BRAUER CORRESPONDENT BLOCKS WITH ONE SIMPLE MODULE

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Abstract. One of the main problems in representation theory is to understand the exact relationship between Brauer corresponding blocks of finite groups. The case where the local correspondent has a unique simple module seems key. We characterize this situation for the principal p-blocks where p is odd.

1. Introduction

Let G be a finite group, let p be a prime, and let $\mathbb{F}$ be an algebraically closed field of characteristic p. The blocks of G are the indecomposable two-sided ideals of the group algebra $\mathbb{F}G$. Richard Brauer associated to each block B of G a p-subgroup D of G, up to conjugation, and a block b of the local subgroup $\mathbf{N}_G(D)$, which is called the Brauer first main correspondent of B. What is the exact relationship between these two algebras, and what invariants they share is one of the main problems in representation theory of finite groups. Our major interest is in the invariants $k(B)$, $k_0(B)$ and $l(B)$ (which are the number of complex irreducible characters in B, those of them which have height zero, and the number of simple modules in B over $\mathbb{F}$, respectively) and their relation with $k(b)$, $k_0(b)$ and $l(b)$. For instance, $k_0(B) = k_0(b)$ is the Alperin-McKay conjecture, and $l(B) \geq l(b)$ would be a consequence of the Alperin weight conjecture.

In this paper, we wish to understand how the local condition $l(b) = 1$ affects B, and the other way around. Already the key case where B and b are the principal blocks (the blocks containing the trivial representation of the group) is hard to handle. This is our main result.

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Theorem A. *Let G be a finite group, let p be an odd prime, and let $P \in \mathrm{Syl}_p(G)$. Let B be the principal block of G, and let b be the principal block of $\mathbf{N}_G(P)$. Then B contains no non-trivial p-rational height zero irreducible character if and only if $l(b) = 1$.*

As we will point out in Conjecture 6.4 below, we have an ad hoc statement for the prime $p = 2$, but to prove it seems presently out of reach. As it seems also out of reach to prove the following.

Conjecture B. *Let G be a finite group, let p be an odd prime, let B be a p-block of G and let b be the Brauer first main correspondent of B. If B contains exactly one p-rational height zero irreducible character, then $l(b) = 1$.*

Outside principal blocks, the converse of Conjecture B is not true, even in blocks with normal maximal defect. For instance, the `SmallGroup`(72,22) in [GAP] is a counterexample for $p = 3$.

There is a related characterization of when $l(b) = 1$ for p-solvable groups. If χ is an ordinary character of G, then χ^0 is the Brauer character obtained by restricting χ to the p-regular elements of G.

Theorem C. *Suppose that G is p-solvable, with p odd. Let B be a p-block with defect group P and let b be its Brauer first main correspondent. Then $l(b) = 1$ if and only if there is exactly one p-rational $\chi \in \mathrm{Irr}(B)$ of height zero and such that $\chi^0 \in \mathrm{IBr}(B)$.*

Unfortunately, the "only if" direction of Theorem C is false outside p-solvable groups. Let $G = \mathsf{A}_6$, $p = 3$ and B the principal block of G. Then B contains a unique p-rational p'-degree irreducible character that lifts an irreducible Brauer character while $l(b) = 4$. In this case, the defect groups of B are abelian. The same situation happens in $\mathrm{PSL}_2(p)$ when $p \geq 5$. In this case, $l(b) = (p-1)/2$ and the defect groups of B are cyclic. It is interesting to speculate to what extent the "only if" direction holds.

For a character theorist it is always pleasant to find new properties of a finite group which can be read off from its character table. By a result of R. Brauer, the principal block of a group has a unique irreducible Brauer character if and only if it has a normal p-complement (see Corollary 6.13 of [N1]). Hence Theorem A is equivalent to the following.

Theorem D. *Let G be a finite group, let p odd, and let $P \in \mathrm{Syl}_p(G)$. Then $\mathbf{N}_G(P)$ has a normal p-complement if and only if there are no non-trivial p-rational p'-degree complex irreducible characters in the principal block of G.*

In general, it is not easy to produce p-rational irreducible characters. Even with the strong hypotheses that $\theta \in \mathrm{Irr}(N)$ is a p-rational character of p'-degree in the principal block of $N \triangleleft G$, G/N is cyclic of p'-order and θ extends to G, then it is not necessarily true that θ has a p-rational extension to G. Our way to produce p-rational characters is indirect, by using some results which we believe are of independent interest. The first of these is a relative to normal subgroups version of the Glauberman correspondence.

If P is a group acting by automorphisms on G, then $\mathrm{Irr}_P(G)$ is the set of P-invariant irreducible characters of G. In Theorem E below, the Glauberman correspondence is obtained when $N = 1$. If χ is a character, then we denote by $\mathbb{Q}(\chi)$ the smallest field containing the values of χ.

Theorem E. *Suppose that a p-group P acts as automorphisms on a finite group G. Let $N \triangleleft G$ be P-invariant such that G/N is a p'-group. Let $C/N = \mathbf{C}_{G/N}(P)$. Then there exists a natural bijection ${}^*: \mathrm{Irr}_P(G) \to \mathrm{Irr}_P(C)$. In fact, if $\chi \in \mathrm{Irr}_P(G)$, then*

$$\chi_C = e\chi^* + p\Delta + \Xi\,,$$

where Δ and Ξ are characters of C or zero, p does not divide e, and no irreducible constituent of Ξ lies over some P-invariant character of N. In fact, $e \equiv \pm 1 \bmod p$. In particular, $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$. Also, if χ has p'-degree, then χ lies in the principal block of G if and only if χ^* lies in the principal block of C.

Several particular cases of Theorem E have appeared previously in the literature (see, for instance, Theorem 5.1 of [IN]).

We will also need a result on extension of characters that generalizes results of Alperin and Dade (see [A] and [D]).

Theorem F. *Suppose that $N \triangleleft G$. Let $\theta \in \mathrm{Irr}(N)$ be p-rational, G-invariant of p'-degree in the principal block of N, where p is odd. Let $Q \in \mathrm{Syl}_p(N)$, and assume that $|G : N\mathbf{C}_G(Q)|$ is a p-power. Then θ uniquely determines a character $\chi \in \mathrm{Irr}(G)$ in the principal block of G such that χ is p-rational and $\chi_N = \theta$.*

Under the hypotheses of Theorem F, it is false that θ has a unique p-rational extension in the principal block of G. For instance, take $p = 3$, $G = C_3 \times \mathsf{S}_3$, $N = C_3$, and θ the principal character of N. In this case, the character χ determined by Theorem F is the trivial character of G, but there is another p-rational extension of θ to G.

2. The Relative Glauberman Correspondence

We follow the notation of [I2] for ordinary characters and the notation of [N1] for modular characters and blocks. In particular, if p is a prime number, and $\mathbf{R}$ is the ring of algebraic integers in $\mathbb{C}$, we choose M a maximal ideal of $\mathbf{R}$ containing $p\mathbf{R}$, with respect to which the Brauer characters of any finite group G are constructed. We also let $^*: \mathbf{R} \to \mathbf{R}/M$ be the canonical ring epimorphism. (Later on, we will also denote by * several character correspondences, but we believe that there is no risk of confusion.) If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G. Also, G_θ is the stabilizer of θ in G. Sometimes, we will denote by $B_0(G)$ the set of the irreducible complex characters of G which lie in the principal p-block of G, where p is a prime. By a block, we mean a p-block. Throughout this paper, we will denote by $\mathcal{G} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the absolute Galois group. By elementary character theory, we know that $\mathcal{G}$ acts on the irreducible complex characters of every finite group G.

We begin by proving the following.

Lemma 2.1. *Suppose that P is a p-group acting as automorphisms on a finite group G. Suppose that $N \triangleleft G$ is P-invariant with G/N a p'-group. Let $\theta \in \mathrm{Irr}(N)$ be P-invariant. If P acts trivially on G/N, then every $\psi \in \mathrm{Irr}(G|\theta)$ is P-invariant.*

Proof. By the Clifford correspondence (Theorem 6.11 of [I2]), we may assume that θ is G-invariant. Let $g \in G$, $x \in P$ and $\psi \in \mathrm{Irr}(G|\theta)$. We want to show that $\psi^x(g) = \psi(g)$. Write $H = N\langle g\rangle$, a P-invariant subgroup of G. By considering the irreducible constituents of χ_H, all of which lie over θ, we may assume that $H = G$. That is to say, we assume that G/N is cyclic. Hence θ extends to G, by Corollary 11.22 of [I2]. By coprime action (Theorem 13.31 of [I2]), there is $\chi \in \mathrm{Irr}(G|\theta)$ which is P-invariant. By Gallagher's theorem (Corollary 6.17 of [I2]), every $\psi \in \mathrm{Irr}(G|\theta)$ is of the form $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$. Since P acts trivially on G/N, then every $\beta \in \mathrm{Irr}(G/N)$ is P-invariant, and the statement follows. □

Lemma 2.2. *Suppose that P is a p-group acting as automorphisms on a finite group G. Suppose that $N \triangleleft G$ is P-invariant with G/N a p'-group, and let $C/N = \mathbf{C}_{G/N}(P)$. Suppose that $\chi \in \mathrm{Irr}(G)$ is P-invariant. Then χ_N has a P-invariant constituent $\theta \in \mathrm{Irr}(N)$, and any two such constituents are C-conjugate.*

Proof. The first part is Theorem 13.27 of [I2]. The second part follows from Corollary 13.9 of [I2]. □

We can now prove Theorem E.

Theorem 2.3 (Relative Glauberman Correspondence)**.** *Suppose that P is a p-group acting as automorphisms on a finite group G. Let $N \triangleleft G$ be P-invariant such that G/N is a p'-group. Let $C/N = \mathbf{C}_{G/N}(P)$. Then there exists a natural bijection*

$$^*: \mathrm{Irr}_P(G) \to \mathrm{Irr}_P(C).$$

In fact,

$$\chi_C = e\chi^* + p\Delta + \Xi\,,$$

where Δ and Ξ are characters of C or zero, $e \equiv \pm 1$ mod p, and no irreducible constituent of Ξ lies over some P-invariant character of N. In particular,

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*).$$

Also, if χ has p'-degree, then χ lies in the principal block of G if and only if χ^* lies in the principal block of C.

Proof. We first prove the part of the statement concerning the existence of a bijection.

Notice that C acts on $\mathrm{Irr}_P(N)$. Indeed, if $\theta \in \mathrm{Irr}_P(N)$, $x \in P$ and $c \in C$, then $c^x = nc$ for some $n \in N$. Hence $(\theta^c)^x = \theta^{x^{-1}cx} = \theta^c$, and θ^c is P-invariant. Let Λ be a complete set of representatives of the C-orbits on $\mathrm{Irr}_P(N)$. We claim that

$$\mathrm{Irr}_P(G) = \bigcup_{\theta \in \Lambda} \mathrm{Irr}_P(G|\theta)$$

is a disjoint union. Let $\chi \in \mathrm{Irr}_P(G)$. By Lemma 2.2 we have that χ_N has a P-invariant irreducible constituent θ, and that all of them are C-conjugate. This proves the claim. By the same argument, we have that

$$\mathrm{Irr}_P(C) = \bigcup_{\theta \in \Lambda} \mathrm{Irr}_P(C|\theta)$$

is a disjoint union. Then it suffices to prove that there are bijections

$$^* : \mathrm{Irr}_P(G|\theta) \to \mathrm{Irr}_P(C|\theta)$$

satisfying the conditions in the statement of the theorem. We prove this by induction on $|G : N|$.

Let $\chi \in \mathrm{Irr}_P(G)$, let $\theta \in \Lambda$ be under χ, let T be the stabilizer of θ in G, and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ. Let $\mathbb{T}$ be a set of representatives of the double cosets of T and C in G with $1 \in \mathbb{T}$. By the Mackey formula, we have that

$$\chi_C = (\psi_{T \cap C})^C + \delta\,,$$

where $\delta = \sum_{1 \neq t \in \mathbb{T}} (\psi^t_{T^t \cap C})^C$. We claim that no irreducible constituent of δ lies over θ. Otherwise, let η be an irreducible constituent of $(\psi^t_{T^t \cap C})^C$ for some $1 \neq t \in \mathbb{T}$ lying over θ. Then η lies over θ and over θ^t. By Clifford's theorem (Theorem 6.2 of [I2]), we have that $\theta = \theta^{tc}$ for some $c \in C$, but this is a contradiction since $1 \neq t \in \mathbb{T}$. This proves the claim.

Notice that, in fact, no irreducible constituent of δ lies over any P-invariant irreducible character $\tau \in \mathrm{Irr}(N)$. Otherwise, τ and θ are P-invariant characters of N lying under χ. By Lemma 2.2 τ is C-conjugate to θ, and thus θ lies under δ, a contradiction.

Suppose now that $T < G$. By induction, there is a bijection

$$^* : \mathrm{Irr}_P(T|\theta) \to \mathrm{Irr}_P(T \cap C|\theta)$$

such that $\psi_{T\cap C} = e\psi^* + p\Delta$, where Δ is a character or zero, and $e \equiv \pm 1 (\operatorname{mod} p)$. Then

$$\chi_C = (\psi_{T\cap C})^C + \delta = e(\psi^*)^C + p\Delta^C + \delta\,.$$

By the Clifford correspondence (Theorem 6.11 of [I2]), we know that induction defines bijections $\mathrm{Irr}_P(T|\theta) \to \mathrm{Irr}_P(G|\theta)$ and $\mathrm{Irr}_P(T\cap C|\theta) \to \mathrm{Irr}_P(C|\theta)$. Since

$$\chi^* = (\psi^*)^C \in \mathrm{Irr}(C|\theta),$$

we conclude that we may assume that θ is G-invariant.

We have to show that for $\chi \in \mathrm{Irr}_P(G|\theta)$, we have that that $\chi_C = e\chi^* + p\Delta$, where $\chi^* \in \mathrm{Irr}(C)$, p does not divide e, and that the map $\chi \mapsto \chi^*$ is a bijection $\mathrm{Irr}_P(G|\theta) \to \mathrm{Irr}_P(C|\theta)$. We consider the semidirect product $\Gamma = GP$ of G by P. Since θ is Γ-invariant, we have that (Γ, N, θ) is a character triple. By Theorem 11.28 of [I2] there is an isomorphism $(\tau, \sigma)\colon (\Gamma, N, \theta) \to (\Gamma^\tau, N^\tau, \theta^\tau)$ of character triples, where N^τ is central in Γ^τ. Recall that $\tau : \Gamma/N \to \Gamma^\tau/N^\tau$ is a group isomorphism. We are going to write $\tau(H/N) = H^\tau/N^\tau$ for every subgroup $N \leq H \leq \Gamma$. Since $(NP)^\tau/N^\tau$ is a p-subgroup of Γ^τ/N^τ, and N^τ is central, then $(NP)^\tau$ has a unique Sylow p-subgroup which we denote by P^τ. Now P^τ acts on G^τ/N^τ the same way as $(PN)^\tau$ acts on G^τ/N^τ. Hence, by the properties of character triple isomorphisms in Definition 11.23 of [I2], it is no loss to assume that $N \leq \mathbf{Z}(\Gamma)$. Hence we may assume that $[N, P] = 1$ and that $N \leq \mathbf{Z}(G)$. In particular, G has a central Sylow p-subgroup N_p, a normal p-complement K, and in particular $C = \mathbf{C}_K(P) \times N_p$. Write $\theta = \theta_{p'} \times \theta_p$, where $\theta_{p'} = \theta_{K\cap N}$ and $\theta_p = \theta_{N_p}$. We have that

$$\mathrm{Irr}_P(G|\theta) = \{\mu \times \theta_p \mid \mu \in \mathrm{Irr}_P(K|\theta_{p'})\}$$

and

$$\mathrm{Irr}_P(C|\theta) = \mathrm{Irr}(C|\theta) = \{\epsilon \times \theta_p \mid \epsilon \in \mathrm{Irr}(C\cap K|\theta_{p'})\}.$$

By Theorem 13.29 of [I2], we have that the Glauberman correspondence

$${}^* : \mathrm{Irr}_P(K) \to \mathrm{Irr}(C\cap K)$$

sends $\mathrm{Irr}_P(K|\theta_{p'})$ bijectively onto $\mathrm{Irr}(C\cap K|\theta_{p'})$. Since $\mu_{C\cap K} = e\mu^* + p\Delta$, where $e \equiv \pm 1 (\operatorname{mod} p)$, the first part of the proof of the statement is now complete.

The action of the absolute Galois group $\mathcal{G}$ on characters commutes with the action of P and with restriction of characters. Hence our map ${}^* : \mathrm{Irr}_P(G) \to \mathrm{Irr}_P(C)$ is $\mathcal{G}$-equivariant. This implies that

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$$

for $\chi \in \mathrm{Irr}_P(G)$.

We finally prove the statement about blocks. Let $\chi \in \mathrm{Irr}(G)$ be P-invariant of p'-degree. We have that $\chi_C = e\chi^* + p\Delta + \Xi$, where p does not divide e and no irreducible constituent of Ξ lies over a P-invariant character of N. We prove that χ

lies in the principal block of G if and only if χ^* lies in the principal block of C. We proceed by induction on $|G : N|$.

Let $\theta \in \mathrm{Irr}(N)$ be P-invariant under χ. Let T be the stabilizer of θ in G, and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ. We have that $\psi(1)$ is a p'-number and ψ is P-invariant. By the first part of the proof $\psi_{T\cap C} = f\psi^* + p\Delta'$, where p does not divide f, and we know that $\psi^* \in \mathrm{Irr}(T \cap C)$ is the Clifford correspondent of χ^*. By induction, if $T < G$, then $\psi \in B_0(T)$ if and only if $\psi^* \in B_0(T \cap C)$. Thus, in this case the statement follows from Corollaries 6.2 and 6.7 of [N1].

We may assume that θ is G-invariant, and therefore we have that

$$\chi_C = e\chi^* + p\Delta$$

and so χ^* has p'-degree. Again, let $\Gamma = GP$ be the semidirect product of G and P. Since NP has p'-index in Γ, we can choose $P \le R$ a Sylow p-subgroup of Γ contained in NP, so that $NP = NR$. Also $\mathbf{N}_{\Gamma/N}(NP/N) = \mathbf{N}_{\Gamma/N}(NR/N)$, and we see that $\mathbf{C}_{G/N}(P) = \mathbf{C}_{G/N}(R)$ and that $\mathrm{Irr}_P(G) = \mathrm{Irr}_R(G)$. Write

$$M/N = \mathbf{N}_{\Gamma/N}(NR/N) = N\mathbf{N}_\Gamma(R)/N,$$

so that $M \cap G = C$. By Corollary 9.6 of [N1], let B be the unique block of Γ covering the block of χ and let b be the unique block of M covering the block of χ^*. Since χ has p'-degree, it enters with p'-multiplicity in $(1_1)^G = ((1_P)^\Gamma)_G$. If $\psi \in \mathrm{Irr}(\Gamma)$ lies over χ, then $[\psi_G, \chi]$ is a p-power, by Corollary 11.29 of [I2] and and using that χ is Γ-invariant. Therefore χ extends to some $\tilde\chi \in \mathrm{Irr}(\Gamma)$. By the same argument χ^* extends to $\tilde\chi^* \in \mathrm{Irr}(M)$. Of course, $B = B_0(\Gamma)$ if and only if χ belongs to the principal block of G and $b = B_0(M)$ if and only if χ^* belongs to the principal block of C (using Corollary 9.6 of [N1]).

Since $\tilde\chi$ and $\tilde\chi^*$ have p'-degree, then we know that B and b have defect group R, by Theorem 4.6 of [N1]. By Problem 4.5 of [N1], we have that $B = B_0(\Gamma)$ if and only if

$$\left(\frac{|\mathrm{cl}_\Gamma(x)|\tilde\chi(x)}{\tilde\chi(1)}\right)^* = |\mathrm{cl}_\Gamma(x)|^*$$

for every p-regular $x \in \Gamma$ such that $R \in \mathrm{Syl}_p(\mathbf{C}_\Gamma(x))$. Similarly, $b = B_0(M)$ if and only if

$$\left(\frac{|\mathrm{cl}_M(y)|\tilde\chi^*(y)}{\tilde\chi^*(1)}\right)^* = |\mathrm{cl}_M(y)|^*$$

for every p-regular $y \in M$ such that $R \in \mathrm{Syl}_p(\mathbf{C}_M(y))$

Suppose that $K = \mathrm{cl}_\Gamma(x)$, where x is p-regular and $R \in \mathrm{Syl}_p(\mathbf{C}_\Gamma(x))$. Notice that $x \in G$, since Γ/G is a p-group. Now

$$\mathbf{C}_G(R)N/N \le \mathbf{C}_{G/N}(R) = \mathbf{C}_{G/N}(P) = C/N,$$

and therefore $x \in \mathbf{C}_G(R) \leq C$. Let $L = \mathrm{cl}_M(x)$. By Lemma 4.16 of [N1] we have that $K \cap \mathbf{C}_\Gamma(R)$ is the conjugacy class of x in $\mathbf{N}_\Gamma(R)$. Also,

$$|K| \equiv |K \cap \mathbf{C}_\Gamma(R)| (\mathrm{mod}\, p)$$

by counting. By the same argument, $L \cap \mathbf{C}_\Gamma(R)$ is the conjugacy class of x in $\mathbf{N}_\Gamma(R)$ and also

$$|L| \equiv |L \cap \mathbf{C}_\Gamma(R)| (\mathrm{mod}\, p).$$

Since $K \cap \mathbf{C}_\Gamma(R) = L \cap \mathbf{C}_\Gamma(R)$, we see that $|K| \equiv |L| (\mathrm{mod}\, p)$. Also, since

$$\chi_C = e\chi^* + p\Delta,$$

we have that $\chi(x) \equiv e\chi^*(x)(\mathrm{mod}\, p)$ and $\chi(1) \equiv e\chi^*(1)(\mathrm{mod}\, p)$, where p does not divide e. Thus $\chi^*(1)\chi(x) \equiv \chi(1)\chi^*(x)(\mathrm{mod}\, p)$ and $\tilde{\chi}^*(1)\tilde{\chi}(x) \equiv \tilde{\chi}(1)\tilde{\chi}^*(x)(\mathrm{mod}\, p)$. Since $|K| = |L| (\mathrm{mod}\, p)$, we deduce that

$$|K|\tilde{\chi}^*(1)\tilde{\chi}(x) \equiv |L|\tilde{\chi}(1)\tilde{\chi}^*(x) \bmod p\,.$$

Using the fact that the degrees of χ and χ^* are p'-numbers, we deduce that

$$\left(\frac{|K|\tilde{\chi}(x)}{\tilde{\chi}(1)}\right)^* = \left(\frac{|L|\tilde{\chi}^*(x)}{\tilde{\chi}^*(1)}\right)^*.$$

The result follows from the discussion in the preceding paragraph using, as we have proved, that $|\mathrm{cl}_\Gamma(x)|^* = |\mathrm{cl}_M(x)|^*$ for every p-regular $x \in \mathbf{C}_\Gamma(R)$. □

3. An extension theorem

The aim of this section is to prove Theorem F. We first need some lemmas.

Lemma 3.1. *Suppose that $N \lhd G$ and that $\psi \in \mathrm{Irr}(G)$ has p'-degree and is such that $\psi_N = \theta \in \mathrm{Irr}(N)$. Assume that ψ_H belongs to the principal block of H whenever H/N is a cyclic p'-group. Then ψ belongs to the principal block of G.*

Proof. Since ψ lies in a block of maximal defect, by Problem 4.5 of [N1], we want to show that

$$\left(\frac{|K|\psi(x)}{\psi(1)}\right)^* = |K|^*,$$

where $K = \mathrm{cl}_G(x)$ is the conjugacy class of a p-regular $x \in G$ with $|G : \mathbf{C}_G(x)|$ a p'-number. Since $\psi(1)$ is not divisible by p, it suffices to show that $\psi(x)^* = \psi(1)^*$. Let $H = N\langle x\rangle$. We know that there is $P \in \mathrm{Syl}_p(G)$ such that $P \leq \mathbf{C}_G(x)$. Let $Q = P \cap N \in \mathrm{Syl}_p(N)$, so that $Q \leq \mathbf{C}_N(x) \leq \mathbf{C}_H(x)$. Since H/N is a p'-number, it follows that $Q \in \mathrm{Syl}_p(H)$. In particular, p does not divide $|L|$, where $L = \mathrm{cl}_H(x)$. By hypothesis ψ_H belongs to the principal block of H, and we conclude that

$$|L|^*\psi(x)^* = |L|^*\psi(1)^*.$$

Then $\psi(x)^* = \psi(1)^*$, as desired. □

We remind the reader that, in general, if $\psi \in \mathrm{Irr}(G)$ lies in the principal block of G and $\psi_H \in \mathrm{Irr}(H)$, then ψ_H needs not to be in the principal block of H. For instance, take $G = \mathsf{A}_4$, $p = 2$, and H is a Sylow 3-subgroup of G. However, the following statement holds.

Lemma 3.2. *Suppose that $\psi \in \mathrm{Irr}(G)$ lies in the principal block of G, and assume that $H \lhd \lhd G$. If $\psi_H \in \mathrm{Irr}(H)$, then ψ_H lies in the principal block of H.*

Proof. Arguing by induction on $|G : H|$, we may assume that $H \lhd G$. Then the result follows by Theorem 9.2 of [N1]. $\square$

Lemma 3.3. *Let $K \lhd G$ with G/K being a p-group and $p > 2$. If $\gamma \in \mathrm{Irr}(K)$ is p-rational and G-invariant, then γ^G contains a unique p-rational irreducible constituent $\hat{\gamma} \in \mathrm{Irr}(G)$. Furthermore, γ lies in the principal block of K if and only if $\hat{\gamma}$ lies in the principal block of G.*

Proof. This is Theorem 6.30 of [I2] together with Corollary 9.6 of [N1]. $\square$

We can now prove Theorem F, which is a variation of Theorem 3.2 of [NT3].

Theorem 3.4. *Suppose that $N \lhd G$. Let $\theta \in \mathrm{Irr}(N)$ be p-rational, G-invariant of p'-degree in the principal block of N, where p is odd. Let $Q \in \mathrm{Syl}_p(N)$, and assume that $|G : N\mathbf{C}_G(Q)|$ is a power of p. Then θ uniquely determines a character $\chi \in \mathrm{Irr}(G)$ in the principal block of G such that χ is p-rational and $\chi_N = \theta$.*

Proof. Let $M = N\mathbf{C}_G(Q)$. By the Frattini argument, we have that $M \lhd G$.

We next show that if $N \leq U \leq M$ and U/N has a normal p-complement, then there exists a unique p-rational extension $\eta_{(U)} \in \mathrm{Irr}(U)$ of θ in the principal block of U. Let V/N be the normal p-complement of U/N. We have that $V = N\mathbf{C}_V(Q)$ and V/N is a p'-group. Since V/N is a p'-group, then $\mathbf{C}_V(Q)/\mathbf{C}_N(Q)$ is a p'-group. By elementary group theory, $\mathbf{Z}(Q)$ is a central Sylow p-subgroup of $\mathbf{C}_V(Q)$, and therefore there exists $Y \leq \mathbf{C}_V(Q)$ of p'-order such that $\mathbf{C}_V(Q) = Y \times \mathbf{Z}(Q)$. By Theorem 3.2 of [NT3], there exists a unique $\hat{\theta} \in \mathrm{Irr}(V)$ in the principal block of V lying over θ. In fact $\hat{\theta}_N = \theta$. By uniqueness, $\hat{\theta}$ is p-rational and U-invariant. (This is a standard argument. For instance, if $\sigma \in \mathcal{G}$ fixes $\mathbb{Q}(\theta)$, then $\hat{\theta}^\sigma$ is a p-rational extension of θ in the principal block, so by uniqueness $\hat{\theta}^\sigma = \hat{\theta}$. Thus $\mathbb{Q}(\theta) = \mathbb{Q}(\hat{\theta})$ and $\hat{\theta}$ is p-rational.) By Lemma 3.3, $\hat{\theta}$ has a unique p-rational extension η to U, which lies in the principal block of U. If η' is another p-rational extension of θ in the principal block of U, then $\eta'_V = \rho \in \mathrm{Irr}(V)$ lies in the principal block of V (by Lemma 3.2), and extends θ. By Theorem 3.2 of [NT3], $\rho = \hat{\theta}$. So η' is a p-rational extension of $\hat{\theta}$, and then $\eta' = \eta$ by Lemma 3.3.

We now define a class function η of M, which is uniquely determined by θ, as follows: for $m \in M$, let $H = N\langle m\rangle \leq M$, and, by the previous paragraph, let $\eta_{(H)} \in \mathrm{Irr}(H)$ be the unique p-rational extension of θ in the principal block of H. Set

$\eta(m) = \eta_{(H)}(m)$. It is straightforward to check that η is a G-invariant class function of M by using that θ is G-invariant and that $\eta_{(H^z)} = (\eta_{(H)})^z$ for $z \in G$. Notice that $\eta(n) = \theta(n)$ for $n \in N$.

Next we prove that η is a generalized character. Suppose that E/N is nilpotent, where $N \leq E \leq M$. By the second paragraph of this proof, there exists a unique p-rational $\psi \in \mathrm{Irr}(E)$ in the principal block extending θ. We prove that $\eta_E = \psi$. Let $g \in E$ and write $H = N\langle g \rangle$. Then ψ_H is p-rational. Since $H \lhd \lhd E$, we have that ψ_H lies in the principal block of H by Lemma 3.2. Since ψ_H extends θ, then $\psi_H = \eta_{(H)}$. Consequently $\psi(g) = \eta(g)$, and $\psi_E = \eta_E$, as wanted. By Theorem 8.4(a) of [I2], we have that η is a generalized character of M. By using Lemma 8.14(c) of [I2] it is easy to prove that $[\eta, \eta] = 1$, so that $\eta \in \mathrm{Irr}(M)$ by Theorem 8.12 of [I2]. Also, $\eta_N = \theta$. By Lemma 3.1, we have that η lies in the principal block of M (because we have shown that if E/N is nilpotent and $N \leq E \leq M$, then η_E is the unique p-rational extension of θ in the principal block of E). Also η is p-rational by definition. We already know that η is G-invariant. By Lemma 3.3, we know that there is a unique p-rational $\chi \in \mathrm{Irr}(G)$ extending η, which lies in the principal block of G. □

The following result is a suitable extension of Theorem 6.1 of [NTT].

Corollary 3.5. *Let $N \lhd G$. Let p be an odd prime and let $P \in \mathrm{Syl}_p(G)$. Suppose that PN/N is self-normalizing in G/N. Suppose that $\nu \in \mathrm{Irr}(N)$ is P-invariant, p-rational, has p'-degree, and lies in the principal block of N. Then there exists a p-rational $\chi \in \mathrm{Irr}(G|\nu)$ of p'-degree lying in the principal block of G.*

Proof. We proceed by induction on $|G : N|$.

We may assume that ν is G-invariant. Indeed, let $T = G_\nu$ be the stabilizer of ν in G. If $T < G$ then, by the inductive hypothesis, there is a p'-degree p-rational $\psi \in \mathrm{Irr}(T)$ lying over ν, in the principal block of T. Then, $\chi = \psi^G \in \mathrm{Irr}(G|\nu)$ is p-rational and has p'-degree (for $PN \leq T$). Also, by Corollary 6.2 and Theorem 6.7 of [N1] χ lies in the principal block, as wanted.

Let M/N be a chief factor of G. We claim that we may assume that $G = MP$. Notice that PM/N has a self-normalizing Sylow p-subgroup. If $MP < G$, then by the inductive hypothesis there is $\eta \in \mathrm{Irr}(MP)$ of p'-degree, p-rational lying over ν, in the principal block of MP. Let $\tau = \eta_M \in \mathrm{Irr}(M)$, which is p-rational of p'-degree, P-invariant, in the principal block of M. Since PM/M is self-normalizing in G/M, again by the inductive hypothesis, there is a p-rational $\chi \in \mathrm{Irr}(G)$ of p'-degree lying over τ and in the principal block of G. Hence the claim follows.

Let Q be a Sylow p-subgroup of N. By the Frattini argument $G = N\mathbf{N}_G(Q)$. Then $N\mathbf{C}_M(Q)$ is normal in G and so, either $M = N\mathbf{C}_M(Q)$ or $\mathbf{C}_M(Q) \leq N$. In the first case, the result follows from Theorem 3.4 since G/M is a p-group.

Assume finally that $\mathbf{C}_M(Q) \leq N$. In this case, by Lemma 3.1 of [NT3], the only block of M covering the principal block of N is the principal block of M, and the only block of G covering the principal block of M is the principal block of G because G/M is a p-group (by Corollary 9.6 of [N1]). Hence the principal block of G is the only block of G covering the principal block of N. By Theorem 6.1 of [NTT], there exists $\chi \in \mathrm{Irr}(G)$ of p'-degree, p-rational lying over ν. Since ν lies in the principal block of N, necessarily χ lies in the principal block of G by Theorem 9.2 of [N1], and the proof of the statement is complete. □

As we have said before, there are examples where G/N is a cyclic p'-group, $\theta \in \mathrm{Irr}(N)$ is p-rational of p'-degree and lies in the principal block of N, the principal block of G is the only block of G, and yet no irreducible constituent of θ^G is p-rational. The smallest counterexample we have found is the `SmallGroup(216,158)` for $p = 3$ (see [GAP]).

4. Proof of the main results

In this section we prove the main results in this paper, assuming Theorem 4.1 below on simple groups, which we will prove in the next section.

Theorem 4.1. *Let p be an odd prime. Let $S \triangleleft G$, where $\mathbf{C}_G(S) = 1$ and S is a non-abelian simple group of order divisible by p. Suppose that G/S is a p-group. Then G has a self-normalizing Sylow p-subgroup if and only if there is no nontrivial p-rational character of p'-degree in the principal block of G.*

In several parts of this paper, we will use the fact that $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$ if $N \triangleleft G$. (See, for instance, the discussion before Theorem 7.6 of [N1].)

Corollary 4.2. *Let p be an odd prime. Suppose that G is a finite group such that $G = NP$, where $P \in \mathrm{Syl}_p(G)$ and $N \triangleleft G$ is a direct product of non-abelian simple groups of order divisible by p. If there are no non-trivial p-rational irreducible characters of p'-degree in the principal block of G, then $P = \mathbf{N}_G(P)$.*

Proof. We proceed by induction on $|G|$. Suppose that $L \neq M$ are proper normal subgroups of G contained in N such that $L \cap M = 1$ (i.e. P is not transitive on the simple normal factors of N). By induction, we have that $\mathbf{N}_G(P)L = PL$ and $\mathbf{N}_G(P)M = PM$. Then

$$\mathbf{N}_G(P) \leq \mathbf{N}_G(P)L \cap \mathbf{N}_G(P)M = PL \cap PM = P(L \cap M) = P.$$

Hence we may assume that N is a minimal normal subgroup of G. Write

$$N = S_1 \times \cdots \times S_t,$$

where $S_i = S_1^{u_i}$ for some $u_i \in P$. Write $H = \mathbf{N}_G(S_1)$, $P_1 = P \cap H$, $Q = P \cap N$ and $Q_1 = Q \cap S_1$. By Lemma 4.1 and Lemma 2.1(ii) of [NTT] we have that: P is self-normalizing in G if, and only if, $\mathbf{C}_{\mathbf{N}_N(Q)/Q}(P) = 1$ if, and only if, $\mathbf{C}_{\mathbf{N}_{S_1}(Q_1)/Q_1}(P_1) = 1$ if, and only if, P_1 is self-normalizing in S_1P_1. Hence it suffices to show that P_1 is self-normalizing in S_1P_1. Assume the contrary. Let $\overline{H} = H/C$, where $C = \mathbf{C}_G(S_1)$.

We have that $S_1 \cong S_1C/C = \overline{S_1} \triangleleft \overline{H}$, $\overline{H}/\overline{S_1}$ is a p-group and $\mathbf{C}_{\overline{H}}(\overline{S_1}) = 1$. We have that $\overline{P_1} = P_1C/C \in \mathrm{Syl}_p(\overline{H})$, and $\overline{H} = \overline{S_1}\,\overline{P_1}$. We can check that $\overline{P_1}$ is not self-normalizing in $\overline{H}$. By Theorem 4.1, $\overline{H}$ has a non-trivial p-rational character γ of p'-degree in the principal block. Let $\gamma_1 = \gamma_{S_1}$. Then $\gamma_1 \in \mathrm{Irr}(S_1)$ is P_1-invariant and lies in the principal block of S_1. By Lemma 4.1 of [NTT], we have that

$$\theta = \gamma_1^{u_1} \times \cdots \times \gamma_1^{u_t} \in \mathrm{Irr}(N)$$

is P-invariant. Of course θ is p-rational of p'-degree and lies in the principal block of N. By Lemma 3.3, we get a contradiction. □

The following easy observation is stated as a lemma for the reader's convenience.

Lemma 4.3. *Let N and M be distinct normal subgroups of a group G. Let P be a Sylow p-subgroup of G. Suppose that $\mathbf{N}_{G/N}(PN/N)$ and $\mathbf{N}_{G/M}(PM/M)$ have a normal p-complement. If $N \cap M = 1$, then $\mathbf{N}_G(P)$ has a normal p-complement.*

Proof. By elementary group theory, $\mathbf{N}_{G/N}(PN/N) = \mathbf{N}_G(P)N/N$. Hence we have that

$$\mathbf{N}_G(P)/\mathbf{N}_N(P) \cong \mathbf{N}_G(P)N/N$$

has a normal p-complement. Similarly, $\mathbf{N}_G(P)/\mathbf{N}_M(P)$ has a normal p-complement. Hence also

$$\mathbf{N}_G(P) = \mathbf{N}_G(P)/(\mathbf{N}_N(P) \cap \mathbf{N}_M(P))$$

has a normal p-complement. □

We are now ready to prove the main result of this paper, which is Theorem D of the introduction (recall that this is equivalent to Theorem A by using Corollary 6.13 of [N1]).

Theorem 4.4. *Let p be an odd prime. Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$. Then $\mathbf{N}_G(P)$ has a normal p-complement if and only if the only p-rational irreducible character of p'-degree lying in the principal block of G is the principal character of G.*

Proof. Suppose that $\mathbf{N}_G(P)$ has a normal p-complement. By Theorem A of [NTV], we have that there is a canonical bijection

$$^* : \mathrm{Irr}_{p'}(B_0(G)) \to \mathrm{Irr}_{p'}(B_0(\mathbf{N}_G(P))).$$

In fact, if $\chi \in \mathrm{Irr}_{p'}(B_0(G))$, then $\chi_{\mathbf{N}_G(P)} = \chi^* + \Delta$, where $\chi^* \in \mathrm{Irr}(\mathbf{N}_G(P))$ is linear in the principal block of $\mathbf{N}_G(P)$, and Δ is zero or a character such that all its irreducible constituents have degree divisible by p. In particular, we see that * commutes with the action of the absolute Galois group $\mathcal{G}$, and therefore $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$. Since $\mathbf{N}_G(P)$ has a normal p-complement X, we have that $\chi^* \in \mathrm{Irr}(\mathbf{N}_G(P)/X)$ is the character of an odd-order p-group P. Hence χ^* is never p-rational, unless $\chi^* = 1$. Therefore, unless $\chi = 1$. This proves one direction.

We assume now that the only p-rational irreducible character of p'-degree lying in the principal block of G is the principal character of G, and we prove that $\mathbf{N}_G(P)$ has a normal p-complement, by induction on $|G|$.

Step 1. We may assume that G has a unique minimal normal subgroup N. Also $\mathbf{N}_G(P)N/N$ has a normal p-complement $V/N \leq K/N = \mathbf{O}_{p'}(G/N)$, and G/K has self-normalizing Sylow p-subgroups.

Let N and M be distinct minimal normal subgroups of G. Since

$$\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G)),\ \mathrm{Irr}(B_0(G/M)) \subseteq \mathrm{Irr}(B_0(G)),$$

by the inductive hypothesis, we have that G/N and G/M have Sylow normalizers with a normal p-complement. By Lemma 4.3, $\mathbf{N}_G(P)$ has a normal p-complement too.

Write $\mathbf{N}_G(P)N/N = PN/N \times V/N$. By Theorem 3.2 of [NTV], we have that $V/N \leq \mathbf{O}_{p'}(G/N) = K/N$. Also, $\mathbf{N}_G(P)K = PK$, and G/K has self-normalizing Sylow p-subgroups.

Step 2. We may assume that N is not a p'-group. In particular $\mathbf{O}_{p'}(G) = 1$.

We know that $\mathbf{N}_{G/N}(PN/N) = PN/N \times V/N$. If N is a p'-group, then V is a normal p-complement of $\mathbf{N}_G(P)N$. Hence $V \cap \mathbf{N}_G(P) \triangleleft \mathbf{N}_G(P)$ is a p-complement of $\mathbf{N}_G(P)$ and we are done.

Step 3. We may assume N is not a p-group. In particular, N is a direct product of isomorphic non-abelian simple groups of order divisible by p.

Suppose that N is a p-group. We know that

$$\mathbf{N}_G(P)/N = \mathbf{N}_{G/N}(P/N) = P/N \times V/N,$$

so that K is a p-solvable group and $\mathbf{O}_p(K) = N$. Recall that $\mathbf{O}_{p'}(K) = 1$, by Step 2. By Hall-Higman Lemma 1.2.3 $\mathbf{C}_K(N) \leq N$. We have that

$$\mathbf{O}_{p'}(\mathbf{N}_G(P)) \leq \mathbf{C}_K(N) \leq N.$$

Hence $\mathbf{O}_{p'}(\mathbf{N}_G(P)) = 1$. By Problem (4.8) of [N1], we have that G has a unique p-block of maximal defect, namely the principal one. Consequently every irreducible character of G of p'-degree lies in $B_0(G)$. We have that $V/K = \mathbf{C}_{K/N}(P/N)$. By the Glauberman correspondence

$$|\mathrm{Irr}_P(K/N)| = |\mathrm{Irr}(V/N)|.$$

If $V/N = 1$, then $\mathbf{N}_G(P) = P$ and we would be done. Hence we may assume that there is some non-trivial $\gamma \in \mathrm{Irr}_P(K/N)$. In particular, γ is p-rational and has p'-degree. Since G/K has a self-normalizing Sylow p-subgroup, we have that Theorem 6.1 of [NTT] produces a p'-degree p-rational character $\chi \in \mathrm{Irr}(G)$ lying over γ. Since $1 \neq \chi$ lies in the principal block of G we get a contradiction.

Step 4. We may assume that $PN \triangleleft G$. Hence $G/N = K/N \times PN/N$.

Recall that by induction

$$\mathbf{N}_{G/N}(PN/N) = PN/N \times V/N,$$

where $V/N \leq \mathbf{O}_{p'}(G/N) = K/N$. Notice that $V/N = \mathbf{C}_{K/N}(PN/N)$. Let $\gamma \in \mathrm{Irr}(PV)$ be p-rational of p'-degree lying in $B_0(PV)$. Hence, $\gamma_V \in \mathrm{Irr}(V)$ lies in $B_0(V)$. By the relative Glauberman correspondence, Theorem 2.3, there is a unique $\tau \in \mathrm{Irr}_P(K)$ such that $\tau^* = \gamma_V$. Also τ lies in $B_0(K)$. By Corollary 3.5, there exists $\chi \in \mathrm{Irr}(G)$ over τ which is p-rational of p'-degree and lies in the principal block. By assumption, χ is the trivial character and hence $\tau = 1$. We conclude $\tau^* = \gamma_V = 1$. Now, $\gamma \in \mathrm{Irr}(PV/V)$ is linear and rational. Since p is odd, it must be $\gamma = 1_{PV}$. We have shown that $PV = \mathbf{N}_G(P)N$ has a unique p-rational irreducible character of p'-degree in its principal block. If $PV < G$, then by induction $\mathbf{N}_{PV}(P) = \mathbf{N}_G(P)$ is p-decomposable. Hence, we may assume $PV = G$. In particular, $PN \triangleleft G$ and $V = K$.

Step 5. Let $Q = P \cap N \in \mathrm{Syl}_p(G)$. We may assume $N\mathbf{C}_K(Q) = K$.

By the Frattini argument $G = N\mathbf{N}_G(Q)$. Then $N\mathbf{C}_K(Q) \triangleleft K$. Assume that $N\mathbf{C}_K(Q) < K$. By Lemma 3.1 of [NT3], we have that $B_0(K)$ is the unique block of K that covers the principal block of $N\mathbf{C}_K(Q)$. Let $1 \neq \gamma \in \mathrm{Irr}(K/N\mathbf{C}_K(Q))$. Then γ lies in $B_0(K)$ and is p-rational of p'-degree. Since $G/N = K/N \times PN/N$, by Lemma 3.3, we have that γ extends to a p-rational character of p'-degree in $B_0(G)$. This is a contradiction because $1 \neq \gamma$.

Step 6. We may assume that $p = 3$ and that N is a direct product of groups isomorphic to $\mathrm{PSL}_2(3^{3^a})$, for some $a \geq 1$. In particular, Q is abelian. Also $NP < G$.

Let $\eta \in \mathrm{Irr}(PN)$ be p-rational of p'-degree lying in $B_0(PN)$. Then $\nu = \eta_N \in \mathrm{Irr}(N)$ is P-invariant, p-rational of p'-degree and lies in $B_0(N)$. By Theorem 3.2 of [NT3], ν extends to a unique $\hat{\nu} \in \mathrm{Irr}(K_\nu)$ in $B_0(K_\nu)$, where K_ν is the stabilizer of ν in K. In particular, by uniqueness, we have that $\hat{\nu}$ is p-rational and P-invariant. Write $\rho = (\hat{\nu})^K \in \mathrm{Irr}(K)$. Then ρ is p-rational, P-invariant and of p'-degree. By Lemma 3.3 we conclude that ρ has an extension to a p'-degree p-rational character in the principal block of G. We conclude that $\rho = 1_K$. This implies $\hat{\nu} = 1_K$. Hence $\nu = 1_N$. Thus $\eta \in \mathrm{Irr}(PN/N)$ is linear and rational. Since p is odd, this implies $\eta = 1$. We have proved that PN has a unique p'-degree p-rational irreducible character in $B_0(PN)$. If $G = NP$, then the theorem follows from Corollary 4.2. Hence, we may assume that $PN < G$. Then, by the inductive hypothesis, $\mathbf{N}_{PN}(P) = P \times Y$. By Theorem 3.2 of [NTV], we have that $Y \leq \mathbf{O}_{p'}(PN) \leq N$. Hence $Y = 1$ (by Step 3). By the main result of [GMN], we have that the non-abelian composition factors of PN are of type $\mathrm{PSL}_2(3^{3^a})$ with $a \geq 1$.

Step 7. The final contradiction.

We have that $\mathbf{C}_K(Q) = Y_0 \times Q$, where Y_0 is a p'-group. If

$$N = S_1 \times \cdots \times S_t,$$

where each S_i is isomorphic to $\mathrm{PSL}_2(3^{3^a})$, then write $Q_i = Q \cap S_i \in \mathrm{Syl}_p(S_i)$. Since Y_0 centralizes $1 \neq Q_i$, it follows that Y_0 normalizes S_i. We have that

$$Y_i = Y_0\mathbf{C}_G(S_i)/\mathbf{C}_G(S_i) \leq \mathrm{Aut}(S_i)$$

centralizes Q_i. By Lemma 3.1(i) of [NTV], it follows that $Y_i = 1$. Thus $Y_0 \leq \mathbf{C}_G(S_i)$ for every i, and so $Y_0 \leq \mathbf{C}_G(N)$. By Step 1, G has a unique minimal normal subgroup, so $\mathbf{C}_G(N) = 1$. Hence $Y_0 = 1$ and $K = N$. This implies that $G = NP$, but this is impossible by Step 6. □

5. Simple Groups

The aim of this section is to prove Theorem 4.1. We begin with some observations.

Lemma 5.1. *Let p be a prime and let S be a normal subgroup of G of p-power index.*
(a) *The principal block $B_0(G)$ is the only block of G that covers the principal block $B_0(S)$ of S.*
(b) *Suppose that $p > 2$ and that $\mathrm{Irr}(S) \cap B_0(S)$ contains a rational G-invariant character α. Then α extends to a rational character $\beta \in B_0(G)$.*
(c) *Theorem 4.1 holds if G has a self-normalizing Sylow p-subgroup P.*

Proof. (a) By Green's Theorem 8.11 of [N1], 1_G is the unique irreducible p-Brauer character of G that lies above 1_S. Hence the statement follows.

(b) By [NT1, Lemma 2.1], α has a unique real extension β to G, whence β is also rational. Since $\alpha \in B_0(S)$, $\beta \in B_0(G)$ by (a).

(c) By [NTT, Theorem A], 1_G is the unique p-rational irreducible character of p'-degree of G, whence the claim follows. □

By virtue of Lemma 5.1(c), it remains to prove the "if" direction of Theorem 4.1.

Lemma 5.2. *Theorem 4.1 holds if S is either ${}^2F_4(2)'$ or one of the 26 sporadic simple groups.*

Proof. Direct computation using [GAP]; note that in this case $G = S$. □

Lemma 5.3. *Theorem 4.1 holds if $S = \mathsf{A}_n$, $n \geq 5$, is an alternating group.*

Proof. As mentioned above, it suffices to prove the "if" direction of Theorem 4.1, that is, $B_0(G)$ contains a nontrivial p-rational irreducible character of p'-degree, where $G = S = \mathsf{A}_n$. Let $H := \mathsf{S}_n$.

Suppose that $p|(n-1)$. Then the character χ of H labelled by the partition $(n-2, 2)$ has degree $n(n-3)/2 \geq 5$ that is coprime to p, and p-core (1), whence $\chi \in B_0(H)$. Since χ_S is irreducible, it has the desired properties.

Assume now that $p \nmid (n-1)$. Write $n - 1 = \sum_{i=1}^{k} a_i p^i$ with $a_i \in \mathbb{Z}$, $0 \leq a_i < p$, and $a_k > 0$. Then the character χ of H labelled by the partition $(n - p^k, 1^{p^k})$ has degree $\binom{n-1}{p^k} > 1$ that is coprime to p, and the same p-core as of 1_H, whence $\chi \in B_0(H)$.

Since $n \neq 2p^k + 1$, the partition $(n - p^k, 1^{p^k})$ is not self-associate, and so χ_S is irreducible and has the desired properties. $\square$

In the case of Theorem 4.1 where S is a simple group of Lie type, we will actually prove more than what is needed for the "if" direction; we believe the established results will be useful in other applications as well. We refer the reader to [C] and [DM] for basics on complex representations of finite groups of Lie type.

Theorem 5.4. *Let $S \not\cong {}^2F_4(2)'$ be a finite simple group of Lie type in characteristic r and $p \neq r$ an odd prime. Then there exists a non-trivial, rational-valued,* $\mathrm{Aut}(S)$-*invariant, unipotent character of p'-degree that belongs to the principal p-block of S.*

Proof. (i) We work in the following setting. Let $\mathbf{G}$ be a simple, simply connected linear algebraic group over an algebraic closure of $\mathbb{F}_r$ with a Steinberg map $F : \mathbf{G} \to \mathbf{G}$ such that $S = G/\mathbf{Z}(G)$ where $G = \mathbf{G}^F$. (This is possible since $S \not\cong {}^2F_4(2)'$.) The unipotent characters of S are then (by definition) precisely the unipotent characters of G (which all have $\mathbf{Z}(G)$ in their kernel).

(ii) We first assume that F is a Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure on $\mathbf{G}$ (i.e., G is not a Suzuki or Ree group). We let d denote the order of q modulo p. By results mainly of Broué, Malle, and Michel, and of Cabanes and Enguehard, summarised in [KM, Theorem A], the unipotent characters in a block of G are unions of d-Harish-Chandra series. Moreover, individual d-Harish-Chandra series are in bijection with irreducible characters of the corresponding relative Weyl groups (see [KM, Theorem B]). Thus by the degree formula for Lusztig induction, the blocks of maximal defect are those parametrized by cuspidal pairs (L, λ) with d-cuspidal $\lambda \in \mathrm{Irr}(L)$ of degree coprime to p, hence with the d-split Levi subgroup L having a d-torus in its centre, so with L being the centralizer $\mathbf{C}_G(\mathbf{T})$ of a Sylow d-torus $\mathbf{T}$ of $\mathbf{G}$. In particular if $\mathbf{C}_{\mathbf{G}}(\mathbf{T})$ is a maximal torus of $\mathbf{G}$, that is, if d is a regular number (in the sense of Springer) for the Weyl group W of G, then there is just one such block, which must be the principal block. In this case, the Steinberg character lies in the principal block, is rational and $\mathrm{Aut}(G)$-invariant (see e.g. [M2, Theorem 2.5]), and its degree is a power of r, hence coprime to p, so we are done.

Consider the case G is of exceptional type. Then all relevant numbers are regular for W unless G is of type E_7 (see the tables given in [BMM]). Hence we only have to consider the latter type. The non-regular numbers are $d = 4, 5, 8, 10, 12$. Here, eight unipotent characters are irrational (those lying in the Harish-Chandra series above the two cuspidal unipotent characters of E_6, those two in the principal series belonging to the non-rational characters of the Hecke algebra, and the two cuspidal unipotent characters). It is immediate from the explicit list of d-Harish-Chandra series in [BMM, Tab. 2] that in each case there exists a unipotent character of p'-degree in the principal block that is $\mathrm{Aut}(S)$-invariant. (This concerns the lines 24, 30, 34, 37 in loc. cit.)

Now assume that S, and hence $\mathbf{G}$, is of classical type. Then the unipotent characters are uniquely determined by their multiplicities in the Deligne–Lusztig characters and hence in particular they are rational. Moreover, all unipotent characters are invariant under all outer automorphisms of S unless either G is of type D_n with $n \geq 4$, or G is of type B_2 in characteristic $r = 2$, see [M2, Theorem 2.5]. Since the relative Weyl group of any non-trivial d-torus is a non-trivial complex reflection group, it has a non-trivial linear character ψ. The unipotent character in the principal block parametrized by ψ then has degree congruent to 1 modulo p by [M1, Theorem 4.2] and is not the trivial character, and hence we are done except for types D_n and B_2. In the cases of types D_4 and B_2, again all relevant d are regular for W, and thus the Steinberg character does the job. So now assume that G is of type D_n with $n \geq 5$. According to [M2, Theorem 2.5] the unipotent characters not stable by outer automorphisms are those labelled by degenerate symbols. On the other hand, the unipotent characters in the principal block are those labelled by symbols with d-core (respectively e-cocore if $d = 2e$ is even) being the symbol of the trivial character (see [BMM, §3A]). Clearly the d-core (respectively e-cocore) of a degenerate symbol is again degenerate, and the symbol for the trivial character is only degenerate when $n = 0$. But in this case, n is divisible by d (respectively by e) and then d is a regular number for W, whence we conclude as before.

(iii) Finally we deal with the case of Suzuki and Ree groups. The theory of d-Harish-Chandra series and p-blocks holds with minor modifications in this case as well, see [BMM] and [M1]. And again all numbers d are regular for the corresponding Weyl groups, whence the Steinberg character has the desired properties. □

Theorem 5.5. *Let S be a finite simple group of Lie type defined over a field of characteristic $p > 2$. If $p = 3$, assume in addition that $S \not\cong \mathrm{PSL}_2(3^{2a+1})$ for any $a \in \mathbb{N}$. Then S has a non-trivial, rational-valued,* $\mathrm{Aut}(S)$*-invariant, irreducible character of p'-degree that belongs to the principal block of S.*

Proof. We keep the notation $(\mathbf{G}, F, G)$ as in Step (i) of the proof of Theorem 5.4. According to [Hum, Theorem, p.69], $B_0(S) = \mathrm{Irr}(S) \setminus \{\mathsf{St}\}$, if St denotes the Steinberg character of G (and S). In particular, any irreducible character of p'-degree of S belongs to $B_0(S)$.

First we note that the result in the case S is an exceptional group of Lie type, respectively $S = \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ with $n \geq 3$, has already been established in Example 5.3(a), (c), Proposition 5.5, and Proposition 5.10 of [NT2], respectively.

In the remaining cases (and viewing $\mathrm{SL}_2(q)$ as $\mathrm{Sp}_2(q)$), we have that $G = \mathrm{Sp}(V)$ or $\mathrm{Spin}(V)$ for a suitable vector space V over $\mathbb{F}_q$. Let the pair $(\mathbf{G}^*, F^*)$ be dual to $(\mathbf{G}, F)$, and set $G^* := (\mathbf{G}^*)^{F^*}$, so that $G^* = \mathrm{SO}(W)$, $\mathrm{PCSp}(W)$, or $\mathrm{PCO}(W)^0$, where $W = \mathbb{F}_q^n$ for a suitable $n \in \mathbb{N}$. If $p \neq 3$, it is easy to see that G^* has a unique conjugacy class of rational elements $s \in [G^*, G^*]$ of order 3 such that an inverse image in $\mathrm{GL}(W)$ of order 3 of s has a fixed point subspace of dimension $n-2$ on W. Likewise, if $p = 3$

and $n \geq 4$, then G^* has a unique conjugacy class of rational elements $s \in [G^*, G^*]$ of order 5 such that an inverse image in $\mathrm{GL}(W)$ of order 5 of s has a fixed point subspace of dimension $n-4$ on W. Finally, if $S = \mathrm{PSp}_2(3^{2a})$ (and so $G^* = \mathrm{SO}_3(q)$ with $q = 3^{2a} \equiv 1 (\bmod 8)$), we can choose $\gamma \in \mathbb{F}_q^\times$ of order 8, $t = \mathrm{diag}(1, \gamma, \gamma^{-1}) \in G^*$, and $s = t^2 \in [G^*, G^*]$. In all cases, s has connected centralizer in $\mathbf{G}^*$. It follows that the corresponding semisimple character χ_s of G is irreducible, trivial at $\mathbf{Z}(G)$, rational-valued, of degree $|G^* : \mathbf{C}_{G^*}(s)|_{p'} > 1$, and $\mathrm{Aut}(S)$-invariant. □

Proof of Theorem 4.1. By Lemma 5.1(c), it remains to prove the "if" direction of the theorem. By Lemmas 5.2 and 5.3 we may assume that $S \not\cong {}^2F_4(2)'$ is a simple group of Lie type. If $(S, p) \neq (\mathrm{PSL}_2(3^{2a+1}), 3)$, then Theorems 5.4 and 5.5 yield a non-trivial $\mathrm{Aut}(S)$-invariant rational irreducible character of p'-degree in $B_0(S)$, whence the same holds for $B_0(G)$ by Lemma 5.1(b). Assume now that $(S, p) = (\mathrm{PSL}_2(3^{2a+1}), 3)$ and that $P \in \mathrm{Syl}_p(G)$ is not self-normalizing. By direct computation (or by using [NTT, Theorem A]), one sees that $\mathrm{Irr}(G)$ contains a non-trivial p-rational irreducible character of p'-degree χ. Since $|G/S|$ is a p-power, χ_S is irreducible and belongs to $B_0(S)$ as we noted in the proof of Theorem 5.5, whence $\chi \in B_0(G)$ by Lemma 5.1(a). □

6. Theorem C and final remarks

We start this section by proving Theorem C of the introduction, which is implied by the deepest parts of the block theory of p-solvable groups. We assume that the reader is familiar with the theory of blocks and normal subgroups (see, for instance, Chapter 9 of [N1]).

Recall that if B is a block of G with defect group P, then B uniquely determines, up to $\mathbf{N}_G(P)$-conjugacy, a p-defect zero character $\theta \in \mathrm{Irr}(P\mathbf{C}_G(P)/P)$ lying in a block b of $P\mathbf{C}_G(P)$ that induces B (see discussion after Theorem 9.12 in [N1]). This character θ is called a canonical character of B. We first need to prove the following lemma.

Lemma 6.1. *Suppose that B is a block of G with normal defect group P. Let $\theta \in \mathrm{Irr}(P\mathbf{C}_G(P)/P)$ be a canonical character of B. Then:*
(a) *All irreducible Brauer characters in B have height zero.*
(b) *$l(B) = 1$ if and only if θ is fully ramified in $G_\theta/P\mathbf{C}_G(P)$.*
(c) *Suppose that p is odd. Then $l(B) = 1$ if and only if there is a unique p-rational $\chi \in \mathrm{Irr}(B)$ such that $\chi^0 \in \mathrm{IBr}(G)$.*

Proof. Let $C = \mathbf{C}_G(P) \triangleleft G$ and $L = PC \triangleleft G$. Let b be a block of L covered by B. We know that $B = b^G$ by Corollary 9.21 of [N1], and that b has defect group P (for instance, by Lemma 4.13 and Theorem 4.18 of [N1]). By Theorem 9.12 of [N1], we may assume that $\mathrm{IBr}(b) = \{\theta^0\}$. By using Theorem 9.2 of [N1], we conclude that $\mathrm{IBr}(B) = \mathrm{IBr}(G|\theta^0)$. Let T be the stabilizer of the Brauer character θ^0 in G. Since

θ vanishes off p-regular elements, we also have that $T = G_\theta$. We even have that T is the stabilizer of b in B, by using Theorem 9.12 of [N1]. Recall that T/L is a p'-group, by Theorem 9.22 of [N1]. By the Fong-Reynolds correspondence (Theorem 9.14 of [N1]) it is enough to prove that the irreducible Brauer characters of T lying over θ^0 have height zero. This is clear, using that T/L is a p'-group. This proves part (a).

Also, we see that $|\mathrm{IBr}(B)| = 1$ if and only if $|\mathrm{IBr}(G|\theta^0)| = 1$. By the Clifford correspondence for Brauer characters (Theorem 8.9 of [N1]), this happens if and only if $|\mathrm{IBr}(T|\theta^0)| = 1$. Since T/L is a p'-group, then every $\psi \in \mathrm{Irr}(T|\theta)$ has p-defect zero, and it follows that restriction to p-regular elements defines a bijection $\mathrm{Irr}(T|\theta) \to \mathrm{Irr}(T|\theta^0)$. We deduce that θ is fully ramified in T, hence proving (b).

In order to prove (c), we claim first that $\mathrm{Irr}(G|\theta)$ is exactly the set of p-rational characters in B that lift irreducible Brauer characters. We already know that $B = b^G$ is the only block of G covering b, so $\mathrm{Irr}(G|\theta) \subseteq \mathrm{Irr}(B)$. Let $\chi \in \mathrm{Irr}(G|\theta)$ and let $\psi \in \mathrm{Irr}(T|\theta)$ be its Clifford correspondent. Since T/L is a p'-group, then ψ has defect zero. In particular $\psi^0 \in \mathrm{IBr}(T)$ and ψ is p-rational. Hence $\chi = \psi^G$ is also p-rational. By the Clifford correspondence for Brauer characters (Theorem 8.9 of [N1]), we have that $\chi^0 = (\psi^0)^G \in \mathrm{IBr}(G)$. Conversely, suppose that $\chi \in \mathrm{Irr}(B)$ is a p-rational character that lifts an irreducible Brauer character. By Lemma X.2.4 of [F], we have that $P \leq \ker \chi$. By Theorem 9.12 of [N1], it follows that χ lies over θ. This proves the claim.

We have that

$$|\mathrm{Irr}(G|\theta)| = |\mathrm{Irr}(T|\theta)| = |\mathrm{Irr}(T|\theta^0)| = |\mathrm{IBr}(G|\theta^0)| = |\mathrm{IBr}(B|\theta^0)|$$

and the proof of the lemma follows. □

Next is Theorem C of the introduction.

Theorem 6.2. *Suppose that G is p-solvable, with p odd. Let B be a block with defect group P and let b be its Brauer first main correspondent. Then $l(b) = 1$ if and only if there is exactly one p-rational $\chi \in \mathrm{Irr}(B)$ of height zero and such that $\chi^0 \in \mathrm{IBr}(B)$.*

Proof. Let $\mathrm{IBr}_0(B)$ be the set of irreducible Brauer characters of B with height zero. By Theorem 23.9 of [MW], we know that $|\mathrm{IBr}_0(B)| = |\mathrm{IBr}_0(b)|$. By Lemma 6.1(b), we have that $|\mathrm{IBr}_0(B)| = |\mathrm{IBr}(b)|$. Hence $|\mathrm{IBr}(b)| = 1$ if and only if $|\mathrm{IBr}_0(B)| = 1$.

By Theorem 10.6 of [N1], for each $\phi \in \mathrm{IBr}(B)$ there exists a unique p-rational character $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 = \phi$. Hence $|\mathrm{IBr}_0(B)|$ is the number of p-rational characters in B of height zero. This concludes the proof of the statement. □

It is interesting to speculate up to what level the local condition $l(b) = 1$ affects the representation theory of its global Brauer correspondent B. As we have proved in this section, this condition implies that B has a unique height zero p-rational character χ lifting an irreducible Brauer character for p-solvable groups, and for blocks with a normal defect groups. It seems that this might be also the case for blocks with abelian defect groups. This would follow from the Alperin weight conjecture together with a conjecture by G. R. Robinson on the uniqueness of p-rational liftings in blocks with a unique simple module (see [MNS]).

Remark 6.3. Let $p > 2$ be a prime and let $\mathcal{O}$ be the (unique up to isomorphism) absolutely unramified complete discrete valuation ring with $\overline{\mathbb{F}}_p$ as its residue field. Let G be any finite group and B a p-block of $\mathcal{O}G$. Suppose that B is Morita equivalent to a p-block B' of $\mathcal{O}H$, where H is a finite p-solvable group, and suppose that the Brauer correspondent b' of B' satisfies $l(b') = 1$. Applying Theorem C to B' and the main result of [K], see also [KL, Corollary 1.7], we see that there is exactly one p-rational $\chi \in \mathrm{Irr}(B)$ of height zero and such that $\chi^0 \in \mathrm{IBr}(B)$.

We have mentioned in the introduction that we believe that there might be a version of Theorem A for the prime $p = 2$. We finish this paper with the following conjecture and some remarks on it.

Conjecture 6.4. *Let G be a finite group. Let $P \in \mathrm{Syl}_2(G)$. Then $\mathbf{N}_G(P)$ has a normal 2-complement if and only if all odd-degree irreducible characters in the principal 2-block of G are σ-invariant, where σ is the Galois automorphism that fixes 2-power roots of unity and squares $2'$-roots of unity.*

Remark 6.5. We offer some evidence in support of Conjecture 6.4, which includes all finite solvable, symmetric, and general linear or unitary groups.

(i) Suppose that G is solvable. Let $L = \mathbf{O}_{2'}(G)$. Then it is well-known that
$$\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G/L).$$
Since $\mathbf{N}_G(P)$ has a normal 2-complement if and only if $\mathbf{N}_{G/L}(PL/L)$ has a normal 2-complement, we may assume that $L = 1$. We know by the main result in [I1] that there is a natural bijection $\mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(\mathbf{N}_G(P))$ that commutes with Galois action. Hence it is no loss to assume that $P \lhd G$. Assume now that G has a normal 2-complement. Then G is a 2-group, and we are done in this case. Conversely, if all the odd-degree irreducible characters of G are σ-invariant, then all characters of G/P are σ-invariant. Then $G = P$ by Lemma 5.1 of [N2].

(ii) Suppose $G = \mathsf{S}_n$. Then $P \in \mathrm{Syl}_2(G)$ is self-normalizing, and certainly all $\chi \in \mathrm{Irr}(G)$ are rational-valued, hence σ-invariant.

(iii) More generally, suppose that G is any finite group with self-normalizing Sylow 2-subgroups. Then a consequence of the Galois refinement of the McKay conjecture [N2] implies that all odd-degree irreducible characters of G are σ-invariant. (A reduction of this statement to quasisimple groups has been given in [NT5, Theorem 5.1] and [Sch, Theorem 3.7].)

(iv) Let $G = \mathrm{GL}_n(q)$ with $2|q$ and $P \in \mathrm{Syl}_2(G)$, chosen to be the subgroup of upper unitriangular matrices in G. Then $\mathbf{N}_G(P) = P \rtimes T$, where T is the subgroup of diagonal matrices in G. In particular, $\mathbf{N}_G(P)$ has a normal 2-complement precisely when $q = 2$. The degree formula for unipotent characters [C, §13.8] shows that the only unipotent character of $\mathrm{GL}_k(q^l)$ of odd degree is the principal character. Hence Lusztig's parametrization of irreducible characters of G [C], [DM] implies that $\chi \in \mathrm{Irr}(G)$ has odd degree precisely when it is the semisimple character χ_s labeled by a semisimple element $s \in G$ (if we identify the dual group G^* with G). Arguing as in the proof of [NT1, Lemma 9.1], one can show that χ_s is σ-invariant exactly when $\chi_s = \chi_{s^2}$, i.e. when s^2 and s are G-conjugate. Furthermore, [Hum, Theorem, p. 69] implies that χ_s belongs to the principal block of G precisely when χ_s is trivial at $\mathbf{Z}(G)$, which, by [NT4, Proposition 4.5], is equivalent to that $s \in [G, G] = \mathrm{SL}_n(q)$. Now it is straightforward to check that s^2 and s are G-conjugate for all semisimple elements $s \in \mathrm{SL}_n(q)$ if and only if $q = 2$. Thus Conjecture 6.4 holds in this case.

A similar argument, applied to $\mathrm{GU}_n(q)$ with $2|q$, shows that Conjecture 6.4 holds in this case as well.

(v) Let $G = \mathrm{GL}_n(q)$ with q odd and $n \geq 2$. By [GKNT, Theorem 2.5], if $\chi \in \mathrm{Irr}(G)$ has odd degree, then

$$\chi = S(s_1, \lambda_1) \circ S(s_2, \lambda_2) \circ \ldots \circ S(s_m, \lambda_m)$$

in James' notation [J], where $s_i \in \mathbb{F}_q^\times$ are pairwise distinct, $\lambda_i \vdash k_i$, $\sum_{i=1}^m k_i = n$, and

$$[n]_2 = [k_1]_2 < [k_2]_2 < \ldots < [k_m]_2,$$

if $[a]_2$ denotes the 2-part of any $a \in \mathbb{N}$. Furthermore, results of Fong and Srinivasan [FS] imply that such a character belongs to the principal 2-block of G only when all s_i are 2-elements. Note that in this case $S(s_i, \lambda_i)$ is a product of the rational-valued (unipotent) character $S(1, \lambda_i)$ of $\mathrm{GL}_{k_i}(q)$ with a linear character of 2-power order of $\mathrm{GL}_{k_i}(q)$, whence it is σ-invariant. Since χ is obtained from the character

$$S(s_1, \lambda_1) \otimes S(s_2, \lambda_2) \otimes \ldots \otimes S(s_m, \lambda_m)$$

of the Levi subgroup

$$\mathrm{GL}_{k_1}(q) \times \mathrm{GL}_{k_2}(q) \times \ldots \times \mathrm{GL}_{k_m}(q)$$

by Harish-Chandra induction, it follows that χ is σ-invariant. On the other hand, $\mathbf{N}_G(P)$ has a normal 2-complement if $P \in \mathrm{Syl}_2(G)$, see e.g. [GKNT, (5.3), (5.5)]).

In fact, we note that [GKNT, Theorem E] implies that Conjecture 6.4 also holds for $\mathrm{GU}_n(q)$ whenever q is odd.

(vi) Let $G = \mathrm{Sp}_{2n}(q)$ with $q \equiv \pm 3 (\mathrm{mod}\, 8)$. As shown in the proof of [Ko, Theorem 1], the normalizer of $P \in \mathrm{Syl}_2(G)$ contains $\mathrm{SL}_2(3)$ as a subgroup, and so $\mathbf{N}_G(P)$ does not have a normal 2-complement. It is well known, see eg. [TZ, §2], that G has a pair of the so-called *Weil characters* $\xi_n, \eta_n \in \mathrm{Irr}(G)$ of degree $(q^n \pm 1)/2$, such that the restriction of ξ_n to $2'$-elements of G equals to the restriction of $1_G + \eta_n$ to $2'$-elements of G. In particular, they belong to the principal 2-block of G, and one of them has odd degree. Inspecting the values of ξ_n and η_n at a transvection $t \in G$ [TZ, Lemma 2.6], one can check that neither ξ_n nor η_n is σ-invariant.

Certainly, the arguments given in (iv)–(vi) also apply to many other finite groups of Lie type. We also note that Conjecture 6.4 has now been reduced to almost simple groups, see [NV].

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