Feigenbaum graphs: a complex network perspective of chaos

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We expose a remarkable relationship between nonlinear dynamical systems and complex networks by means of the horizontal visibility (HV) algorithm[1,2] that transforms time series into graphs. In low-dimensional dissipative systems chaotic motion develops out of regular motion in a few number of ways or routes, amongst which the period-doubling bifurcation cascade or Feigenbaum scenario is perhaps the better known and most famous mechanism. This route to chaos appears infinitely many times amongst the family of attractors spawned by unimodal maps within the so-called periodic windows that interrupt stretches of chaotic attractors. In opposition, a route out of chaos accompanies each period-doubling cascade by a chaotic band-splitting cascade, and their shared bifurcation accumulation points form transitions between order and chaos that are known to possess universal properties. Low-dimensional maps have been extensively studied from a purely theoretical perspective, but systems with many degrees of freedom used to study diverse problems in physics, biology, chemistry, engineering, and social science, are known to display low-dimensional dynamics.

The horizontal visibility (HV) algorithm converts the information stored in a time series into a network, setting the nature of the dynamical system into a different context that requires complex network tools to extract its properties. Relevant information can be obtained through this methodology, including the characterization of fractal behavior[3] or the discrimination between random and chaotic series[1,4], and it finds increasing applications in separate fields, from geophysics[5], to finance[6] or physiology[7]. Here we offer a distinct view of the Feigenbaum scenario through the HV formalism, making a complete study of the HV graphs associated to orbits extracted from unimodal maps, which in this context we will call Feigenbaum graphs. We first characterize their topology via order-of-visit and self-affinity properties of the maps. Additionally, a matching renormalization group (RG) procedure leads via its flows to or away from network fixed-points to a comprehensive view of the entire family of attractors. Furthermore, the optimization of the entropy obtained from the degree distribution coincides with the RG fixed points and reproduces the essential features of the map's Lyapunov exponent independently of its sign. A general observation is that the visibility algorithm extracts only universal elements of the dynamics, free of peculiarities of the individual unimodal map, but also of universality classes characterized by the degree of nonlinearity.

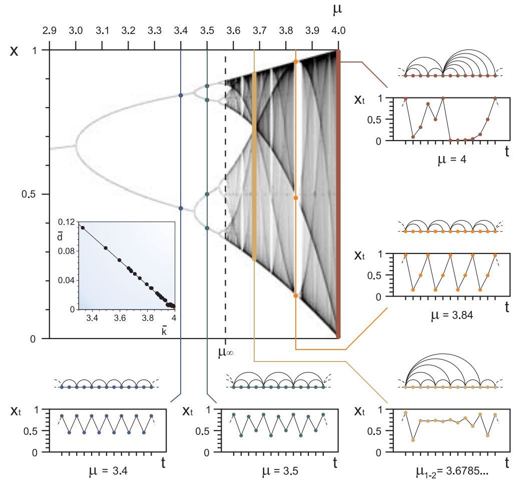

Figura 1. Feigenbaum diagram of the Logistic map $x_{t+1} = \mu x_t(1-x_t)$, indicating a transition from periodic to chaotic behavior at $\mu_\infty = 3.569946...$ through period-doubling bifurcations. For $\mu \geq \mu_\infty$ the map shows a chaotic mirror image of the period-doubling tree, where aperiodic behavior appears interrupted by periodic windows. Surrounding the central figure, for several values of μ we show time series and their associated Feigenbaum graphs. *Inset:* Numerical values of the mean normalized distance $\bar{d}$ as a function of mean degree $\bar{k}$ of the Feigenbaum graphs for $3 < \mu < 4$ (associated to time series of 1500 data after a transient, and a step $\delta\mu = 0.05$), in good agreement with the theoretical linear relation: $\bar{d}(\bar{k}) = \frac{1}{6}(4-\bar{k})$.

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