MR3644110 37B15 05B45 68-02 68Q80 82D05 92C15 92D15 92D30
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★**Cellular automata: analysis and applications.**
Springer Monographs in Mathematics.
Springer, Cham, 2017. *xi*+467 *pp. ISBN* 978-3-319-53042-0; 978-3-319-53043-7

Cellular automata constitute one of the basic notions of computer science, with important applications in many fields such as mathematical logic, the theory of dynamical systems, biology, chemistry, physics and the social sciences. The concept was introduced by John von Neumann and Stanislaw Ulam in the 1940s, but it was not until John Conway's discovery of the Game of Life (a 2D cellular automaton that has the computational power of a universal Turing machine) that cellular automata became popular. Cellular automata are maps acting on a state space consisting of a grid, where each grid point assumes one of a finite number of possible local states as a function of the neighboring grid points. These maps incorporate certain properties, in particular locality and translational invariance (homogeneity).

Although there is a vast literature on cellular automata, most of information is scattered in journal articles and conference proceedings, and there are, comparatively, few textbooks on this subject. They usually fall into three classes: (1) popular books and light introductions to cellular automata addressing a wide readership and having little or no mathematical content; (2) books concentrating on specific applications and simulation results addressing specialists in different fields of the applied sciences, and (3) advanced mathematical texts and monographs appealing to pure mathematicians and theoretical computer scientists. While the barrier between classes (1) and (2) is diffuse and can be smoothly surmounted by the interested reader, this is not so with the one existing between classes (2) and (3), where there exists a significant gap. The book under review constitutes a welcome addition to the literature on the subject since it contains both rigorous mathematical results and selected applications, thus aiming to cover this gap. The main theme of the present work is, in the words of the authors, "the quest for efficient approaches that allow for an insight beyond pure simulations". In my opinion, the authors largely succeed in their enterprise. Important results in the field are presented with clarity and rigor and, without sacrificing any depth, the reader is gently introduced to useful and general tools to handle subtle aspects of cellular automata while showing a keen eye for applications of the formalism.

After a light introduction with interesting examples, Chapter 2, with its basic definitions, is already excellent in setting the scene for the developments to come later. The presentation is algebraic and combinatorial; everything is rigorously and clearly formulated for the working mathematician. The examples, illustrations and the pace at which the definitions are introduced make the text quite accessible for the general interested reader, who has a good opportunity here to appreciate the beauty of what is being discussed. This is a general feature of this book.

This chapter concisely presents all concepts necessary to understand how cellular automata work in their full generality: regular and non-Abelian grids, Cayley graphs, neighborhoods, elementary and global states, the function specifying the local and global dynamics, etc. The chapter culminates with the introduction of the growth function of a Cayley graph, which is used in later sections. The authors reproduce Milnor's proof on the existence of an asymptotic growth rate of a Cayley graph and the relation between its numerical value and the underlying group.

The authors devote Chapters 3 and 4 to estimating the degree of generality that is inherent to cellular automata and to acquainting the reader with some fundamental properties. They approach these tasks from the topological dynamical systems point of view. The authors give a proof of the fundamental Curtis-Hedlund-Lyndon theorem, the most important result in this field, for a Cantor topology (Chapter 3). The result establishes that the global functions of cellular automata are exactly the continuous functions on a metrizable Cantor space that commute with all shift operators. This result implies that cellular automata are by far too general to allow for many useful theorems to be valid for all of them. One of the authors, Müller, has contributed an extension of the Curtis-Hedlund-Lyndon theorem to Besicovitch and Weyl topologies. The original proof of this result is presented in Chapter 4.

The set of cellular automata is much too large and inhomogeneous. With this in mind, it is necessary, to gain deeper insight, to define small, homogeneous classes of cellular automata. This can be done algebraically, selecting a certain aspect and proceeding in a top-down approach classifying cellular automata with respect to this aspect by partitioning cellular automata into disjoint sets. Classes of cellular automata can also be defined with an orientation towards certain applications in a bottom-up manner. These applications are useful as guiding tools since they force the cellular automaton to behave in a rather special way to resemble some real-world phenomenon.

Chapters 5–7 focus on the top-down approach. In each of these chapters a classification scheme of cellular automata according to a different aspect is given. In Chapter 5, the attractors of the long-term dynamics of cellular automata are analyzed and classified in terms of Hurley classes, which address the set of all Conley attractors for a given cellular automaton. Gilman's classification scheme is then discussed in Chapter 6, based on the fact that, by the Curtis-Hedlund-Lyndon theorem, the global function of a cellular automaton is continuous and, therefore, a function can be continuous or equicontinuous at a certain point under iteration. Gilman's identification of Lyapunov-continuous states is helpful to discriminate in the long-term behavior between chaotic and uniform cellular automata. In Chapter 7 Kůrka's classification of cellular automata is presented. This classification addresses the complexity of a cellular automaton, relating it to that of the grammar necessary to generate the words obtained by finitely iterating the cellular automaton.

In Chapter 8, Turing machines and tessellations are discussed. The authors then present Kari's proof of the undecidability of the question whether a given tile set allows for the tessellation of $\mathbb{Z}^2$. This important result provides a tool to analyze the decidability of certain properties of cellular automata, in view of the connection of the latter to tessellations. This is investigated in Chapter 9, where Garden-of-Eden theorems, which relate injectivity to surjectivity in cellular automata, are discussed. It is shown that many properties of one-dimensional structures are decidable. However, in two dimensions, even very basic properties such as injectivity, surjectivity or the existence of stationary states are not decidable anymore.

In Chapters 10–13, cellular automata classes are defined in a bottom-up approach, motivated also by applications. Chapter 10 contains a good treatment of linear cellular automata, although there is a vast literature on this topic, and it would have been a good opportunity to mention some classical papers of the 1970's and 1980's, such as the ones by Amoroso, Yamada, Cooper, Barto and some others and, most importantly, the article by Martin, Odlyzko and Wolfram on the algebraic theory of linear cellular automata. Although the Fermat property is discussed in detail, a treatment of Lucas' correspondence theorem would have also been great in this context, since this theorem is a very powerful tool to draw conclusions on the spatiotemporal evolution of linear cellular automata. Chapter 11 contains a nice discussion on particle motion and cellular automata models for diffusion. The discussion of the intriguing ultradiscretization method that allows one to relate certain partial differential equations to cellular automata is particularly valuable. Chapter 12 discusses Turing patterns and the Greenberg-Hastings model for excitable media, and the book culminates with a chapter devoted to applications to self-organized criticality (sandpile automata), epidemiology and evolution.

I have enjoyed reading this book. It provides a clear account, with many examples and nice proofs, of the most important and general rigorous results of cellular automata in a way that is accessible to a wide readership. Advanced undergraduate and beginning graduate students of several fields, including theoretical computer science, applied mathematics and the modelling of natural phenomena, will find here a valuable toolbox. The book is also valuable for self-study and as a reference, and does a great service in bridging the gap between applications/simulations and rigorous mathematical results.

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