Let us consider a model of demand and supply with stocks. Let Q_t be the level of stocks at the end of period t, so

$$Q_t - Q_{t-1} = q_t^s - q_t^d.$$

Interpret the above equation.

Q_t donotes the level of stocks at the end of period t and $Q_{t-1}$is the level of stocks at the end of period $t-1$, so $Q_t - Q_{t-1}$ represents the change in stocks over period t, which is equal to the differece between supply and demand over period t. In other words, Q_t=Q_{t-1}+(q_t^s-q_t^d), that is, the level of stoks at the end of period t will be equal to the level of stocks at the end of period $t-1$ plus the excess supply over period t (if q_t^s>q_t^d) or minus the excess demand over period t (if q_t^d>q_t^s).

We consider lineal functions of demand and supply:

$$q_t^s = u + a\, p_{t-1}$$
$$q_t^d = v - b\, p_t$$

Notice that the supply provided by producers for the year t is that corresponding to the price of the previous year.
We assume that suppliers alter prices in response to stock changes according to the equation:

$$p_{t+1} = p_t - \lambda(Q_t - Q_{t-1}), \qquad \lambda > 0$$

Discuss the economic meaning of this equation.

This equation explains how suppliers alter prices in response to stock changes. They will raise the prise in period $t+1$ if the level of stocks at the end of period t is lower than the level of stocks at the end of period $t-1$ and they will lower the price if the level of stocks at the end of period t is greater than the level of stocks at the end of period $t-1$. The rise or fall in price is set proportional to the change in the level of stocks.

Check that these assumptions lead to the following second-order difference equation:

$$p_{t+1} = (1 - \lambda\, b)\, p_t - \lambda\, a\, p_{t-1} + \lambda\,(v - u)$$

$$p_{t+1} = p_t - \lambda(Q_t - Q_{t-1}) \;=\; p_t - \lambda\left(q_t^s - q_t^d\right) = p_t - \lambda(u + a\, p_{t-1} - v + b\, p_t) = (1 - \lambda\, b)\, p_t - \lambda\, a\, p_{t-1} + \lambda\,(v - u)$$

Calculate the equilibrium price and check that it is the limit of the convergent orbits.

$u + a\, p = v - b\, p \quad \longleftrightarrow \quad a\, p + b\, p = v - u \quad \longleftrightarrow \quad p\,(a + b) = v - u \quad \longleftrightarrow \quad p = \frac{v-u}{a+b}$

On the other hand, for a general linear second-order difference equation of the form $x_{t+1} = A\, x_t + B\, x_{t-1} + C$, we know that the limit of the convergent orbits is $\frac{-C}{A+B-1}$. In this case:

$A = 1 - \lambda\, b \qquad B = -\lambda\, a \qquad C = \lambda(v - u)$ so the limit is

$$\frac{-\lambda(v-u)}{1-\lambda\, b-\lambda\, a-1} = \frac{-\lambda(v-u)}{1-\lambda(b+a)-1} = \frac{-\lambda(v-u)}{-\lambda(b+a)} = \frac{v-u}{b+a}$$

From now on consider the following specific functions:

$$q_t^s = 2 + 4\, p_t$$
$$q_t^d = 20 - 2.5\, p_t.$$

Determine for which values of λ the price policy leads to the equilibrium price.

We apply the stability conditions ($A + B < 1 \qquad B - A < 1 \qquad B > -1$) taking into account that now $a = 4$ and $b = 2.5$.

$1 - 2.5\,\lambda - 4\,\lambda = 1 - \lambda\,(6.5) < 1$ This condition is always satisfied since $\lambda > 0$.

$-4\,\lambda - 1 + 2.5\,\lambda = \lambda(-1.5) - 1 < 1 \quad \longleftrightarrow \quad \lambda(-1.5) < 2$ This conditions is always satisfied

$-4\lambda > -1 \quad \longleftrightarrow \quad 4\lambda < 1 \quad \longleftrightarrow \quad \lambda < \frac{1}{4}$

All in all, the price orbits will be convergent when $\lambda < \frac{1}{4}$.

Set $\lambda = 0.2$. Write the characteristic equation for this particular case and find its roots.

```
a = 4; b = 2.5; u = 2; v = 20; λ = 0.2;
x^2 - (1 - λ b) x + λ a
Solve[x^2 - (1 - λ b) x + λ a == 0, x]
```

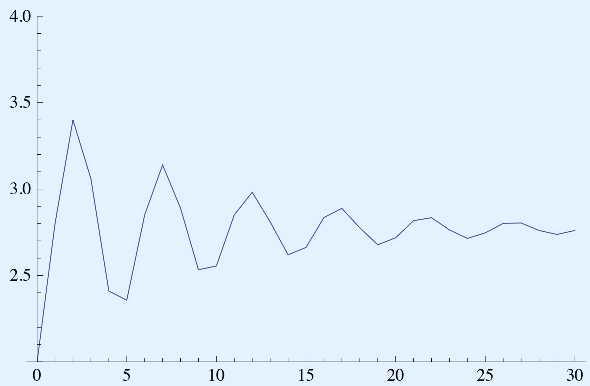
$0.8 - 0.5\,x + x^2$

```
{{x → 0.25 - 0.858778 i}, {x → 0.25 + 0.858778 i}}
```

So the characteristic equation in this case is $x^2 - 0.5\,x + 0.8 = 0$ and its roots are 0.25-0.858778 i and 0.25+0.858778 i (complex conjugate numbers). We plot the orbit of the price for initial prices $p_0 = 2$ and $p_1 = 2.8$.

```
RSolve[{p[t + 1] == (1 - λ b) p[t] - λ a p[t - 1] + λ (v - u), p[0] == 2, p[1] == 2.8}, p, t];
pp = p /. %[[1]];
FullSimplify[pp[t]]
ListPlot[Table[Re[pp[i]], {i, 0, 30}],
 DataRange → {0, 30}, Joined → True, PlotRange → {2, 4}]
(v - u) / (a + b)
```

$\left(2.76923 + 1.23924 \times 10^{-16}\, i\right) - (0.384615 - 0.12988\, i)\,(0.25 - 0.858778\, i)^t - (0.384615 + 0.12988\, i)\,(0.25 + 0.858778\, i)^t$



2.76923

Since $\lambda = 0.2 < 1/4$, we knew that the orbit would converge to the equilibrium price, which is equal to 2.769 in this case.

Now plot the orbit of the price for $\lambda = 3.32$ and the same initial values. Would this price policy be possible in practice in the long run?

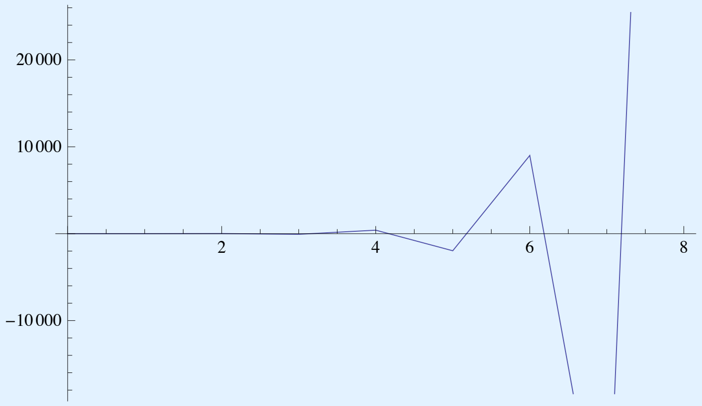
```
a = 4; b = 2.5; u = 2; v = 20; λ = 3.32;
(v - u) / (a + b)
RSolve[{p[t + 1] == (1 - λ b) p[t] - λ a p[t - 1] + λ (v - u), p[0] == 2, p[1] == 2.8}, p, t];
pp = p /. %[[1]];
FullSimplify[pp[t]]
x^2 - (1 - λ b) x + λ a
Solve[x^2 - (1 - λ b) x + λ a == 0, x]
ListPlot[Table[Re[pp[i]], {i, 0, 8}], DataRange → {0, 8}, Joined → True]
```

```
2.76923
```

$2.76923 + 6.35041\ (-3.85616)^t - 7.11964\ (-3.44384)^t$

$13.28 + 7.3\ x + x^2$

```
{{x → -3.85616}, {x → -3.44384}}
```



This price policy would not be possible, since we see in the figure that prices would become negative.

Determine for which values of λ the roots of the characteristic equation are real and the orbits of the price are convergent. Choose a specific value for λ among them and plot the orbit of the price for the initial values given in 2.7.

We know that the orbits are convergent when $\lambda < 1/4 = 0.25$. Moreover, the roots of the characteristic equation will be real when the discriminant of the characteristic equation $(A^2 + 4B)$ is positive or 0.

```
Reduce[(1 - x b)^2 - 4 x a >= 0, x]
```

Reduce::ratnz :
Reduce was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result. ≫

```
x ≤ 0.0483138 || x ≥ 3.31169
```

So λ should be less than (or equal to) 0.048. We choose, for example, $\lambda = 0.04$.

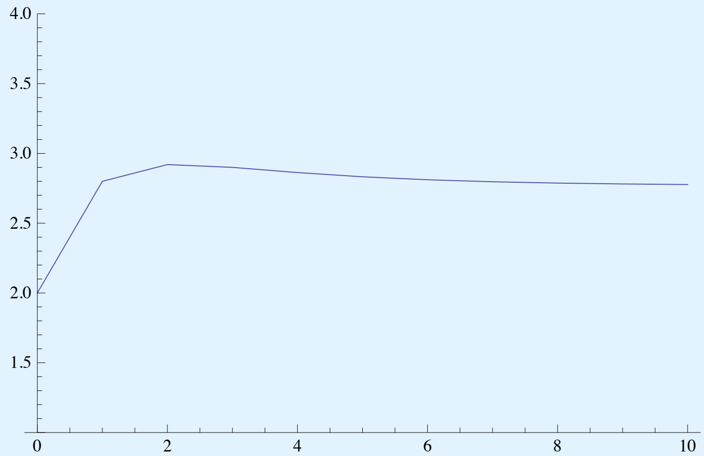
```
a = 4; b = 2.5; u = 2; v = 20; λ = 0.04;
(v - u) / (a + b)
RSolve[{p[t + 1] == (1 - λ b) p[t] - λ a p[t - 1] + λ (v - u), p[0] == 2, p[1] == 2.8}, p, t];
pp = p /. %[[1]];
FullSimplify[pp[t]]
x^2 - (1 - λ b) x + λ a
Solve[x^2 - (1 - λ b) x + λ a == 0, x]
ListPlot[Table[Re[pp[i]], {i, 0, 10}],
 DataRange → {0, 10}, Joined → True, PlotRange → {1, 4}]
```

```
2.76923
```

$2.76923 - 1.29879 \times 0.243845^{t} + 0.529557 \times 0.656155^{t}$

$0.16 - 0.9\,x + x^{2}$

$\{\{x \to 0.243845\}, \{x \to 0.656155\}\}$



We see that the orbit is convergent and the roots of the characteristic equation are real.