CERTAIN MONOMIAL CHARACTERS AND THEIR NORMAL CONSTITUENTS

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Abstract

1 Introduction

There are few results guaranteeing that a single irreducible complex character $\chi \in \mathrm{Irr}(G)$ of a finite group G is monomial. Recall that $\chi \in \mathrm{Irr}(G)$ is **monomial** if there is $\lambda \in \mathrm{Irr}(U)$ linear such that $\lambda^G = \chi$. It is known that every irreducible character of a supersolvable group is monomial, for instance, but this result depends more on the structure of the group rather than on the properties of the characters themselves. An exception is a theorem by R. Gow of 1975 ([3]): an odd degree real valued irreducible character of a solvable group is monomial. Recently, we gave in [9] an extension of this theorem which also dealt with the degree and the field of values of the character. (Yet another similar monomiality criterium was given in [10]: if the field of values $\mathbb{Q}(\chi)$ of χ is contained in the cyclotomic field $\mathbb{Q}_n$ and $(\chi(1), 2n) = 1$, then χ is monomial if G is solvable.) In this note, we apply non-trivial Isaacs π-theory of solvable groups to give new proofs of the above results at the same time that we gain some new information about the subnormal constituents of the characters, among other things. It does not seem easy at all how to prove these new facts without using this deep theory.

Recall that for every solvable group and any set of primes π, M. Isaacs defined a canonical subset $B_\pi(G)$ of $\mathrm{Irr}(G)$ with remarkable properties ([4]).

Theorem 1.1 *Let p be a prime, let G be a p-solvable finite group, and let $P \in \mathrm{Syl}_p(G)$. Let $\chi \in \mathrm{Irr}(G)$ be such that p does not divide $\chi(1)$ and such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{p^a}$ for some $a \geq 0$. If $|\mathbf{N}_G(P)/P|$ is odd, then $\chi \in B_p(G)$. In particular, if $N \lhd\lhd G$ and θ is an irreducible constituent of χ_N, then θ is monomial.*

We obtain the following consequence, in which a global invariant of a finite group is calculated locally.

Corollary 1.2 *Let p be a prime, let G be a p-solvable finite group, and let $P \in \mathrm{Syl}_p(G)$. Assume that $\mathbf{N}_G(P)/P$ has odd order. Then the number of irreducible characters χ of G such that $\chi(1)$ is not divisible by p and $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|_p}$ is the number of orbits of the natural action of $\mathbf{N}_G(P)$ on P/P'.*

2 Proofs

The notation for characters is that of [6]. The π-special characters were defined in [2], while B_π-characters were defined in [4].

Lemma 2.1 *Suppose that G is a finite p-solvable group. Let $P \in \mathrm{Syl}_p(G)$, and assume that $\mathbf{N}_G(P)/P$ has odd order. If $\alpha \in \mathrm{Irr}(G)$ is p'-special and real, then α is the trivial character.*

Proof We argue by induction on $|G|$. Let $K = \mathbf{O}_p(G)$. If $K > 1$, then $K \subseteq \ker(\alpha)$ by Corollary (4.2) of [2], and we apply induction in G/K. Otherwise, let $K = \mathbf{O}_{p'}(G)$. Since α has p'-degree, then there is some P-invariant $\theta \in \mathrm{Irr}(K)$ under α, and all of them are $\mathbf{N}_G(P)$-conjugate, using the Clifford correspondent and the Frattini argument. Since α is real, then $\bar\theta$ is also under α, and therefore there is $g \in \mathbf{N}_G(P)$ such that $\bar\theta = \theta^g$. Now g^2 fixes θ, and since $\mathbf{N}_G(P)/P$ has odd order, we see that $\bar\theta = \theta$. Since $\mathbf{N}_G(P)/P$ has odd order, we have that $\mathbf{C}_K(P)$ has odd order. Let $\theta^* \in \mathrm{Irr}(\mathbf{C}_K(P))$ be the P-Glauberman correspondent of θ. Since the Glauberman correspondence commutes with Galois automorphisms, we have that θ^* is a real irreducible character of a group of odd order. By Burnside's theorem, $\theta^* = 1$ and $\theta = 1$ by the uniqueness of the Glauberman correspondence. Thus $K \subseteq \ker(\theta)$, and we apply induction. □

Proof [Proof of Theorem 1.1 and Corollary 1.2] By Theorem (3.6) of [7], there exists a subgroup $P \subseteq W \subseteq G$ and a p-special linear character $\lambda \in \mathrm{Irr}(W)$, such that: $\psi = \lambda^G \in \mathrm{Irr}(G)$ is a B_p-character, and (W, λ) is a nucleus of ψ. Also, there is a p'-special character $\alpha \in \mathrm{Irr}(W)$ such that $\chi = (\lambda\alpha)^G$. By Theorem 4.2 of [8], the pair $(W, \lambda\alpha)$ is unique up to G-conjugacy. Now, let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_p})$ be the unique Galois automorphism that complex conjugates the p'-roots of unity and fixes p-power roots of unity. Since χ and λ are fixed by σ, then we deduce that there is $g \in G$ such that $(W^g, \lambda^g\alpha^g) = (W, \lambda\alpha^\sigma)$. Hence $\alpha^g = \alpha^\sigma$ by Proposition (7.1) of [2]. Since $P, P^g \subseteq W$, then $P^{gw} = P$ for some $w \in W$, and we may assume that $g \in \mathbf{N}_G(P)$. Also, $\alpha^{g^2} = \alpha$, and therefore since $\mathbf{N}_G(P)/P$ has odd order, we see that $\alpha^\sigma = \alpha$. Now, let H be a p-complement of W. Then

$$\bar\alpha_H = \overline{\alpha_H} = (\alpha^\sigma)_H = \alpha_H$$

and we deduce that $\bar\alpha = \alpha$, by using Proposition (6.1) of [2]. Since $\mathbf{N}_W(P)/P$ has odd order, by Lemma (2.1), we have that $\alpha = 1$. Thus $\chi = \psi \in B_p(G)$ and χ is monomial. Now, to prove the second part of the theorem, use that subnormal constituents of B_p-characters are B_p-characters (Corollary (7.5) of [4]), and the second part of Theorem 2.2 of [1] , that asserts that B_p-characters of p'-degree are monomial.

Finally, Corollary 1.2 follows from Theorem 2.2 and 2.4 of [1]. □

Finally, we also give another proof of [10] that uses Isaacs theory,

Theorem 2.2 *Suppose that G is solvable. Suppose that $\chi \in \mathrm{Irr}(G)$ has odd degree. Suppose that $(\chi(1), n) = 1$, where $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_n$. Then χ is monomial.*

Proof Let π be the set of primes (possibly empty) dividing n. Then χ has π'-degree. Let H be a Hall π-subgroup. By Theorem (3.6) of [5], then we have that there is a linear character $\lambda \in \mathrm{Irr}(H)$ a subgroup $H \subseteq W \subseteq G$, such that λ has a π-special extension $\hat{\lambda}$, there exists a π'-special character α such $\chi = (\hat{\lambda}\alpha)^G$. Also, $\hat{\lambda}^G = \psi \in B_\pi(G)$ and $(W, \hat{\lambda})$ is a nucleous of ψ. Also by Theorem 4.2 of[8], the pair $(W, \hat{\lambda}\alpha)$ is unique up to G-conjugacy. We have that

$$\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{|G|} \cap \mathbb{Q}_n = \mathbb{Q}_{\gcd(n,|G|)} \subseteq \mathbb{Q}_{|G|_\pi}.$$

Now, let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|_{\pi'}}/\mathbb{Q})$ which we may extend to some $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_\pi})$. Then σ fixes χ and $\hat{\lambda}$. Thus $(W, \hat{\lambda}\alpha^\sigma) = (W, \hat{\lambda}\alpha)^g$. Thus $g \in \mathbf{N}_G(W)$ and $\alpha^\sigma = \alpha^g$. Let $\beta = \alpha^{\mathbf{N}_G(W)}$. Now β is not necessarily irreducible but has all of its values in $\mathbb{Q}_{|G|_{\pi'}}$. Then we have that β is fixed by σ and therefore β is rational valued (of odd) degree. Since $\alpha^{\mathbf{N}_G(W)} = (\bar{\alpha})^{\mathbf{N}_G(W)}$, it follows that some irreducible constituent ψ over α and $\bar{\alpha}$. Now, it follows that $\bar{\alpha} = \alpha^g$ for some $g \in \mathbf{N}_G(W)$ of 2-power degree. Since $\mathbf{N}_G(W)/W$ has odd order, it follows that $g \in W$. Then α is real, and by Gow it is monomial. Hence χ is monomial. $\square$

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