Cognition and competence in mental calculation

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Abstract

The present article deals with the performance of Spanish students in a Teacher Training College when solving problems of mental computation with natural and decimal numbers before and after receiving instruction based on formal methods (i.e. methods taken from the written tradition found in arithmetic textbooks). The study seeks to establish the cognitive processes underlying lack of competence and to establish a typology of the different errors of the students.

Resumen

Este artículo trata sobre el desempeño de estudiantes españoles para futuros profesores al resolver ejercicios de cálculo mental con números naturales y decimales, antes y después de recibir enseñanza de los métodos formales (entiéndase que son aquellos que recoge la tradición escrita en los libros de Aritmética). El estudio se orientó hacia la determinación de los procesos cognitivos involucrados en la falta de competencia, y en particular en la tipología de los errores que cometieron .

Introduction

Recently there has been a growing interest in the sources of students systematic errors. Matz (1982), in an article about errors in the solution of algebra problems, suggests that they stem from adaptations, which though reasonable do not always work, of previously acquired knowledge to new situations and that some of these errors in algebra are rooted in a lack of mastery and understanding of arithmetic procedures.

On the other hand it has been empirically established that one of the problems in mental computation is the generalized lack of competence and more specifically in a diminished competence when problems involving decimal numbers, rather than integers, are to be solved. It has been claimed that this is because the concepts are not well developed or because learners lack valid strategies (Reys, Trafton, Reys and Zawolesky, 1984, *cit.* Reys, 1985, p.25).

It has also been argued (Butlen & Pezard, 1991) that mental computation is an excellent method for bringing out and checking students' conceptions of arithmetic, which normally remain concealed when standard pen and paper calculations are performed.

An experimental study based on these ideas has been carried out to determine how far the lack of strategies, insufficiently developed concepts, lack of mastery and understanding of arithmetic procedures and faulty adaptations of previously acquired procedures are the cause of the generalized lack of competence in students who have already demonstrated adequate capacity in arithmetic and algebra at school.

Methodology

The study was carried out throughout the academic year 1992-93 with 144 Spanish students training to be future teachers in their normal venue of study the "Escuela Universitaria de Profesorado" of the University of Valencia.

The students were administered two tests: one prior to instruction in formal methods of mental calculation and another after the period of instruction. Both tests were divided into two one-hour sessions, one session of each test consisted of subtraction problems and the other of multiplication problems. The problems were selected after a number of pilot experiments and were characterised by the following features: they were not excessively difficult, it was possible to use alternative strategies to solve them, they allowed for comparison between data gathered for natural numbers and decimals, the number of problems in each test was such that students were able to solve them in a one hour session.

Pre-instruction Problems

547-189	243-75	1300-875	461-166	265-199	13-8,75	2,23-1,58
$3{,}1^2$	37x0,25	28x35	47x99	41x42	3,4x0,15	64x25

Post-instruction Problems

737-289	634-75.	1400-675.	481-186	245-197	14-7,75.	2,32-1,67	54,7-18,9	24,3-7,5	46,1-16,6
$4{,}1^2$	47x0,25	26x35	37x98	31x42	2,4x0,15	64x75	2,8x0,45	4,9x5,1	19x18

Students were required to solve the problems in their head and although there was no time limit for each item, the time assigned to each session was one hour, which proved to be sufficient. They were instructed to write only the result of their computation so that they had to complete the operation before writing anything. On the same sheet, they then had to explain in their own words how they had obtained the answer. Subsequently the answers and the students' explanations were analyzed and a typology of student errors was established, which was refined by means of individual interviews.

The instruction in methods of mental computation was developed following an experimental program which had been previously devised by the author (Gomez, b 1994) and based on a broad study of the history of mental calculation in the school curriculum.

Unlike the program which is currently popular in school textbooks and which is analytic and case dependant, the experimental program is a global and simultaneous synthesis of formal methods for the four operations based on levels of generalization and common guiding principles.

The students were introduced to this synthesis. The basis of the synthesis was discussed in class to bring out the basic facts of the number system and the properties and concepts of the four operations and to promote flexibility rather than mere training or pointless discussion of which is the best method.

Results and conclusions

Before receiving instruction in formal methods, most of the students were restricted to column methods and the most common errors were those that seemed to stem from carelessness or lapses of memory, however some errors were also observed that appeared to be symptomatic or evidence of more specific problems.

After instruction students showed more flexibility and there was a greater number of error types in particular for symptomatic errors . In the subtraction problems 15 different types of error were identified, of which 7 could be put down to carelessness or lapse of memory and 8 seemed symptomatic; in the multiplication problems, of the 27 types of error 10 seemed to be due carelessness and 17 symptomatic.

The high number of error types may be explained by the way error types have been distinguished, since we have considered as belonging to different types both errors that appear in different methods and errors that appear in the same method but which seem to require a different explanation.

This wide range of error types can, however, be arranged in terms of three categories: rule extrapolation, generalized method, or centring on the modified element instead of on the effect of modification.

1. Examples of Rule extrapolation:

3.1^2 = "96.1. I multiplied 3.1x3.1 ->

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  3.1
  3.1
  3 1
93  .
96.1 "
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This error seems to be an extrapolation of the rule for placing the decimal point from the column based algorithms for adding and subtracting. Here it seems that the student applies the same rule to multiplication so that he brings the decimal point down respecting its initial position.

The mistake of misplacing the decimal point by counting final zeroes in a decimal number:

2.4x0.15= "0.036; 2.4x0.15=1.2x0.30=0.6x0.60=0.036"

seems to be an extrapolation of the rule which counts decimal places of the data in order to place the decimal point in the product. In this case the student seems to count the zeroes to the right of the final significant decimal place, as though he considered 0.60 to be different from 0.6.

The error of eliminating or recovering improperly the decimal point after moving it in both factors

3.4x0.15=10.5 "I multiplied 34 by 1.5 and got 6.0+4.5 =10.5"
(In this example there is the additional error of not recognising the place value of 3 in 34.)

seems to be an extrapolation of the rule used in the division algorithm to eliminate the decimal point in the divisor by moving the decimal point simultaneously in the dividend and the divisor.

2. Examples of errors that seem stem from generalizing methods from one operation to another.

The error of decomposing a number and multiplying its first digit by the other number and subsequently multiplying the result by the second digit:

28x35= "I multiplied 5x28 and then multiplied the result by 3".

appears to be a generalization of the method of addition by decomposition combined with a failure to recognise the place value of the 3.

The error of invalid "adding and taking away" which involves multiplying, not the data given, but the results of adding a number to one of the data and taking away the same number from the other datum.

64x75=4830. "I take 5 away from 75 and add 5 to 64 so I get 69x70 (I multiply by an easier number and I get the same result)

seems to be a generalization of the compensation approach to adding, whereby a quantity is added to one datum and taken away from the other.

The mistake of "front-ending" or place grouping the elements of a multiplication without recognising the intermediate products.

<u>Completed digits</u> (i.e. including their place order zero)

28x35=640 "I multiplied 20x30=600 and then 8x5=40 and I added 600+40 = 640".

appears to be a generalization the method for adding whereby digits with different place values are added separately: units with units, tens with tens.

<u>Isolated digits</u>

3.1^2="9.1. I squared 3 and then 1 (I multiplied 3.1x3.1)"

This variation seems to be evidence of the same theorem implicit in the previous example, but here the student is not aware of the initial decimal form 0. (zero point) before the 1 of 3.1 when this is decomposed into 3+0.1.

3. Examples of errors that seem to result from centring on the element modified rather than on the effect of the modification.

The error of increasing the multiplier by x and subtracting the product of the multiplier and x instead of subtracting the product of x and the multiplicand:

28x35= "994. I multiply 35x30 and I subtract 56" seems to be evidence of centring on the modified element. It seems that here the student does the following: 28x35=(30-2)x35=30x35 -2x28.

The error of subtracting x from the multiplier and adding the product of the multiplier and x instead of adding the product of x and the multiplicand:

41x32=1294. "41 times 30 = 1230 and I add 64".

seems again to indicate centring on the modified element. Here it appears that the student multiplies each of the elements of the decomposition (30+2) by each of the original numbers: 30x41 and 2x32.

The error of miscounting the zeroes after multiplying numbers ending in zero:

47x99="423; 47x100=470-47=423".

appears to indicate centring on a hundred when multiplying by a hundred and not on the effect that it produces.

Implications

The problems identified in student errors seem to originate not so much in the existence of badly developed concepts as in rigid rule applications, generalizations, extrapolations and centring, that could be the consequence of teaching which lays too much emphasis on the creation of automatism at the expense of and of promoting reflection on procedures and on the effect that modifications to data have on the results.

To avoid these undesirable consequences it seems reasonable to propose a change in teaching so as to improve students' conceptions of arithmetic procedures as a meaningful, non-automatic actions on numbers.

References.

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