McKay Natural Correspondences

Carolina Vallejo Rodríguez

Universitat de València

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This is a joint work with G. Navarro and P. H. Tiep.

Introduction

Let G be a finite group, p prime, $P \in \mathrm{Syl}_p(G)$.

We write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid p \text{ does not divide } \chi(1)\}$.

Conjecture (McKay)

Let G be a finite group, p prime. Then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|.$$

How to prove it? Isaacs, Malle and Navarro proved in 2007 a reduction theorem for the McKay conjecture.

The problem reduces to show that the finite simple groups satisfy the so called inductive McKay condition.

However, it is fundamental to understand as much as possible the nature of the problem.

A natural case to consider: $\mathbf{N}_G(P) = P$.

In 2003, G. Navarro proved that if G is p-solvable, there exists a canonical bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}(P/P')$$

$$\chi \mapsto \chi^* \, .$$

In fact, if $\chi \in \mathrm{Irr}_{p'}(G)$, then

$$\chi_P = \chi^* + \Delta \, ,$$

where χ^* is the unique linear constituent of χ_P.

Does this hold without the solvability assumption?

- For $p = 2$, S_5 is a counterexample to the previous statement.
- If $p > 3$, Guralnick, Malle and Navarro proved (in 2003) that the groups having a self-normalizing Sylow p-subgroup are solvable. Therefore, the problem is completely solved for $p > 3$.
- What about $p = 3$? We have now a positive answer.

Main Result

In fact, we have proved more. Recall that we denote by $B_0(G)$ the set of irreducible characters of G in the principal block.

Theorem A (Main)

Let G be a group, p odd, $P \in \mathrm{Syl}_p(G)$. Suppose $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$. If $\chi \in \mathrm{Irr}_{p'}(G)$ lies in the principal block then $\chi_{\mathbf{N}_G(P)}$ has a unique p'-degree constituent χ^*. Furthermore, the map $\chi \mapsto \chi^*$ is a bijection $\mathrm{Irr}_{p'}(B_0(G)) \to \mathrm{Irr}_{p'}(B_0(\mathbf{N}_G(P)))$.

Remarks:

- If $\mathbf{N}_G(P) = P$, then $B_0(G)$ is the only block of maximal defect, so we get a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}(P/P')$.
- Theorem A is not true for non-principal blocks ($p = 3$ and $G = SL_2(27) \cdot C_3$).

Why the case $p = 3$ has been solved 10 years after it was originally considered by Navarro?

It is not only very difficult to control restriction of characters, but it has been necessary to use a recent and deep result by Navarro and Sp¬*äth*.

Some ideas about the proof of Theorem A:

We need to use a strong induction over normal subgroups.

Theorem A'

Let G be a group, p odd, $P \in \mathrm{Syl}_p(G)$. Suppose $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$.

Let $L \triangleleft G$. If $\chi \in \mathrm{Irr}_{p'}(G)$ lies in the principal block, then $\chi_{L\mathbf{N}_G(P)}$ has a unique p'-degree irreducible constituent χ^*.

The map $\chi \mapsto \chi^*$ is a bijection $\mathrm{Irr}_{p'}(B_0(G)) \to \mathrm{Irr}_{p'}(B_0(L\mathbf{N}_G(P)))$.

Theorem

Assume p is odd and $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$. Then $\mathbf{O}_{p'}(\mathbf{N}_G(P)) \subseteq \mathbf{O}_{p'}(G)$.

Theorem (Extension)

Assume p is odd and $\mathbf{N}_G(P) = P$. Let $L \triangleleft G$. If $\theta \in \mathrm{Irr}_{p'}(L)$ is G-invariant and extends to PL, then θ extends to G.

Some Applications

Theorem A provides a surprising characterization (p odd) of when a group has a self-normalizing Sylow subgroup or a decomposable Sylow normalizer in terms of the degrees of the irreducible constituents of a permutation character.

Corollary B

Let G be a group, p odd and $P \in \mathrm{Syl}_p(G)$.

(a) $\mathbf{N}_G(P) = P$ if and only if 1_G is the unique p'-degree irreducible constituent of $(1_P)^G$.

(b) $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$ if and only 1_G is the only irreducible constituent of $(1_{P\mathbf{C}_G(P)})^G$ that belongs to $\mathrm{Irr}_{p'}(B_0(G))$.

Corollary B is not true for $p = 2$. Let $G = S_5$. The character $(1_P)^G$ contains 1_G and an irreducible constituent of degree 5.

Part (a) of the previous corollary is the exact (!) opposite of a recent result by Malle and Navarro.

Theorem (Malle, Navarro; 2012)

A finite group G has a normal Sylow p-subgroup P if and only if all the irreducible constituents of $(1_P)^G$ have degree not divisible by p.

Notation: Recall, $\chi \in \mathrm{Irr}(G)$ is said to be p-rational if the values of χ lie in $\mathbb{Q}_n$ for some integer n not divisible by p. We write

$$\mathrm{Irr}_{\mathbb{Q}_{p'}}(G) = \{\chi \in \mathrm{Irr}(G) | \chi \text{ is } p\text{-rational}\}.$$

Conjecture (Navarro, 2004)

Let G be a group, p a prime and $P \in \mathrm{Syl}_p(G)$. If $B \in \mathrm{Bl}(G|P)$ and $b \in \mathrm{Bl}(\mathbf{N}_G(P)|P)$ is its Brauer correspondent, then

$$|\mathrm{Irr}_{p',\mathbb{Q}_{p'}}(B)| = |\mathrm{Irr}_{p',\mathbb{Q}_{p'}}(b)|.$$

The previous conjecture has the following consequence:

Consequence

Let G be a group, p odd, $P \in \mathrm{Syl}_p(G)$. Then, $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$ if and only if 1_G is the unique p-rational, p'-degree character lying in the principal block of G.

- If $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$, by Theorem A the fields of values of the p'-degree nontrivial irreducible characters in the principal block of G are $\mathbb{Q}_{p^a}$ for $a > 0$.Thus, no nontrivial p'-degree irreducible character in the principal block is p-rational.
- The other half is still open.

Future

Open problems:

- If $\mathbf{N}_G(P) = \mathbf{C}_G(P)P$ (and perhaps p odd), is there a canonical bijection
$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))?$$
Aside of the p odd principal block case, we have proved there is a canonical bijection provided that G is p-solvable.
- Theorem A suggests to study blocks $B \in \mathrm{Bl}(G|D)$ such that $\mathbf{N}_G(D, b_D) = \mathbf{C}_G(D)D$. (If G is p-solvable, we have an analog of Theorem A in this case).

Merci beaucoup!

Thanks for your attention!