventions,

$$
\begin{aligned}
\int dx &\;\rightarrow\; \epsilon\sum_x \\
\delta(x-y) &\;\rightarrow\; \frac{\delta_{x,y}}{\epsilon} \\
\frac{\delta}{\delta\phi(x)} &\;\rightarrow\; \frac{1}{\epsilon}\frac{\partial}{\partial\phi} \\
\phi(x)' &\;\rightarrow\; \frac{\phi(x_{i+1})-\phi(x_i)}{\epsilon} \\
(\omega)^2 &\;\rightarrow\; \frac{\sum_i\left(2-2\cos(\epsilon\omega_i)\right)}{\epsilon^2}
\end{aligned}
$$

The VEV of the Hamiltonian in the large L limit can be approximated by the integral,

$$\langle 0|\hat{H}_{\rm matt}(x)|0\rangle = \int_0^{2\pi/\epsilon} d\omega \frac{-\omega^2 x^2 + 2\omega\cos(\omega x)\sin(\omega x) - \sin^2(\omega x)}{2x^2\pi\omega}.$$

And can be computed in closed form using integral cosine functions. It is more instructive to study the expansion in ε, the lattice spacing,

$$\langle 0|\hat{H}_{\rm matt}(x)|0\rangle = \frac{\pi}{\epsilon^2} - \frac{\sin^2(2\pi x/\epsilon)}{\pi x^2} + \frac{\ln(x/\epsilon)}{4x^2\pi} + O(\epsilon^0).$$

The first term has dimensions of energy density (in one dimension) and is a “cosmological constant”. Except that in 1+1d the role of this type of cosmological constant is not to curve space locally but to produce a global deficit solid angle.

To see better the last point, we rewrite the Hamiltonian as,

$$
\begin{aligned}
H_{\text{vac}} &= \left(-x(1-2\Lambda) - xK_\varphi^2 + \frac{x^3}{(E^\varphi)^2}\right)', \\
H_{\text{matt}} &= \frac{P_\phi^2}{(E^\varphi)^2} + \frac{x^4(\phi')^2}{(E^\varphi)^2} - 2\frac{xK_\varphi P_\phi \phi'}{E^\varphi} - 16\pi\rho_{\text{vac}},
\end{aligned}
$$

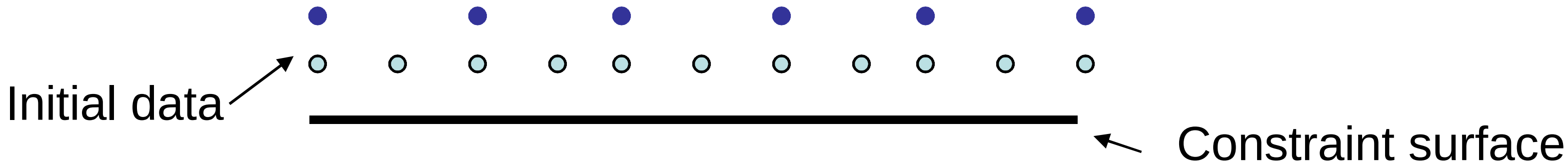
$$
\Lambda = 8\pi\rho_{\text{vac}} \text{ and } \rho_{\text{vac}} = \pi/\epsilon^2.
$$

So we cancel the divergent part of the matter part. The gravitational part suffers what amounts to a rescaling of x, which was the factor in front of the solid angle portion of the spherical metric of space-time. That is how a solid deficit angle ("cosmic texture") appears.

Full quantum theory:

We will use the uniform discretization procedure, in which one discretizes space and time and the evolution is given by a discrete version of the master constraint. The continuum limit is achieved when the master constraint vanishes.

Graphically,



So our goal is to minimize the master constraint. We will use a variational technique to seek for a minimum.

Variational technique to minimize the master constraint

We will write the master constraint in self-adjoint fashion: $\mathbb{H} = H^\dagger H$.

And introduce a fiducial space $\mathcal{H}_{\rm aux}$ of square integrable functions of the configuration variables.

We consider a one-parameter family of elements of $\mathcal{H}_{\rm aux}$, $|\psi_\epsilon\rangle$ with a suitable ε<1 and

$$\langle\psi_\epsilon|\mathbb{H}|\psi_\epsilon\rangle = O(\epsilon^2).$$

We also consider a subset of "test" functions Φ of $\mathcal{H}_{\rm aux}$ that is infinitely differentiable and of compact support.

We will assume that the sequence $\langle\psi_\epsilon|/\epsilon^p$ with some suitable p<1 is, in the limit ε going to 0, a distribution in the dual of F.

That is, $\lim_{\epsilon\to 0} \frac{1}{\epsilon^p}\langle\psi_\epsilon|\phi\rangle$ is finite and non vanishing with $|\phi\rangle \in \Phi.$

Since $\lim_{\epsilon \to 0} \frac{1}{\epsilon^{2p}} \langle \psi_\epsilon | \mathbb{H} | \psi_\epsilon \rangle = 0,$ then $H|\psi_\epsilon\rangle/\epsilon^p$ has zero norm

Therefore $\lim_{\epsilon \to 0} \frac{1}{\epsilon^{p}} \langle \psi_\epsilon | \mathbb{H} | \phi \rangle = 0$

And $\lim_{\epsilon \to 0} \langle \psi_\epsilon | / \epsilon^p$ is a distribution in the dual of Φ that is annihilated by the master constraint, if the limit exists. If the limit does not exist one gets a state that has small values of the master constraint for small values of ε.

The technique in an example:

We consider a mechanical system with two degrees of freedom q_1, $q_{2,}$ $p_{1,}$ $p_{2.}$ and two constraints p_1 =0 and p_2 =0. The physical space will be given by the distribution,

$$\delta(p_1)\delta(p_2)$$

Let us see how our technique reproduces this result. We start by fixing a gauge, $q_1 - q_2$ =0. This is not needed in this example, but in more complicated cases one may have to fix a gauge, so we want to show it does not cause problems. The conjugate variable to the gauge fixing, $p_1 - p_2$ is strongly zero.

We start with a two-parameter set of states in $\mathcal{H}_{\rm aux}$ choosing as configuration variables $q_1 - q_2$ and $p_1 + p_2$.

$$\psi_{\sigma\pm,\beta} = \frac{1}{\sqrt{\pi\sqrt{\sigma_+\sigma_-}}} \exp\left(-\frac{(q_1-q_2)^2}{2\sigma_-}\right) \exp\left(-\frac{(p_1+p_2)^2}{2\sigma_+}\right) \exp\left(i\beta\left(p_1+p_2\right)\right)$$

These states describe wavepackets centered around the classical solutions of the constraints $q_1 - q_2$=0, $p_1 - p_2$=0 and $p_1 + p_2$=0 .

The expectation value of the master constraint $\mathbb{H} = p_1^2 + p_2^2$ on these states is,

$$\langle \psi_{\sigma_\pm,\beta} | \mathbb{H} | \psi_{\sigma_\pm,\beta} \rangle = \frac{8 + \sigma_+ \sigma_-}{16\sigma_+} - \frac{1}{4\sigma_+}$$

And one sees that the expectation value is not zero for finite values of the sigmas (they are positive). However if one takes $\sigma_- = \varepsilon^2$ and $\sigma_+ = 1/\varepsilon^2$ then in the limit $\varepsilon \to 0$ the master constraint is $< \mathbb{H} >= O(\epsilon^2)$ and the states become,

$$\langle q_1 - q_2, p_1 + p_2 | \psi_\epsilon \rangle = \frac{1}{\sqrt{\pi}} \exp\left(-\left(q_1 - q_2\right)^2 \epsilon^2\right) \exp\left(-\frac{\left(p_1 + p_2\right)^2}{\epsilon^2}\right),$$

So the physical states are,

$$\langle q_1 - q_2, p_1 + p_2 | \psi \rangle_{\rm ph} = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \langle q_1 - q_2, p_1 + p_2 | \psi_\epsilon \rangle \to \delta(p_1 + p_2)\delta(p_1 - p_2) = \boxed{\delta(p_1)\delta(p_2)}$$

It is interesting to “break” the model a little bit by making the constraints be second class. This mimics to what happens to a real theory with first class constraints when it is discretized on a lattice. So if we take

$$p_1 + \beta q_2 = 0 \text{ and } p_2 = 0 \text{ with } \beta \text{ a small parameter}$$

And we now repeat the previous procedure, one gets,

$$\langle \psi_{\sigma_\pm,\beta} | \mathbb{H} | \psi_{\sigma_\pm,\beta} \rangle = \frac{3}{32}\sigma_- + \frac{3}{32\sigma_+} + \frac{\alpha^2}{8}\left(\sigma_+ + 2\beta^2 + \frac{1}{\sigma_-}\right)$$

And now there is no limit in parameter space in which the master constraint vanishes. One can choose values of the sigmas that minimize the master constraint. The resulting states approach the correct physical states in the limit β->0 and the master constraint vanishes in that limit.

So the message is that one can ignore the problem that discretized constraints fail to be first class, but one will have to live with a small but non-vanishing master constraint.

Quantization of the full model:

We will consider only situations with weak fields, so to construct the ansatz for the classical solution we ignore the matter Hamiltonian and solve the gravitational part, which just yields flat space with a solid deficit angle, $K_\varphi=0$ and $E^x=x/(1-2\Lambda)^{1/2}$.

We take the complete Hamiltonian constraint,

$$
\begin{aligned}
H_{\rm vac} &= \left(-x(1-2\Lambda) - xK_\varphi^2 + \frac{x^3}{(E^\varphi)^2}\right)', \\
H_{\rm matt} &= \frac{P_\phi^2}{(E^\varphi)^2} + \frac{x^4(\phi')^2}{(E^\varphi)^2} - 2\frac{xK_\varphi P_\phi \phi'}{E^\varphi} - 16\pi\rho_{\rm vac},
\end{aligned}
$$

We discretize it and polymerize the gravitational variables,

$$
\begin{aligned}
H(i) = &-(1-2\Lambda)\epsilon - x(i+1)\frac{\sin^2\left(\rho K_\varphi(i+1)\right)}{\rho^2} + x(i)\frac{\sin^2\left(\rho K_\varphi(i)\right)}{\rho^2} + \frac{x(i+1)^3\epsilon^2}{(E^\varphi(i+1))^2} - \frac{x(i)^3\epsilon^2}{(E^\varphi(i))^2} \\
&+\epsilon\frac{(P^\varphi(i))^2}{(E^\varphi(i))^2} + \epsilon\frac{x(i)^4\left(\phi(i+1)-\phi(i)\right)^2}{(E^\varphi(i))^2} - 2x(i)\frac{\sin\left(\rho K_\varphi(i)\right)}{E^\varphi(i)\rho}\left(\phi(i+1)-\phi(i)\right)P^\phi(i) - 16\pi\rho_{\rm vac}\epsilon
\end{aligned}
$$

We now construct the master constraint,

$$\mathbb{H} = \frac{1}{2}\int d^3x \frac{H(x)^2}{\sqrt{g}}, \qquad \text{or} \qquad \mathbb{H} = \frac{1}{2}\int dx \frac{H(x)^2}{(E^\varphi)\sqrt{E^x}},$$

And upon discretization,

$$\mathbb{H}^\epsilon = \sum_i \mathbb{H}(i) \text{ with } \qquad \mathbb{H}(i) = \frac{1}{2}\frac{H(i)^2}{\sqrt{E^x(i)}E^\varphi(i)}$$

Expanding out one has,

$$\begin{aligned}\mathbb{H}(i) \;=\; \ell_{\rm P} \Big[& c_{11}(i)\left(H^{(1)}_{\rm matt}(i)\right)^2 + c_{22}(i)\left(H^{(2)}_{\rm matt}(i)\right)^2 \\ & + c_1(i)H^{(1)}_{\rm matt}(i) + c_2(i)H^{(2)}_{\rm matt}(i) + c_{33}(i)\left(H^{(3)}_{\rm matt}(i)\right)^2 + c_3(i)H^{(1)}_{\rm matt}(i) \\ & + c_{12}(i)H^{(1)}_{\rm matt}(i)H^{(2)}_{\rm matt}(i) + c_{13}(i)H^{(1)}_{\rm matt}(i)H^{(3)}_{\rm matt}(i) + c_{23}(i)H^{(2)}_{\rm matt}(i)H^{(3)}_{\rm matt}(i) + c_{00}(i)\Big]\end{aligned}$$

$$\begin{aligned}H^{(1)}_{\rm matt}(i) \;&=\; \left(\epsilon\left(P^\varphi(i)\right)^2 + \epsilon x(i)^4\left(\phi(i+1)-\phi(i)\right)^2\right)\ell_P^2 \\ H^{(2)}_{\rm matt}(i) \;&=\; \left(-2x(i)\left(\phi(i+1)-\phi(i)\right)P^\varphi(i)\right)\ell_P^2 \\ H^{(3)}_{\rm matt}(i) \;&=\; 16\pi\rho_{\rm vac}\epsilon\ell_P^2\end{aligned}$$

And the c's are rather lengthy, but all can be written in terms of a few significant quantum operators, e.g.,

$$\hat{c}_{11}(i) = \frac{\epsilon}{2\hat{E}^{\varphi}(i)^4 x(i)^2}$$

$$\hat{c}_{12}(i) = \frac{\epsilon}{x(i)^2\rho} \frac{1}{\left(\hat{E}^{\varphi}(i)\right)^{3/2}} \sin\left(\rho K_{\varphi}(i)\right) \frac{1}{\left(\hat{E}^{\varphi}(i)\right)^{3/2}}$$

$$\hat{c}_{13}(i) = -\frac{2\epsilon}{x(i)^2\rho} \frac{1}{\left(\hat{E}^{\varphi}(i)\right)^2}$$

$$\hat{c}_{22}(i) = \frac{\epsilon}{4x(i)^2\rho^2} \left(\frac{1}{\left(\hat{E}^{\varphi}(i)\right)^2} - \frac{1}{\hat{E}^{\varphi}(i)} \cos\left(2\rho\hat{K}_{\varphi}(i)\right) \frac{1}{\hat{E}^{\varphi}(i)} \right)$$

$$\hat{c}_{23}(i) = -\frac{2\epsilon}{x(i)^2\rho} \frac{1}{\sqrt{\hat{E}^{\varphi}(i)}} \sin\left(\rho K_{\varphi}(i)\right) \frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}$$

$$\hat{c}_{33}(i) = \frac{2\epsilon}{x(i)^2}$$

$$\hat{c}_{1}(i) = -x(i)\epsilon^2 \frac{1}{\left(\hat{E}^{\varphi}(i)\right)^4} + \frac{1}{2\left(x(i)\hat{E}^{\varphi}(i)\right)^2} \left(-2\epsilon\left(1-2\Lambda\right) - \frac{\epsilon}{\rho^2} + x(i+1)\frac{\cos\left(2\rho\hat{K}_{\varphi}(i+1)\right)}{\rho^2} \right)$$
$$- \frac{1}{\hat{E}^{\varphi}(i)} \frac{\cos\left(2\rho\hat{K}_{\varphi}(i)\right)}{2x(i)\rho^2} \frac{1}{\hat{E}^{\varphi}(i)} + \frac{\epsilon^2 x(i+1)^3}{x(i)^2} \frac{1}{\left(\hat{E}^{\varphi}(i)\hat{E}^{\varphi}(i+1)\right)^2}$$

$$
\begin{aligned}
\hat{c}_2(i) &= \frac{1}{x(i)^2}\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}\frac{\sin(\rho\hat{K}_{\varphi}(i))}{\rho}\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}\left(\epsilon\,(2\Lambda-1)-\frac{x(i+1)}{2\rho^2}\left(1-\cos(2\rho\hat{K}_{\varphi}(i+1))\right)\right. \\
&\left.+\frac{3x(i)}{4\rho^2}+x(i+1)^3\left(\hat{E}^{\varphi}(i+1)\right)^2\right) \\
&-\frac{1}{x(i)\rho^3}\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}\sin\left(3\rho\hat{K}_{\varphi}(i)\right)\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}-4\frac{x(i)}{\rho^3}\frac{1}{\left(\hat{E}^{\varphi}(i)\right)^{3/2}}\sin\left(\rho\hat{K}_{\varphi}(i)\right)\frac{1}{\left(\hat{E}^{\varphi}(i)\right)^{3/2}} \\
\hat{c}_3(i) &= \frac{2}{x(i)\rho^2}-\frac{2\cos(\rho\hat{K}_{\varphi}(i))^2}{x(i)\rho^2}+2\frac{x(i)\epsilon^2}{\left(\hat{E}^{\varphi}(i)\right)^2}-2\frac{\epsilon^2 x(i+1)^3}{x(i)^2\left(\hat{E}^{\varphi}(i+1)\right)^2} \\
&-4\frac{\epsilon\Lambda}{x(i)^2}+2\frac{\epsilon}{x(i)^2}+2\frac{x(i+1)}{x(i)^2\rho^2}-2\frac{x(i+1)\cos^2\left(\rho\hat{K}_{\varphi}(i+1)\right)}{x(i)^2\rho^2}
\end{aligned}
$$

$\hat{c}(i) =$ Lengthy (trust me…)

Loop representation for the spherically symmetric case:

Manifold is a line. “Graph” is a set of edges $g = \bigcup_i e_i$. The variables are scalars, so in the Loop representation one uses “point holonomies” to represent them. What essentially goes on is a “loop quantum cosmology at every point”:

$$\mathcal{H} = L^2\left(\otimes_N R_{\rm Bohr}, \otimes_N d\mu_0\right) \qquad T_{\vec{\mu}} = \exp\left(i\sum_j \mu_j K_\varphi(j)\right) = \langle K_\varphi | \vec{\mu}\,\rangle.$$

$$\hat{E}^\varphi_m = -i\ell^2_{\rm Planck}\frac{\partial}{\partial K_{\varphi,m}}, \qquad \hat{E}^\varphi_m T_{g,\vec{\mu}} = \sum_{v\in V(g)} \mu_m \gamma \ell^2_{\rm Planck}\delta_{m,n(v)} T_{g,\vec{\mu}},$$

The quantum state we pick as guess for the variational method is a Gaussian centered around the classical solution $E^\varphi(i) = \epsilon x_1(i) = \epsilon x(i)/\sqrt{1-2\Lambda}$,

$$\langle \vec{\mu}\,|\psi_{\vec{\sigma}}\rangle = \prod_i \sqrt[4]{\frac{2}{\pi\sigma(i)}}\exp\left(-\frac{1}{\sigma(i)}\left(\mu_i - \frac{x_1(i)\epsilon}{\ell_P^2}\right)^2\right)$$

on this state $\langle E^\varphi(i)\rangle = \epsilon x_1(i)$ and $\langle K_\varphi(i)\rangle = 0.$

Finding the vacuum:

We will now take the expectation value of the matter portion of the Hamiltonian constraint on the above state. The result is an operator acting on the matter variables. We will construct the Fock vacuum for that operator. What we are doing therefore is studying a quantum field theory on the curved space-time given by the expectation values of the triad and the curvature given by the previous state.

Why the Fock vacuum? If you wish, this is just a trial, given that it is the first time the problem is addressed. In the future, one should polymerize the matter part of the theory and show that the Fock vacuum arises from the polymerized theory. Some hints of this already exist (Sahlmann and Thiemann, Ashtekar, Fairhurst, Willis), although in different contexts.

To realize the matter part of the Hamiltonian constraint,

$$H_{\rm matt} \;=\; \epsilon\frac{(P^{\varphi}(i))^2}{(E^{\varphi}(i))^2}+\epsilon\frac{x(i)^4\,(\phi(i+1)-\phi(i))^2}{(E^{\varphi}(i))^2}-2x(i)\frac{\sin\left(\rho K_{\varphi}(i)\right)}{E^{\varphi}(i)\rho}\left(\phi(i+1)-\phi(i)\right)P^{\phi}(i)-16\pi\rho_{\rm vac}\epsilon$$

We need to realize two operators

The first one is,

$$\frac{1}{\left(\hat{E}^{\varphi}(i)\right)^2}\langle\, \mu(i)\,|\psi_{\sigma(i)}\rangle \;=\; \left(\frac{2}{3}\right)^{12} |\mu(i)|\,\left((|\mu(i)+\rho|)^{3/4}-(|\mu(i)-\rho|)^{3/4}\right)^{12}\sqrt[4]{\frac{2}{\pi\,\sigma(i)}}\exp\left(-\frac{\left(\mu(i)-\frac{\epsilon\, x_1(i)}{\ell_P^2}\right)^2}{\sigma(i)}\right)$$

And people may recognize the similarity with an operator realized by Ashtekar, Pawlowski and Singh in the context of LQC. With this operator one can compute,

$$\langle\,\psi_{\vec\sigma}|\frac{1}{\left(\hat{E}^{\varphi}(i)\right)^2}|\psi_{\vec\sigma}\rangle = \frac{1-2\,\Lambda}{\epsilon^2 x(i)^2}+\frac{5}{8}\,\frac{{\ell_P}^4\,(1-2\,\Lambda)^2\,\rho^2}{\epsilon^4 x(i)^4}+\frac{3}{4}\,\frac{\sigma\,{\ell_P}^4\,(1-2\,\Lambda)^2}{\epsilon^4 x(i)^4}$$

The computation is done integrating over μ. This can be done exactly but the result is lengthy, here we show only the expansion in powers of ℓ_P/ε.

The second operator is

$$\langle\,\psi_{\vec\sigma}|\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}\frac{\sin(\rho\hat{K}_\varphi(i))}{\rho}\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}|\,\psi_{\vec\sigma}\rangle = 0.$$

Which is easily seen to vanish since the Gaussian is even and sin is odd.

With the previous results the expectation value of the matter part of the Hamiltonian on the gravitational states yields and "effective" matter Hamiltonian that is an operator acting on the matter states,

$$\hat{H}^{\rm eff}_{\rm matt} = \langle\, \psi_{\vec\sigma} | \hat{H}_{\rm matt}(x,t) |\, \psi_{\vec\sigma}\rangle = \frac{(1-2\,\Lambda)\left(\hat{P}^{\phi}(x,t)\right)^2}{x^2 g(x)^2} + \frac{x^2\,(1-2\,\Lambda)\left(\hat{\phi}'(x,t)\right)^2}{g(x)^2} - 16\pi\rho_{\rm vac}.$$

Where,

$$g(x) = 1 + \frac{5}{16}\,\frac{{\ell_P}^4\rho^2(1-2\Lambda)}{x^2\epsilon^2} + \frac{3}{8}\,\frac{\sigma(x)\,\ell_P^4(1-2\Lambda)}{x^2\epsilon^2}$$

And the second term in g(x) is a correction due to polymerization and the last term another quantum correction.

With this Hamiltonian we can get the equations of motion, yielding the effective "wave equation" for the fields,

$$\frac{2}{x}\,\frac{\partial\phi(x,t)}{\partial x} - \frac{2}{g(x)}\,\frac{\partial\phi(x,t)}{\partial x}\,\frac{\partial g(x)}{\partial x} + \frac{\partial^2\phi\,(x,t)}{\partial x^2} - \frac{1}{4}\,\frac{g(x)^4}{(1-2\Lambda)^2}\,\frac{\partial^2\phi(x,t)}{\partial t^2} = 0.$$

Since the background is time independent , there is no obstruction to introducing positive and negative frequency modes. The resulting equation can be cast in Sturm-Liouville form,

$$(2B(x)\phi'(x,\omega))' + \frac{\omega^2}{2}\phi(x,\omega)A(x) = 0$$

$$\begin{aligned} A(x) &= \epsilon^2 \frac{x^2}{1-2\Lambda} + \frac{3}{8}\rho^2 {\ell_P}^4 - \frac{3}{4}\sigma\,{\ell_P}^4, \\ B(x) &= \frac{(1-2\Lambda)x^2}{\epsilon^2} - \frac{3}{8}\frac{(1-2\Lambda)^2\ell_P^4}{\epsilon^4}\left(\rho^2 - 2\sigma\right) \end{aligned}$$

With solution,

$$\begin{aligned} \phi(x,w) &= \frac{\sin\left(\frac{\omega\,x}{2}\right)}{x} + \frac{1}{3x^3}\left(-\frac{5}{16}\frac{\rho^2\,lp^4(1-2\Lambda)}{\epsilon^2} - \frac{3}{8}\frac{\sigma\,lp^4(1-2\Lambda)}{\epsilon^2}\right)\times \\ &\times\left\{\cos\left(\frac{\omega\,x}{2}\right)\omega\,x + 2\,\sin\left(\frac{\omega\,x}{2}\right)Ci\,(\omega\,x)\,x^2\omega^2 - 2\,\sin\left(\frac{\omega\,x}{2}\right) - 2\,\cos\left(\frac{\omega\,x}{2}\right)Si\,(\omega\,x)\,x^2\omega^2\right\} \\ \phi(x,t) &= \int_0^\infty d\omega\phi(x,\omega)\left(C(\omega)e^{i(1-2\Lambda)\,\omega\,t} + \bar{C}(\omega)e^{-i(1-2\Lambda)\,\omega\,t}\right) \end{aligned}$$

Where $Si(x) \equiv \int_0^x dt\sin(t)/t,\ \ Ci(x) \equiv \gamma + \ln(x) + \int_0^x dt(\cos(t)-1)/t$

$\gamma = 0.5772156649.$

Using Hamilton's equations one gets, $P^{\varphi}(x,t) = \frac{x^2 g(x)^2}{2(1-2\Lambda)} \frac{\partial \phi(x,t)}{\partial t}$

And at the end of the day the effective Hamiltonian is

$$\hat{H}^{\text{eff}}_{\text{matt}} = (1-2\Lambda) \int_0^{2\pi/\epsilon} d\omega \omega \hat{\bar{C}}(\omega) \hat{C}(\omega).$$

And we can immediately construct a Fock space and a vacuum for it.

We now will proceed to address the full theory using the variational method and using as trial state,

$$|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = |\psi_{\vec{\sigma}}\rangle \otimes |0\rangle$$

And we will look for σ's that minimize the master constraint. Notice that the state is a direct product since we are in an eigenstate. If we were studying excitations one would expect entanglement.

We need to realize as a quantum operator the master constraint. This requires eight operators. We already realized one. The others are,

$$\langle\psi^{\rm trial}_{\vec\sigma}|\cos\left(2\rho\hat K_\varphi(i)\right)|\psi^{\rm trial}_{\vec\sigma}\rangle = \exp\left(-\frac{2\rho^2}{\sigma(i)}\right),$$
$$\langle\psi^{\rm trial}_{\vec\sigma}|\cos\left(4\rho\hat K_\varphi(i)\right)|\psi^{\rm trial}_{\vec\sigma}\rangle = \exp\left(-\frac{8\rho^2}{\sigma(i)}\right).$$

$$\langle\psi^{\rm trial}_{\vec\sigma}|\frac{1}{\left(\hat E^\varphi(i)\right)^4}|\psi^{\rm trial}_{\vec\sigma}\rangle = \frac{(1-2\,\Lambda)^2}{\epsilon^4 x(i)^4} + \frac{5}{4}\frac{{\ell_P}^4\,(1-2\,\Lambda)^3\,\rho^2}{\epsilon^6 x(i)^6} + \frac{5}{2}\frac{\sigma\,{\ell_P}^4\,(1-2\,\Lambda)^3}{\epsilon^6 x(i)^6},$$

$$\langle\psi^{\rm trial}_{\vec\sigma}|\frac{1}{\hat E^\varphi(i)}\cos\left(2\rho\hat K_\varphi(i)\right)\frac{1}{\hat E^\varphi(i)}|\psi^{\rm trial}_{\vec\sigma}\rangle = \frac{(1-2\Lambda)}{\epsilon^2 x(i)^2\exp\left(2\frac{\rho^2}{\sigma}\right)}\times$$
$$\times\left[1+\frac{(1-2\Lambda)\,\ell_P^4}{\epsilon^2 x(i)^2}\left(\rho^2\left(3-6\sqrt{\frac{2}{\pi\sigma}}\right)-\rho+\frac{3}{4}\sigma\right)\right]$$

$$\langle\psi^{\rm trial}_{\vec\sigma}|\frac{1}{\left(\hat E^\varphi(i)\right)^{3/2}}\sin\left(\rho\hat K_\varphi(i)\right)\frac{1}{\left(\hat E^\varphi(i)\right)^{3/2}}|\psi^{\rm trial}_{\vec\sigma}\rangle = 0,$$

$$\langle\psi^{\rm trial}_{\vec\sigma}|\frac{1}{\sqrt{\hat E^\varphi(i)}}\sin\left(\rho\hat K_\varphi(i)\right)\frac{1}{\sqrt{\hat E^\varphi(i)}}|\psi^{\rm trial}_{\vec\sigma}\rangle = 0,$$

$$\langle\psi^{\rm trial}_{\vec\sigma}|\frac{1}{\sqrt{\hat E^\varphi(i)}}\sin\left(3\rho\hat K_\varphi(i)\right)\frac{1}{\sqrt{\hat E^\varphi(i)}}|\psi^{\rm trial}_{\vec\sigma}\rangle = 0.$$

With the previous results we can compute the expectation value of the master constraint on the gravitational state. The result is an operator acting on matter states. The non-vanishing terms are,

$$\langle\psi_{\vec{\sigma}}|\hat{\mathbb{H}}(i)|\psi_{\vec{\sigma}}\rangle = \ell_{\rm P}\left[\langle\hat{c}_{11}(i)\rangle\left(\widehat{H^{(1)}_{\rm matt}(i)}\right)^2 + \langle\hat{c}_{22}(i)\rangle\left(\widehat{H^{(2)}_{\rm matt}(i)}\right)^2 + \langle\hat{c}_{1}(i)\rangle\widehat{H^{(1)}_{\rm matt}(i)} + \langle\hat{c}_{33}(i)\rangle\left(\widehat{H^{(3)}_{\rm matt}(i)}\right)^2 \right.$$
$$\left. +\langle\hat{c}_{3}(i)\rangle\widehat{H^{(1)}_{\rm matt}(i)} + \langle\hat{c}_{13}(i)\rangle\widehat{H^{(1)}_{\rm matt}(i)}\widehat{H^{(3)}_{\rm matt}(i)} + \langle\hat{c}_{00}(i)\rangle\right].$$

To compute the expectation value on the matter states it is useful to use a trick in which discretized derivatives and integrals are replaced by continuum expressions to avoid dealing with summations and discrete equations. The integrals in ω are done with a cutoff nevertheless. For instance, the result for the first operator is,

$$\langle 0|\hat{H}^{(1)}_{\rm matt}(x)|0\rangle = \ell_P^2\left(-\frac{A(x)^2}{8x^4}\cos^2\left(\frac{\pi x}{\epsilon}\right) + \frac{A(x)^2}{8x^4} + \frac{\pi^2 A(x)^2}{8x^2\epsilon^2} - \frac{A(x)^2\pi}{4x^3\epsilon}\cos\left(\frac{\pi x}{\epsilon}\right)\sin\left(\frac{\pi x}{\epsilon}\right)\right)(1-2\Lambda)$$
$$+\ell_P^2\left(\frac{\ln(2)}{4} - \frac{5}{8}\sin^2\left(\frac{\pi x}{\epsilon}\right) + \frac{\pi}{4\epsilon x}\cos\left(\frac{\pi x}{\epsilon}\right)\sin\left(\frac{\pi x}{\epsilon}\right) + \frac{x^2\pi^2}{8\epsilon^2} - \frac{1}{4}\,Cin\left(2\,\frac{\pi x}{\epsilon}\right)\right)(1-2\Lambda)^{-1}$$

With $Cin(x) = \gamma + \ln x - Ci(x).$

A more manageable expression can be obtained by ignoring corrections of ℓ_P^4 and and the highly oscillating terms involving sin(πx/ε) and cos(πx/ε) and Cin(πx/ε). The result is,

$$\langle 0|\hat{H}^{(1)}_{\rm matt}(x)|0\rangle = \frac{\ell_P^2}{4(1-2\Lambda)}\left(-\frac{3}{4}+\gamma+\ln(2)+\frac{\pi^2 x^2}{\epsilon^2}+\ln\left(\frac{\pi x}{\epsilon}\right)\right)$$

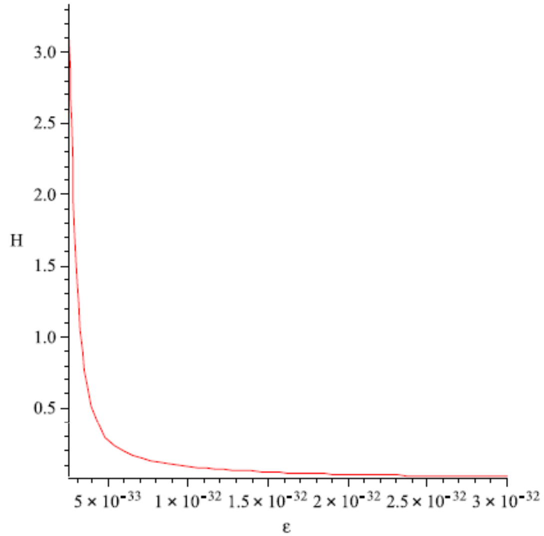
Or, reverting to the discrete theory,

$$\langle 0|\hat{H}^{(1)}_{\rm matt}(i)|0\rangle = \frac{\ell_P^2\epsilon^2}{4(1-2\Lambda)}\left(-\frac{3}{4}+\gamma+\ln(2)+\frac{\pi^2 x(i)^2}{\epsilon^2}+\ln\left(\frac{\pi x}{\epsilon}\right)\right).$$

Completing the calculation of the master constraint is lengthy but manageable. The result is,

$$
\begin{aligned}
\langle \hat{\mathbb{H}} \rangle \; = \; & \frac{\sigma_0 \, {\ell_P}^3}{\epsilon \, x^2} + \left(8 \, \frac{\pi^2}{\epsilon^3 x^2} + \frac{32}{\epsilon x^4} \ln\left(\frac{L}{\epsilon}\right) - \frac{\left(\gamma - 2 + \ln\left(\frac{2\pi x}{\epsilon}\right)\right)\pi}{\epsilon \, x^4 (1 - 2\Lambda)} + \frac{1}{96} \, \frac{\pi^3}{\epsilon^5 x^2 \sigma_0 \, (1 - 2\Lambda)^2} \right. \\
& - \frac{1}{48} \, \frac{\Lambda \, \pi^3}{\epsilon^5 x^2 \sigma_0 \, (1 - 2\Lambda)^2} - \frac{43}{128} \, \frac{\Lambda \, \pi}{\epsilon^3 x^4 \sigma_0 \, (1 - 2\Lambda)^2} \\
& + \frac{\epsilon \left(\gamma - 2 + \ln\left(\frac{2\pi x}{\epsilon}\right)\right)\pi}{x^4 (1 - 2\Lambda) \, L^2} + 8 \, \frac{\epsilon \, \pi^2}{x^2 L^4} - \frac{2 \, \pi}{\epsilon x^4 (1 - 2\Lambda)} \ln\left(\frac{L}{\epsilon}\right) - 16 \, \frac{\pi^2}{\epsilon \, x^2 L^2} + \frac{1}{32} \, \frac{\pi}{\epsilon \, x^4 (1 - 2\Lambda)^2} \\
& + \frac{1}{48} \, \frac{\pi^3}{\epsilon^3 x^2 (1 - 2\Lambda)^2} + \frac{32 \, \epsilon}{x^6 \pi^2} \left(\ln\left(\frac{L}{\epsilon}\right)\right)^2 - \frac{\pi^3}{\epsilon^3 x^2 (1 - 2\Lambda)} + 4 \, \frac{\pi}{\sigma_0 \, \epsilon^3 x^2} \\
& - \frac{32}{x^4 L^2} \, \epsilon \, \ln\left(\frac{L}{\epsilon}\right) - 3 \, \frac{\sigma_0 \, \pi}{x^3 L^2} + \frac{43}{256} \, \frac{\pi}{\epsilon^3 x^4 \sigma_0 \, (1 - 2\Lambda)^2} + \frac{8}{\sigma_0 \epsilon x^4 \pi} \ln\left(\frac{L}{\epsilon}\right) \\
& - \frac{1}{4(1 - 2\Lambda)} \frac{\left(4 x^2 \epsilon \sigma_0 + 4 \, x {\sigma_0}^3 \epsilon^2 + 4 \, \sigma_0 \, \epsilon^2 x + 7 \, {\sigma_0}^3 \epsilon^3\right)}{(x + \epsilon) \sigma_0^2 x^5 \epsilon^4} \times \left(\frac{1}{4} \pi^2 x^2 + \frac{1}{4} \epsilon^2 \left(\gamma - 2 + \ln\left(\frac{2\pi x}{\epsilon}\right)\right)\right) \\
& - 3 \, \frac{\sigma_0 \, \pi}{(x + \epsilon) \, x^2 \epsilon^2} + 3 \, \frac{\sigma_0 \, \pi}{(x + \epsilon) \, x^2 L^2} - \frac{6 \, \sigma_0}{(x + \epsilon) x^4 \pi} \ln\left(\frac{L}{\epsilon}\right) - 4 \, \frac{\pi}{x^2 \epsilon \, \sigma_0 \, L^2} + \frac{\pi^3}{\epsilon \, x^2 (1 - 2\Lambda) \, L^2} \\
& \left. - 2 \, \frac{\epsilon}{x^6 (1 - 2\Lambda) \pi} \left(\gamma - 2 + \ln\left(\frac{2\pi x}{\epsilon}\right)\right) \ln\left(\frac{L}{\epsilon}\right) + 3 \, \frac{\sigma_0 \, \pi}{x^3 \epsilon^2} + 6 \, \frac{\sigma_0}{x^5 \pi} \ln\left(\frac{L}{\epsilon}\right) - 1/16 \, \frac{\epsilon}{x^6 (1 - 2\Lambda)^2 \pi} \right) {\ell_P}^5 .
\end{aligned} \qquad (
$$

One can analyze the behavior of the master constraint numerically. Here is the minimum as a function of ε, the lattice spacing (in cms),



So we see that in this approximation the theory does not seem to have a good continuum limit, but one can have very small values of the master constraint for lattice spacings that are very small compared to the characteristic lengths of all other non-gravitational interactions.

Two final observations:

a) what would have happened if instead of choosing for the gravitational variables the state peaked around flat space we had chosen "zero loop vacuum" state. Such a state corresponds to degenerate triads and is annihilated by the matter part of the Hamiltonian. One can compute the expectation value of the master constraint in closed form within the approximations used here,

$$\langle \hat{\mathbb{H}} \rangle = \frac{1}{8} \frac{L \ell_P}{\epsilon^2 \rho}.$$

And one sees that it has a much larger expectation value of the master constraint than the one we found for the other state.

b) The cosmological constant: Goes as $\Lambda \sim \ell_P^2 / \epsilon^2$ and as we need to take e much larger than ℓp, this means that Λ will be much smaller than the Planck scale.

Summary:

- One can study spherically symmetric gravity coupled to a spherical scalar field using techniques of loop quantum gravity.

- The lack of a Lie algebra of constraints is correctly handled by the uniform discretization technique, but the master constraint has a non-vanishing minimum that approximates the continuum well.
- The Fock vacuum in the matter portion appears to be compatible with the loop states for the gravitational variables.
- The cosmological constant is suggestively suppressed.