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# Vacuum type I spacetimes and aligned Papapetrou fields: symmetries

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**Abstract**

We analyse type I vacuum solutions admitting an isometry whose Killing 2-form is aligned with a principal bivector of the Weyl tensor, and we show that these solutions belong to a family of type I metrics which admit a group $G_3$ of isometries. We give a classification of this family and study the Bianchi type for each class. The classes compatible with an aligned Killing 2-form are also determined. The Szekeres–Brans theorem is extended to non-vacuum spacetimes with vanishing Cotton tensor.

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## 1. Introduction

Several methods have been developed to simplify Einstein equations in order to look for new exact solutions. These approaches usually imply imposing conditions that restrict the space of possible solutions. Thus, a notable number of known solutions have been obtained under the hypothesis that they admit a fixed isometry or conformal group. It has also been fruitful to impose restrictions on the algebraic structure of the Weyl tensor. Indeed, wide families of algebraically special solutions of Einstein equations have been found by considering coordinates or frames adapted to the multiple Debever direction that these spacetimes admit. Nevertheless, there is a lack of knowledge about algebraically general solutions, and they have usually been obtained by imposing spacetime symmetries. One way to correct this situation is to opt for imposing a complementary condition on a type I Weyl tensor which allows us to simplify Einstein equations. This means considering subclassifications of algebraically general spacetimes and looking for solutions in every defined class.

Debever [1] was the first to suggest a classification of type I metrics based on the nullity of one of the Weyl algebraic invariant scalars. A similar kind of condition is satisfied by the

Weyl eigenvalues in the purely electric or purely magnetic solutions [2], as well as in some classes defined by McIntosh and Arianrhod [3] where the positive or negative real character of a Weyl invariant scalar is imposed. These classes may also be reinterpreted in terms of some kind of degeneracy of the four Debever directions that a type I metric admits [3, 4]. In this latest work we have shown that these classes contain some families of metrics which can be interpreted as generalized purely electric or purely magnetic spacetimes.

Besides those algebraic classifications quoted above one can define other families of type I spacetimes by imposing conditions on the first-order Weyl concomitants. For example, in the normal shear-free perfect fluid solutions [5] one restricts the kinematic coefficients associated with the unique timelike Weyl principal direction. Another classification that imposes first-order differential conditions on the type I canonical frame was given by Edgar [6]. Here we have changed this approach slightly by imposing invariant conditions that are symmetric with respect to the three spacelike principal directions. When the spacetime Cotton tensor vanishes, the Bianchi identities imply that in both Edgar's and our degenerate classes, the Weyl eigenvalues are functionally dependent. The present work is concerned with the vacuum solutions of the more degenerate class which we call $I_1$. We show that in this case a three-dimensional group of isometries exists and all the Weyl-invariant scalars depend on a unique real function. We also offer an invariant subclassification of the type $I_1$ metrics and prove that the classes can be characterized by the Bianchi type of the $G_3$. Some of these results were communicated without proof at the Spanish Relativity Meeting 1998 [7].

In an algebraically general spacetime the Weyl canonical frame determines three timelike principal 2-planes associated with a frame of orthonormal eigenbivectors of the Weyl tensor. Here we use the complex Cartan formalism adapted to this orthonormal frame of eigenbivectors under the hypothesis of a vanishing Cotton tensor. In this case the Cartan structure equations and their integrability conditions, the Bianchi identities, do not constitute a completely integrable system of equations. The integrability conditions for this system were considered by Bell and Szekeres [8] and were investigated by Brans [9] who named them the *post-Bianchi equations*. Edgar [10] analysed in detail these post-Bianchi equations that we obtain easily here using a plain covariant approach. On the other hand, the complex formalism that we use allows us to generalize easily the Szekeres–Brans theorem on the non-existence of vacuum type I solutions with a zero Weyl eigenvalue [11, 12]. This extension is similar to that known for the Goldberg–Sachs theorem and it implies that the Szekeres–Brans result also applies to the non-vacuum solutions with vanishing Cotton tensor. This fact has already been used in analysing the restrictions on the existence of purely magnetic solutions [13]. It is worth pointing out that the set of the metrics with vanishing Cotton tensor contains all the vacuum solutions as well as other non-vacuum metrics with specific restrictions on the energy content. For example, in the case of perfect fluid solutions, this condition leads to non-shearing irrotational flows [14].

When a spacetime admits an isometry, the Killing vector $\xi$ plays the role of an electromagnetic potential satisfying the Lorentz gauge, and its covariant derivative $\nabla\xi$ is, in the vacuum case, a solution of the source-free Maxwell equations. Because this fact was first observed by Papapetrou [15], the *Killing 2-form* $\nabla\xi$ has also been called the *Papapetrou field* [16]. Metrics admitting an isometry were studied by considering the algebraic properties of the associated Killing 2-form [17, 18], and this approach was extended to the spacetimes with an homothetic motion [19, 20].

More recently, Fayos and Sopuerta [16, 21] have developed a formalism that improves the use of the Killing 2-form and its underlined algebraic structure in the analysis of the vacuum solutions with an isometry. They consider two new viewpoints that permit a more accurate classification of these spacetimes: (i) the differential properties of the principal directions of the Killing 2-form, and (ii) the degree of alignment of the principal directions of the Killing 2-form with those of the Weyl tensor. The Fayos and Sopuerta approach uses the Newman–Penrose formalism and several extensions have been built for homothetic and conformal motions [22, 23] and for non-vacuum solutions [24].

In the Kerr geometry the principal directions of the Killing 2-form associated with the timelike Killing vector are the two double principal null (Debever) directions of the Weyl tensor [16]. But in a type D spacetime these two Debever directions are the principal directions of the principal 2-plane and, consequently, in the Kerr spacetime, the Killing 2-form is an eigenbivector of the Weyl tensor. This fact has also been remarked by Mars [25] who has shown that this property characterizes the Kerr solution under an asymptotic flatness behaviour. Elsewhere [26] we have shown that the type D vacuum solutions having a timelike Killing 2-form aligned with the Weyl geometry are the Kerr–NUT spacetimes.

In a Petrov type I spacetime a principal direction of a Weyl principal 2-plane never coincides with one of the four null Debever directions [27, 28]. Therefore, in a type I metric two different kinds of alignment of a Killing 2-form and the Weyl tensor can be considered. On one hand, we can impose, as Fayos and Sopuerta do [21], the alignment between a principal direction of the Killing 2-form and a principal Debever direction of the Weyl tensor. On the other hand, we can consider that a principal 2-plane of the Weyl tensor is the principal 2-plane of the Killing 2-form. In this work we adopt this second point of view and show that all the vacuum type I solutions having a Killing 2-form which is an eigenbivector of the Weyl tensor belong to the class $\mathrm{I}_1$ quoted above. In consequence, these spacetimes admit a group $\mathrm{G}_3$ of isometries. We also study the Bianchi types compatible with this kind of alignment.

The paper is organized as follows. In section 2 we present the general notation of the complex formalism used in this work and apply this formalism to write the Cartan structure equations and to characterize the geometric properties of a 2+2 almost-product structure. In section 3 we adapt the Cartan complex formalism to the canonical frame of a type I Weyl tensor and write the Bianchi identities for the spacetimes with a vanishing Cotton tensor. Some direct consequences of these equations lead us to present a classification of the type I spacetimes. The generalization of a theorem by Brans and a tensorial expression for the post-Bianchi equations are easily obtained too. In section 4 we study general properties of an aligned Papapetrou field and show that every type I vacuum solution with an isometry having an aligned Killing 2-form belongs to the more degenerate class $\mathrm{I}_1$ in the classification given in the previous section. Some basic results on the type $\mathrm{I}_1$ vacuum solutions are presented in section 5 where a subclassification of these metrics is also offered. In section 6 we show that a type $\mathrm{I}_1$ vacuum metric with a non-constant eigenvalue admits a group $\mathrm{G}_3$ of isometries and, for every subclass presented in previous section, we study its Bianchi type. The Bianchi types which are compatible with an aligned Killing 2-form are also determined. Finally, in section 7 we study the vacuum case with constant eigenvalues and prove that the spacetime has a group $\mathrm{G}_4$ of isometries, obtaining in this way an intrinsic characterization of the vacuum homogeneous Petrov metric.

## 2. Cartan equations in complex formalism

Let $(V_4, g)$ be an oriented spacetime of signature $(-, +, +, +)$ and let $\eta$ be the volume element. If we denote by $*$ the Hodge dual operator, we can associate with every 2-form $F$ the self-dual 2-form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - \mathrm{i} * F)$. Hereafter we shall say bivector to indicate a self-dual 2-form.

We can associate with every oriented orthonormal frame $\{e_\alpha\}$ the orthonormal frame $\{\mathcal{U}_i\}$ of the bivector space defined as $\mathcal{U}_i = \frac{1}{\sqrt{2}}[e_0 \wedge e_i - \mathrm{i} * (e_0 \wedge e_i)]$. This frame satisfies

$2\mathcal{U}_i^2 = g$, and it has the induced orientation given by

$$\mathcal{U}_i \times \mathcal{U}_j = -\frac{\mathrm{i}}{\sqrt{2}}\epsilon_{ijk}\mathcal{U}_k, \qquad i \neq j, \tag{1}$$

where $\times$ means the contraction of adjacent indices in the tensorial product and $\mathcal{U}_i^2 = \mathcal{U}_i \times \mathcal{U}_i$.

On the other hand, if $\{\mathcal{U}_i\}$ is an oriented orthonormal frame of bivectors, for every (complex) non-null vector $x$, the four directions $\{x, \mathcal{U}_i(x)\}$ are orthogonal and can be normalized to get an oriented orthonormal frame. In terms of these vectors, the metric tensor $g$ is written

$$g = \frac{1}{x^2}\left[x \otimes x - 2\sum_{i=1}^{3}\mathcal{U}_i(x) \otimes \mathcal{U}_i(x)\right]. \tag{2}$$

Moreover, the self-dual 2-forms $\mathcal{U}_i$ can also be written as

$$\mathcal{U}_i = -\frac{1}{x^2}\left(x \wedge \mathcal{U}_i(x) + \frac{\mathrm{i}}{\sqrt{2}}\epsilon_{ijk}\mathcal{U}_j(x) \wedge \mathcal{U}_k(x)\right). \tag{3}$$

Nevertheless, if $x$ is a null vector the four orthogonal directions $\{x, \mathcal{U}_i(x)\}$ cannot be independent. So, one of the directions is a linear combination of the other ones, $\mathcal{U}_i(x) = ax + b^j\mathcal{U}_j(x)$. Contracting this equation with $\mathcal{U}_i$ we obtain the following

**Lemma 1.** *If $\{\mathcal{U}_i\}$ is an orthonormal frame of bivectors and $x$ is a null vector, then scalars $b^i$ exist such that*

$$x = \sum b^i\mathcal{U}_i(x).$$

Every unitary bivector $\mathcal{U}_i$ defines a 2+2 almost product structure. These kinds of structures have been considered elsewhere [29] and have been used to classify the type D spacetimes attending to the geometric properties of the Weyl principal structure. The 1-form $\delta\mathcal{U}_i \equiv -\operatorname{tr}\nabla\mathcal{U}_i$ collects the information about the minimal and the foliation character of the 2-planes that $\mathcal{U}_i$ defines. More precisely, we have [29, 30]:

**Lemma 2.** *Let $\mathcal{U}_i$ be a unitary bivector and $\lambda_i = \mathcal{U}_i(\delta\mathcal{U}_i)$, and let us consider the 2+2 almost product structure that $\mathcal{U}_i$ defines. It holds:*

*(i) Both planes are minimal if, and only if, $Re(\lambda_i) = 0$.*
*(ii) Both planes are foliations if, and only if, $Im(\lambda_i) = 0$.*

The umbilical nature of the 2-planes defined by $\mathcal{U}_i$ can also be characterized in terms of the covariant derivative of $\mathcal{U}_i$ [29, 30]. This property is equivalent to the null principal directions of $\mathcal{U}_i$ being shear-free geodesics and can be stated in terms of the 1-forms $\lambda_j$ as:

**Lemma 3.** *Let us consider the 2+2 structure defined by a bivector $\mathcal{U}_1$, and let us take $\{\mathcal{U}_2, \mathcal{U}_3\}$ to complete an orthonormal frame. The principal directions of $\mathcal{U}_1$ are shear-free geodesics if, and only if, $\lambda_2 = \lambda_3$.*

For a given orthonormal frame $\{e_\alpha\}$ six connection 1-forms $\omega_\alpha^\beta$ $\left(\omega_\alpha^\beta = -\omega_\beta^\alpha\right)$ are defined by $\nabla e_\alpha = \omega_\alpha^\beta \otimes e_\beta$. These equations are equivalent to the first structure equations in the Cartan formalism. If we consider $\{\mathcal{U}_i\}$ the orthonormal frame of bivectors associated with $\{e_\alpha\}$, the six connection 1-forms can be collected into three complex ones $\Gamma_i^j$ $\left(\Gamma_i^j = -\Gamma_j^i\right)$ that are defined by

$$\Gamma_i^j = \omega_i^j - \mathrm{i}\,\epsilon_{ijk}\,\omega_0^k.$$

In terms of these complex 1-forms, the first Cartan structure equations are equivalent to

$$\nabla\mathcal{U}_i = \Gamma_i^j \otimes \mathcal{U}_j. \tag{4}$$

The second Cartan structure equations follow by applying the Ricci identities to the bivectors $\{\mathcal{U}_i\}$, $\nabla_{[\alpha}\nabla_{\beta]}\mathcal{U}_{i\,\epsilon\delta} = \mathcal{U}_{i\,\epsilon}{}^{\mu} R_{\mu\delta\beta\alpha} + \mathcal{U}_i{}^{\mu}{}_{\delta} R_{\mu\epsilon\beta\alpha}$, and they can be written as

$$\mathrm{d}\Gamma_i^k - \Gamma_i^j \wedge \Gamma_j^k = \mathrm{i}\sqrt{2}\epsilon_{ikm}\,\mathrm{Riem}(\mathcal{U}_m), \tag{5}$$

where $\mathrm{Riem}(\mathcal{U}_m)_{\alpha\beta} = \frac{1}{2}R_{\alpha\beta\mu\nu}(\mathcal{U}_m)^{\mu\nu}$. Taking into account the invariant decomposition of the Riemann tensor into the trace-free part, the Weyl tensor, and the trace, the Ricci tensor, we have

$$\mathrm{Riem} = W + Q \wedge g, \qquad Q = \tfrac{1}{2}\left[\mathrm{Ric} - \tfrac{1}{6}(\mathrm{tr}\,\mathrm{Ric})g\right].$$

As the self-dual Weyl tensor $\mathcal{W} = \frac{1}{2}(W - \mathrm{i} * W)$ satisfies $\mathcal{W}(\mathcal{U}) = W(\mathcal{U})$ for every self-dual 2-form $\mathcal{U}$, the second term of equations (5) becomes

$$\mathrm{Riem}(\mathcal{U}_m) = \mathcal{W}(\mathcal{U}_m) + \mathcal{U}_m \times Q + Q \times \mathcal{U}_m. \tag{6}$$

The three complex 1-forms $\lambda_i = \mathcal{U}_i(\delta\mathcal{U}_i)$ contain the 24 independent connection coefficients as the $\Gamma_i^j$ do. In fact, by using (1) and the first structure equations (4), both sets $\left\{\Gamma_i^j\right\}$ and $\{\lambda_i\}$ can be related by

$$\lambda_i \equiv \mathcal{U}_i(\delta\mathcal{U}_i) = -\frac{\mathrm{i}}{\sqrt{2}}\epsilon_{ijk}\mathcal{U}_k\left(\Gamma_i^j\right). \tag{7}$$

And the inverse of these expressions says that for $i, j, k$ different

$$\mathcal{U}_k\left(\Gamma_i^j\right) = \frac{\mathrm{i}}{\sqrt{2}}\epsilon_{ijk}(\lambda_i + \lambda_j - \lambda_k), \qquad (i, j, k \neq). \tag{8}$$

There is a subset of the second structure equations (5) that can be concisely stated in terms of the 1-forms $\lambda_i$. Indeed, if we calculate $\left(\mathrm{d}\Gamma_i^j, \mathcal{U}_k\right)$ from the second structure equations (5) and use (6) to replace $\mathrm{Riem}(\mathcal{U}_k, \mathcal{U}_k)$, we get that for $i, j, k$ different, it holds:

$$\nabla\cdot\lambda_i = \lambda_i^2 - (\lambda_j - \lambda_k)^2 - \mathcal{W}(\mathcal{U}_i, \mathcal{U}_i) - \tfrac{1}{2}\,\mathrm{tr}\,Q, \qquad (i, j, k \neq), \tag{9}$$

where we have denoted $\nabla\cdot \equiv \mathrm{tr}\,\nabla$ and $\lambda_i^2 = g(\lambda_i, \lambda_i)$.

## 3. Type I metrics with vanishing Cotton tensor: generalized Szekeres–Brans theorem

In a type I spacetime the Weyl tensor defines an orthonormal frame $\{\mathcal{U}_i\}$ of eigenbivectors. If we denote by $\alpha_i$ the corresponding eigenvalues, the self-dual Weyl tensor takes the canonical expression [35]

$$\mathcal{W} = -\sum \alpha_i \mathcal{U}_i \otimes \mathcal{U}_i. \tag{10}$$

So, the structure equations (4), (5) can be written in this frame of principal bivectors. The integrability conditions for the second structure equations (5) are the Bianchi identities which equal the divergence of the Weyl tensor with the Cotton tensor $P$:

$$\nabla\cdot W = P, \qquad P_{\mu\nu,\beta} \equiv \nabla_{[\mu}Q_{\nu]\beta}. \tag{11}$$

Hereafter we will consider spacetimes with vanishing Cotton tensor. Then, the Bianchi identities (11) state that the Weyl tensor is divergence-free, $\nabla\cdot\mathcal{W} = 0$. If we use the canonical expression (10) to compute the divergence of the Weyl tensor and we take into account that $*\mathcal{U}_j = \mathrm{i}\,\mathcal{U}_j$, the Bianchi identities become

$$\mathrm{d}\alpha_i = (\alpha_j - \alpha_k)(\lambda_j - \lambda_k) - 3\alpha_i\lambda_i, \qquad (i, j, k \neq), \tag{12}$$

where $i, j, k$ take different values. Let us suppose now that one of the eigenvalues, say $\alpha_i$, takes the value zero. So, from (12) we have $0 = (\alpha_j - \alpha_k)(\lambda_j - \lambda_k)$. As $\alpha_j \neq \alpha_k$, it must be $\lambda_j - \lambda_k = 0$. This condition is equivalent to the principal directions of $\mathcal{U}_i$ being shear-free geodesics as lemma 3 states, and by the generalized Goldberg–Sachs theorem [31] the spacetime must be algebraically special. Thus we have generalized a previous result, which can be inferred from a paper by Szekeres [11] and was stated by Brans [12], on the non-existence of vacuum type I solutions with a vanishing Weyl eigenvalue to the case of non-vacuum solutions with vanishing Cotton tensor.

**Theorem 1.** *There is no type I spacetime with vanishing Cotton tensor for which one of the eigenvalues of the Weyl tensor is zero.*

The Weyl tensor is trace-free, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, and consequently only two of the three equations (12) are independent. So, we can write Bianchi identities explicitly as

$$\begin{aligned} \mathrm{d}\alpha_1 &= (\alpha_1 + 2\alpha_2)(\lambda_2 - \lambda_3) - 3\alpha_1\lambda_1 \\ \mathrm{d}\alpha_2 &= (2\alpha_1 + \alpha_2)(\lambda_1 - \lambda_3) - 3\alpha_2\lambda_2. \end{aligned} \tag{13}$$

From these equations a direct calculation leads us to the following:

**Proposition 1.** *In a type I spacetime with vanishing Cotton tensor the scalars* $\{\alpha_i\}$ *depend on a function* $(\mathrm{d}\alpha_i \wedge \mathrm{d}\alpha_j = 0)$ *if, and only if, one of the following conditions holds:*

*(i)* $\lambda_i \wedge \lambda_j = 0, \forall i, j,$
*(ii)* $\sum p^i \lambda_i = 0$, *where the scalars* $p^i$ *satisfy* $\displaystyle\sum_{i,j,k\neq} p^i(\alpha_j - \alpha_k)^2 = 0.$

If we take into account lemmas 2 and 3, the last proposition states that the functional dependence of the Weyl eigenvalues is related to restrictions on the geometric properties of the principal 2-planes. Of course we are under the hypothesis of a vanishing Cotton tensor. We can find a similar situation in type D spacetimes where some families determined by imposing conditions on the gradient of the Weyl eigenvalue turn out to be those classes defined attending the geometric properties of the Weyl principal structure [29]. The above result for the case of type I metrics leads us to the following classification [7]:

**Definition 1.** *We will say that a type I spacetime is of class* $I_a$ $(a = 1, 2, 3)$ *if the dimension of the space that the* $\lambda_i$ *generate is a.*

Differential conditions of this kind were imposed by Edgar [10] on the type I spacetimes, and he showed that in the vacuum case his classification also has consequences on the functional dependence of the Weyl eigenvalues. We have slightly modified the Edgar approach in order to obtain a classification that is symmetric in the principal structures of the Weyl tensor. We stress the invariant nature of this classification. It is based on the vector Weyl invariants $\lambda_i$ which have a precise geometric meaning: they contain the information about the properties of the Weyl principal planes (see lemmas 2 and 3). On the other hand, these geometric properties can be interpreted in terms of the kinematical behaviour of the null principal directions of these planes [29]. Let us take into account that $\lambda_i$ cannot all be zero because this fact implies that all of the connection coefficients are so, and the spacetime would be plane. So, after definition 1, proposition 1 can be stated as:

**Proposition 2.** *In a type I spacetime with vanishing Cotton tensor the Weyl eigenvalues depend on a function* $(\mathrm{d}\alpha_i \wedge \mathrm{d}\alpha_j = 0)$ *if, and only if, it is class* $I_1$ *or it is class* $I_2$ *and the second condition of proposition 1 is satisfied.*

Different authors [10, 12] have shown that when the Cartan structure equations in vacuum are referred to the Weyl principal frame the Bianchi identities have non-trivial integrability conditions. First considered by Bell and Szekeres [8], these integrability conditions were called the post-Bianchi equations [12] and they have usually been written in the NP formalism [10, 12]. Here we can easily obtain these post-Bianchi equations in tensorial formalism for spacetimes with vanishing Cotton tensor. Indeed, taking the exterior derivative of (13) we obtain

$$\begin{aligned} &\frac{1}{\alpha_2-\alpha_3}\,\mathrm{d}\lambda_1+\frac{1}{\alpha_3-\alpha_1}\,\mathrm{d}\lambda_2+\frac{1}{\alpha_1-\alpha_2}\,\mathrm{d}\lambda_3=0\\ &\frac{\alpha_1}{\alpha_2-\alpha_3}\,\mathrm{d}\lambda_1+\frac{\alpha_2}{\alpha_3-\alpha_1}\,\mathrm{d}\lambda_2+\frac{\alpha_3}{\alpha_1-\alpha_2}\,\mathrm{d}\lambda_3+4(\lambda_1\wedge\lambda_2+\lambda_2\wedge\lambda_3+\lambda_3\wedge\lambda_1)=0. \end{aligned} \tag{14}$$

If we take into account the Bianchi identities (13), the integrability conditions of the post-Bianchi equations (14) are now an identity and so, no new equations are obtained as other authors claimed [10, 12]. Moreover, only 9 of these 12 complex equations are independent of the Cartan structure equations [12].

## 4. Aligned Papapetrou fields: the vacuum type I case

If $\xi$ is a (real) Killing vector its covariant derivative $\nabla\xi$ is named *Killing 2-form* or *Papapetrou field* [15, 16]. The Papapetrou fields have been used to study and classify spacetimes admitting an isometry or an homothetic or conformal motion (see [16–24]). In this way, some classes of vacuum solutions with a principal direction of the Papapetrou field aligned with a (Debever) null principal direction of the Weyl tensor have been considered [21]. Also, the alignment between the Weyl principal plane and the Papapetrou field associated with the timelike Killing vector has been shown in the Kerr geometry [21, 25].

Is it possible to determine all the vacuum solutions having this property of the Kerr metric? Elsewhere [26] we give an affirmative answer to this question for the case of type D spacetimes by showing that *the type D vacuum solutions with a timelike Killing 2-form aligned with the Weyl geometry are the Kerr–NUT metrics*. Here we analyse the type I case and will see that the spacetime necessarily admits a group $G_3$ of isometries. We begin by showing in this section that these solutions belong to class $\mathrm{I}_1$ of definition 3.

If $\{\mathcal{U}_i\}$ is an orthonormal basis of the self-dual 2-form space, the Papapetrou field $\nabla\xi$ becomes

$$\nabla\xi=\sum\Omega_i\mathcal{U}_i+\sum\tilde{\Omega}_i\tilde{\mathcal{U}}_i, \tag{15}$$

where $\Omega_i$ are three complex functions and $\tilde{}$ means complex conjugate. Let us suppose that $\mathcal{U}_1$ is $\xi$-invariant, that is, $\mathcal{L}_\xi\mathcal{U}_1=0$. If we denote by $A^t$ the transpose of the tensor $A$, this condition reads

$$\mathrm{i}(\xi)\nabla\mathcal{U}_1+(\nabla\xi\times\mathcal{U}_1)-(\nabla\xi\times\mathcal{U}_1)^t=0. \tag{16}$$

Contracting this equation with $\mathcal{U}_2$ and $\mathcal{U}_3$, we get

$$\Omega_2=-\frac{\mathrm{i}}{\sqrt{2}}\big(\xi,\Gamma_3^1\big),\qquad \Omega_3=-\frac{\mathrm{i}}{\sqrt{2}}\big(\xi,\Gamma_1^2\big). \tag{17}$$

So, if $\mathcal{U}_1$ is $\xi$-invariant, two complex components (or four real ones) of the Killing 2-form $\nabla\xi$ are determined by $\xi$. If, in addition, $\mathcal{U}_2$ (and so $\mathcal{U}_3$) are invariant, then $\nabla\xi$ is totally determined by $\xi$. As a consequence of this result, a group that acts on a spacetime admitting an invariant frame must be simply transitive. In a type I spacetime the Weyl tensor defines an invariant orthonormal frame $\{\mathcal{U}_i\}$ of eigenbivectors. Thus, we have the following:

**Proposition 3.** *Let $\{\mathcal{U}_i\}$ be the principal 2-forms of a type I spacetime and $\left\{\Gamma_i^j\right\}$ the associated complex connection 1-forms. If $\xi$ is a Killing field, then the Papapetrou field becomes $\nabla\xi = \sum \Omega_i \mathcal{U}_i + \sum \tilde{\Omega}_i \tilde{\mathcal{U}}_i$ where, for every cyclic permutation,*

$$\Omega_i \equiv -(\nabla\xi, \mathcal{U}_i) = -\frac{\mathrm{i}}{\sqrt{2}}\left(\xi, \Gamma_j^k\right). \tag{18}$$

In order to clarify what kind of alignment between the Killing 2-form and the Weyl tensor is analysed in this work we give the following:

**Definition 2.** *We say that a Papapetrou field $\nabla\xi$ is aligned with a bivector $\mathcal{U}$ if both 2-forms have the same principal 2-planes, that is, $\nabla\xi = \Omega\mathcal{U} + \tilde{\Omega}\tilde{\mathcal{U}}$. We say that a Papapetrou field $\nabla\xi$ is aligned (with the Weyl tensor) if it is aligned with a Weyl principal bivector.*

After this definition, a corollary immediately follows from proposition 3:

**Corollary 1.** *Let $\xi$ be a (real) Killing field in a type I spacetime, and let $\left\{\Gamma_i^j\right\}$ be the (complex) connection 1-forms associated with the principal bivectors of the Weyl tensor. The necessary and sufficient condition for the Papapetrou field $\nabla\xi$ to be aligned with the principal bivector $\mathcal{U}_i$ ($\nabla\xi = \Omega\mathcal{U}_i + \tilde{\Omega}\tilde{\mathcal{U}}_i$) is for $\xi$ to be orthogonal to the two complex connection 1-forms $\Gamma_i^j$.*

This result is independent of the Ricci tensor. Now we will analyse the case of vacuum solutions in detail.

In a type I spacetime, the connection 1-forms associated with the principal bivectors of the Weyl tensor must also be $\xi$-invariant. This condition means that $\mathrm{d}\left(\xi, \Gamma_i^j\right) = -i(\xi)\,\mathrm{d}\Gamma_i^j$. Moreover, if the Ricci tensor is zero, we can use the second structure equations (5) with $Q = 0$ to substitute the differential of the connection 1-forms and we obtain that every Killing field $\xi$ must satisfy

$$\mathrm{d}\Omega_i = \Omega_j \Gamma_i^j + \alpha_i \mathcal{U}_i(\xi). \tag{19}$$

Thus, if the Killing 2-form $\nabla\xi$ is aligned with a principal bivector, let us say $\mathcal{U}_3$, $\xi$ must be orthogonal to $\Gamma_3^1$ and $\Gamma_3^2$ and so $\Omega_1 = 0 = \Omega_2$. In this particular case, equations (19) become

$$\mathrm{d}\Omega_3 = \alpha_3 \mathcal{U}_3(\xi), \qquad \Omega_3\Gamma_2^3 = -\alpha_2\mathcal{U}_2(\xi), \qquad \Omega_3\Gamma_1^3 = -\alpha_1\mathcal{U}_1(\xi). \tag{20}$$

Taking into account that $\mathcal{U}_i^2 = \frac{1}{2}g$, from the second of the equations above we obtain $\xi = 2\frac{\Omega_3}{\alpha_2}\mathcal{U}_2\left(\Gamma_3^2\right)$. So, it follows that if $\xi_1$ and $\xi_2$ are two Killing fields which are orthogonal to the same pair of connection 1-forms, then $\xi_1 \wedge \xi_2 = 0$. This result can be stated as

**Proposition 4.** *Let $\{\mathcal{U}_i\}$ be the principal bivectors of a type I vacuum solution. Then, for every $\mathcal{U}_i$, there is at most, one (real) Killing field $\xi_i$ such that its associated Papapetrou field $\nabla\xi_i$ is aligned with $\mathcal{U}_i$.*

Equations (20) can be written in the equivalent form

$$\xi = \frac{2}{\alpha_3}\mathcal{U}_3(\mathrm{d}\Omega_3), \qquad \Gamma_3^2 = \mathrm{i}\sqrt{2}\frac{\alpha_2}{\alpha_3}\mathcal{U}_1(\mathrm{d}\ln\Omega_3), \qquad \Gamma_1^3 = \mathrm{i}\sqrt{2}\frac{\alpha_1}{\alpha_3}\mathcal{U}_2(\mathrm{d}\ln\Omega_3). \tag{21}$$

Taking into account these expressions and (7), from Bianchi identities (13) we obtain

$$\mathrm{d}(\alpha_3)^2 = 4\left(\alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2\right)\mathrm{d}\ln\Omega_3. \tag{22}$$

So $\mathrm{d}\alpha_3 \wedge \mathrm{d}\ln\Omega_3 = 0$, and if we differentiate (22) we have $(2\alpha_2 + \alpha_3)\,\mathrm{d}\alpha_2 \wedge \mathrm{d}\ln\Omega_3 = 0$. Then, as $(2\alpha_2 + \alpha_3) \neq 0$ we conclude that

$$\mathrm{d}\alpha_2 \wedge \mathrm{d}\ln\Omega_3 = 0. \tag{23}$$

Moreover, if $\alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2 = 0$, from (22) we have $\mathrm{d}\alpha_3 = 0$ and so $\mathrm{d}\alpha_2 = 0$. Thus, the eigenvalues are constant and the Bianchi identities lead to $\lambda_i \wedge \lambda_j = 0$. So, the spacetime belongs to class $\mathrm{I}_1$. On the other hand, if $\alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2 \neq 0$, from (22) and (23) we have $\mathrm{d}\alpha_2 \wedge \mathrm{d}\alpha_3 = 0$. But (21) and (13) imply

$$\mathrm{d}\alpha_2 \wedge \mathrm{d}\alpha_3 = \frac{2}{\alpha_1}(\alpha_1 - \alpha_2)\big(\alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2\big)\mathcal{U}_2\big(\Gamma_1^3\big) \wedge \mathcal{U}_3\big(\Gamma_1^2\big)$$

and so, it follows that $\mathcal{U}_2\big(\Gamma_1^3\big) \wedge \mathcal{U}_3\big(\Gamma_1^2\big) = 0$. From expressions (7), this last condition states that $\lambda_i \wedge \lambda_j = 0$, and so we have established:

**Theorem 2.** *A vacuum type I spacetime which admits a Killing field with an aligned Papapetrou field belongs to class $I_1$.*

In the following sections we study the symmetries of the class $\mathrm{I}_1$ spacetimes and will show that they admit more than one Killing field. Consequently, a unique symmetry with an aligned Papapetrou field implies that other symmetries exist. More precisely, from theorem 2 above and theorem 3 and proposition 12 that we will show in following sections, we can state:

**Corollary 2.** *A vacuum type I spacetime which admits a Killing field with an aligned Papapetrou field admits, at least, a three-dimensional group of isometries.*

## 5. Type $\mathrm{I}_1$ vacuum solutions: basic properties and classification

In this section we analyse some basic properties of type $\mathrm{I}_1$ vacuum metrics which lead us to a natural subclassification. Afterwards, in the following sections, we study the symmetries of these spacetimes. We shall start our analysis of type $\mathrm{I}_1$ vacuum metrics for the case of solutions having a non-constant eigenvalue $\alpha_1$. The case of all the eigenvalues being constant will be dealt with in the last section.

Type $\mathrm{I}_1$ vacuum solutions satisfy $\lambda_i \wedge \lambda_j = 0$. Then, if $\mathrm{d}\alpha_1 \neq 0$, Bianchi identities (13) show $\mathrm{d}\alpha_1 \wedge \lambda_i = 0$. Now expressions (8) say that three functions $\gamma_i$ exist such that

$$\Gamma_i^j = \gamma_k \epsilon_{ijk}\, \mathcal{U}_k(\mathrm{d}\alpha_1). \tag{24}$$

In this case, the second structure equations (5) can be written as

$$\mathrm{d}\mathcal{U}_i(\mathrm{d}\alpha_1) = -\mathrm{d}\ln\gamma_i \wedge \mathcal{U}_i(\mathrm{d}\alpha_1) - \frac{\gamma_j\gamma_k}{\gamma_i}\mathcal{U}_j(\mathrm{d}\alpha_1) \wedge \mathcal{U}_k(\mathrm{d}\alpha_1) + \mathrm{i}\sqrt{2}\frac{\alpha_i}{\gamma_i}\mathcal{U}_i \tag{25}$$

for every cyclic permutation $i, j, k$. In terms of these functions $\gamma_i$, the Bianchi identities (13) become

$$\begin{aligned}
\mathrm{d}\alpha_1 &= \frac{1}{\sqrt{2}}(\gamma_3(\alpha_1 - \alpha_2) + \gamma_2(\alpha_1 - \alpha_3))\,\mathrm{d}\alpha_1 \\
\mathrm{d}\alpha_2 &= \frac{1}{\sqrt{2}}(\gamma_3(\alpha_2 - \alpha_1) + \gamma_1(\alpha_2 - \alpha_3))\,\mathrm{d}\alpha_1.
\end{aligned} \tag{26}$$

The second equation above says that $\alpha_2$ depends on $\alpha_1$. Now we will prove that the functions $\gamma_k$ depend on $\alpha_1$ too. From equations (24) we have $\Gamma_1^2 = \gamma_3\mathcal{U}_3(\mathrm{d}\alpha_1)$, and we can calculate $\nabla\Gamma_1^2$ and make use of (4) to substitute $\nabla\mathcal{U}_3$. Thus, we obtain

$$\nabla\Gamma_1^2 = \mathrm{d}\ln\gamma_3 \otimes \Gamma_1^2 - \gamma_3(\nabla\mathrm{d}\alpha_1) \times \mathcal{U}_3 - \frac{\gamma_3}{\gamma_1}\Gamma_1^3 \otimes \Gamma_2^3 + \frac{\gamma_3}{\gamma_2}\Gamma_2^3 \otimes \Gamma_1^3.$$

Contracting this equation with $\mathcal{U}_1$ and $\mathcal{U}_2$ and taking into account (24) and that $\nabla\mathrm{d}\alpha_1$ is symmetric, we get

$$\mathcal{U}_j\big(\Gamma_1^2, \mathrm{d}\ln\gamma_3\big) = -\big(\mathcal{U}_j, \mathrm{d}\Gamma_1^2\big) = 0 \qquad j = 1, 2,$$

where, in the last equality, we have used the second structure equations (25) and expressions (24). So we find that $d\gamma_3$ is orthogonal to $\mathcal{U}_1(\Gamma_1^2)$ and $\mathcal{U}_2(\Gamma_1^2)$, or from (24), $d\gamma_3$ is orthogonal to $\mathcal{U}_1(d\alpha_1)$ and $\mathcal{U}_2(d\alpha_1)$. If we repeat this calculation replacing $\Gamma_1^2$ by $\Gamma_1^3$, we also see that $d\gamma_2$ is orthogonal to $\mathcal{U}_3(d\alpha_1)$ and $\mathcal{U}_1(d\alpha_1)$. According to (26) $\gamma_3$ depends on $\alpha_1$ and $\gamma_2$, so we have that $d\gamma_3$ is orthogonal to every $\mathcal{U}_j(d\alpha_1)$. Equations (26) say that the same applies for $d\gamma_2$ and $d\gamma_1$. Thus we have established that, for all $i$, $j$,

$$(d\gamma_i, \mathcal{U}_j(d\alpha_1)) = 0. \tag{27}$$

If $d\alpha_1$ is not a null vector, $\{d\alpha_1, \mathcal{U}_i(d\alpha_1)\}$ is an orthogonal frame and (27) proves that $d\gamma_j \wedge d\alpha_1 = 0$. So, we can state:

**Lemma 4.** *Let $g$ be a class $I_1$ vacuum solution with the Weyl tensor having an eigenvalue $\alpha_1$ such that $(d\alpha_1)^2 \neq 0$. Then the functions $\gamma_j$ given in (24) satisfy $d\gamma_j \wedge d\alpha_1 = 0$.*

On the other hand, the second structure equations (25) give us the differential of the 1-forms $\mathcal{U}_i(d\alpha_1)$. If we consider an arbitrary (complex) vector field $\chi$ and the 1-forms $\{\mathcal{U}_i(\chi)\}$, expressions (1) and (4) can be used to compute the Lie derivative $\mathcal{L}_\chi g$ in terms of the exterior differentials $\{d\mathcal{U}_i(\chi)\}$ and the connection coefficients associated with $\{\mathcal{U}_i\}$. More precisely, we have:

$$\mathcal{L}_\chi g \equiv (\nabla\chi + {}^t\nabla\chi) = 2(\mathcal{U}_1 \times \Lambda_1 + \Lambda_2 \times \mathcal{U}_2 + i\sqrt{2}\,\mathcal{U}_1 \times \Lambda_3 \times \mathcal{U}_2), \tag{28}$$

where $(\Lambda_i)_{\alpha\beta} = -d(\mathcal{U}_i(\chi))_{\alpha\beta} - \chi^\epsilon\left(\nabla_\alpha(\mathcal{U}_i)_{\epsilon\beta} - \nabla_\beta(\mathcal{U}_i)_{\epsilon\alpha}\right)$. As $\nabla\, d\alpha_1$ is a symmetric tensor and $d(d\alpha_1)^2 = 2(\nabla d\alpha_1, d\alpha_1)$, we can use expression (28) with $\chi = d\alpha_1$ and the second structure equations (25) to compute $d(d\alpha_1)^2$. Thus we obtain

$$\begin{aligned} d(d\alpha_1)^2 = &-\frac{1}{\gamma_3}(d\alpha_1)^2\left(d\gamma_3 + \frac{1}{2}(\gamma_1\gamma_2 - \gamma_2\gamma_3 - \gamma_1\gamma_3)\, d\alpha_1\right) \\ &- (d\ln\gamma_1, d\alpha_1)\, d\alpha_1 + i\sqrt{2}\left(\frac{\alpha_3}{\gamma_3} - \frac{\alpha_1}{\gamma_1} - \frac{\alpha_2}{\gamma_2}\right) d\alpha_1. \end{aligned} \tag{29}$$

If $d\alpha_1$ is not a null vector lemma 4 applies and $d\gamma_3 \wedge d\alpha_1 = 0$. Then, from (29) we have:

**Lemma 5.** *Let $g$ be a class $I_1$ vacuum solution with the Weyl tensor having an eigenvalue $\alpha_1$ such that $(d\alpha_1)^2 \neq 0$. Then the function $(d\alpha_1)^2$ satisfies $d(d\alpha_1)^2 \wedge d\alpha_1 = 0$.*

Finally we will prove now that $d\alpha_1$ cannot be a null vector. Let us suppose $(d\alpha_1)^2 = 0$. Then, from lemma 1, $d\alpha_1$ would be a combination of the vectors $\mathcal{U}_i(d\alpha_1)$, and from (27) we have

$$(d\alpha_1, d\gamma_i) = 0. \tag{30}$$

Using (7) to express $\lambda_i$ in terms of $\gamma_i$ and $d\alpha_1$, the equations (9) for the vacuum case can be written as

$$-\frac{i}{\sqrt{2}}(d(\gamma_j + \gamma_k), d\alpha_1) - \frac{i}{\sqrt{2}}(\gamma_j + \gamma_k)\Delta\alpha_1 + \gamma_j\gamma_k(d\alpha_1)^2 = 2\alpha_i$$

for every cyclic permutation of $i, j, k$. Now, if $(d\alpha_1)^2 = 0$ and taking into account (30), we have $(\gamma_j + \gamma_k)\Delta\alpha_1 = i\,2\sqrt{2}\alpha_i$. Solving these equations we obtain

$$\gamma_2 = \frac{\alpha_2}{\alpha_3}\gamma_3, \qquad \gamma_1 = \frac{\alpha_1}{\alpha_3}\gamma_3. \tag{31}$$

On the other hand, taking into account (30) and $(d\alpha_1)^2 = 0$, equation (29) says that $\frac{\alpha_3}{\gamma_3} - \frac{\alpha_1}{\gamma_1} - \frac{\alpha_2}{\gamma_2} = 0$. But from (31) we arrive at $\alpha_3 = 0$, which is not compatible with the vacuum condition as a consequence of the Szekeres–Brans theorem. Therefore, $\mathrm{d}\alpha_1$ cannot be a null vector. We summarize these results and lemmas 4 and 5 in the following

**Proposition 5.** *Let $g$ be a class $I_1$ vacuum solution with a non-constant Weyl eigenvalue $\alpha_1$. Then, it holds:*

*(i)* $(\mathrm{d}\alpha_1)^2 \neq 0$
*(ii)* $\mathrm{d}(\mathrm{d}\alpha_1)^2 \wedge \mathrm{d}\alpha_1 = 0$
*(iii)* $\mathrm{d}\gamma_k \wedge \mathrm{d}\alpha_1 = 0$ *where $\gamma_k$ are the functions given in (24).*

The properties of the vacuum solutions of class $\mathrm{I}_1$ summarized in proposition 5 give us the basic elements to analyse this family of metrics when a non-constant eigenvalue $\alpha_1$ exists. Indeed, $\mathrm{d}\alpha_1$ being a non-null vector, we can use (3) with $x = \mathrm{d}\alpha_1$ to eliminate $\mathcal{U}_i$ from the second structure equations (25) and we obtain that those equations become

$$\mathrm{d}\mathcal{U}_i(\mathrm{d}\alpha_1) = \bar{\mu}_i(\alpha_1)\,\mathrm{d}\alpha_1 \wedge \mathcal{U}_i(\mathrm{d}\alpha_1) + \bar{\nu}_i(\alpha_1)\,\mathcal{U}_j(\mathrm{d}\alpha_1) \wedge \mathcal{U}_k(\mathrm{d}\alpha_1), \tag{32}$$

where the functions $\bar{\mu}_i$ and $\bar{\nu}_i$ are given by

$$\bar{\mu}_i = \bar{\mu}_i(\alpha_1) \equiv -\frac{\mathrm{d}\ln\gamma_i}{\mathrm{d}\alpha_1} + \frac{\sqrt{2}\alpha_i}{\gamma_i(\mathrm{d}\alpha_1)^2}, \qquad \bar{\nu}_i = \bar{\nu}_i(\alpha_1) \equiv -\mathrm{i}\left(\frac{2\alpha_i}{\gamma_i(\mathrm{d}\alpha_1)^2} + \frac{\gamma_j\gamma_k}{\gamma_i}\right).$$

If $\mathcal{U}_i(\mathrm{d}\alpha_1)$ satisfy the second structure equations (32), we can use (2) to find the metric tensor as

$$g = \frac{1}{(\mathrm{d}\alpha_1)^2}\left(\mathrm{d}\alpha_1 \otimes \mathrm{d}\alpha_1 - 2\sum \mathcal{U}_i(\mathrm{d}\alpha_1) \otimes \mathcal{U}_i(\mathrm{d}\alpha_1)\right). \tag{33}$$

This seems a hard task, not because of the procedure, but because real coordinates must be adapted to the complex 1-forms $\mathcal{U}_i(\mathrm{d}\alpha_1)$. In this work we do not go on to the explicit integration of the vacuum Einstein equation, but some results of this kind will be presented elsewhere [32]. At this point, it is clear that the integration of the system (32) and, consequently, the gravitational field which is a solution of it, depends strongly on the number of the $\mathcal{U}_i(\mathrm{d}\alpha_1)$ that are integrable 1-forms. We will see in the next section that this condition determines the group of isometries of the spacetime. So it seems suitable to give an invariant classification of type $\mathrm{I}_1$ spacetimes that takes into account these restrictions:

**Definition 3.** *We will say that a type $I_1$ vacuum metric with $\mathrm{d}\alpha_1 \neq 0$ is of class $I_{1A}$ $(A = 0, 1, 2, 3)$ if there are exactly $A$ integrable 1-forms in the set $\{\mathcal{U}_i(\mathrm{d}\alpha_1)\}$.*

## 6. Class $\mathrm{I}_1$ vacuum solutions with non-constant eigenvalues: symmetries

Let us consider now a type I metric that admits a Weyl eigenvalue $\alpha_1$ with non-null gradient, $(\mathrm{d}\alpha_1)^2 \neq 0$. The orthogonal frame $\{\mathrm{d}\alpha_1, \mathcal{U}_i(\mathrm{d}\alpha_1)\}$ is built up with invariants and so these 1-forms and their square $(\mathrm{d}\alpha_1)^2$ must be invariant with respect to every Killing field $\xi$. On the other hand, if $\xi$ is a vector field such that $\mathrm{d}\alpha_1, \mathcal{U}_i(\mathrm{d}\alpha_1)$ and $(\mathrm{d}\alpha_1)^2$ are $\xi$-invariant, then it must be a Killing field because of (33). Thus, we arrive at the following result:

**Lemma 6.** *Let $g$ be a type I metric such that an eigenvalue of the Weyl tensor exists satisfying $(\mathrm{d}\alpha_1)^2 \neq 0$, and let $\mathcal{U}_i$ be the principal bivectors. Then, $\xi$ is a Killing field if, and only if, it satisfies*

$$\mathcal{L}_\xi\,\mathrm{d}\alpha_1 = \mathcal{L}_\xi\,\mathcal{U}_i(\mathrm{d}\alpha_1) = \mathcal{L}_\xi(\mathrm{d}\alpha_1)^2 = 0.$$

In the case of a vacuum class $\mathrm{I}_1$ spacetime with a non-constant eigenvalue $\alpha_1$, the function $(\mathrm{d}\alpha_1)^2 \neq 0$ depends on $\alpha_1$ as proposition 5 states. So, the vector fields orthogonal to $\mathrm{d}\alpha_1$ are those that leave invariant the scalar $(\mathrm{d}\alpha_1)^2$. But as the space orthogonal to $\mathrm{d}\alpha_1$ is generated by the vectors $\mathcal{U}_i(\mathrm{d}\alpha_1)$, every Killing field $\xi$ must be a combination of them. Then, lemma 6 can be stated for the $\mathrm{I}_1$ case as:

**Proposition 6.** *Let g be a class $I_1$ vacuum metric such that* $\mathrm{d}\alpha_1 \neq 0$. *A vector $\xi$ is a Killing field if, and only if, it satisfies:*

*(i)* $\xi \wedge \mathcal{U}_1(\mathrm{d}\alpha_1) \wedge \mathcal{U}_2(\mathrm{d}\alpha_1) \wedge \mathcal{U}_3(\mathrm{d}\alpha_1) = 0$.
*(ii)* $\mathcal{L}_\xi \mathcal{U}_i(\mathrm{d}\alpha_1) = 0$.

If we consider three functions depending on $\alpha_1$, $m_i(\alpha_1)$, then the 1-forms $M_i = m_i(\alpha_1)\mathcal{U}_i(\mathrm{d}\alpha_1)$ also satisfy the conditions (i) and (ii) of proposition 6. Moreover $m_i(\alpha_1)$ exist such that the equations (32) can be written for the 1-forms $M_i = m_i(\alpha_1)\mathcal{U}_i(\mathrm{d}\alpha_1)$ as the exterior system

$$\mathrm{d}M_1 = \delta_1 M_2 \wedge M_3, \qquad \mathrm{d}M_2 = \delta_2 M_3 \wedge M_1, \qquad \mathrm{d}M_3 = \delta_3 M_2 \wedge M_1, \tag{34}$$

where $\delta_i$ takes the value 0 if $\mathcal{U}_i(\mathrm{d}\alpha_1)$ is integrable and 1 if this does not hold. Thus, the 1-forms $M_i = m_i(\alpha_1)\mathcal{U}_i(\mathrm{d}\alpha_1)$ can be considered as the dual 1-forms of the reciprocal group of a transitive group $G_3$ of isometries. Moreover, the (complex) Bianchi type will depend on the integrable character of every $\mathcal{U}_1(\mathrm{d}\alpha_1)$. Thus, taking into account definition 3 we have:

**Theorem 3.** *The class $I_1$ vacuum solutions with a non-constant Weyl eigenvalue admit a $G_3$ group of isometries. The Bianchi type depends on the subclasses $I_{1A}$.*

These results allow us to analyse in detail the Bianchi type of every class $\mathrm{I}_{1A}$ and to study when a Killing 2-form can be aligned with the Weyl tensor.

### *6.1. Class $I_{10}$ vacuum solutions*

If none of the directions $M_i$ is vorticity free, we can choose every $\delta_i$ of (34) to take value 1. So, complex coordinates exist such that the 1-forms $M_i$ take the canonical expression of the reciprocal group of the (complex) Bianchi type VIII that corresponds to the real types VIII and IX [31]. But the system can also be integrated in complex coordinates to get

$$\begin{aligned} M_1 &= -\frac{\mathrm{i}}{2}[\mathrm{e}^x\,\mathrm{d}z + \mathrm{e}^{-x}(2\,\mathrm{d}y - y^2\,\mathrm{d}z)] \\ M_2 &= \frac{1}{2}[\mathrm{e}^x\,\mathrm{d}z - \mathrm{e}^{-x}(2\,\mathrm{d}y - y^2\,\mathrm{d}z)] \\ M_3 &= \mathrm{i}(\mathrm{d}x - y\,\mathrm{d}z). \end{aligned} \tag{35}$$

In this coordinate system the (complex) Killing fields can be expressed as

$$\xi = (k_2 + 2k_1 z)\partial_x + (y(2k_1 + k_2) - 2k_1)\partial_y + (k_1 z^2 + k_2 z + k_3)\partial_z. \tag{36}$$

To see if an aligned Killing 2-form exists we must impose a Killing field to be orthogonal to two connection 1-forms. As every connection 1-form takes the direction of one of the $M_i$, we must see if there is a Killing field which is orthogonal to two of the 1-forms $M_i$. But from the general expression of a Killing field (36) if $\xi$ is orthogonal to $M_3$ then $\xi = 0$. The only possibility is $\xi$ to be orthogonal to $M_1$ and $M_2$. But as $\mathrm{e}^x$ and $\mathrm{e}^{-x}$ are independent, we also obtain $\xi = 0$. So, we can conclude:

**Proposition 7.** *Every vacuum solution of class $I_{10}$ admits a $G_3$ of Bianchi type VIII or IX. In such a spacetime there is no Killing 2-form aligned with the Weyl tensor.*

### 6.2. *Class $I_{11}$ vacuum solutions*

Let us suppose that one of the 1-forms $M_i$, say $M_2$, is integrable. Then, the exterior system (34) becomes the system satisfied by the dual 1-forms of the reciprocal group of the (complex) Bianchi type VI which corresponds to the real types VI or VII. Both cases can be integrated at once in complex coordinates $\{x, y, z\}$ to obtain

$$M_1 + M_3 = \mathrm{e}^{-z}\,\mathrm{d}x, \qquad M_1 - M_3 = \mathrm{e}^{z}\,\mathrm{d}y, \qquad M_2 = \mathrm{d}z. \tag{37}$$

In this coordinate system, the field

$$\xi = (k_1 + k_3 x)\partial_x + (k_2 - k_3 y)\partial_y + k_3\partial_z \tag{38}$$

is a (complex) Killing field for arbitrary values of the constants $k_i$.

To see if there is a Killing field with an aligned Killing 2-form, we must impose a Killing field to be orthogonal to two of the 1-forms $M_i$. A straightforward calculation shows that in this case

$$(\xi, M_2) = k_3$$
$$(\xi, M_1) = (k_1 + k_3 x)\,\mathrm{e}^{-z} + (k_2 - k_3 y)\,\mathrm{e}^{z}$$
$$(\xi, M_3) = (k_1 + k_3 x)\,\mathrm{e}^{-z} - (k_2 - k_3 y)\,\mathrm{e}^{z}.$$

From here it is easy to show that there is no Killing field that is orthogonal to two of the connection 1-forms. So, we have shown:

**Proposition 8.** *Every vacuum solution of class $I_{11}$ admits a $G_3$ of Bianchi type VI or VII. In such spacetime there is no Killing 2-form aligned with the Weyl tensor.*

### 6.3. *Class $I_{12}$ vacuum solutions*

Let us suppose that $M_1$ and $M_2$ are integrable. The exterior system (34) can be easily integrated in complex coordinates to get

$$M_1 = \mathrm{d}x, \qquad M_2 = \mathrm{d}y, \qquad M_3 = -x\,\mathrm{d}y + \mathrm{d}z \tag{39}$$

which correspond to the reciprocal group of a Bianchi type II. Moreover, for arbitrary values of the constants $k_i$ the field

$$\xi = k_1\partial_x + k_2\partial_y + (k_1 y + k_3)\partial_z$$

is a (complex) Killing field.

A straightforward calculation shows that the only Killing field that is orthogonal to two connection 1-forms is $\xi = \partial_z$. But as $z$ is a complex coordinate we cannot conclude that it defines a unique real Killing field and so we cannot ensure that an aligned (real) Killing 2-form exists. Thus, at this point we can state:

**Proposition 9.** *Every vacuum solution of class $I_{12}$ admits a $G_3$ of Bianchi type II. If $\mathcal{U}_j\,(\mathrm{d}\alpha_1)$ is the unique non-integrable 1-form, then $\mathcal{U}_j$ is the only principal bivector that could be aligned with a Killing 2-form.*

### 6.4. *Class $I_{13}$ vacuum solutions*

If all of the 1-forms $M_i$ are integrable, the system (34) says that three complex functions $\{x_i\}$ exist, such that

$$M_i = \mathrm{d}x_i. \tag{40}$$

This corresponds to a commutative group $G_3$ of isometries.

Moreover, for every pair of connection 1-forms a complex Killing field exits such that its Killing 2-form is aligned with it. But as happened before we cannot conclude here that a real aligned Killing 2-form exists. At this point we can state:

**Proposition 10.** *Every vacuum solution of class $I_{13}$ admits a $G_3$ of Bianchi type I. Every principal bivector $\mathcal{U}_i$ could be aligned with a Killing 2-form.*

## 7. Type I vacuum solutions with constant eigenvalues

If a type I metric satisfies $\mathrm{d}\alpha_i = 0$, the Bianchi identities (13) imply

$$\lambda_2 = \frac{(2\alpha_1 + \alpha_2)^2}{(\alpha_1 + 2\alpha_2)^2}\lambda_1, \qquad \lambda_3 = \frac{(\alpha_1 - \alpha_2)^2}{(\alpha_1 + 2\alpha_2)^2}\lambda_1 \tag{41}$$

and so we have that the metric always belongs to class $\mathrm{I}_1$. This fact can be stated as

**Proposition 11.** *Every type I spacetime with vanishing Cotton tensor and constant Weyl eigenvalues is of class $I_1$.*

Using (8), conditions (41) can be stated in terms of the connection 1-forms as

$$\Gamma_1^3 = -\mathrm{i}\sqrt{2}\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}\mathcal{U}_1\big(\Gamma_1^2\big), \qquad \Gamma_2^3 = -\mathrm{i}\sqrt{2}\frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}\mathcal{U}_2\big(\Gamma_1^2\big). \tag{42}$$

Putting (41) in (9) with $Q = 0$, we obtain

$$\alpha_1{}^2 + \alpha_1\alpha_2 + \alpha_2{}^2 = 0, \qquad (\lambda_1, \lambda_1) = -\frac{\alpha_1}{4}, \qquad \nabla\cdot\lambda_1 = 0. \tag{43}$$

The first of these equations states that $\mathrm{tr}\,\mathcal{W}^2 = 0$ and so, the four Debever vectors define a symmetric frame as was established in [4]. Moreover, as $(\lambda_1, \lambda_1) = -\frac{\alpha_1}{4}$ and using (7) we deduce

$$\big(\Gamma_1^2\big)^2 = -\frac{\alpha_2{}^2}{\alpha_1} = -\alpha_3$$

where the first expression of (43) has been used to eliminate $\alpha_2^2$. So, $\big(\Gamma_1^2\big)^2 \neq 0$, and we can consider the orthogonal frame $\{\Gamma_1^2, \mathcal{U}_i\big(\Gamma_1^2\big)\}$, where every 1-form has, up to a factor 2, the same constant modulus $\sqrt{-\alpha_3}$. Taking into account (42), the second structure equations (5) with $Q = 0$ just become

$$\begin{aligned}
&\mathrm{d}\Gamma_1^2 = \mathrm{i}\sqrt{2}\,\Gamma_1^2 \wedge \mathcal{U}_3\big(\Gamma_1^2\big)\\
&\mathrm{d}\mathcal{U}_1\big(\Gamma_1^2\big) = \mathrm{i}\sqrt{2}\,\frac{\alpha_3}{\alpha_2}\,\mathcal{U}_1\big(\Gamma_1^2\big) \wedge \mathcal{U}_3\big(\Gamma_1^2\big)\\
&\mathrm{d}\mathcal{U}_2\big(\Gamma_1^2\big) = \mathrm{i}\sqrt{2}\,\frac{\alpha_1}{\alpha_2}\,\mathcal{U}_2\big(\Gamma_1^2\big) \wedge \mathcal{U}_3\big(\Gamma_1^2\big).
\end{aligned} \tag{44}$$

As the Weyl eigenvalues are constant, the integrability conditions of (44) state $\mathrm{d}\mathcal{U}_3\big(\Gamma_1^2\big) = 0$. Then, the exterior system can easily be integrated in complex coordinates $\{x, y, z, w\}$ to obtain

$$\begin{aligned}
&\Gamma_1^2 = \exp(-\mathrm{i}\sqrt{2}w)\,\mathrm{d}x, &&\mathcal{U}_1\big(\Gamma_1^2\big) = \exp\left(-\mathrm{i}\sqrt{2}\frac{\alpha_3}{\alpha_2}w\right)\mathrm{d}y\\
&\mathcal{U}_2\big(\Gamma_1^2\big) = \exp\left(-\mathrm{i}\sqrt{2}\frac{\alpha_1}{\alpha_2}w\right)\mathrm{d}z, &&\mathcal{U}_3\big(\Gamma_1^2\big) = \mathrm{d}w.
\end{aligned} \tag{45}$$

The 1-forms (45) define an orthogonal frame built up with invariants. So, $\xi$ is a Killing field if, and only if, it leaves the frame unchanged, $\mathcal{L}_\xi\Gamma_1^2 = 0$, $\mathcal{L}_\xi\,\mathcal{U}_i\big(\Gamma_1^2\big) = 0$. If we write $\xi$ as a linear combination of the coordinate fields $\{\partial_x, \partial_y, \partial_z, \partial_w\}$, we find that for arbitrary values of the constants $k_i$, the fields

$$\xi = k_4\partial_w + (-\mathrm{i}\sqrt{2}k_4 x + k_1)\partial_x + \left(-\mathrm{i}\sqrt{2}\,\frac{\alpha_3}{\alpha_2}k_4 y + k_2\right)\partial_y + \left(-\mathrm{i}\sqrt{2}\,\frac{\alpha_1}{\alpha_2}k_4 z + k_3\right)\partial_z \tag{46}$$

are Killing fields, and so, a $G_4$ exists. Then, the spacetime is the Petrov homogeneous vacuum solution [31, 33]:

**Proposition 12.** *The only type I vacuum solution with Weyl constant eigenvalues is the Petrov solution, and so it admits a four-dimensional group of isometries.*

This proposition provides a *intrinsic* (depending solely on the metric tensor) characterization of the Petrov homogeneous vacuum solution.

Indeed, the Petrov solution can be found as the only one satisfying [33]: (i) vacuum, (ii) existence of a simply transitive group $G_4$ of isometries. The first condition is intrinsic because it imposes a restriction on a metric concomitant, the Ricci tensor. Nevertheless, the second one imposes equations that mix up, in principle, elements other than the metric tensor (Killing vectors of the isometry group). Proposition 12 substitutes this last non-intrinsic condition for an intrinsic one: the Weyl tensor is Petrov type I with constant eigenvalues. Moreover, the characterization can also become *explicit* because the metric concomitants admit known explicit expressions in terms of the metric tensor [35]. More precisely, if we consider that Weyl constant eigenvalues is equivalent to constant Weyl symmetric scalars, we have:

**Theorem 4.** *Let $Ric(g)$ and $\mathcal{W} \equiv \mathcal{W}(g)$ be the Ricci and the self-dual Weyl tensors of a spacetime metric $g$. The necessary and sufficient conditions for $g$ to be the Petrov homogeneous vacuum solution are*

$$Ric(g) = 0, \qquad (\mathrm{tr}\,\mathcal{W}^2)^3 \neq 6(\mathrm{tr}\,\mathcal{W}^3)^2, \qquad \mathrm{d}\,\mathrm{tr}\,\mathcal{W}^2 = \mathrm{d}\,\mathrm{tr}\,\mathcal{W}^3 = 0. \tag{47}$$

If we want to study the alignment of the Killing 2-forms with the Weyl tensor, we can compute the product of the Killing tensors with the connection 1-forms taking into account (45) and (42). Thus, we obtain

$$\left(\xi, \Gamma_1^2\right) = 0 \Longleftrightarrow -\mathrm{i}\sqrt{2}k_4 x + k_1 = 0$$
$$\left(\xi, \Gamma_1^3\right) = 0 \Longleftrightarrow -\mathrm{i}\sqrt{2}\frac{\alpha_3}{\alpha_2}k_4 y + k_2 = 0$$
$$\left(\xi, \Gamma_2^3\right) = 0 \Longleftrightarrow -\mathrm{i}\sqrt{2}\frac{\alpha_1}{\alpha_2}k_4 z + k_3 = 0.$$

So, for every pair of connection 1-forms there is a Killing field which is orthogonal to them. In order to see if an aligned Killing 2-form exists, we should have to prove if the complex Killing field defines only a real one. At this point we can state:

**Proposition 13.** *The Petrov homogeneous vacuum solution could admit a Killing 2-form aligned with every Weyl principal bivector.*

## 8. Concluding remarks

The results in this work show that the alignment of the Papapetrou field associated with the Killing vector of a type I vacuum solution with an isometry imposes strong complementary restrictions on the metric tensor, namely, it admits a $G_3$ or a $G_4$ group of isometries. In contrast, we know [26] that in the type D vacuum case the Kerr–NUT family has a Papapetrou field aligned with the Weyl principal 2-planes, and this family admits only a $G_2$, the minimum group of isometries of a type D vacuum solution.

Our study is based in showing that the type I vacuum metrics with an aligned Papapetrou field belong to the more degenerate class $I_1$ of an invariant classification of type I spacetimes. All these metrics admit at least a group $G_3$ of isometries and the Bianchi type can also be characterized in terms of invariant conditions imposed on the Weyl tensor. The full integration of the vacuum equations is an ongoing study which will be presented elsewhere [32]. We are obtaining some known spatially homogeneous vacuum solutions, such as the Kasner or Taub metrics [31], as well as their counterparts with timelike orbits. Some solutions with timelike orbits which are not orthogonal to a Weyl principal direction have also been found. The explicit expression of these metrics in a coordinate system is necessary in order to complete the results obtained here on the type I metrics with aligned Papapetrou fields.

We have shown here that a type I vacuum metric with Weyl constant eigenvalues admits a group $G_4$ of isometries and, consequently, it is the Petrov homogeneous vacuum solution [33]. This result allows us to give an intrinsic and explicit identification of the Petrov solution in theorem 4. Elsewhere [34] we have pointed out the interest in obtaining a fully intrinsic and explicit characterization of a metric or a family of metrics. We have also explained the role that the covariant determination of the Ricci and Weyl eigenvalues and eigenvectors plays in this task [35]. In a natural sequel to the present paper [32] we will integrate vacuum equations by using a method which allows us to label every solution. In this way we will obtain an intrinsic and explicit algorithm to identify every type I vacuum metric admitting an aligned Papapetrou field.

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