The coalitional value in finite-type continuum games

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Abstract

The coalitional value [Owen, Values of games with a priori unions. In: Hein R, Moeschlin O (Eds), Essays in Mathematical Economics and Game Theory. Springer Verlag, 1977] is defined for the class of continuos games with a finite type of players. A formula for its computation is provided jointly with an axiomatic characterization of it. The properties used are a natural extension in this setting of the properties used in the characterization of the Owen's coalitional value for games with a finite set of players.

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Introduction

Many economic situations in which agents can cooperate partially among them can be modeled as coalitional games with transferable utility. In this case one of the most applied

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solution concept is the Shapley value [15]. When the number of players in the game becomes large, the computation of the Shapley value becomes impracticable. In this case the best option is to make a continuous approximation of the large game with a nonatomic game and apply the continuum extension of the Shapley value that was provided by Aumann and Shapley [2]. One of the examples of this approach is the problem of allocating joint costs when output can vary. It can be summarized as follows: there are n goods (or facilities) that are jointly produced and $\boldsymbol{v}(x)$ is the joint cost of producing the bundle $x = (x_1, ..., x_n)$, where x_i is the quantity of good i. In this set-up a *cost allocation method* is a function $\psi(x, \boldsymbol{v}) = p$ where p is a vector of prices satisfying $\sum p_i x_i = \boldsymbol{v}(x)$. The first application of the Aumann-Shapley value in this setting appears in Billera, Heath and Raanan [4], in what has been known as the *Aumann-Shapley prices*, which compute the following cost share Φ of good i by:

$$\Phi_i(\bar{x}, \boldsymbol{v}) = \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_i}(t\bar{x})dt. \tag{1}$$

Here Φ is an average of the marginal costs of good i along the "diagonal" (i.e., the straight line from 0 to $\bar{x}$). The two first axiomatic characterizations of this formula in economic cost allocation setting were given by Billera and Heath [3] and Mirman and Tauman [10]. For comprehensive surveys about this topic the reader is referred to Tauman [16] and Young [20].

Sometimes players organize themselves into groups for the purpose of the payoffs' bargaining. This includes the instances of syndicates, unions, cartels, parliamentary coalitions, cities, countries, etc. This action can be reflected by including a *coalition structure* into the game, which is done by an exogenous partition of players into a set of groups or unions. Games with coalition structures where first considered by Aumann and Drèze [1] who extend the Shapley value in such a manner that the game splits into subgames played by the unions isolate from each other, and every player receives his Shapley value in the subgame he is playing within his union. A different approach was used by Owen [14]. In this case, the unions play a quotient game among themselves, and each one receives a payoff which is shared among its players in an internal game. Both payoffs, in the quotient game between unions and within each union for its players, are given by the Shapley value.

When the number of players in the game enlarge, the Owen value has the same calculus difficulties as the Shapley value has. Nevertheless in the literature there is not an analogous "Owen prices" for continuum games as the Aumann-Shapley prices is for the Shapley value.

We illustrate the interest of having such a concept by using a variation of the example provided in Young [20]. Two towns are considering wether to build a joint water distribution system. The water must be also recycled before be distributed, and assume that only two types of quality are needed: for human and industrial consumptions. The joint cost will be a function of the total different types of water demanded. Denote by x_1 and x_2 the amounts of water for human and industrial consumption respectively in town A, and x_3 and x_4 the amounts for human and industrial consumption in town B. Let the joint cost function $\boldsymbol{v}$ given by

$$\boldsymbol{v}(x_1, x_2, x_3, x_4) = (4x_1 + x_2 + 4x_3 + x_4)\left(1 + e^{-0.2(x_1+x_2+x_3+x_4)}\right).$$

Suppose that the total demands are $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = 1$. This result in a charge of 19.518. If city A builds the facility by itself the cost is $\boldsymbol{v}(1, 2, 0, 0) = 9.2929$, and for city B alone $\boldsymbol{v}(0, 0, 2, 1) = 13.939$. The savings of cooperation are $(9.293 + 12.939) - 19.518 = 3.714$, and if they want to share it equally, city A has to pay: $9.293 - 1.857 = 7.436$ million and city B: $13.939 - 1.857 = 12.082$. This is the standard solution for two-players cooperative games, and we can use the Shapley value if we want to allocate the total costs when we have more than two cities (players). But still we have the problem of how to share the costs within each city between the two types (human and industrial) of water consumers.

When the number of consumers of the two types in a city becomes large, instead of considering all water consumers as individual agents with their own demands and try to compute how much has each one to pay, it is better to aggregate all of them into two types of consumers with mass equal to the sum of all of their respective demands. Now in the continuum game we can compute the price of each type of water unit by using the Aumann-Shapley prices.

For example, if city A do not cooperate with city B, and want to share its own total cost $\boldsymbol{v}(1,2,0,0) = 9.2929$ among their human and industrial consumers, the AS-prices Φ are

$$\Phi_1\left((1,2,0,0),\boldsymbol{v}\right) = \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_1}(t,2t,0,0)dt = 6.6016$$
$$\Phi_2\left((1,2,0,0),\boldsymbol{v}\right) = \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_2}(t,2t,0,0)dt = 1.3456$$

which cover the total costs:

$$1\cdot\Phi_1\left((1,2,0,0),\boldsymbol{v}\right) + 2\cdot\Phi_2\left((1,2,0,0),\boldsymbol{v}\right) = \boldsymbol{v}(1,2,0,0) =$$
$$= 1\cdot 6.6016 + 2\cdot 1.3456 = 9.2929.$$

If city A cooperates with city B, its total costs become 7.436. What should be the prices now?

Note that the AS-prices applied to the total demands $(1,2,2,1)$ are

$$\Phi_1\left((1,2,2,1),\boldsymbol{v}\right) = \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_1}(t,2t,2t,t)dt = 5.6265,$$
$$\Phi_2\left((1,2,2,1),\boldsymbol{v}\right) = \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_2}(t,2t,2t,t)dt = 0.87948,$$

which yield different costs of what should correspond to city A under the cooperation with B, i.e.:

$$1 \cdot 5.6265 + 2 \cdot 0.87948 = 7.3855 \neq 7.436.$$

In this paper we illustrate how to compute the "Owen prices" in the continuum setup.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary definitions and notation. Then, Section 3 presents the continuum finite type setting and an asymptotic approach is used in order to find the coalitional value prices. In Section 4, the tools of the multilinear extension and the potential of a game are used to easily obtain an explicit formula for the prices. Finally, in Section 5 the *coalitional values are* [coalitional value is] characterized axiomatically by using the same set of axioms that as the ones used in the finite setup.

1 Preliminaries

Let N be a finite set and v: $2^N \longrightarrow \mathbb{R}$ be a real function satisfying $v(\emptyset) = 0$, where 2^N is the set of all subsets of N. We say that (N, v) is a *transferable utility (TU)-game* with player set N and *characteristic function* v. Denote by G^N the space of all games with finite player set N.

For any coalition $S \subseteq N$, $v(S)$ is called the *worth* of S and (S, v) is *the restriction of* (N, v) *to* S, i.e. a TU-game in which S is the set of players and the characteristic function is the restriction of v to 2^S.

A *value* is a function γ which assigns to every TU-game (N, v) and every player $i \in N$, a real number $\gamma_i(N, v)$.

The *Shapley value* [15] is defined as:

$$\phi_i(N,v) = \sum_{S \subset N \backslash i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S) \right], \quad (i \in N, (N,v) \in G^N). \qquad (2)$$

Given N, a coalition structure B is a partition of N, that is $B = \{B_1, ..., B_m\}$, such that $\underset{1 \le k \le m}{\cup} B_k = N$, and $B_l \cap B_k = \emptyset$ when $l \neq k$. We also assume $B_k \neq \emptyset$. Denote by $\mathcal{B}(N)$, the set of all coalition structures on N. The sets $B_k \in B$ are called "unions" or "blocks", and we denote by $M = \{1, 2, ..., m\}$ the index set of unions. Define the *quotient game* (M, v_M) induced by the coalition structure B, considering the coalitions of B as players and $v_M : 2^M \to \mathbb{R}$, defined by $v_M(\emptyset) := 0$, and $v_M(T) := v\left(\underset{k \in T}{\cup} B_k\right)$, for all $T \subset M$. Therefore $(M, v_M) \in G^M$. Let $B_k \in B$, for all $S \subset B_k$, define $B\mid_S$ as the new coalition structure defined on $(\cup_{j \neq k} B_j) \cup S$, which appears when the complementary of S in B_k leaves the game. That is,

$$B\mid_S = \{B_1, ..., B_{k-1}, S, B_{k+1}, ..., B_m\}.$$

For any $S \subset B_k$, we have the corresponding quotient game $\left(M\mid_S, v_{M\mid_S}\right) \in G^{M\mid_S}$, where the index k represents the union S instead of B_k. Now, given a fixed $B_k \in B$, let $(B_k, v_{B_k}) \in G^{B_k}$ be the game between the players in B_k, with characteristic function $v_{B_k} : 2^{B_k} \to \mathbb{R}$, defined by $v_{B_k}(\emptyset) := 0$, and $v_{B_k}(S) := \phi_k\left(M\mid_S, v_{M\mid_S}\right)$, for all $S \subset B_k$.

A game$(N,v) \in G^N$ with a coalition structure $B \in \mathcal{B}(N)$, is denoted by (B, N, v). A *coalitional value* is a function ψ which assigns to every $(B, N, v) \in \mathcal{B}(N) \times G^N$, and every player $i \in N$, a real number $\psi_i(B, N, v)$. The *Owen value* [14] is defined as

$$\varphi_i(B, N, v) := \phi_i\left(B_k, v_{B_k}\right), \quad (i \in B_k \in B). \qquad (3)$$

The Owen value satisfies the *quotient game property*:

$$\sum_{i \in B_k} \varphi_i(B, N, v) = \phi_k\left(M, v_M\right), \quad (k \in M).$$

Therefore, first the union k plays the quotient game (M, v_M) among the unions, and the payoff obtained is shared between its members by playing the subgame (B_k, v_{B_k}). In both levels of bargaining, the payoffs are obtained by using the Shapley value ϕ.

2 Asymptotic Approach

The coalitional value can be obtained following an asymptotic approach. Here a continuum finite type game is regarded as a limit of a sequence of finite games. These games are finite approaches to the original one, allowing only a finite number, instead of a continuum, of players. If the coalitional values of these finite games converge to the same limit, regardless of the sequence used, this limit is called the *asymptotic coalitional value* of the original game. The formal treatment is as follows.

Let $N = \{1, .., n\}$ be the set of types. A vector $x \in \mathbb{R}^N_+$ represents a coalition and x_i is the number (or "mass") of players of type i. A *finite-type continuum game* $\boldsymbol{v}$ is given by a mapping $\boldsymbol{v} : \mathbb{R}^N_+ \to \mathbb{R}$, with $\boldsymbol{v}(0) = 0$[1]. Let C^N be the family of finite-type continuum games where $\boldsymbol{v}$ has continuous first partial derivatives on $\mathbb{R}^N_+$ (it is understood that the derivatives are one sided when x belongs to the boundary of $\mathbb{R}^N_+$). A *type-symmetric value* for $\boldsymbol{v}$ is a function $\Gamma(\cdot, \boldsymbol{v}) : \mathbb{R}^N_{++} \to \mathbb{R}^N$, where $\Gamma_i(x, \boldsymbol{v})$ represents the per-capita payoff of players of type i, so $x_i \Gamma_i(x, \boldsymbol{v})$ are the aggregate patoffs that players of type i receive.

Suppose now that we have a coalition structure B on the set of types N. A *coalitional value* is a vector function Ψ which assigns to every $(B, x, \boldsymbol{v}) \in \mathcal{B}(N) \times \mathbb{R}^N_{++} \times C^N$, and

[1] We use bold letters to denote continuum games to differentiate from finite games.

every type of player $i \in N$, a real number $\Psi_i(B, x, \boldsymbol{v})$. This allows us to interpret Ψ as a vector of *prices*.

Given a continuum game $(B, \bar{x}, \boldsymbol{v})$, with $\bar{x} \in \mathbb{R}^N_{++}$, we can associated a nonatomic game $(I, \mathcal{C})$, where $I = \cup_{i \in N} I_i$, $I_i = [i-1, i]$ for each $i \in N$, and $\mathcal{C}$ is the σ-field of Borel subsets of I. The elements of $\mathcal{C}$ are the coalitions. Given a coalition $T \in \mathcal{C}$, let $\bar{x}(T) \in \mathbb{R}^N_+$ be such that $\bar{x}_i(T) = \bar{x}_i \lambda(T \cap I_i)$ for each $i \in N$, where λ denotes the Lebesgue measure. The nonatomic game $f_{(\bar{x},v)}$ is defined by $f_{(\bar{x},v)}(T) = \boldsymbol{v}(\bar{x}(T))$ for all $T \in \mathcal{C}$.

Let $\{\Pi^\tau\}_{\tau=1,2,...}$ be a sequence of partitions of I into measurable sets (i.e., members of $\mathcal{C}$), such that each $\Pi^{\tau+1}$ refines the previous partition Π^τ (i.e., each member of Π^τ is a union of members of $\Pi^{\tau+1}$). Assume also that the sequence is *separating*, that is, if s and t are distinct points in I, then for τ sufficiently large, s and t are in different members of Π^τ. We say that $\{\Pi^\tau\}_{\tau=1,2,...}$ is an *admissible sequence* of I if it satisfies the above conditions and $\Pi^1 = \{[0,1], (1,2], (2,3], ..., (n-1, n]\}$.

For each τ, the grand coalition N^τ is given by the members of Π^τ, i.e., $N^\tau := \cup_{i \in N} N_i^\tau$, where $N_i^\tau := \{j : \pi_j^\tau \in \Pi^\tau \text{ and } \pi_j^\tau \subset I_i\}$, for each $i \in N$. For any coalition $S \subset N^\tau$ and vector $\bar{x} \in \mathbb{R}^N_{++}$, let $\bar{x}_i[S] := \bar{x}_i \lambda\left(\left(\cup_{k \in S} \pi_k^\tau\right) \cap I_i\right)$, for all $i \in N$. Therefore, each i-component of the vector $\bar{x}[S] \in \mathbb{R}^N_+$ is the total sum of the masses of members of type i in coalition S. We define the finite game $v^\tau \in G^{N^\tau}$ by

$$v^\tau(S) := \boldsymbol{v}\left(\bar{x}[S]\right), \text{ for all } S \subset N^\tau.$$

When $\boldsymbol{v} \in C^N$ it holds that $f_{(\bar{x},v)}$ belongs to the $pNAD$ class of nonatomic games (see [9]), and this is a sufficient condition to have an *asymptotic value* $\Phi f_{(\bar{x},v)}$(Aumann and Shapley[2]). Then, if we want to compute the total payoffs of coalition I_i,i.e. $\Phi f_{(\bar{x},v)}(I_i)$, we compute the Shapley value of the finite game (N^τ, v^τ) for each τ, and we take the aggregate payoffs of the members of $[N_i^\tau]$, that is $\phi(N^\tau, v^\tau)[N_i^\tau] = \sum_{j \in N_i^\tau} \phi_j(N^\tau, v^\tau)$. Now let $\tau \to \infty$, then it holds that $\phi(N^\tau, v^\tau)[N_i^\tau]$ has a limit, and this limit is independent of the admissible sequence $\{\Pi^\tau\}_{\tau=1,2,...}$chosen, hence

$$\lim_{\tau\to\infty} \phi(N^\tau, v^\tau)[N_i^\tau] = \Phi f_{(\bar{x},v)}(I_i).$$

Moreover, as $f_{(\bar{x},v)}$ belongs to the class of $pNAD$ games, this limit reduces to

$$\Phi f_{(\bar{x},v)}(I_i) = \bar{x}_i \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_i}(t\bar{x})\, dt = \bar{x}_i \Phi_i(\bar{x}, \boldsymbol{v}), \quad \text{for all } i \in N,$$

where the type-symmetric vector of payoffs $\Phi(\bar{x}, \boldsymbol{v})$ is called the *Aumann-Shapley prices* of $\boldsymbol{v}$ at $\bar{x}$.

Assume now that we have a coalition structure $B = \{B_1, ..., B_m\}$ on N. For each finite approximation (N^τ, v^τ), we can associate the corresponding coalition structure $B^\tau = \{B_1^\tau, ..., B_m^\tau\}$, defined by $B_k^\tau = \cup_{i \in B_k} N_i^\tau$, for all $k \in M = \{1, ..., m\}$, and we can also compute the Owen value, $\varphi(B^\tau, N^\tau, v^\tau)$. The *asymptotic coalitional value* for finite-type continuum games can be defined using the same limit approach:

Definition 1 *An asymptotic coalitional value is a function Ψ such that for all $(B, \bar{x}, \boldsymbol{v}) \in \mathcal{B}(N) \times \mathbb{R}_{++}^N \times C^N$ and all admissible sequence $\{\Pi^\tau\}_{\tau=1,2,...}$ we have*

$$\lim_{\tau\to\infty} \varphi(B^\tau, N^\tau, v^\tau)[N_i^\tau] = \bar{x}_i \Psi_i(B, \bar{x}, \boldsymbol{v}), \quad \textit{for all } i \in N,$$

where φ is the Owen value.

The main result of this paper shows that, when $(B, \bar{x}, \boldsymbol{v})$ belongs to $\mathcal{B}(N) \times \mathbb{R}_{++}^N \times C^N$, the continuous game $\boldsymbol{v}$ has an asymptotic coalitional value Ψ at $\bar{x}$. This theorem follows from previous asymptotic results in nonatomic games.

For its statement we need some previous definitions.

For any $x \in \mathbb{R}^N$, and $S \subset N$, we denote by x^S the vector in $\mathbb{R}^S$ such that $x_i^S = x_i$, for all $i \in S$. Given $(B, \bar{x}, \boldsymbol{v})$ we define the quotient game between unions $(M, v_{\bar{x}}) \in G^M$, by $v_{\bar{x}}(S) = \boldsymbol{v}(\bar{x}^{\cup_{k\in S}B_k}, 0^{\cup_{k\in M\setminus S}B_k})$ for all $S \subset M$, and $v_{\bar{x}}(\emptyset) = \boldsymbol{v}(0^{\cup_{k\in M}B_k}) = 0$. Given a union $B_k \in B$, and a fixed vector $\bar{x}^{N\setminus B_k} \in \mathbb{R}^{N\setminus B_k}_{++}$, the continuum game $\left(x^{B_k}, \boldsymbol{v}_{(B_k,\bar{x})}\right)$ within the types of B_k is defined as follows:

$$\boldsymbol{v}_{(B_k,\bar{x})}(x^{B_k}) := \phi_k\left(M, v_{\left(x^{B_k},\bar{x}^{N\setminus B_k}\right)}\right), \quad (x^{B_k} \in \mathbb{R}^{B_k}_{+}). \tag{4}$$

That is, for each coalition x^{B_k} it is computed the value that this coalition will obtain in the quotient game with the rest of unions $N\setminus B_k$ at $\bar{x}^{N\setminus B_k}$. Because $\boldsymbol{v} \in C^N$ and formula (2) is a polynomial, it holds that $\boldsymbol{v}_{(B_k,\bar{x})} \in C^{B_k}$, and this fact will guarantee that $\left(\bar{x}^{B_k}, \boldsymbol{v}_{(B_k,\bar{x})}\right)$ has an asymptotic value, which is the key step in the existence proof of the coalitional value.

Theorem 1 *For all $(B, \bar{x}, \boldsymbol{v}) \in \mathcal{B}(N) \times \mathbb{R}^N_{++} \times C^N$, it holds that $\boldsymbol{v}$ has an asymptotic coalitional value Ψ at $\bar{x}$. Moreover,*

$$\Psi_i\left(B, \bar{x}, v\right) := \Phi_i\left(\bar{x}^{B_k}, \boldsymbol{v}_{(B_k,\bar{x})}\right), \quad \textit{for all } i \in B_k \in B, \tag{5}$$

where Φ is the vector of Aumann-Shapley prices.

Proof. let $(B, \bar{x}, \boldsymbol{v}) \in \mathcal{B}(N) \times \mathbb{R}^N_{++} \times C^N$, and for any τ let (B^τ, N^τ, v^τ) be its corresponding finite approximation, and let $(M^\tau, v^\tau_{M^\tau})$ be the associated quotient game induced by the coalition structure B^τ. As the Owen value φ verifies the quotient game property, we know that

$$\varphi(B^\tau, N^\tau, v^\tau)[B_k^\tau] = \phi_k(M^\tau, v^\tau_{M^\tau}), \text{ for all } k \in M^\tau.$$

Take any $S \subset M^\tau$, as

$$\bigcup_{k\in S} B_k^\tau = \bigcup_{k\in S}\left(\bigcup_{i\in B_k^\tau} N_i^\tau\right),$$

and $\lambda\left(\cup_{j\in N_i^\tau}\pi_j^\tau\right)=\lambda\left(I_i\right)=1$, it holds that

$$\bar{x}\left[\bigcup_{k\in S}B_k^\tau\right]=\bar{x}\left[\bigcup_{k\in S}\left(\bigcup_{i\in B_k^\tau}N_i^\tau\right)\right]=\left(\bar{x}^{\cup_{k\in S}B_k},0^{\cup_{k\in M\setminus S}B_k}\right).$$

Hence,

$$v_{M^\tau}^\tau(S)=v^\tau\left(\bigcup_{k\in S}B_k^\tau\right)=\boldsymbol{v}\left(\bar{x}\left[\bigcup_{k\in S}B_k^\tau\right]\right)=\boldsymbol{v}\left(\bar{x}^{\cup_{k\in S}B_k},0^{\cup_{k\in M\setminus S}B_k}\right)=v_{\bar{x}}\left(S\right).$$

Moreover, the index set of unions M^τ in the coalition structure B^τ is, by construction, always equal to index set of the original coalition structure, i.e. $M^\tau=M$. Therefore $(M^\tau,v_{M^\tau}^\tau)=(M,v_{\bar{x}})$ for all τ, and then

$$\lim_{\tau\to\infty}\varphi(B^\tau,N^\tau,v^\tau)[B_k^\tau]=\lim_{\tau\to\infty}\phi_k(M^\tau,v_{M^\tau}^\tau)=\phi_k\left(M,v_{\bar{x}}\right),\ \text{ for all } k\in M.$$

This implies that in case that Ψ exists, it must satisfy the *quotient game property*:

$$\sum_{i\in B_k}\bar{x}_i\Psi_i\left(B,\bar{x},\boldsymbol{v}\right)=\phi_k\left(M,v_{\bar{x}}\right). \tag{6}$$

Now we compute the Owen value of the players set $N_i^\tau\subset B_k^\tau$. Let $\left(B_k^\tau,v_{B_k^\tau}^\tau\right)\in G^{B_k^\tau}$ be the game between the players in B_k^τ. For any $S\subset B_k^\tau$, let $\bar{x}_i\left[S\right]:=\bar{x}_i\lambda\left(\bigcup_{j\in N_i^\tau}\pi_j^\tau\right)$, for all $i\in B_k$. Note that $\bar{x}\left[S\right]^{B_k}\leq\bar{x}^{B_k}$. Hence the characteristic function $v_{B_k^\tau}^\tau$ is defined by $v_{B_k^\tau}^\tau\left(S\right)=\phi_k\left(M^\tau\mid_S,v_{M^\tau|_S}^\tau\right)$. By construction, $M^\tau\mid_S=M$ and

$$v_{M^\tau|_S}^\tau\left(S\right)=\boldsymbol{v}\left(\bar{x}\left[S\right]^{B_k},\bar{x}^{N\setminus B_k}\right).$$

The game $\left(B_k^\tau,v_{B_k^\tau}^\tau\right)$ is a finite approximation of the continuum game $\left(\bar{x}^{B_k},\boldsymbol{v}_{(B_k,\bar{x})}\right)$, and because ϕ_k is a polynomial expression, it follows that $\boldsymbol{v}_{B_k}\in C^{B_k}$. Therefore the Aumann-Shapley value Φ has an asymptotic value at $\bar{x}^{B_k}$, and then

$$\lim_{\tau\to\infty}\varphi\left(B^\tau,N^\tau,v^\tau\right)\left[N_i^\tau\right]=\lim_{\tau\to\infty}\phi\left(B_k^\tau,v_{B_k^\tau}^\tau\right)\left[N_i^\tau\right]=\bar{x}_i\Phi_i\left(\bar{x}^{B_k},\boldsymbol{v}_{(B_k,\bar{x})}\right).$$

■

3 Multilinear Extension and Potential

3.1 Multilinear extension

We can use the fact that the Shapley value of any finite game can be computed by using the multilinear extension of the game in order to obtain an alternative formula for the coalitional value. In this way the computation becomes easier.

For any $S \subset N$, let $e^S = (1, ..., 1) \in \mathbb{R}^S$. Given a game (N, v) the *multilinear extension* is a function $E : \mathbb{R}^N \to \mathbb{R}$ defined by:

$$E[y] := \sum_{S \subset N} \left[\prod_{i \in S} y_i \prod_{i \in N \setminus S} (1 - y_i) \right] v(S).$$

The Shapley value can be obtained by using the multilinear extension, as it was shown in Owen [13], i.e.,

$$\phi_i(N, v) = \int_0^1 \frac{\partial}{\partial y_i} E\left[te^N\right] dt, \quad (i \in N, (N, v) \in G^N).$$

Now, in continuous games we know that the coalitional value satisfies (4), and ϕ_k can be obtained by using the multilinear extension applied to the game (M, v_x). Therefore, by definition

$$\Psi_i(B, \bar{x}, \boldsymbol{v}) = \Phi_i\left(\bar{x}^{B_k}, \boldsymbol{v}_{B_k}\right) = \int_0^1 \frac{\partial}{\partial x_i} \boldsymbol{v}_{B_k}\left(t\bar{x}^{B_k}\right) dt = \int_0^1 \frac{\partial}{\partial x_i} \phi_k\left(M, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N \setminus B_k}\right)}\right) dt.$$

The multilinear extension of the game $\left(M, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N \setminus B_k}\right)}\right)$ is

$$E\left[y, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N \setminus B_k}\right)}\right] = \sum_{S \subseteq M} \left[\prod_{k \in S} y_k \prod_{k \in M \setminus S} (1 - y_k) \right] v_{\left(t\bar{x}^{B_k}, \bar{x}^{N \setminus B_k}\right)}(S),$$

and then it holds that

$$\phi_k\left(M, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N\setminus B_k}\right)}\right) = \int_0^1 \frac{\partial}{\partial y_k} E\left[se^M, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N\setminus B_k}\right)}\right] ds.$$

Which yields finally

$$\Psi_i\left(B, \bar{x}, \boldsymbol{v}\right) = \int_0^1 \int_0^1 \frac{\partial^2}{\partial x_i \partial y_k} E\left[se^M, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N\setminus B_k}\right)}\right] dsdt, \quad \left(i \in B_k \in B\right). \tag{7}$$

It is instructive to illustrate this approach by the example given in the introduction. The set of types is $N = \{1, 2, 3, 4\}$, an the coalition structure is $B = \{B_a, B_b\}$, where $B_a = \{1, 2\}$, and $B_b = \{3, 4\}$, hence $M = \{a, b\}$. The characteristic function is given by

$$\boldsymbol{v}(x_1, x_2, x_3, x_4) = (4x_1 + x_2 + 4x_3 + x_4)\left(1 + e^{-0.2(x_1+x_2+x_3+x_4)}\right).$$

We first compute the Aumann-Shapley value of $\boldsymbol{v}$ in $x = (1, 2, 2, 1) \in \mathbb{R}^N_{++}$

For type 1:

$$\frac{\partial \boldsymbol{v}}{\partial x_1} = 4 + (4 - 0.8x_1 - 0.2x_2 - 0.8x_3 - 0.2x_4)\, e^{-0.2(x_1+x_2+x_3+x_4)},$$

and

$$\Phi_1\left((1, 2, 2, 1), \boldsymbol{v}\right) = \int_0^1 \frac{\partial \boldsymbol{v}}{\partial x_1}\left(t, 2t, 2t, t\right) dt = \int_0^1 \left(4 + (4 - 3t)\, e^{-1.2t}\right) dt = 5.6265.$$

In a similar way we obtain:

$$\Phi_2\left((1, 2, 2, 1), \boldsymbol{v}\right) = 0.87948,\ \Phi_3\left((1, 2, 2, 1), \boldsymbol{v}\right) = 5.6265,\ \Phi_4\left((1, 2, 2, 1), \boldsymbol{v}\right) = 0.87948.$$

For the continuum Owen value, given the coalition structure $B = \{B_a = \{1, 2\}, B_b = \{3, 4\}\}$, the multilinear extension is

$$E\left[y, v_x\right] = y_a\left(1 - y_b\right) v_x(\{a\}) + y_b(1 - y_a)v_x(\{b\}) + y_a y_b v_x(\{a, b\}).$$

Then, it holds that

$$\frac{\partial}{\partial y_a} E\left[y, v_x\right] = (1 - y_b)\, v_x(\{a\}) - y_b v_x(\{b\}) + y_b v_x(\{a, b\}).$$

Note that, by definition, $v_x(\{b\}) = \boldsymbol{v}(0, 0, x_3, x_4)$. Therefore,

$$\frac{\partial}{\partial x_1} v_x(\{b\}) = 0,$$

and then

$$\frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial y_a} E\left[y, v_x\right]\right] = (1 - y_b)\left[4 + (4 - 0.8x_1 - 0.2x_2)\, e^{-0.2(x_1+x_2)}\right] + \\ + y_b\left[4 + (4 - 0.8x_1 - 0.2x_2 - 0.8x_3 - 0.2x_4)\, e^{-0.2(x_1+x_2+x_3+x_4)}\right],$$

and

$$\frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial y_a} E\left[y, v_x\right]\right] = (1 - y_b)\left[1 + (1 - 0.8x_1 - 0.2x_2)\, e^{-0.2(x_1+x_2)}\right] + \\ + y_b\left[1 + (1 - 0.8x_1 - 0.2x_2 - 0.8x_3 - 0.2x_4)\, e^{-0.2(x_1+x_2+x_3+x_4)}\right].$$

For the calculus of Ψ_1 and Ψ_2 at $x = (1, 2, 2, 1)$, we made $x_1 = t$, $x_2 = 2t$, $x_3 = 2$, and $x_4 = 1$, in the multilinear extension. Then

$$\frac{\partial^2}{\partial x_1 \partial y_a} E\left[(s, s), v_{(t,2t,2,1)}\right] = 4 + (1 - s)\,(4 - 1.2t)\, e^{-0.6t} + s\,(2.2 - 1.2t)\, e^{-0.6t-0.6},$$

$$\frac{\partial^2}{\partial x_2 \partial y_a} E\left[(s, s), v_{(t,2t,2,1)}\right] = 1 + (1 - s)\,(1 - 1.2t)\, e^{-0.6t} + s\,(-0.8 - 1.2t)\, e^{-0.6t-0.6},$$

which yield as payoffs

$$\Psi_1(B, (1, 2, 2, 1), \boldsymbol{v}) = \int_0^1 \left(\int_0^1 \left(4 + (1 - s)\,(4 - 1.2t)\, e^{-0.6t} + s\,(2.2 - 1.2t)\, e^{-0.6t-0.6}\right) dt\right) ds = 5.6433,$$

$$\Psi_2(B, (1, 2, 2, 1), \boldsymbol{v}) = \int_0^1 \left(\int_0^1 \left(1 + (1 - s)\,(1 - 1.2t)\, e^{-0.6t} + s\,(-0.8 - 1.2t)\, e^{-0.6t-0.6}\right) dt\right) ds = 0.89624.$$

We can check the quotient game property:

$$\Psi_1(B,(1,2,2,1),\boldsymbol{v})+2\Psi_2(B,(1,2,2,1),\boldsymbol{v})=5.6433+2\cdot 0.89624=$$
$$=\Phi_a(\{a,b\},v_{(1,2,2,1)})=7.436.$$

Making the same approach for types 3 and 4, it can be checked that

$$\Psi_3(B,(1,2,2,1),\boldsymbol{v})=\int_0^1\left(\int_0^1\left(4+(1-s)(4-1.8t)\,e^{-0.6t}+s\,(2.8-1.8t)\,e^{-0.6t-0.6}\right)dt\right)ds=5.6097,$$
$$\Psi_4(B,(1,2,2,1),\boldsymbol{v})=\int_0^1\left(\int_0^1\left(1+(1-s)(1-1.8t)\,e^{-0.6t}+s\,(-0.2-1.8t)\,e^{-0.6t-0.6}\right)dt\right)ds=0.86272,$$
$$2\Psi_3(B,(1,2,2,1),\boldsymbol{v})+\Psi_4(B,(1,2,2,1),\boldsymbol{v})=2\cdot 5.6097+0.86272=$$
$$=\Phi_b(\{a,b\},v_{(1,2,2,1)})=12.082.$$

If we wish to compute directly $\Psi_1(B,(1,2,2,1),\boldsymbol{v})$ by using (5), first note that

$$\phi_a\left(M,v_{(tx_1,tx_2,\bar{x}_3,\bar{x}_4)}\right)=\frac{1}{2}\left[\boldsymbol{v}(tx_1,tx_2,0,0)-\boldsymbol{v}(0,0,0,0)\right]$$
$$+\frac{1}{2}\left[\boldsymbol{v}(tx_1,tx_2,\bar{x}_3,\bar{x}_4)-\boldsymbol{v}(0,0,\bar{x}_3,\bar{x}_4\right]=\boldsymbol{v}_{(B_a,\bar{x})}\left(tx^{B_a}\right),$$

and hence

$$\frac{\partial}{\partial x_1}\boldsymbol{v}_{(B_a,\bar{x})}\left(tx^{B_a}\right)=(0.2-0.4tx_1-0.1tx_2-0.4\bar{x}_3-0.1\bar{x}_4)\,e^{-0.2(tx_1+tx_2+\bar{x}_3+\bar{x}_4)}$$
$$+(0.2-0.4tx_1-0.1tx_2)\,e^{-0.2(tx_1+tx_2)}+0.4.$$

Therefore

$$\Psi_1(B,(1,2,2,1),\boldsymbol{v})=\int_0^1\left((1.1-0.6t)\,e^{(-0.6t-0.6)}+(2.0-0.6t)\,e^{-0.6t}+4.0\right)dt=5.6433.$$

3.2 Potential

In Hart and Mas-Colell [7] it was shown that the Shapley value has an associated potential function such that the marginal contribution of each player to the potential is just its Shapley value. In the same paper it is shown that the Aumann-Shapley value for continuum games has also a potential. In particular, given a game $\boldsymbol{v}$ and a coalitional vector profile $\bar{x} \in \mathbb{R}^N_{++}$, the potential associated, $P(\bar{x}, \boldsymbol{v}) \in \mathbb{R}$, is defined by

$$P(\bar{x}, \boldsymbol{v}) = \int_0^1 \frac{1}{t} \boldsymbol{v}(t\bar{x}) dt,$$

and it holds that

$$\Phi_i(\bar{x}, \boldsymbol{v}) = \frac{\partial}{\partial x_i} P(\bar{x}, \boldsymbol{v}), \quad (i \in N).$$

In Winter [17] it was shown that the Owen value for finite games has also an associated potential function, but now the potential is a real vector function, having as many components as unions have the coalition structure. Now, the marginal contribution of each player in a union to the corresponding potential component is just its Owen value. We can find a parallel result for continuum games.

Given $(B, \bar{x}, \boldsymbol{v})$, with a set of unions M, the potential will be a real vector function $P(B, \bar{x}, \boldsymbol{v}) \in \mathbb{R}^M$, where for every $k \in M$, we have that

$$P^k (B, \bar{x}, \boldsymbol{v}) = \int_0^1 \frac{1}{t} \left[\int_0^1 \frac{\partial}{\partial y_k} E \left[se^M, v_{\left(t\bar{x}^{B_k}, \bar{x}^{N \setminus B_k}\right)} \right] ds \right] dt,$$

and

$$\Psi_i (B, \bar{x}, \boldsymbol{v}) = \frac{\partial}{\partial x_i} P^k (B, \bar{x}, \boldsymbol{v}), \quad (i \in B_k \in B).$$

4 Axiomatic approach

We show in this section that the coalitional value can also be characterized with the same system of properties that characterize the Owen value in the finite setting. Several alternative system of axioms can be used. We choose the simplest one, although the reader will find at the end of this Section that the strategy used in the proof is easily adaptable to any system of axioms that characterize the Owen value.

A value γ on G^N is said to be *efficient* if $\sum_{i \in N} \gamma_i (N, v) = v(N)$.

A value γ on G^N satisfies *balanced contributions* if

$$\gamma_i (N, v) - \gamma_i (N \backslash j, v) = \gamma_j (N, v) - \gamma_j (N \backslash i, v), \quad (\{i, j\} \subset N, (N, v) \in G^N).$$

Efficiency is a natural budget restriction, and balanced contributions is a fair-marginal rule: The player j's marginal contribution to player i $(\gamma_i(N, v) - \gamma_i (N \backslash j, v))$ must be equal to i's marginal contribution to player j $(\gamma_j (N, v) - \gamma_j (N \backslash i, v))$. Myerson [8] gave an axiomatic characterization of the Shapley value in terms of these two properties:

Theorem 2 *[Myerson, 1980]. A value γ on G^N satisfies efficiency and balanced contributions if and only if $\gamma \equiv \phi$.*

In the setting of games with coalition structures, the balanced contributions property is applied in two levels: Firstly, how to share the aggregate payoffs among the unions and secondly, how to share the individual payoffs within the members of each union. For any $x \in \mathbb{R}^N$ and $S \subset N$, denote by $x [S] = \sum_{i \in S} x_i$.

Axiom 1 (BCwU) *A solution γ on $\mathcal{B}(N) \times G^N$, is said to satisfy* balanced contributions between unions *if, for all* $\{B_i, B_j\} \subset B$*, we have that*

$$\gamma(B,N,v)[B_i] - \gamma(B \backslash B_j, N \backslash B_j, v)[B_i] = \gamma(B,N,v)[B_j] - \gamma(B \backslash B_i, N \backslash B_i, v)[B_j].$$

Axiom 2 (BCwU) *A solution* γ *on* $\mathcal{B}(N) \times G^N$*, is said to satisfy* balanced contributions within unions *if, for all* $\{i,j\} \subset B_k \in B$*, we have that*

$$\gamma_i(B,N,v) - \gamma_i\left(B\mid_{B_k \backslash j}, N \backslash j, v\right) = \gamma_j(B,N,v) - \gamma_j\left(B\mid_{B_k \backslash i}, N \backslash i, v\right).$$

Given formula (3) and Theorem 1, applying repeatedly balanced contributions within players in each union, and between unions, we have the following characterization of the Owen value:

Theorem 3 (Calvo et al., 1996) *A solution* γ *on* $\mathcal{B}(N) \times G^N$ *satisfies efficiency, BCwU, and BCbU, if, and only if* $\gamma(B,N,v) = \varphi(B,N,v)$*.*

Given two vectors $x, y \in \mathbb{R}^N$ denote by $x \cdot y = \sum_{i \in N} x_i y_i$. Let Δ^N be the family of all values continuously differentiable on $\mathbb{R}^N_{++} \times C^N$. In the context of continuous games, a value Γ on Δ^Nsatisfies the properties of *efficiency* if

$$\bar{x} \cdot \Gamma(\bar{x}, \boldsymbol{v}) = \boldsymbol{v}(\bar{x}), \quad \left(\bar{x} \in \mathbb{R}^N_{++}\right),$$

and *balanced contributions* if

$$\frac{\partial}{\partial x_j}\Gamma_i(\bar{x}, \boldsymbol{v}) = \frac{\partial}{\partial x_i}\Gamma_j(\bar{x}, \boldsymbol{v}), \quad \left(\{i,j\} \subset N, \ \bar{x} \in \mathbb{R}^N_{++}\right).$$

The following theorem can be found either in Calvo and Santos [6], or Ortmann [12].

Theorem 4 *A value* Ψ *on* Δ^N *satisfies efficiency and balanced contributions if, and only if* $\Psi = \Phi$*.*

Consider now continuous games with a coalition structure. Let $(\mathcal{B}\Delta)^N$ be the family of all coalitional values continuously differentiable on $\mathcal{B}(N) \times \mathbb{R}^N_{++} \times C^N$.

We will extend the balanced contributions property in the same way as in the finite case. Given $B_i \in B$, $\boldsymbol{v} \in C^N$, and $x \in \mathbb{R}^N_{++}$, let $\left(B \backslash B_j, x^{N \backslash B_j}, \boldsymbol{v}\right) \in \mathcal{B}(N \backslash B_i) \times \mathbb{R}^{N \backslash B_i}_{++} \times C^{N \backslash B_i}$ be the new game that appears when the union B_i leaves the game. Given $S \subset N$, define $\Gamma(B, x, \boldsymbol{v})[S] = \sum_{i \in S} x_i \Gamma_i(B, x, \boldsymbol{v})$.

Axiom 3 (BCbUT) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* balanced contributions between unions of types *if, for all* $\{B_i, B_j\} \subset B$*, we have that*

$$\Gamma(B, \bar{x}, \boldsymbol{v})[B_i] - \Gamma\left(B \backslash B_j, \bar{x}^{N \backslash B_j}, \boldsymbol{v}\right)[B_i] = \Gamma(B, \bar{x}, \boldsymbol{v})[B_j] - \Gamma\left(B \backslash B_i, \bar{x}^{N \backslash B_i}, \boldsymbol{v}\right)[B_j].$$

Axiom 4 (BCwUT) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* balanced contributions within unions of types *if, for all* $\{i, j\} \subset B_k \in B$*, we have that*

$$\frac{\partial}{\partial x_j} \Gamma_i(B, \bar{x}, \boldsymbol{v}) = \frac{\partial}{\partial x_i} \Gamma_j(B, \bar{x}, \boldsymbol{v}).$$

Theorem 5 *A solution* Γ *on* $(\mathcal{B}\Delta)^N$ *satisfies efficiency, BCwUT, and BCbUT, if, and only if* $\Gamma(B, \bar{x}, v) = \Psi(B, \bar{x}, v)$.

Proof. *Existence.* It is easy to see that Ψ satisfies the axioms by construction:

Efficiency: As Ψ satisfies the quotient game property, and ϕ satisfies efficiency, it holds that

$$\sum_{i \in N} \bar{x}_i \Psi_i(B, \bar{x}, \boldsymbol{v}) = \sum_{k \in M} \sum_{i \in B_k} \bar{x}_i \Psi_i(B, \bar{x}, \boldsymbol{v}) = \sum_{k \in M} \phi_k(M, v_{\bar{x}}) = \boldsymbol{v}(\bar{x}).$$

BCwUT: It follows from the fact that $\Psi_i(B, x, v) = \Phi_i\left(x^{B_k}, \boldsymbol{v}_{(B_k, x)}\right)$, for all $i \in B_k \in B$, and Φ satisfies balanced contributions on Δ^{B_k}.

BCbUT: It follows from the fact that $\sum_{i \in B_k} \bar{x}_i \Psi_i(B, \bar{x}, \boldsymbol{v}) = \phi_k(M, v_{\bar{x}})$, for all $k \in M$, and ϕ satisfies balanced contributions on G^M.

Uniqueness. Let Γ be a coalitional value that satisfies efficiency and BCbUT, then by Theorem (2) it must hold that $\Gamma(B, \bar{x}, \boldsymbol{v})[B_k] = \phi_k(M, v_{\bar{x}}) = \boldsymbol{v}_{B_k}(\bar{x}^{B_k})$, for all $B_k \in B$. As ϕ_k is a polynomial expression, $\boldsymbol{v}_{B_k} \in C^{B_k}$. Now, by efficiency and BCwUT, it must hold by Theorem (4) that

$$\Gamma_i(B, \bar{x}, \boldsymbol{v}) = \Phi_i\left(\bar{x}^{B_k}, \boldsymbol{v}_{B_k}\right) = \Psi_i(B, \bar{x}, \boldsymbol{v}), \quad (i \in B_k).$$

■

When players correspond to different types of consumers and the unions are cities, the balanced contributions, applied between unions and within players of each union, is a natural property to deal with this situation. In this context it is a fair rule to balance the relative bargaining power between individuals and groups of individuals.

On the other hand, if we wish to apply the coalitional prices as a cost allocation rule, it is preferable to set axioms which have an interpretation in pure economic terms. For that purpose we follow the marginalistic Young's approach (see [18,19]), adapting the axioms to the coalitional structure setting. Now x_i is the quantity of good $i \in N$ which is jointly produced, and $B = \{B_1, ..., B_m\}$ is the coalition structure defined over the the set of goods. We could face different situations, for example, each B_k is a set of goods produced by the same line or division within a firm, or goods produced by a firm which belongs to a joint venture with a set of firms, etc.

The first axiom says that if each product is rescaled by a positive factor then the prices should be scaled by the same factor.

Axiom 5 (R) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* rescaling *if, when for two cost functions* $\boldsymbol{v}$ *and* $\boldsymbol{w}$*, and a vector* $\lambda \in \mathbb{R}^N_{++}$ *it holds that* $\boldsymbol{v}(x) = \boldsymbol{w}(\lambda * x)$*, where* $\lambda * x = (\lambda_i x_i)_{i \in N}$*, then we have that* $\Gamma_i(B, \bar{x}, \boldsymbol{v}) = \lambda_i \Gamma_i(B, \bar{x}, \boldsymbol{w})$*.*

The second axiom was introduced in Young [19]. It is an incentive-compatible property in the sense that decreasing the marginal costs of a good should not increase its price. This guarantees that the introduction of a new technology that decreases uniformly the cost of production of some good never will be penalized.

Axiom 6 (ISM) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* individual strong monotonicity *if, when for two cost functions* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *such that* $\frac{\partial \boldsymbol{v}(x)}{\partial x_i} \leq \frac{\partial \boldsymbol{w}(x)}{\partial x_i}$ *for all* $x \in \mathbb{R}^N_+$*, then* $\Gamma_i(B, \bar{x}, \boldsymbol{v}) \leq \Gamma_i(B, \bar{x}, \boldsymbol{w})$.

The next axiom is a symmetry property that help us to compare costs of goods inside the same union. It says that commodities which have the same effect on the costs should have the same price. Let $S \subset B_k \in B$ and $x_s = \sum_{i \in S} x_i$; $\boldsymbol{v} \in C^N$ and $\boldsymbol{v}' \in C^{N'}$, where $N' = (N \backslash S) \cup \{s\}$, and $B' = \{B_1, ..., B'_k, ..., B_m\}$, where $B'_k = (B_k \backslash S) \cup \{s\}$.

Axiom 7 (WCwU) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* weak consistency within a union *if, when* $\boldsymbol{v}(x) = \boldsymbol{v}'(x_s, x^{N \backslash S})$ *for all* $x \in \mathbb{R}^N_+$*, it holds that*

$$\Gamma_i(B, \bar{x}, \boldsymbol{v}) = \Gamma_s\left(B', (\bar{x}_s, \bar{x}^{N \backslash S}), \boldsymbol{v}'\right), \textit{ for all } i \in S.$$

The following two properties are concerned with the problem of how sharing the aggregate costs between the unions.

Firstly, when the marginal contributions of two groups B_k and B_l are equal in the quotient game, the sum of the costs associated to each group must be equal.

Axiom 8 (ETbU) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* equal treatment between unions *if, when for* $\{B_k, B_l\} \subset B$*, it holds that* $v_{\bar{x}}(S \cup k) = v_{\bar{x}}(S \cup l)$*, for all* $S \subset M \backslash \{k, l\}$*, then* $\Gamma(B, \bar{x}, \boldsymbol{v})[B_k] = \Gamma(B, \bar{x}, \boldsymbol{v})[B_l]$

The last property is also a monotonicity property for the aggregate costs of a union. This property was introduced in Young [18] for finite games.

Axiom 9 (ASM) *A solution* Γ *on* $(\mathcal{B}\Delta)^N$*, is said to satisfy* Aggregate strongly monotonicity *if, when for two cost functions* $\boldsymbol{v}$ *and* $\boldsymbol{w}$*, and* $B_k \in B$*, it holds that* $v_{\bar{x}}(S\cup k) - v_{\bar{x}}(S) \leq w_{\bar{x}}(S\cup k) - w_{\bar{x}}(S)$*, for all* $S \subset M\backslash k$*, then we have that* $\Gamma(B, \bar{x}, \boldsymbol{v})[B_k] \leq \Gamma(B, \bar{x}, \boldsymbol{w})[B_k]$*.*

Theorem 6 *A solution* Γ *on* $(\mathcal{B}\Delta)^N$ *satisfies efficiency, rescaling, ISM, WCwU, ETbU, and ASM if, and only if* $\Gamma(B, \bar{x}, v) = \Psi(B, \bar{x}, v)$*.*

Proof. *Existence.* Efficiency is straightforward. Now, by the quotient game property (6), ETbU and ASM are satisfied because the Shapley value satisfies both equal treatment and strong monotonicity (see Young [18]) in the quotient game $(M, v_{\bar{x}})$. For checking the rest of the properties we use an explicit expression of the coalitional value Γ_i. Let $i \in B_k \in B$, and for a fixed $\bar{x}^{N\backslash B_k}$,because

$$\boldsymbol{v}_{B_k}\left(x^{B_k}\right) = \phi_k\left(M, v_{\left(x^{B_k}, \bar{x}^{N\backslash B_k}\right)}\right) = \sum_{S\subset M\backslash k} \frac{s!(n-s-1)!}{n!}\left[v_{\left(x^{B_k}, \bar{x}^{N\backslash B_k}\right)}(S\cup k) - v_{\left(x^{B_k}, \bar{x}^{N\backslash B_k}\right)}(S)\right],$$

and for $S \subset M\backslash k$ and $i \in B_k$ we have that

$$\frac{\partial}{\partial x_i} v_{\left(x^{B_k}, \bar{x}^{N\backslash B_k}\right)}(S) = \frac{\partial}{\partial x_i}\boldsymbol{v}\left(x^{\left(\cup_{j\in S} B_j\right)}, 0^{N\backslash\left(\cup_{j\in S} B_j\right)}\right) = 0,$$

then it holds that

$$\Gamma_i(B, \bar{x}, \boldsymbol{v}) = \Phi_i\left(\bar{x}^{B_k}, \boldsymbol{v}_{B_k}\right) = \int_0^1 \frac{\partial}{\partial x_i}\boldsymbol{v}_{B_k}\left(t\bar{x}^{B_k}\right) dt = \tag{8}$$

$$= \sum_{S\subset M\backslash k} \frac{s!(n-s-1)!}{n!}\int_0^1 \frac{\partial}{\partial x_i}\boldsymbol{v}\left(t\bar{x}^{B_k}, \bar{x}^{\left(\cup_{j\in S} B_j\right)}, 0^{N\backslash\left(\cup_{j\in S\cup k} B_j\right)}\right) dt.$$

From (8) it follows easily that Γ satisfies rescaling, individual strong monotonicity and weak consistency within a union.

Uniqueness. From efficiency, equal treatment between unions and aggregate monotonicity it follows that Γ must satisfy the quotient game property (see Young [18]), i.e.

$\Gamma(B, \bar{x}, \boldsymbol{v})[B_k] = \phi_k(M, v_{\bar{x}}) = \boldsymbol{v}_{B_k}(\bar{x}^{B_k})$, for all $B_k \in B$, and for all $\bar{x}^{B_k} \in \mathbb{R}_{++}^{B_k}$. As $\boldsymbol{v}_{B_k} \in C^{B_k}$, by efficiency, rescaling, weak consistency within a union and individual strong monotonicity, it follows that $\Gamma_i(B, \bar{x}, \boldsymbol{v})$ must be the Aumann-Shapley price of the game $\boldsymbol{v}_{B_k}$ at x^{B_k}, that is, $\Gamma_i(B, \bar{x}, \boldsymbol{v}) = \Phi_i\left(\bar{x}^{B_k}, \boldsymbol{v}_{B_k}\right)$ (see Young [19] and Monderer and Neyman [11]). ■

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