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# The asymptotic behavior of the solutions of the Cauchy problem generated by $\phi$-accretive operators

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**Abstract**

The purpose of this paper is to study the asymptotic behavior of the solutions of certain type of differential inclusions posed in Banach spaces. In particular, we obtain the abstract result on the asymptotic behavior of the solution of the boundary value problem

$$\begin{cases} u_t - \Delta_p(u) + |u|^{\gamma-1}u = f & \text{on } ]0,\infty[\times\Omega, \\ -\frac{\partial u}{\partial\eta} \in \beta(u) & \text{on } [0,\infty[\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega$ is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $f(t,x)$ is a given $L^1$-function on $]0,\infty[\times\Omega$, $\gamma \geqslant 1$ and $1 \leqslant p < \infty$. $\Delta_p$ represents the $p$-Laplacian operator, $\frac{\partial}{\partial\eta}$ is the associated Neumann boundary operator and $\beta$ a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0 \in \beta(0)$.


*Keywords:* Accretive operator; Asymptotic behavior; Nonexpansive semigroup; Stationary point

## 1. Introduction

Let $X$ be a real Banach space. A mapping $A : X \to 2^X$ will be called an operator on $X$. The domain of $A$ is denoted by $D(A)$ and its range by $\mathcal{R}(A)$. An operator $A$ on $X$ is said

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to be *accretive* if the inequality $\|x-y+\lambda(z-w)\| \geqslant \|x-y\|$ holds for all $\lambda \geqslant 0$, $z \in Ax$, and $w \in Ay$. If, in addition, $\mathcal{R}(I+\lambda A)$ is for one, hence for all, $\lambda > 0$, precisely $X$, then $A$ is called $m$-accretive. We say that $A$ satisfies the range condition if $\overline{D(A)} \subset \mathcal{R}(I+\lambda A)$ for all $\lambda > 0$. (See, for instance, [10,17] to find sufficient properties which imply the range condition.) Accretive operators were introduced by F.E. Browder [9] and T. Kato [15] independently. Those accretive operators which are $m$-accretive or satisfy the range condition play an important role in the study of nonlinear semigroups, differential equations in Banach spaces, and fully nonlinear partial differential equations. For example, it is well known that if $X$ is Banach space and $A : D(A) \to 2^X$ is an accretive operator which satisfies the range condition, then the initial value problem of the form

$$u'(t) + A\big(u(t)\big) \ni 0, \quad u(0) = x_0, \tag{1}$$

has a unique integral solution for each $x_0 \in \overline{D(A)}$, which is given by the Crandall–Liggett exponential formula [12]:

$$u(t) := \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n}(x_0).$$

Moreover, the family

$$\mathcal{F} := \big\{S(t) : \overline{D(A)} \to \overline{D(A)}\colon\ t \geqslant 0\big\},$$

where $S(t)x = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}(x)$, is a nonexpansive semigroup.

Concerning the strong convergence of semigroups, Brézis in [7] (see also [8,19]) proved that in Hilbert spaces, if the interior of the stationary points set of the semigroup generated by $-A$ is nonempty then, for each $x \in \overline{D(A)}$, $S(t)x$ converges strongly to a zero of $A$ as $t \to \infty$ (this result has been subsequently extended in [18] and [14]).

On the other hand, in [20] Pazy introduced a general condition on the generator of a semigroup $\mathcal{F}$ in a Hilbert space $H$, which guarantees the strong convergence of $S(t)x$ as $t \to \infty$ for each $x$ in the domain of $\mathcal{F}$.

This convergence condition was subsequently extended by Nevanlinna and Reich [18] in 1979 and recently by Xu [22] to a Banach space setting.

In this paper, we study a special class of accretive operators which have a unique zero and our goal is to show that for this kind of accretive operators the integral solution of problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \\ u(0) = x_0 \end{cases} \tag{2}$$

converges as $t \to \infty$, to the zero of $A$. Moreover, we should mention that, in general, the above results cannot be applied in this case.

## 2. Preliminaries

Throughout this paper we assume that $X$ is a real Banach space and denote by $X^\star$ the dual space of $X$. We define the normalized duality mapping by

$$J(x) := \big\{j \in X^\star\colon\ \langle x, j\rangle = \|x\|^2,\ \|j\| = \|x\|\big\}.$$

Let $\langle y, x\rangle_+ := \max\{\langle y, j\rangle\colon\ j \in J(x)\}$. It is well known that an operator $A$ on $X$ is *accretive* if and only if $\langle u - v, x - y\rangle_+ \geqslant 0$ for all $(x, u), (y, v) \in A$. (We refer the reader to [4,6,11] for background material on accretivity.)

Let $\mathcal{F} = \{T(t): C \to C,\ t \geqslant 0\}$ be a family of self-mappings of $C \subset X$. We recall that $\mathcal{F}$ is said to be a nonexpansive semigroup acting on $C$ if the following conditions are satisfied:

(a) $T(0) := I$, where $I$ is the identity mapping on $C$.
(b) $T(s+t)x = T(s)T(t)x$ for all $s, t \in [0, \infty[$ and $x \in C$.
(c) $\|T(t)x - T(t)y\| \leqslant \|x - y\|$ for all $x, y \in C$ and $t \in [0, \infty[$.
(d) $t \to T(t)x$ is continuous in $t \in [0, \infty[$ for each $x \in C$.

Given $x \in C$, the orbit of $x$ under $\mathcal{F}$ will be the function

$$\gamma : [0, \infty[ \to C \quad \text{defined by } \gamma(t) := T(t)x.$$

Let $A : D(A) \to 2^X$ be an accretive operator with the range condition and $f \in L^1(0, \infty, X)$. If we consider the following initial value problem:

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \\ u(0) = x_0, \end{cases} \tag{3}$$

we say that a continuous function $u : [0, \infty[ \to X$ is an integral solution of (3) if $u(0) = x_0$ and the inequality

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \leqslant 2 \int_s^t \langle f(\tau) - y, u(\tau) - x\rangle_+ \, d\tau$$

holds whenever $0 \leqslant s \leqslant t$, and $(x, y) \in A$.

This concept of solution was introduced by Bénilan, who showed that for each $x_0 \in \overline{D(A)}$ problem (3) has a unique integral solution $u$ such that $u(t) \in \overline{D(A)}$ for all $t$.

The following facts about nonexpansive semigroups can be found in [16].

A continuous function $u : [0, \infty[ \to C$ is called an almost-orbit of $\mathcal{F}$ if

$$\lim_{s \to \infty} \Big( \sup_{t \in [0, \infty[} \|u(t+s) - T(t)u(s)\| \Big) = 0.$$

Of course, every orbit is an almost-orbit.

**Lemma 1** [16]. *Let $X$ be a Banach space and let $\mathcal{F}$ be a nonexpansive semigroup on a subset $C$ of $X$. If $u, v$ are almost-orbits of $\mathcal{F}$, then we have*:

(a) $\lim_{t \to \infty} \|u(t) - v(t)\|$ *exists.*
(b) *If $A$ is an accretive operator in $X$ with the range condition, then the integral solution of the initial value problem*

$$u'(t) + Au(t) \ni f(t), \quad t \geqslant 0, \qquad u(0) = x \in \overline{D(A)},$$

*with $f(.) \in L^1(0, \infty, X)$ is an almost-orbit of the nonexpansive semigroup generated by $-A$.*

## 3. $\phi$-accretive operators

In order to proceed, we shall first give the following definition.

**Definition 2.** Let $X$ be a Banach space, let $\phi : X \to [0, \infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ and which satisfies the following condition:

For every sequence $(x_n)$ in $X$ such that $(\|x_n\|)$ is decreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$.

An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is said to be $\phi$-accretive at zero whenever the inequality

$$\langle u, x - z\rangle_+ \geqslant \phi(x - z), \quad \text{for all } (x, u) \in A, \tag{4}$$

holds.

**Remark 3.** The uniqueness of a zero for an operator $\phi$-accretive at zero is an immediate consequence of (4).

On the other hand, it is an easy consequence of [13, Theorem 8] that every $m$-$\psi$-strongly accretive operator is $\phi$-accretive at zero with $\phi = \psi \circ \|.\|$.

Recall that given a continuous function $\psi : \mathbb{R}^+ \to [0, \infty)$ such that $\psi(0) = 0$ and $\psi(x) > 0$ for $x \neq 0$. An accretive operator $A$ on $X$ is said to be $\psi$-strongly accretive if for each $(x, u), (y, v) \in A$ the inequality

$$\langle u - v, x - y\rangle_+ \geqslant \psi\big(\|x - y\|\big)\|x - y\|$$

holds.

**Proposition 4.** *Let $A : D(A) \to 2^X$ be an $m$-accretive operator on $X$ such that there exists $z \in X$ satisfying expression* (4). *Then $A$ is $\phi$-accretive at zero.*

**Proof.** Since $A$ is $m$-accretive, it is enough to consider the operator $\tilde{A}$ defined by

$$\tilde{A} : D(A) \cup \{z\} \to 2^X,$$

$$x \mapsto \tilde{A}(x) = \begin{cases} A(x), & x \in D(A) \setminus \{z\}, \\ A(z) \cup \{0\}, & x = z \in D(A), \\ 0, & x = z \notin D(A). \end{cases}$$

It is obvious, from expression (4), that $\tilde{A}$ is an accretive operator and therefore $A = \tilde{A}$. $\square$

Our results are stated for operators $\phi$-accretive at zero, which happen to form a wider family of operators than the $\psi$-strongly accretive ones.

**Example 5.** Let $X$ be a Banach space. Consider the following operator on $X$:

$$T : X \to 2^X,$$
$$x \mapsto T(x) = \begin{cases} \frac{x}{\|x\|}, & x \neq 0, \\ B_X, & x = 0, \end{cases}$$

where $B_X$ denotes the unit ball of $X$. It is easy to see that this operator is $m$-accretive on $X$, $\phi$-accretive at zero for $\phi(x) = \|x\|$ but it fails to be $\psi$-expansive for any $\psi$, and hence it cannot be $\psi$-strongly accretive.

**Proposition 6.** *Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $1 < q < \infty$. Consider $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a function such that*

(a) *$g(.,.)$ satisfies Carathéodory's conditions (i.e., the map $\zeta \to g(x, \zeta)$ is continuous for almost all $x$ and the map $x \to g(x, \zeta)$ is measurable for every $\zeta$) and there exist $\lambda > 0$ and $R > 0$ such that $g(x, \zeta)\zeta \geqslant \lambda|\zeta|^2$ whenever $|\zeta| > R$.*
(b) *$g(x, .) : \mathbb{R} \to \mathbb{R}$ is increasing, $g(x, 0) = 0$ and $g(x, \zeta) \neq 0$ whenever $\zeta \neq 0$.*
(c) *The mapping $u \in L^q(\Omega) \to g(x, u(x)) \in L^q(\Omega)$ is well defined.*

*Then the operator $B : L^q(\Omega) \to L^q(\Omega)$ defined by $B(u)(x) := g(x, u(x))$, $x \in \Omega$, is $m$-$\phi$-accretive at zero on $L^q(\Omega)$.*

**Proof.** It is well know that $B$ is an $m$-accretive operator on $L^q(\Omega)$.

Thus, we will only prove that $B$ is $\phi$-accretive at zero. To see this, consider the function

$$\phi(u) = \|u\|_q^{2-q} \int\limits_{\Omega} g\big(x, u(x)\big)u(x)\big|u(x)\big|^{q-2}\,dx.$$

Hence, since (see, for instance, [11]) the normalized duality map on $L^q(\Omega)$ is given by

$$J(u) = \|u\|_q^{2-q}|u|^{q-2}u,$$

we have that

$$\big\langle B(u), u\big\rangle_+ = \phi(u).$$

Having this in mind, it will be sufficient to see that $\phi$ satisfies the condition of Definition 2. It is not difficult to see that $\phi$ satisfies the following:

(i) $\phi(0) = 0$ and $\phi(x) > 0$ whenever $x \neq 0$.
(ii) With respect to the continuity of $\phi$ we argue as follows: Let $(u_n)$ be a sequence in $L^q(\Omega)$ such that converges to $u \in L^q(\Omega)$ in $L^q(\Omega)$. We have to see that $\lim_{n\to\infty} \phi(u_n) = \phi(u)$.

We know that given $(u_k)$ a subsequence of $(u_n)$, there exists $(u_{k_s})$ subsequence of $(u_k)$ such that

(a) $u_{k_s}(x) \to u(x)$ a.e.
(b) $|u_{k_s}(x)| \leqslant h(x)$ for all $s \in \mathbb{N}$ and a.e. with $h \in L^q(\Omega)$.

We can use that $g(.,.)$ satisfies Carathéodory's conditions and thus we derive that

$$g\big(x, u_{k_s}(x)\big)u_{k_s}(x)\big|u_{k_s}(x)\big|^{q-2} \to g\big(x, u(x)\big)u(x)\big|u(x)\big|^{q-2} \quad \text{a.e.}$$

On the other hand, it is clear that

$$\Big|g\big(x, u_{k_s}(x)\big)u_{k_s}(x)\big|u_{k_s}(x)\big|^{q-2}\Big| \leqslant g\big(x, h(x)\big)h(x)\big|h(x)\big|^{q-2} \quad \text{a.e.}$$

Since, by Hölder's inequality, the right-hand side of the above inequality is an integrable function, then the dominated convergence theorem allows us to conclude the continuity of $\phi$.

Now, we only need to show that if $(u_n)$ is a sequence of $L^q(\Omega)$ such that $(\|u_n\|_q)$ is decreasing and $\phi(u_n) \to 0$ as $n \to \infty$, then $\|u_n\|_q \to 0$.

Indeed, suppose

$$\lim_{n\to\infty} \phi(u_n) = \lim_{n\to\infty} \|u_n\|_q^{2-q} \int_\Omega g\big(x, u_n(x)\big)u_n(x)\big|u_n(x)\big|^{q-2}\,dx = 0. \tag{5}$$

Let us notice that (5) implies that either $\|u_n\|_q \to 0$, as $n \to \infty$, or

$$\lim_{n\to\infty} \int_\Omega g\big(x, u_n(x)\big)u_n(x)\big|u_n(x)\big|^{q-2}\,dx = 0. \tag{6}$$

Since clearly the results hold if $\|u_n\|_q \to 0$, we may assume (6). In this case,

$$\|u_n\|_q^q = \int_\Omega |u_n|^q = \int_{|u_n|>R} |u_n|^q + \int_{|u_n|\leqslant R} |u_n|^q,$$

where $R$ is given in the hypothesis of the proposition.

Now, we shall check both terms of the right-hand side of the above equality.

Since $g(x,\zeta)\zeta \geqslant \lambda|\zeta|^2$ whenever $|\zeta| > R$. We have

$$\int_{|u_n|>R} \big|u_n(x)\big|^q \leqslant \frac{1}{\lambda} \int_{|u_n|>R} \big|u_n(x)\big|^{q-2}u_n(x)g\big(x, u_n(x)\big), \tag{7}$$

which, by (6) and (7), means that

$$\lim_{n\to\infty} \int_{|u_n|>R} |u_n|^q = 0.$$

Concerning the other term, we have

$$\int_{|u_n|\leqslant R} |u_n|^q \leqslant \int_{|u_n|\leqslant R} R^{q-1}|u_n| \leqslant R^{q-1} \int_\Omega |u_n|. \tag{8}$$

Thus to obtain the proof it is sufficient to prove that

$$\lim_{n\to\infty} \int_\Omega |u_n| = 0, \tag{9}$$

which is true.

Otherwise, we can find a subsequence $(u_{n_k})$ of $(u_n)$ and a positive number $r > 0$ such that $\lim_{k\to\infty}\int_\Omega |u_{n_k}| = r$. Nevertheless, by (6) and the properties of $g$ it is easy to see that there exists a subsequence $(u_{n_{k_s}})$ of $(u_{n_k})$ such that $u_{n_{k_s}} \to 0$ a.e., moreover, since $(\|u_n\|_q)$ is decreasing, then $(\|u_{n_k}\|_q)$ is bounded, and thus we may apply Vitali's theorem to obtain that $\lim_{s\to\infty}\int_\Omega |u_{n_{k_s}}| = 0$, which is a contradiction. □

**Remark 7.** It is an easy consequence of both Theorem 16.4 of [6] and Definition 2 that if $X$ is a smooth Banach space, $A : D(A) \to 2^X$ an $m$-accretive operator on $X$ such that $0 \in Az$ and $B : X \to X$ is a continuous operator on $X$ with $0 = Bz$ which is $\phi$-accretive at zero. Then $A + B : D(A) \to 2^X$ is an $m$-$\phi$-accretive at zero.

## 4. Strong asymptotic behavior

Let $A : D(A) \to 2^X$ be a $\phi$-accretive operator at zero on $X$ with the range condition. If we consider the initial value problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, & t \in [0,\infty[, \\ u(0) = x_0, \end{cases} \tag{10}$$

it is well known that if $x_0 \in \overline{D(A)}$, then such problem has a unique integral solution. Such solution is given by Crandall–Liggett's formula, so we have that

$$u(t) := S(t)(x_0) = \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n}(x_0).$$

Although a useful method to study the asymptotic behavior of semigroups of nonlinear contractions is the Lyapunov method introduced by Pazy in [21], we will use the ideas developed in [18,22] to obtain the following result. For such result we shall also need the concept of strong solution for problem (10). That is (see [4, p. 110]):

A continuous function $u : [0,\infty[ \to X$ is said to be a strong solution of problem (10) if it is Lipschitz on every bounded sub-intervals of $[0,\infty[$, a.e. differentiable on $\mathbb{R}^+$, $u(0) = x_0$, $u(t) \in D(A)$ a.e., and $u'(t) + A(u(t)) \ni 0$ for almost every $t \in \mathbb{R}^+$.

**Theorem 8.** *Let $X$ be a Banach space, if $A$ is an operator on $X$ $\phi$-accretive at zero with the range condition and such that problem* (10) *has a strong solution for each $x \in D(A)$, and $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} :\ t \geqslant 0\}$ is the nonexpansive semigroup generated by $-A$ via the exponential formula, then every almost-orbit of $\mathcal{F}$ is strongly convergent to the zero of $A$.*

**Proof.** Since $A$ is $\phi$-accretive at zero, then $A$ has a unique zero $z \in D(A)$.

Let $u : [0,\infty[ \to X$ be an almost-orbit of $\mathcal{F}$ and consider the following initial value problem:

$$\begin{cases} w_s'(t) + A(w_s(t)) \ni 0, \\ w_s(0) = u(s). \end{cases} \tag{11}$$

If we assume that $u(s) \in D(A)$ for a fixed $s \geqslant 0$, then the unique solution of problem (11) will be $w_s(t) = S(t)u(s)$ and moreover, by hypothesis, it will be a strong solution. Therefore, there exists $w_s'(t)$ a.e. and moreover it satisfies $-w_s'(t) \in Aw_s(t)$ a.e. Then, there exists $j(t) \in J(w_s(t) - z)$ such that

$$\begin{aligned}\langle -w_s'(t), w_s(t) - z\rangle_+ &= \langle -w_s'(t), j(t)\rangle \\ &= \frac{1}{h}\langle w_s(t-h) - w_s(t), j(t)\rangle + \langle \xi(t,h), j(t)\rangle,\end{aligned}$$

where $\lim_{h\to 0} \xi(t,h) = 0$.

Since $\|j(t)\| = \|w_s(t) - z\|$ an elemental calculus yields:

$$\begin{aligned}\langle w_s(t-h) - w_s(t), j(t)\rangle &= \langle w_s(t-h) - z - w_s(t) + z, j(t)\rangle \\ &= -\|w_s(t) - z\|^2 + \langle w_s(t-h) - z, j(t)\rangle \\ &\leqslant \frac{1}{2}\left(\|w_s(t-h) - z\|^2 - \|w_s(t) - z\|^2\right).\end{aligned}$$

On the other hand, since the mapping $t \to \|w_s(t) - z\|$ is Lipschitzian, it will be also differentiable almost everywhere. Consequently,

$$0 \leqslant \langle -w_s'(t), j(t)\rangle \leqslant -\frac{1}{2}\frac{d}{dt}\|w_s(t) - z\|^2. \tag{12}$$

Moreover, since $t \to \|w_s(t) - z\|$ is decreasing, the function $t \to \frac{1}{2}\frac{d}{dt}\|w_s(t) - z\|^2$ is Lebesgue integrable on $[0,\infty)$. Hence by (12) we know that the function $t \to \langle -w_s'(t), j(t)\rangle$ is also Lebesgue integrable on $[0,\infty)$. Then

$$\liminf_{t\to\infty}\langle -w_s'(t), j(t)\rangle = 0,$$

which means that there exists a sequence $(t_n)$ with $t_n \to \infty$ such that

$$\lim_{n\to\infty}\langle -w_s'(t_n), j(t_n)\rangle = 0. \tag{13}$$

Since $A$ is $\phi$-accretive at zero, we know that

$$\phi(w_s(t) - z) \leqslant \langle -w_s'(t), j(t)\rangle$$

and, since the sequence $(\|w_s(t_n) - z\|)$ is decreasing, by (13) we derive

$$\lim_{n\to\infty}\|w_s(t_n) - z\| = 0.$$

Finally, since the function $t \to \|w_s(t) - z\|$ is decreasing, we conclude that

$$\lim_{t\to\infty}\|w_s(t) - z\| = 0.$$

If we suppose that $u(s) \in \overline{D(A)}$, then there exists a sequence $(x_n) \subseteq D(A)$ such that $x_n \to u(s)$. If we call $u_n(t) = S(t)x_n$, by the above argument we have

$$\lim_{t\to\infty} u_n(t) = z.$$

Now, let us see that $\lim_{t\to\infty}\|w_s(t) - z\| = 0$. Indeed, given $\epsilon > 0$ we know that there exists $n_1 \in \mathbb{N}$ such that

$$\|u(s) - x_{n_1}\| < \frac{\epsilon}{2}.$$

Consequently, taking $t \geqslant t_0$,

$$\begin{aligned}\|w_s(t) - z\| &\leqslant \|w_s(t) - u_{n_1}(t)\| + \|u_{n_1}(t) - z\| \\ &\leqslant \|u(s) - x_{n_1}\| + \|u_{n_1}(t) - z\| \leqslant \epsilon.\end{aligned}$$

The above argument shows that $\lim_{t\to\infty} \|w_s(t) - z\| = 0$ for any fixed $s > 0$.

On the other hand, since $u$ is an almost-orbit of $\mathcal{F}$, we have that

$$\|S(t)u(s) - u(s+t)\| \leqslant \varphi(s) \to 0 \quad \text{as } s \to \infty.$$

Hence,

$$\|u(t+s) - z\| \leqslant \|u(t+s) - S(t)u(s)\| + \|S(t)u(s) - z\| \leqslant \varphi(s) + \|w_s(t) - z\|.$$

Therefore,

$$\limsup_{t\to\infty} \|u(t+s) - z\| \leqslant \varphi(s) \to 0 \quad \text{as } s \to \infty. \tag{14}$$

Now, since $u(.)$ and $z$ are almost-orbits of $\mathcal{F}$, by Lemma 1 we know that $\lim_{t\to\infty} \|u(t) - z\|$ exists.

Consequently, by using this fact and (14) we obtain that

$$\lim_{t\to\infty} \|u(t) - z\| = 0. \qquad \square$$

**Corollary 9.** *Let $X$ be a Banach space. Suppose that $A \subseteq X \times X$ is an $m$-$\psi$-strongly accretive operator on $X$. Suppose that problem* (10) *has a strong solution for each $x \in D(A)$. Then, for each $x \in \overline{D(A)}$, the integral solution $u(.)$ of the problem*

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), & t \in [0, \infty[, \\ u(0) = x, \end{cases} \tag{15}$$

*where $f(.) \in L^1(0, \infty, X)$, converges strongly to the zero of $A$ as $t \to \infty$.*

**Proof.** This corollary is a consequence of Lemma 1 and both [13, Theorem 8], and Theorem 8. $\square$

**Corollary 10.** *Let $X$ be a Banach space with the Radon–Nikodym property* (*RN for short*)*. Suppose that $A \subseteq X \times X$ is an $m$-accretive operator satisfying condition* (4) *for some $z \in X$. Then for each $x \in \overline{D(A)}$ the integral solution $u(.)$ of problem* (15) *converges strongly to $z$ as $t \to \infty$.*

**Proof.** First, we may notice that since $A$ is $m$-accretive then, by Proposition 4, $0 \in A(z)$.

Second, since $X$ has the RN property, then the integral solution of problem (10) is in fact a strong solution whenever the initial data is in $D(A)$ (see [4]).

Third, since $f(.) \in L^1(0, \infty, X)$, by Lemma 1, the integral solution of problem (15) is an almost-orbit of the semigroup generated by $-A$ via Crandall–Liggett.

Finally, we may apply Theorem 8 and thus we obtain the result. $\square$

As an immediate consequence of Proposition 6 and the above corollary, we obtain

**Corollary 11.** *Let $B : L^q(\Omega) \to L^q(\Omega)$ be the operator given in Proposition* 6*. Then, for each $x \in L^q(\Omega)$, the integral solution $u(.)$ of the problem*

$$\begin{cases} u'(t) + B(u(t)) = f(t), & t \in [0, \infty[, \\ u(0) = x, \end{cases} \tag{16}$$

*where $f(.) \in L^1(0, \infty, L^q(\Omega))$, converges strongly in $L^q(\Omega)$ to* 0 *as $t \to \infty$.*

Ph. Bénilan and M.G. Crandall introduce in [5] the concept of completely accretive operator. This class of operators, in the particular case of $L^1(\Omega)$ with $\Omega$ bounded, can be defined as follows: An operator $A \subseteq L^1(\Omega) \times L^1(\Omega)$, is said to be *completely accretive* if one of the following conditions holds:

(i) For $\lambda > 0$, $(u, v), (x, y) \in A$ and $j \in J_0$,

$$\int_\Omega j(u - x) \leqslant \int_\Omega j\big(u - x + \lambda(v - y)\big),$$

where

$$J_0 = \big\{\text{convex lower-semicontinuous maps } j : \mathbb{R} \to [0, \infty] \\ \text{satisfying } j(0) = 0\big\}.$$

(ii) For $(u, v), (x, y) \in A$ and $p \in P_0$,

$$\int_\Omega p(u - x)(v - y) \geqslant 0,$$

where

$$P_0 = \big\{p \in \mathcal{C}^\infty(\mathbb{R}):\ 0 \leqslant p' \leqslant 1,\ \operatorname{supp}(p') \text{ is compact and } 0 \notin \operatorname{supp}(p)\big\}.$$

**Corollary 12.** *Let $\Omega$ be a bounded subset in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the Banach space $X = L^1(\Omega)$. If $A : D(A) \subseteq X \to 2^X$ is $m$-completely accretive and $\phi$-accretive at zero, then, for each $x \in \overline{D(A)}$, the integral solution $u(.)$ of problem* (15) *converges strongly to the zero of $A$.*

**Proof.** This is a consequence of Theorem 8, since in this case the homogeneous problem has a strong solution whenever the initial data belongs to $D(A)$ (see [5, Theorem 4.2]). □

## 5. Application

The present section is devoted to apply the abstract results of the previous sections to a concrete example of an initial value problem for a partial differential equation.

Throughout this section we will assume that $\Omega$ is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. It will be further assumed that $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

(a) For almost all $x \in \Omega$, $r \to \varphi(x,r)$ is continuous and nondecreasing.
(b) For every $r \in \mathbb{R}$, $x \to \varphi(x,r)$ is in $L^1(\Omega)$.
(c) $\varphi(x,0) = 0$, $\varphi(x,r) \neq 0$ whenever $r \neq 0$ and there exist $\lambda > 0$ and $\alpha \geqslant 2$ such that $\varphi(x,r)r \geqslant \lambda |r|^\alpha$.

**Example 13.** The function $\varphi(x,r) = |r|^{\gamma-1}r$ satisfies the above conditions whenever $\gamma \geqslant 1$.

Consider the following nonlinear boundary value problem:

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) + \varphi(x,u) = f & \text{on } ]0,\infty[ \times \Omega, \\ -\frac{\partial u}{\partial \eta} \in \beta(u) & \text{on } [0,\infty[ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{17}$$

where $f(t,x)$ is a given $L^1$-function on $]0,\infty[ \times \Omega$, $1 \leqslant p < \infty$, $\frac{\partial}{\partial \eta}$ is the associated Neumann boundary operator, i.e., $\frac{\partial u}{\partial \eta} = \langle |Du|^{p-2}Du, \eta\rangle$, with $\eta$ the unit outward normal on $\partial\Omega$, $Du$ the gradient of $u$, $\beta$ a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (a)–(c) as above.

In order to obtain the asymptotic behavior of the solution of problem (17), we shall first study a perturbation result on completely accretive operators which will be useful for our goal.

**Proposition 14.** *Let $A$ be an $m$-completely accretive operator in $L^1(\Omega)$ such that* $0 \in A(0)$, *and let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (a) *and* (b) *as above. If we define the single-valued operator $B_\varphi$ in $L^1(\Omega)$ as follows*: $D(B_\varphi) := \{u \in L^1(\Omega)\colon \varphi(.\,,u(.)) \in L^1(\Omega)\}$ *and for every $u \in D(B_\varphi)$, $B_\varphi(u(x)) := \varphi(x,u(x))$, then $A + B_\varphi$ is an $m$-completely accretive operator on $L^1(\Omega)$. Moreover, if $\overline{D(A)} = L^1(\Omega)$ we have that $\overline{D(A+B_\varphi)} = L^1(\Omega)$.*

**Proof.** First, we will prove that $B_\varphi$ is completely accretive on $L^1(\Omega)$.

Indeed, consider $p \in P_0$, and $(u, \varphi(.\,,u(.))), (v, \varphi(.\,,v(.))) \in B_\varphi$.

Since, for almost all $x \in \Omega$, $\varphi(x,.)$ is nondecreasing, we have

$$\bigl(\varphi\bigl(x,u(x)\bigr) - \varphi\bigl(x,v(x)\bigr)\bigr)\bigl(u(x)-v(x)\bigr) \geqslant 0 \quad \text{a.e.} \tag{18}$$

On the other hand, since $p(0) = 0$ and $p$ is nondecreasing we know that $p(x)x \geqslant 0$, hence $p(u(x)-v(x))(u(x)-v(x)) \geqslant 0$ and therefore by (18) it is clear that

$$\bigl(\varphi\bigl(x,u(x)\bigr) - \varphi\bigl(x,v(x)\bigr)\bigr)p\bigl(u(x)-v(x)\bigr) \geqslant 0 \quad \text{a.e.,}$$

thus we may conclude that

$$\int_\Omega p\bigl(u(x)-v(x)\bigr)\bigl(\varphi\bigl(x,u(x)\bigr) - \varphi\bigl(x,v(x)\bigr)\bigr) \geqslant 0.$$

Second, by [5, Corollary 2.4], we have that $A + B_\varphi$ is completely accretive on $L^1(\Omega)$.

Third, since $A$ is $m$-completely accretive and $0 \in A(0)$, then by [5, Proposition 2.2], $A$ satisfies the conditions of [1, Corollary 3.1], and therefore we can conclude that $A + B_\varphi$ is $m$-completely accretive.

Finally, let us see that if $\overline{D(A)} = L^1(\Omega)$ then $\overline{D(A + B_\varphi)} = L^1(\Omega)$. For this purpose, it will be enough to show that $L^\infty(\Omega) \subseteq \overline{D(A + B_\varphi)}$.

For $u \in L^\infty(\Omega)$, since $A + B_\varphi$ is an $m$-accretive operator, given $n \in \mathbb{N}$ there exists $u_n \in D(A + B_\varphi)$ such that $u_n = (I + \frac{1}{n}(A + B_\varphi))^{-1}u$, then $u_n = (I + \frac{1}{n}A)^{-1}(u - \frac{1}{n}B_\varphi(u_n))$. Since $A + B_\varphi$ is completely accretive we know (see [5]) that $\|u_n\|_\infty \leqslant \|u\|_\infty$ therefore $u - \frac{1}{n}B_\varphi(u_n) \to u$ in $L^1(\Omega)$, as $n \to \infty$.

On the other hand, if we denote $J^A_{1/n} := (I + \frac{1}{n}A)^{-1}$, since $\overline{D(A)} = L^1(\Omega)$, there exists a sequence $(s_m)$ in $D(A)$ such that $s_m \to u$ in $L^1(\Omega)$, as $m \to \infty$, hence

$$\begin{aligned}
\|u_n - u\|_1 &= \left\| J^A_{1/n}\left(u - \frac{1}{n}B_\varphi u_n\right) - u \right\|_1 \\
&\leqslant \left\| J^A_{1/n}\left(u - \frac{1}{n}B_\varphi u_n\right) - J^A_{1/n}u \right\|_1 + \left\| J^A_{1/n}u - u \right\|_1 \\
&\leqslant \left\| \frac{1}{n}B_\varphi u_n \right\|_1 + \left\| J^A_{1/n}u - J^A_{1/n}s_m \right\|_1 + \left\| J^A_{1/n}s_m - u \right\|_1 \\
&\leqslant \left\| \frac{1}{n}B_\varphi u_n \right\|_1 + \|u - s_m\|_1 + \left\| J^A_{1/n}s_m - u \right\|_1.
\end{aligned}$$

Consequently

$$\limsup_{n\to\infty} \|u_n - u\|_1 \leqslant 2\|s_m - u\|_1,$$

which means that $u_n \to u$ in $L^1(\Omega)$, as $n \to \infty$. □

**Theorem 15.** *Let $A$ be an $m$-completely accretive operator in $L^1(\Omega)$ such that* $0 \in A(0)$, *and let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying conditions* (a)–(c) *as above. Then $A + B_\varphi$ is an $m$-$\phi$-completely accretive at zero operator on $L^1(\Omega)$.*

**Proof.** It is clear that $0 \in (A + B_\varphi)(0)$. Moreover, since by Proposition 14, $A + B_\varphi$ is $m$-completely accretive on $L^1(\Omega)$, we only have to see that $A + B_\varphi$ is $\phi$-accretive at zero on $L^1(\Omega)$.

Indeed, consider $(x, u) \in A + B_\varphi$, then $u = v + B_\varphi(x)$, where $(x, v) \in A$. Since $0 \in A(0)$ we know that $\langle v - 0, x - 0\rangle_+ \geqslant 0$, which means that there exists

$$j \in J(x) = \|x\|_1 \big\{ j\colon\ j \in L^\infty(\Omega),\ |j| \leqslant 1,\ \text{and } jx = |x| \text{ a.e.} \big\}$$

such that $\langle v, j\rangle \geqslant 0$.

Consequently, Hölder's inequality yields $K > 0$ such that

$$\begin{aligned}
\langle u - 0, x - 0\rangle_+ \geqslant \langle u, j\rangle \geqslant \big\langle B_\varphi(x), j\big\rangle &= \|x\|_1 \int_\Omega \varphi\big(t, x(t)\big) j \\
&= \|x\|_1 \int_{\{t\in\Omega:\ x(t)\neq 0\}} \varphi\big(t, x(t)\big)\frac{x(t)}{|x(t)|} \\
&\geqslant \lambda \|x\|_1 \int_\Omega \big|x(t)\big|^{\alpha-1} \geqslant K\|x\|_1^\alpha.
\end{aligned}$$

Therefore, if we define $\phi(x) = K\|x\|_1^{\alpha}$, we obtain that $A + B_\varphi$ is $\phi$-accretive at zero.

In [2,3] the following problem is studied:

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 & \text{on } ]0,\infty[\times\Omega,\\ -\frac{\partial u}{\partial\eta} \in \beta(u) & \text{on } [0,\infty[\times\partial\Omega,\\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{19}$$

In order to obtain the solution of (19) for $p > 1$ in [2, Theorem 2.1], an $m$-completely accretive operator $A_{\beta,p}$ in $L^1(\Omega)$ is introduced such that $0 = A_{\beta,p}(0)$ and with dense domain. On the other hand, in [3, Theorem 5], it is studied the above problem for $p = 1$ by using an $m$-completely accretive operator $A_{\beta,1}$ such that $0 = A_{\beta,1}(0)$ and with dense domain. Thus the abstract Cauchy problem in $L^1(\Omega)$ corresponding to (19) reads as follows:

$$\begin{cases} u'(t) + A_{\beta,p}u(t) = 0, & 0 < t < \infty,\\ u(0) = u_0. \end{cases} \tag{20}$$

On the other hand, given a function $\varphi$ which satisfies conditions (a)–(c) as above, by Theorem 15 we know that $B := A_{\beta,p} + B_\varphi$, where $D(B) = D(A_{\beta,p}) \cap D(B_\varphi)$, is an $m$-$\phi$-completely accretive at zero operator on $L^1(\Omega)$.

Thus problem (17) may be rewritten as

$$\begin{cases} u'(t) + Bu(t) = f(t), & 0 < t < \infty,\\ u(0) = u_0, \end{cases} \tag{21}$$

where $u(.)$ is regarded as a function from $[0,\infty[$ to $L^1(\Omega)$. $\square$

**Theorem 16.** *If $u_0 \in L^q(\Omega)$, $f \in L^1((0,\infty), L^q(\Omega))$ and $u$ is the integral solution of problem* (21), *then $u(t)$ converges in $L^q(\Omega)$ to* 0, *as $t \to \infty$.*

**Proof.** *Case $q = 1$.* It is clear that the operator $B$ is under the conditions of Theorem 15 and it has dense domain (see Proposition 14), therefore we may apply Corollary 12 to obtain the result.

*Case* $1 < q < \infty$. First, since $\overline{D(B)} = L^1(\Omega)$ and $B$ is $m$-completely accretive by [5, Proposition 3.4], it is clear that $L^q(\Omega) = \overline{D(B)} \cap L^q(\Omega) = \overline{D(B)}^{L^q(\Omega)}$.

Now, we have to notice that if $A$ is an $m$-completely accretive operator on $L^1(\Omega)$ and $1 \leqslant q < \infty$, then the restriction $A_q$ of $A$ to $L^q(\Omega)$ is $m$-accretive on $L^q(\Omega)$ (see [5]).

Therefore, since $B$ is in such conditions, we know that its restriction $B_q$ to $L^q(\Omega)$ is $m$-accretive. Thus, following the argument in the proof of Theorem 15, by Corollary 10 we only need to show that $B_{\varphi,q}$ (it means the restriction of $B_\varphi$ to $L^q(\Omega)$) is $\phi$-accretive at zero in $L^q(\Omega)$.

Given $u \in D(B_{\varphi,q})$, we obtain

$$\begin{aligned} \langle B_{\varphi,q}(u), u\rangle_+ &= \|u\|_q^{2-q} \int_\Omega \varphi\big(x, u(x)\big)u(x)\big|u(x)\big|^{q-2}\,dx\\ &\geqslant \lambda\|u\|_q^{2-q} \int_\Omega \big|u(x)\big|^{q+\alpha-2}\,dx. \end{aligned}$$

Now by Hölder's inequality, we obtain that there exists $K > 0$ such that

$$\langle B_{\varphi,q}(u), u\rangle_+ \geqslant K\|u\|_q^\alpha.$$

Then, it is sufficient to take the function $\phi(x) = K\|x\|_q^\alpha$. $\square$

To finish this section, we are going to study the abstract result on the asymptotic behavior of the solutions of the following problem:

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) + g(x,u) = f & \text{on } ]0,\infty[\, \times \Omega, \\ -\frac{\partial u}{\partial \eta} \in \beta(u) & \text{on } [0,\infty[\, \times \partial\Omega, \\ u(0,x) = u_0 \in L^q(\Omega), \end{cases} \tag{22}$$

where ($1 \leqslant p < \infty$ and $1 < q < \infty$), $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is as in Proposition 6.

Problem (22) can be read as the following abstract Cauchy problem:

$$\begin{cases} u'(t) + (A_{\beta,p,q} + B)u(t) = f(t), \quad 0 < t < \infty, \\ u(0) = u_0, \end{cases} \tag{23}$$

where $A_{\beta,p,q}$ means the restriction of the operator $A_{\beta,p}$ to $L^q(\Omega)$, $B$ is the operator given in Proposition 6 and $f \in L^1((0,\infty), L^q(\Omega))$. From Proposition 6 and Remark 7 it is clear that $A_{\beta,p,q} + B$ is an $m$-$\phi$-accretive at zero operator on $L^q(\Omega)$. Hence since $0 = (A_{\beta,p,q} + B)(0)$, we can apply Corollary 10 and thus we can conclude that the integral solution of problem (23) goes to zero as $t \to \infty$.

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