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Replication of spatial patterns with reversible and additive cellular automata

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Abstract

In this article, the replication of arbitrary patterns by reversible and additive cellular automata is reported. The orbit of an 1D cellular automaton operating on p symbols that is both additive and reversible is explicitly given in terms of coefficients that appear in the theory of Gegenbauer polynomials. It is shown that if p is an odd prime, the pattern formed after $(p-1)/2$ time steps from any arbitrary initial condition (spatially confined to a region of side less than p) replicates after $p+(p-1)/2$ time steps in a way that resembles budding in biological systems.

Keywords: cellular automata, reversibility, replication, budding

(Some figures may appear in colour only in the online journal)

1. Introduction

An open question in complexity science is how to construct reduced models involving the spatiotemporal dynamics of a few appropriate coarse-grained 'mesoscopic' degrees of freedom so that emergent properties of complex systems with a huge number of microscopic degrees of freedom can be captured [1, 2]. Cellular automata (CAs) can be thought as the kind of mesoscopic models that would result after performing such a reduction. CAs are discrete dynamical systems in which a discrete lattice of p symbols, that can be labeled with the set $\mathcal{A}_p = \{0, 1, \ldots p-1\}$ called the alphabet, is iteratively updated according to a specified local rule [3–5]. Locally, each site $i \in \mathbb{Z}$ in the lattice is characterized by the variable $x_t^i \in \mathcal{A}_p$ at time t. An additive cellular automaton (ACA) of radius $(l+r)/2$ is a CA whose update rule is a linear function modulo p of the dynamical states $x_t^{i+l}, x_t^{i+l-1}, \ldots x_t^i, \ldots, x_t^{i-r+1}, x_t^{i-r}$. The spatiotemporal evolutions of ACAs are well known to give rise to nested patterns and fractals (when infinite lattices are considered) [3].

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ACAs have found many technological applications in, for example, image processing [6] and the synthesis of cryptographic interleaved sequences [7]. ACAs were the subject of intense research in the late 60s [8] and during the 70s because of the discovery by Amoroso and Cooper [9] of their ability to replicate any pattern. A series of works were published where this work was extended to several dimensions [10], arbitrary neighborhood indices [11] and reproduction in quiescent environments [12]. Barto [13] provided a nice summary of these works and showed the connection between the replication problem and local transformations that are linear over fields of nonzero characteristic. This series of works culminated with the article by Itô *et al* [14] who generalized these results beyond finite fields to finite rings, establishing rigorous results on the surjectivity of additive CA rules in those cases. The crucial work of Martin *et al* [15] established some algebraic properties of ACAs, showing also the connection of the spatiotemporal evolution of the latter with generating functions. ACAs are straightforwardly related to arithmetic triangles and simplices (e.g. Pascal pyramids) modulo an integer number p [16, 17].

ACAs admit analogs of Green's functions [3]: given an initial condition, the resulting evolution can be found by means of a convolution of the discrete evolution of a single non-zero site (analogous to an integral kernel) with the initial condition [3]. For example, the ACA of radius 1/2 given by

$$x_{t+1}^{i} = x_t^i + x_t^{i+1} \bmod p \tag{1}$$

has solution

$$x_t^i = \binom{t}{i} \bmod p \tag{2}$$

for an initial condition consisting of a single seed of value one at $i = 0$ and zero elsewhere, i.e. $x_0^i = \delta_{0i}$. This corresponds to the Pascal triangle modulo p [18–20]. For an initial condition, consisting of an array of $2d-1$ arbitrary site values $x_0^{-d+1}, x_0^{-d+2}, \ldots, x_0^0, \ldots, x_0^{d-1} \ldots$ surrounded by zeros, the solution is a convolution of the solution for a single seed with this more general initial condition and one has

$$x_t^i = \sum_{k=-d+1}^{d-1} \binom{t}{i-k} x_0^{i-k} \bmod p. \tag{3}$$

Equation (2) provides a simple, particular instance in which the replication property is manifested. The ACA starts with a single seed at $i = 0$. At time p, two seeds are formed at $i = 0$ and $i = p$, since

$$\binom{p}{0} = \binom{p}{p} = 1 \tag{4}$$

and, if p is prime,

$$\binom{p}{i} = 0 \bmod p \quad \text{if } 1 \leqslant i \leqslant p-1. \tag{5}$$

This latter equation holds because

$$\binom{p}{i} = \frac{p!}{i!(p-i)!} \tag{6}$$

is divisible by p because the numerator contains a factor p that cannot be divided by the factors in the denominator (because of p being prime). Two seeds separated by sites in the quiescent state are thus formed at $t = p$ and these initiate spatiotemporal evolutions that are similar to the ones found for $t \leqslant p-1$. If one thinks in the patterns arising from the spatiotemporal evolution of the ACA as modeling the shapes of organisms, this process does not resemble any replication process found in biological systems.

Although replication is a transparent property of ACAs, this property is highly nontrivial for non-additive CA [21]. It is also highly nontrivial for second-order reversible ACAs (RACAs) [22–24], since initial conditions have to be specified not only at an initial time $t = 0$ but also on a previous time $t = -1$ and the resulting combinatorial solution is much more complex. A reversible CA is a CA in which every configuration has a unique predecessor. Because of the time-reversibility of microscopic physical laws, reversible CAs are natural candidates for coarse-grained models in physical modeling, and, indeed they are used to model particle motion or the alignment of magnetic dipoles [25]. Reversible CAs pose formidable mathematical problems and, it is known, for example, that it is undecidable to determine whether a given CA in two or more dimensions is reversible [26]. RACAs share the nice convolution property of ACAs but their spatiotemporal evolution is much more complex and their ability to replicate patterns has never been reported before, to the best of our knowledge.

In this work we report on a robust replication property for certain RACAs under very general initial conditions. We first consider 1D RACAs acting on a local neighborhood of unit radius (three sites) and whose state space is an odd prime number p of symbols. Our result shows that replication in RACAs is, indeed, possible and much more subtle than in ACAs. Furthermore, the replication process resembles budding in biological systems: a new pattern/organism develops from a bud at one particular site. When mature, the buds develop into tiny individuals detaching from the parent body and becoming new independent individuals. Our main result, theorem 2, establishes the mathematical details of this replication process. The dynamical coarse-grained variable x_t^i at time t and position i, can be seen as a crude model of the cellular mass at each particular location. Our result can be generalized to a larger number of dimensions (we briefly sketch the rigorous argument that leads to the proof).

The outline of this article is as follows. In section 2 we present the RACA under study and establish a rigorous mathematical result that yields its exact orbit. This part draws heavily on previous work by Dilcher [27] from which it constitutes an application. It is needed to obtain our main, original result, which is presented in section 3: we establish the replication property of the RACA under general initial conditions. Finally, in section 4 we discuss our result, presenting some simulations to illustrate it visually, as well as some generalizations.

2. The CA map and its exact spatiotemporal evolution

The spatiotemporal dynamics of the RACA here considered is given by the map

$$x_{t+1}^i = x_t^{i+1} + x_t^i + x_t^{i-1} - x_{t-1}^i \bmod p, \tag{7}$$

where $i \in \mathbb{Z}$ labels the site on a 1D discrete lattice and $t \in [-1, \infty)$, $t \in \mathbb{Z}$ is the discrete time. The dynamic variable x_t^i is restricted to an integer value in the interval $[0, p-1]$, where p shall be considered an odd prime number. The initial condition at times $t = -1$ and $t = 0$ is assumed to be the same, i.e. $x_{-1}^i = x_0^i$ a crucial condition that is needed for the result. We first study the solution of the map for the initial conditions $x_{-1}^i = x_0^i = \delta_{i0}$ consisting of a single site with value 1 surrounded by zeros. The result for this particular case easily generalizes to an arbitrary initial condition by means of the convolution property. The reversibility of the map is clear from the fact that it remains invariant after exchanging $t+1 \leftrightarrow t-1$. We note that, besides being time reversible, the rule given by equation (7) is invariant under a exchange $i \mp 1 \rightarrow i \pm 1$ (left–right reversibility). This property is absent in equation (1) and in the time-reversible version of it. However, we regard this property as important in modeling physical systems, because it constitutes a discrete version of the isotropy found in the physical laws of homogeneous systems.

Let us first leave aside the mod p operation in equation (7). Then, the map

$$x_{t+1}^{i} = x_t^{i+1} + x_t^{i} + x_t^{i-1} - x_{t-1}^{i} \tag{8}$$

with the initial conditions $x_{-1}^{i} = x_0^{i} = \delta_{i0}$ can be expressed in terms of a generating function as

$$\frac{1-zy}{1-y(1+z+z^2)+y^2z^2} = 1 + \sum_{t=1}^{\infty}\sum_{i=-t}^{t} x_t^{i} z^{t-i} y^{t}. \tag{9}$$

This latter expression is obtained from equation (8) by multiplying both sides by $z^{t-i}y^{t}$ and then summing over $\sum_{t=0}^{\infty}\sum_{i=-t}^{t}(\cdots)$ and using that $x_{-1}^{i} = x_0^{i} = \delta_{i0}$ and the geometric series $\sum_{k=0}^{\infty} x^{-k} = 1/(1-x)$. Equation (9) can be checked by formally expanding the lhs of equation (9) in powers of z and y and equating powers of the latter quantities with same exponents on both sides.

For the generating function

$$\frac{1}{1-y(1+z+z^2)+y^2z^2} = \sum_{t=0}^{\infty}\sum_{i=-t}^{t} C_{t,i}^{1,1} z^{t-i} y^{t}. \tag{10}$$

Dilcher [27] found that the coefficients $C_{t,i}^{1,1}$ are given by the expression

$$C_{t,i}^{1,1} = \sum_{s=0}^{\lfloor \frac{t-|i|}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-|i|-2s}{2} \rfloor} (-1)^{s} \binom{t-s}{s} \binom{2j+|i|}{j} \binom{t-2s}{2j+|i|} \tag{11}$$

(see theorem 1 in [27]). The superscripts 1, 1 in the coefficients $C_{t,i}^{1,1}$ follow the notation by Dilcher in [27] since these are particular instances of the coefficients $C_{t,i}^{\nu,\lambda}$, with $\nu = \lambda = 1$ as defined in [27]. These coefficients, related to the theory of Gegenbauer (ultraspherical) polynomials [27], obey a recurrence relation similar to equation (8)

$$C_{t+1,i}^{1,1} = C_{t,i+1}^{1,1} + C_{t,i}^{1,1} + C_{t,i-1}^{1,1} - C_{t-1,i}^{1,1}. \tag{12}$$

We then have, by using equation (10)

$$\begin{aligned}\frac{1-zy}{1-y(1+z+z^2)+y^2z^2} &= \sum_{t=0}^{\infty}\sum_{i=-t}^{t} [C_{t,i}^{1,1} - zyC_{t,i}^{1,1}] z^{t-i} y^{t} \\ &= 1 + \sum_{t=1}^{\infty}\sum_{i=-t}^{t} [C_{t,i}^{1,1} - C_{t-1,i}^{1,1}] z^{t-i} y^{t} \\ &= 1 + \sum_{t=1}^{\infty}\sum_{i=-t}^{t} x_t^{j} z^{t-i} y^{t},\end{aligned} \tag{13}$$

where equation (9) has also been used. We thus find that, for $t \geqslant 1$

$$x_t^j = C_{t,i}^{1,1} - C_{t-1,i}^{1,1} \bmod p \tag{14}$$

for initial conditions $x_{-1}^i = x_0^i = \delta_{i0}$. Equation (14) follows also immediately from additivity. We gather now all above observations.

Theorem 1. *The solution for the orbit x_t^i of the one-dimensional reversible cellular automaton rule*

$$x_{t+1}^i = x_t^{i+1} + x_t^i + x_t^{i-1} - x_{t-1}^i \bmod p, \tag{15}$$

where $x_0^i = x_{-1}^i = \delta_{i,0}$, $\delta_{i,j}$ is the Kronecker delta and p is any odd prime number, is given for $t \geqslant 1$ by:

$$x_t^i = C_{t,i}^{1,1} - C_{t-1,i}^{1,1} \bmod p, \tag{16}$$

where

$$C_{t,i}^{1,1} = \sum_{s=0}^{\lfloor \frac{t-|i|}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-|i|}{2} - s \rfloor} (-1)^s \binom{t-s}{s} \binom{2j+|i|}{j} \binom{t-2s}{2j+|i|} \tag{17}$$

with $\lfloor \cdots \rfloor$ denoting the floor function.

We note that, by writing the binomial coefficients in terms of factorials, equation (17) can equivalently be rewritten as

$$C_{t,i}^{1,1} = \sum_{s=0}^{\lfloor \frac{t-|i|}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{t-|i|-2s}{2} \rfloor} (-1)^s \binom{t-s}{s, j, j+|i|, t-2(s+j)-|i|}, \tag{18}$$

where we have introduced the multinomial coefficient

$$\binom{i+j+k+l}{i, j, k, l} = \frac{(i+j+k+l)!}{i!j!k!l!}. \tag{19}$$

The interest of the structure of equation (18) lies in the fact that we can understand $C_{t,i}^{1,1}$ as a sum over multinomial coefficients in a Pascal pyramid. In higher dimensions, this generalizes to a sum over multinomial coefficients on a Pascal simplex.

3. Main result: replication of spatially extended structures

It is interesting to compare the spatiotemporal evolution of the map equation (8) before and after the mod p operation. This is shown in figure 1 where both evolutions are shown. Without performing the mod p operation, the values obtained by the recurrence in equation (8) with $x_0^i = x_{-1}^i = \delta_{i,0}$ arrange in the arithmetic triangle shown in figure 1 (a). There, each element in the tth row is the sum of the three closest elements in the $(t-1)$th row, minus the closest element in the $(t-2)$th row. Remarkably, if one now picks any p prime, and performs the mod p operation, the structure originated after $(p-1)/2$ time steps is duplicated at $p + (p-1)/2$ time steps as can be seen in figure 1 (b) for $p = 5$. The two copies of the structure are neatly separated by sites in the quiescent state. For other initial conditions which are nonzero only for sites within a region of size lower than $(p-1)/2$, an analogous behavior is observed.

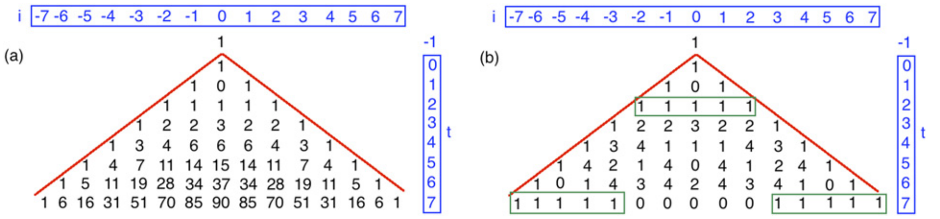

Figure 1. Arithmetic triangle formed by the values obtained by iterating (a) the map given by equation (8), (b) the CA rule equation (15) with $x_0^i = x_{-1}^i = \delta_{i,0}$ for $p = 5$. The structure formed at $t = (p-1)/2 = 2$ (shown in a green box) duplicates at time $t = p + (p-1)/2 = 7$, both structures being separated by sites in the quiescent state. This duplication process grows out of 'buds' that are created at time $t = p = 5$.

Considering two dimensions and other neighborhoods the same fact is computationally observed.

We now mathematically substantiate this fact. We first study the orbit of the RACA and prove two lemmas before our main result, the theorem on the replication of complex structures. Lemma 1 establishes some overall symmetries of the arithmetic triangle arising from the spatiotemporal evolution of the RACA. Lemma 2 is necessary to establish that the sites separating the two copies of the same structure are in the quiescent state.

Lemma 1. *The following relationships, for x_t^i given by equation (16), hold*

(*a*) $x_t^i = x_t^{-i}$
(*b*) $x_{t+p}^{i+p} = x_t^i = x_{t+p}^{-i-p}$ *for* $0 \leqslant t \leqslant p-1$ *and* $|i| \leqslant t$
(*c*) $x_t^i = 0$ *for* $|i| > t$

Proof. Result (a) is trivial to prove from the symmetry relationships of the coefficients $C_{t,i}^{1,1} = C_{t,-i}^{1,1}$ in equations. Equations (16) and (17) since all the i-dependence of the latter is through the absolute value $|i|$.

Result (b) is a remarkable property of equations (16) and (17). To prove it we first recall Lucas' correspondence theorem. This theorem, proved in [28], establishes that if two integer numbers n and m have the base p representation $n = n_0 + n_1 p + \cdots + n_k p^k$, $m = m_0 + m_1 p + \cdots + m_k p^k$ with k integer, then

$$\binom{n}{m} \bmod p = \prod_{h=0}^{k} \binom{n_h}{m_h} \bmod p. \tag{20}$$

We now observe that, since $0 \leqslant s \leqslant s_{\max} < t \leqslant p-1$, $0 \leqslant j \leqslant s_{\max} < t \leqslant p-1$, $0 \leqslant 2j + |i| \leqslant t \leqslant p-1$ and $0 \leqslant t-s \leqslant t \leqslant p-1$, we have, from Lucas' theorem

$$\binom{t-s+p}{s} \bmod p = \binom{t-s}{s}\binom{1}{0} \bmod p = \binom{t-s}{s} \bmod p \tag{21}$$

$$\binom{t-2s+p}{2j+|i|+p} \bmod p = \binom{t-2s}{2j+|i|}\binom{1}{1} \bmod p = \binom{t-2s}{2j+|i|} \bmod p \tag{22}$$

and

$$\binom{2j+|i|+p}{j} \bmod p = \binom{2j+|i|}{j}\binom{1}{0} \bmod p = \binom{2j+|i|}{j} \bmod p. \tag{23}$$

We thus have, from equation (17)

$$\begin{aligned} C^{1,1}_{t+p,i+p} \bmod p &= \sum_{s=0}^{\lfloor\frac{t-|i|}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{t-|i|-2s}{2}\rfloor} (-1)^s \binom{t-s+p}{s}\binom{2j+|i|+p}{j} \\ &\quad\times \binom{t-2s+p}{2j+|i|+p} \bmod p \\ &= \sum_{s=0}^{\lfloor\frac{t-|i|}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{t-|i|-2s}{2}\rfloor} (-1)^s \binom{t-s}{s}\binom{2j+|i|}{j} \\ &\quad\times \binom{t-2s}{2j+|i|} \bmod p \\ &= C^{1,1}_{t,i} \bmod p, \end{aligned} \tag{24}$$

where equations (21)–(23) have been used to replace the binomial coefficients within the sum. We thus have as well

$$x^{i+p}_{t+p} = C^{1,1}_{t+p,i+p} - C^{1,1}_{t-1+p,i+p} \bmod p = C^{1,1}_{t,i} - C^{1,1}_{t-1,i} \bmod p = x^i_t \tag{25}$$

as we wanted to prove. The second equality in (b) comes then after applying result (a).

Result (c) is directly obtained from equation (17), since $C^{1,1}_{t,i} = 0$ for $|i| > t$. □

Lemma 2. *Let $|i| \leqslant (p-1)/2$ and $p > 1$ be any odd prime number. Then*

$$C^{1,1}_{p+\frac{p-1}{2},i} \bmod p = C^{1,1}_{p+\frac{p-1}{2}-1,i} \bmod p \tag{26}$$

and, therefore, for the map given by equation (15), we have that $x^i_{p+(p-1)/2} = 0$, $\forall |i| \leqslant (p-1)/2$.

Proof. From equation (17), we have, for $t = T := p + (p-1)/2$,

$$C^{1,1}_{T,i} = \sum_{s=0}^{\left\lfloor\frac{T-|i|}{2}\right\rfloor}\sum_{j=0}^{\left\lfloor\frac{T-|i|}{2}-s\right\rfloor} c_{T,i}(s,j), \tag{27}$$

where

$$c_{T,i}(s,j) := (-1)^s \binom{T-s}{s}\binom{2j+|i|}{j}\binom{T-2s}{2j+|i|} \tag{28}$$

$$= (-1)^s \frac{(T-s)!}{s!j!(j+|i|)!\big(T-2s-2j-|i|\big)!}. \tag{29}$$

Clearly, from equation (29) all $c_{T,i}(s,j)$'s for which

$$T - p < 2s \leqslant p - 1 \tag{30}$$

do not contribute to the sum in equation (27) because they are equal to 0 mod p. This can be seen from the fact that the factorial $(T-s)!$ in the numerator of $c_{T,i}(s,j)$ contains a factor p that is absent in the denominator, since $s < p$, $j < p$ and because $|i| \leqslant (p-1)/2$ we also have $j + |i| \leqslant \left\lfloor \frac{T+|i|}{2} \right\rfloor < p$. Thus, the only terms that contribute to the sum in equation (27) are those for which $0 \leqslant 2s \leqslant T - p$ and $p + 1 \leqslant 2s \leqslant 2\left\lfloor \frac{T+|i|}{2} \right\rfloor$. If one considers now time $T-1$, these inequalities for the values of s read now $0 \leqslant 2s \leqslant T - p - 1$ and $p - 1 \leqslant 2s \leqslant 2\left\lfloor \frac{T-1+|i|}{2} \right\rfloor$. This means that for any $s = s'$ such that $c_{T,i}(s',j)$ is not divisible by p a bijection can be established to another value $s'' = s' + \frac{p-1}{2}$ mod p for which $c_{T-1,i}(s'',j)$ is divisible by p neither. We now show that $c_{T,i}(s',j) \bmod p = c_{T-1,i}(s'',j) \bmod p$. Note that, by using the symmetries of binomial coefficients and Lucas' correspondence theorem we have, on one hand,

$$\begin{aligned} c_{T,i}(s',j) \bmod p &= (-1)^{s'} \binom{T-s'}{T-2s'} \binom{2j+|i|}{j} \binom{T-2s'}{2j+|i|} \bmod p \\ &= (-1)^{s'} \binom{p+T-s'}{T-2s'} \binom{2j+|i|}{j} \binom{T-2s'}{2j+|i|} \bmod p \end{aligned} \tag{31}$$

and, on the other hand, using again Lucas' correspondence theorem twice

$$c_{T-1,i}(s'',j) \bmod p = (-1)^{s'+\frac{p-1}{2}} \binom{2p-1-s'}{T-2s'} \binom{2j+|i|}{j} \binom{T-2s'}{2j+|i|} \bmod p. \tag{32}$$

We now note that

$$\begin{aligned} \binom{p+T-s'}{T-2s'} &= \frac{(p+T-s')}{p+s'} \frac{(p+T-s'-1)}{(p+s'-1)} \dots \\ &\quad \times \frac{(2p-s')}{\left(p+s'-\frac{p-1}{2}\right)} \binom{2p-1-s'}{T-2s'} \end{aligned} \tag{33}$$

and, therefore,

$$\binom{p+T-s'}{T-2s'} \bmod p = (-1)^{\frac{p-1}{2}} \binom{2p-s'-1}{T-2s'} \bmod p. \tag{34}$$

By replacing this result in equation (31), we obtain

$$c_{T,i}(s',j) \bmod p = (-1)^{s'+\frac{p-1}{2}} \binom{2p-s'-1}{T-2s'} \binom{2j+|i|}{j} \binom{T-2s'}{2j+|i|} \bmod p \tag{35}$$

from which, by comparing with equation (32),

$$c_{T,i}(s',j) \bmod p = c_{T-1,i}(s'',j) \bmod p \tag{36}$$

and, therefore

$$C^{1,1}_{T,i} \bmod p = \sum_{s'}\sum_{j} c_{T,i}(s',j) = \sum_{s''}\sum_{j} c_{T-1,i}(s'',j) = C^{1,1}_{T-1,i} \bmod p \tag{37}$$

as we wanted to prove. Now, since $x^i_T = C^{1,1}_{T,i} - C^{1,1}_{T-1,i} \bmod p$, we finally have

$$x^i_{p+\frac{p-1}{2}} = 0 \bmod p \quad \forall |i| \leqslant \frac{p-1}{2} \tag{38}$$

and the proof of the lemma is completed. □

We now state and give a proof of the main result of this article.

Theorem 2 (Replication of spatially extended structures). *Let an arbitrary initial condition be given for an integer variable $x^i_t \in [0, p-1]$, with p and odd prime at $t = 0$ and $t = -1$, which is nonzero only on a collection of $2\delta + 1 < p$ adjacent sites centered at $i = 0$ and such that $x^{-\delta}_0 = x^{-\delta}_{-1}$, $x^{-\delta+1}_0 = x^{-\delta+1}_{-1}$, ..., $x^0_0 = x^0_{-1}$, ..., $x^{\delta-1}_0 = x^{\delta-1}_{-1}$, $x^{\delta}_0 = x^{\delta}_{-1}$. Then, at time $t'' = p + (p-1)/2$, two copies arise that are identical to the pattern obtained at time $t' = (p-1)/2$. The copies are separated by sites in the quiescent state (i.e. sites with value 0).*

Proof. Because of the linearity of the map, equation (15), the orbit is given by the superposition (convolution) of arithmetic triangles obtained in theorem 1 with the initial condition as

$$x^i_t = \sum_{k=-\delta}^{\delta} x^{i-k}_0 \left(C^{1,1}_{t,i-k} - C^{1,1}_{t-1,i-k}\right) \bmod p \tag{39}$$

with $C^{1,1}_{t,i}$ given by equation (17). At time $t = \frac{p-1}{2}$ a pattern is obtained

$$x^i_{\frac{p-1}{2}} = \sum_{k=-\delta}^{\delta} x^{i-k}_0 \left(C^{1,1}_{\frac{p-1}{2},i-k} - C^{1,1}_{\frac{p-1}{2}-1,i-k}\right) \bmod p. \tag{40}$$

Because of lemma 1, (a) and (b) two copies of this pattern are produced at time $t = p + \frac{p-1}{2}$. Let $|i| \leqslant t$, i.e. $-t \leqslant i \leqslant t$. The two copies are given by

$$x^{i+p}_{p+\frac{p-1}{2}} = \sum_{k=-\delta}^{\delta} x^{i-k}_0 \left(C^{1,1}_{\frac{p-1}{2},i-k} - C^{1,1}_{\frac{p-1}{2}-1,i-k}\right) \bmod p \tag{41}$$

$$x^{-i-p}_{p+\frac{p-1}{2}} = x^{i+p}_{p+\frac{p-1}{2}} \tag{42}$$

and are centered at positions p and $-p$, respectively, occupying positions $[p - \frac{p-1}{2} - \delta, p + \frac{p-1}{2} + \delta]$ and $[-p - \frac{p-1}{2} - \delta, -p + \frac{p-1}{2} + \delta]$. These structures are externally surrounded by sites in the quiescent state (because of lemma 1 (c)). The sites occupying positions $[-\frac{p-1}{2} + \delta, \frac{p-1}{2} - \delta]$ are also in the quiescent state, because of lemma 2. Note that the properties of the coefficients $C^{1,1}_{t,i}$ when taken modulo p alone suffice to establish all this. □

If two dimensions are considered, but the dynamics takes place only along one dimension, the map can be rewritten in a straightforward manner as

$$x^{i,j}_{t+1} = x^{i,j-1}_t + x^{i,j}_t + x^{i,j+1}_t - x^{i,j}_{t-1} \bmod p. \tag{43}$$

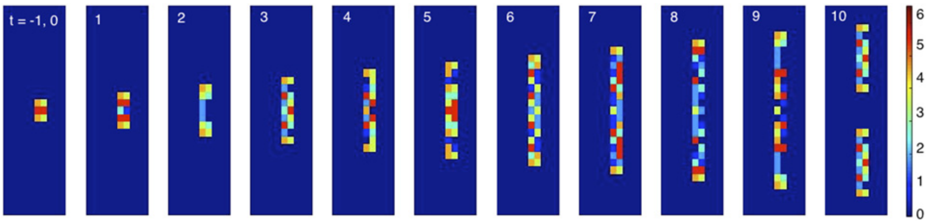

Figure 2. Spatiotemporal evolution of map equation (43) for the initial condition shown in the leftmost panel and $p = 7$. Subsequent time steps are shown and it is observed that the pattern obtained at $t = (p - 1)/2 = 3$ is replicated at time step $t = p + (p - 1)/2 = 10$, yielding two copies separated by sites in the quiescent state (dark blue).

Here $i, j \in \mathbb{Z}$ give the coordinates of a site on a discrete, planar lattice. The dynamics takes place only along the j direction. By selecting an initial condition $x_{-1}^{i,j} = x_0^{i,j}$ where the nonzero sites are contained on a rectangle and such that along the j direction the conditions of theorem 2 are satisfied (the nonzero sites occupy a region of side δ such that $2\delta + 1 < p$), the solution of the above map is simply given for $t \geqslant 1$ as

$$x_t^{i,j} = \sum_{k=-\delta}^{\delta} x_0^{i,j-k} \left(C_{t,j-k}^{1,1} - C_{t-1,j-k}^{1,1} \right). \tag{44}$$

Theorem 2 can be applied in this case and, for a given odd prime p the structure obtained at time step $(p - 1)/2$ will yield two copies at time $p + (p - 1)/2$ separated by sites in the quiescent state. This is shown in figure 2 for $p = 7$ and an initial condition contained in a rectangle satisfying the conditions of theorem 2.

4. Discussion and generalizations

Computer simulations show that the replication process described by theorem 2 extends to von Neumann and Moore neighborhoods. In two dimensions the RACA for a von Neumann neighborhood has the form

$$x_{t+1}^{i,j} = x_t^{i,j-1} + x_t^{i,j} + x_t^{i,j+1} + x_t^{i-1,j} + x_t^{i+1,j} - x_{t-1}^{i,j} \bmod p. \tag{45}$$

In this case, the pattern obtained after $(p - 1)/2$ time steps starting from an arbitrary initial condition of side, at most p, is replicated at time $T = p + (p - 1)/2$. However, four copies are produced for the pattern in this case, that compare to the two copies produced by the 1D RACA, equation (15). This is observed in figure 3 in which the spatiotemporal evolution of equation (45) is shown for $p = 13$, starting from an initial condition $x_{-1}^{i,j} = x_0^{i,j}$ contained in a small rectangle, as shown in the leftmost, uppermost panel. Subsequent time steps are shown and it is observed that four copies are produced at $t = p + (p - 1)/2 = 19$ of the pattern obtained at $t = (p - 1)/2 = 6$ through a process that resembles budding in biological systems. Four buds are originated at $t = p = 13$ and these grow until the patterns representing the mature individuals are obtained, being then separated from the main body by sites in the quiescent state.

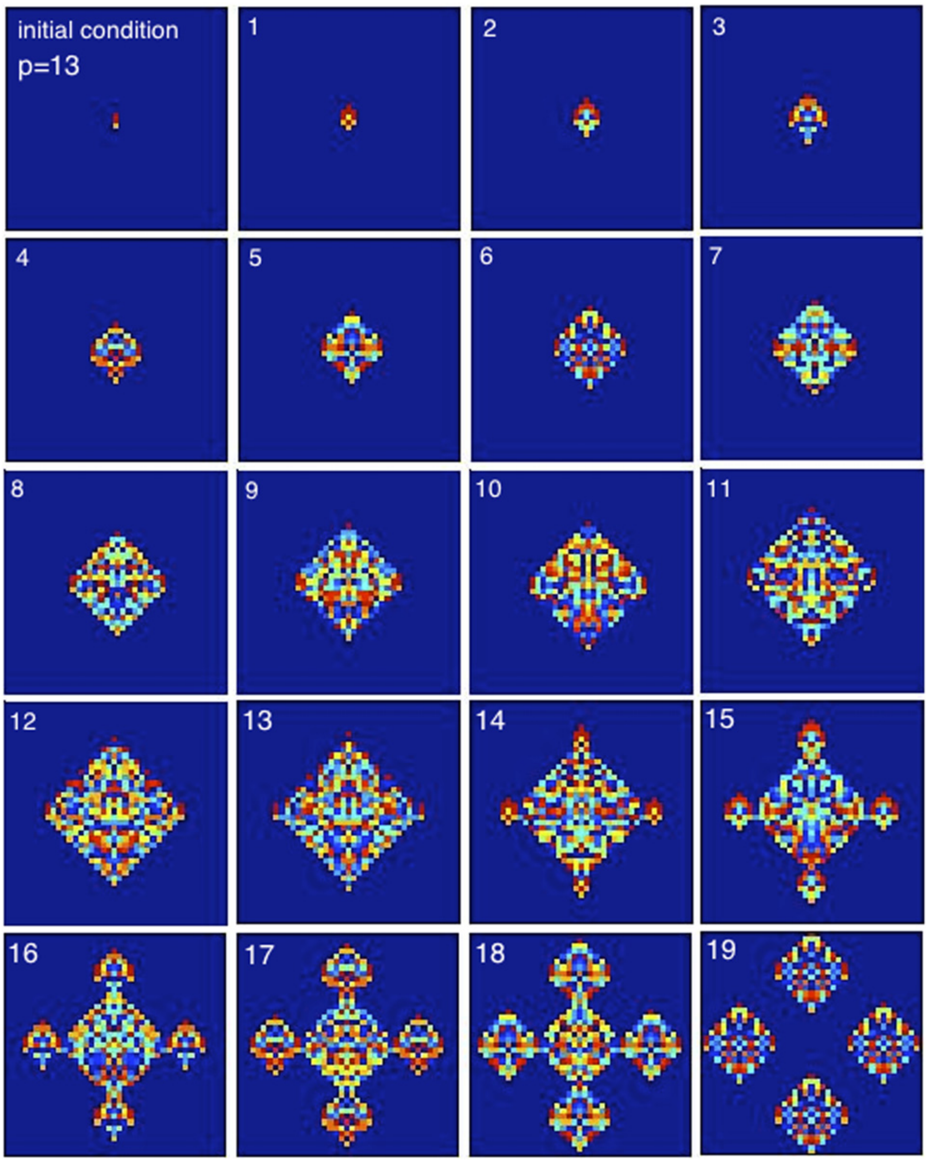

Figure 3. Spatiotemporal evolution of map equation (45) for the initial condition at $t=-1$ and $t=0$ shown in the leftmost and uppermost panel and $p=13$. Subsequent time steps (indicated in the panels) are shown and it is observed that the pattern obtained at $t=(p-1)/2=6$ is replicated at time $t=p+(p-1)/2=19$ from four 'buds' generated at $t=p$, yielding four copies separated by sites in the quiescent state (dark blue).

For a Moore neighborhood the RACA

$$x_{t+1}^{i,j}=\sum_{k=-1}^{1}\sum_{m=-1}^{1}\left(x_t^{i-k,j-m}\right)-x_{t-1}^{i,j} \bmod p \tag{46}$$

computer explorations also show the fact that the pattern obtained at time $(p-1)/2$ replicates at time $T = p + (p-1)/2$. This time, eight copies of the pattern are obtained.

Our main result can be rigorously extended to these topologies. Although the proof is rather technically involved, it essentially follows the same steps given above in detail for the 1D dynamics. We sketch the rigorous argument to retrace the proof for these cases. For the von Neumann neighborhood the respective generating function, generalizing equation (9) is

$$\frac{1-zy}{1-y(1+z+z^2+w+w^2)+y^2z^2} = 1 + \sum_{t=1}^{\infty}\sum_{i=-t}^{t}\sum_{j=-t}^{t} x_t^{i,j} w^{t-j} z^{t-i} y^t. \tag{47}$$

For the Moore neighborhood we have, instead,

$$\frac{1-zy}{1-y(1+z+z^2)(1+w+w^2)+y^2z^2} = 1 + \sum_{t=1}^{\infty}\sum_{i=-t}^{t}\sum_{j=-t}^{t} x_t^{i,j} w^{t-j} z^{t-i} y^t. \tag{48}$$

These generating functions can again be brought in relationship with the coefficients of Gegenbauer polynomials with the methods in [27] and a generalized version of equation (16) again holds, with equation (18) now generalized to multinomial coefficients on a Pascal simplex. We now note that any multinomial coefficient modulo p satisfies a quick corollary of Lucas' correspondence theorem (used in lemma 1) which is called Dickson's theorem. Let $i_1, i_2, \ldots, i_n$ have the base p representations $i_1 = i_{0,1} + i_{1,1}p + \cdots + i_{m,1}p^m$, $i_2 = i_{0,2} + i_{1,2}p + \cdots + i_{m,2}p^m$, $\ldots$, $i_n = i_{0,n} + i_{1,n}p + \cdots + i_{m,n}p^m$ with m integer, then [29]

$$\binom{i_1+i_2+\cdots+i_n}{i_1, i_2, \ldots, i_n} \bmod p = \prod_{h=0}^{m} \binom{(i_1+i_2+\cdots+i_n)_h}{i_{1,h}, i_{2,h}, \ldots, i_{n,h}} \bmod p, \tag{49}$$

where $(x)_h$ denotes the hth digit in the base p representation of x. Dickson's theorem is then used to generalize lemma 1 to higher dimensions. Since there are now the spatial symmetries $i \leftrightarrow -i$, $j \leftrightarrow -j$, the symmetry of the resulting Pascal pyramid modulo p leads to the production of four copies (in the case of a Moore neighborhood there are the symmetries $i \leftrightarrow -i$, $j \leftrightarrow -j$, $i+j \leftrightarrow -i-j$ and $i-j \leftrightarrow -i+j$ leading to the production of eight copies). Finally, lemma 2 (cells in quiescent states) also generalizes to these topologies, by realizing that the relevant factors within the multinomial coefficients in the Pascal simplex are only those that explicitly contain the time t variable and that those factors behave exactly as in the 1D case.

Let $f(\mathbf{x}_t)$ denote the right-hand side of equations (15), (43) and (45) or (46). Further modifications can be introduced to any of these maps by following the methods in [30] to create (conditional) predictability, mutations on the patterns, or fixed points. For example, if we wish that the map given by equation (43) does not evolve further after the replication has taken place at $T = p + (p-1)/2$ (so that the replicated structures are fixed points) we can modify the map as follows

$$x_{t+1}^{i,j} = x_t^{i,j} + \left(x_t^{i,j-1} + x_t^{i,j+1} - x_{t-1}^{i,j}\right)\mathcal{B}\left(t - \frac{p-1}{2}, p + \frac{1}{2}\right) \bmod p, \tag{50}$$

where the boxcar function

$$\mathcal{B}(x,y) := \frac{1}{2}\left(\frac{x+y}{|x+y|} - \frac{x-y}{|x-y|}\right) \tag{51}$$

was introduced in [19] to formulate a universal map for CA. equation (50) behaves as equation (43) for $t \leqslant T$ and yields $x_{t+1}^{i,j} = x_t^{i,j}$ for $t > T$. Note that the creation of this fixed point involves breaking the reversibility of equation (43) since, by its very definition, a reversible CA does not allow for fixed points.

Let $S_t := x_t^{i,j} + x_t^{i,j-1} + x_t^{i,j+1}$. The following modification of equation (43)

$$x_{t+1}^{i,j} = x_t^{i,j} + \left(x_t^{i,j-1} + x_t^{i,j+1} - x_{t-1}^{i,j}\right)\left[1 - \mathcal{B}\left(S_t, \frac{1}{2}\right)\right] \bmod p \tag{52}$$

has a behavior similar to equation (43) except in those places where $x_t^{i,j} = x_t^{i,j-1} = x_t^{i,j+1} = S_t = 0$, in which case, from equation (52) $x_{t+1}^{i,j} = 0$ regardless of the value of $x_{t-1}^{i,j}$. Again this breaks the reversibility of the rule: if a quiescent neighborhood is attained reversibly, the quiescent state is not left. This is a realistic, physical modification because once the dynamics leads to a quiescent neighborhood it is physical to require that no activity will arise *ex nihilo*. Once structures are separated, they will not coalesce again in the next time step. It is observed in the simulations that, for certain initial conditions and prime numbers of the form $p = 4n + 3$, $n \in \mathbb{N}$ defects are introduced in the replication process leading to 'mutations' in the copies of the patterns replicated.

5. Conclusions

In this article it has been rigorously shown that certain RACAs are able to replicate spatial patterns found in their spatiotemporal evolution starting from quite arbitrary initial conditions. The dynamics takes place along an axis in one dimension. The RACAs here studied can, indeed, be adapted to model the main features of life [31]: (a) *multiplication (or replication)*, i.e. the ability of an individual to produce two; (b) *heredity*, i.e. there are different kinds of individuals and these produce offsprings like themselves and (c) *variation*: heredity is not perfect so that, occasionally, the replicated structures have mutations. We have shown that the RACAs here presented satisfy property (a) naturally giving rise to the replicated patterns in a way that resembles biological processes. Property (b) is also naturally incorporated, since different initial conditions will lead to different patterns at time $(p-1)/2$ from which copies like themselves will be produced at time $p + (p-1)/2$. Finally, property (c) can be achieved by means of appropriate adaptations of the RACAs, using the methods in [30], as explained in the modification given by equation (52) of the original RACA, equation (43).

It is also possible to embed the RACA here presented in continuous structures (coupled map lattices [32]) so that, although the dynamics is still dictated by the discrete evolution of the RACA, continuous structures emerge and replicate. This embedding, called nonlinear $\mathcal{B}_\kappa$-embeddings allow to construct complex shapes out of a discrete amount of information [33]. If the shapes are constructed at each time of the RACA evolution, continuous complex structures can be thus replicated. These lattice functions for CAs will be discussed elsewhere.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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