Models, main problem in TSG10[1]

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The models are present in several of the contributions from the TSG10, who are concerned about the difficulties presented in teaching and learning fractions. Some of the difficulties are related to the generalisation of multiplying and dividing operations. In this study, the alternative historical approaches to tackle this generalisation are analysed. In one of them, connections are sought between models of operations with natural numbers and those with fractions, in order to facilitate the conceptual change, and in the other this conceptual change is avoided.

Key words. Models, difficulties, fractions, division.

1 Important issues in TSG10

1.1 The main lines

In any discipline, the concept that researchers have of their discipline as well as their professional interest and work is what determines the scope of their study. In TSG-10 and in Mathematical Education in general, this scope of study may include three main lines of investigation, which involve a wide variety of problems related to different topics. These lines are:

1. The study of the theoretical foundations of the teaching and learning of mathematics.
2. The study, development, implementation and evaluation of the knowledge involved in mathematics classroom practice.
3. The analysis and concretion of knowledge and practice that supports plans for professionally qualifying and improving mathematics teachers.

1.2 The main topics

In the TSG-10, these three main lines involve the following key mathematical topics: number systems and arithmetic, including operations in the number systems, ratio and proportion, and rational numbers**.**
Although this is not the time nor place to focus the debate on what “arithmetic” really includes or should include, it is worth pointing out that the relation of key topics is given by the internal logic of mathematics, as opposed to what one see in the Handbook of the NCTM that have recently appeared (Lester, 2007), where the research logic aimed at building and understanding mathematical concepts is adopted. Namely: Whole number concepts (Single-digit computation, multi-digit computation, estimation and number sense, word problems, the

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structure of the whole-number system, and rational number concepts (fractions and decimals, ratio and proportions) (see, Lester, 2007).

The reason for pointing this out is to highlight two ideas. One is that the fundamental nucleus of the research effort of the TSG-10, ICME 11, is the mathematical content with respect to objects that must be taught and learnt.[2] And the other is the direction taken by the reflection that follows.

1.3 The main problems

Almost thirty years ago, in a Plenary Meeting of the ICME 4, Freudenthal, one of those who helped to re-launch the International Congress on Mathematical Education in 1969, stated what in his opinion were the main problems in mathematics education.

Allow me to start with the most down to earth problem I can think of. Among the major ones it is the most urgent. There is even the problem of how to formulate it correctly and unmistakably. Let us try a pre-formulation. It runs (thus*): Why can Johnny not do arithmetic?*

Freudenthal (1981, p, 134) was referring to how to approach the question as to why many children do not learn arithmetic as they are expected to.

Since then, the research carried out has shown that the problem is so complex that the solution to general problems of teaching and learning mathematics is a long way off. This has led the concerns of a good many researchers to still be focussed on understanding and approaching didactic problems, and on developing trustworthy criteria to evaluate their eventual advances and relevance, etc. It is difficult for there to be plenty of results directly applicable to the classroom (Hitt, 2001, p. 166).

1.4 Issues in TSG-10

The different papers submitted by participants, reviewed and accepted by the TSG-10 organizational team, consider the three lines of research and the key topics that have been indicated in the previous points. For the oral presentation of these, they have been divided into two sessions; one of these includes issues, which could refer to the wide domain of research known as the conceptual multiplicative field; in this way a variety of key topics can be tied in, from the multiplication of natural numbers to fractions (thanks to the proportional nature of rational numbers).

The papers provide information about aspects of individual performance on, and understanding of, these key topics from different angles (discontinuities of models, linearity, number sense, posing problems) and different perspectives (children’s, prospective teachers, and in-service teachers). In several of these, models and modelling are implicitly or explicitly present, related to the limited competence and conceptions in the domain of fractions

2 Models and the division of fractions

2.1 Models and modeling

Although the different meanings of the terms ‘models” and “modelling” have been widely dealt with in the literature (for example, Gravemeijer, Lehrer, & Vereschaffel, 2002), in what follows the term “model” is used in the sense that is compatible with what is called ‘mathematical modelling”.

[2] Seeking to identify and characterise the learning problems they bring with them and the mathematical discourse that is best adapted to the students in each educational circumstance.

And mathematical modelling is used to refer to the process of building mathematical objects that symbolically reproduce essential characteristics of a phenomenon or situation from the real world that one intends to study.

The purpose of these mathematical models of reality is to explain, deduce or predict results, draw conclusions, or answer questions about the real phenomenon or situation that they model. This is what is done, for example, with the derivative, when it is taken as a mathematical model of instantaneous velocity.

2.2 The reversible nature of modelling in elemental arithmetic. Two sides of a coin.

In the teaching of elemental arithmetic, the modelling process can be seen as the reverse of what usually appears in mathematics in general. What is first provided is a physical phenomenon situation that acts as a model for studying the mathematical object, such that instead of modelling the phenomenon or situation by means of the mathematical object, the mathematical object is modelled by means of the phenomenon or situation[3].

Thus, for example, temperature is taken as a phenomenon to model integers, a pizza or a cake is taken as a situation to model fractions, and certain word problems are taken as models of elemental mathematics operations.

When the teacher considers that the child, aided by the physical model, has built up a “good enough” conception of the mathematical object, he or she then inverts the terms again and proposes different questions that can be modelled by this mathematical object, thus showing that it has «applications» that «are useful for solving real problems».

As the phenomena or situations are closer to the child’s daily experience, they produce significant interpretations about the arithmetic notions that one wants to teach. However, the limitations of teaching means that only one or several of the modelling phenomena or situations are chosen, not all the possible ones. As a consequence, a restriction is produced in the semantic field and a conceptual limitation.

2.3 Difficulties with the multiplication and division of fractions

Traditional teaching in the multiplication and division of fractions emphasizes drill and practice with the focus on algorithms, on numerical examples, not on word problems (Only one side of a coin). When teachers propose resolving different word problems, the children’s activity is reduced to the selection and execution of the operation to be modelled.

But, to date, several studies have shown children’s difficulty in the selection of an operation for solving a big variety of fraction, multiplication and division problems.

The efforts aimed at understanding how children attempt to solve multiplicative word problems have shown that children’s ability to solve these problems is influenced by a large variety of factors interacting in multiple ways. Among these factors it is worth noting the following:

a) The presence of certain key words in the problem text, the association between the situation described in the problem and some of the primitive models of operations, the type, size and structure of numbers embedded in the problem text and their relation with the result of this calculation.

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[3] *We learn applications in order to learn multiplication* (Usiskin, this issue).

b) The way in which teachers conceive and treat problems in the classroom. Their very poor preparation to plan their fraction lessons; limitations of the models which they use to represent the concept of division, and their “remarkable dependence on the official teaching books, which reduced her educational creativity and autonomy” (Valdemoros, this issues)

In the different theoretical approaches that exist for explaining difficulties in students’ competences and conceptions in the domain of fractions, one common aspect of several approaches is the emphasis on discontinuities between natural and fractional numbers.

To explain students’ difficulties with these discontinuities, one theoretical approach is the conceptual change approach and another is the theoretical approach which emphasizes the importance of underlying mental models (Prediger, in this issue).

With multiplication and division operations, on changing the numeric field the operations are no longer the same, though their name does not change. Using the terminology of “levels” used by Predinger, it may be said that on the formal level the definition changes, on the algorithmic level the rules change, and on the intuitive level the meanings change.

Predinger, attempts to reconcile (or integrate) both theoretical focuses by relating conceptual change with the change in some mental models of multiplication.

not all mental models for multiplication have to be changed in the transition from natural to fractional numbers. The interpretation as an area of a rectangle or as scaling up can be continued for fractions as well as the multiplicative comparison. In contrast, the basic ‘repeated addition’ model is not sustainable for fractions, nor the combinatorial interpretation. Vice versa: the basic model of the multiplication of a fraction, the part-of-interpretation, has no direct correspondence for natural numbers (Predinger, this issue).

The same could be said of the division of fractions, where it does not work in the partition model, whereas the measurement model works well enough in both directions: writing a mathematical expression for a problem and problem posing for a mathematical expression (see Carbone & Eaton, this issue, related to a prospective teacher posing problems that shows th the meaning of $2\frac{1}{2} \div \frac{1}{2}$).

2.4 The historical approach in old textbooks

How should children learn? Or, how do people learn?

This is the question with which Freudenthal (ob. cit. p, 137) drew up his second major problem. To which he answered that

The way to answer it would be: by observing learning processes, analysing them and reporting paradigms.

And that, amongst the learners,

the biggest one, mankind, is also a learner. Observing its learning process is what we call history

To observe history one must turn to textbooks of the past. There, one may track the process of generalizing the division of whole numbers to the division of fractions.

1. In the first arithmetic texts printed, which followed Arabic traditions, the definition of multiplication and division operations were introduced through the proportional model.

Multiplication

To multiply one number by itself or by another is to find from two given numbers a third number which contains one of these numbers as many times as there are units in the other (Treviso Arithmetic, 1478/1989, p. 67).

Division, with two options: $\frac{\boldsymbol{D}}{\boldsymbol{d}} \div \frac{\boldsymbol{c}}{\boldsymbol{1}}$ (1) and $\frac{\boldsymbol{D}}{\boldsymbol{c}} \div \frac{\boldsymbol{d}}{\boldsymbol{1}}$ (2)

1. *I say that division is the operation of finding, from two given numbers, a third number which is contained as many times in the greater number as unity is contained in the lesser number* (Treviso Arithmetic, 1989, p. 85).

2. *Dividing one number by another means looking for another third number which is to be found with unity in such a proportion as the number that we divide with the divisor* (Pérez de Moya, 1562/1998, p. 197).

Influential authors like Lacroix (1825, p. 47) accompanied these definitions with commentaries aimed at explaining the conceptual change, in terms that intended to extend "to all cases" the definition that they had previously adopted of multiplying or dividing (repeated addition or partition).

In these commentaries the authors began to show the discontinuity of the model, beyond natural numbers, and later they made an effort to create connections between the previous definition and the new one, making us see that the first one is a specific case of the second one.

Multiplication

The doctrine of fractions enables us to generalize the definition of multiplication given in article 21. When the multiplicand is a whole number, it shows how many times the multiplicand is to be repeated; but the term multiplication, extended to fractional expressions, does not always imply augmentation, as in the case of whole numbers. To comprehend in one statement every possible case, it may be said that to multiply one number by another is to form a number by means of the first, in the same manner as the second is formed, by means of unity. In reality, when it is necessary to multiply by 2, by 3, etc. the product consists of twice, three times, etc. the multiplicand, in the same way as the multiplier consists of two, three, etc. units; and to multiply any number by a fraction, 1/5 for example, is to take the fifth part of it, because the multiplier 1/5 being the fifth part of unity, shows that product ought to be the fifth part of the multiplicand. Also, to multiply any number by 4/5 is to take out of this number or the multiplicand a part which shall be four fifths of it, or equal to four times one fifth.

Division

The word 'contain', in its strict sense, is not more proper in the different cases presented by division, than the word 'repeat' in those presented by multiplication; for it cannot be said that the dividend contains the divisor, when it is less than the latter; the expressions is generally used, but only by analogy and extension.

To generalize division, the dividend must be considered as having the same relation to the quotient, that the divisor has to unity, because the divisor and quotient are the two factors of the dividend. This consideration is applicable to every case that division can present. When, for instance, the divisor is 5, the dividend is equal to 5 times the quotient, and consequently, the latter is the fifth part of the dividend. If the divisor is a fraction, ½ for instance, the dividend cannot be but half of the quotient, or the latter must be double that of the former.

2. In more recent textbooks, some authors like Rey Pastor and Puig Adam (1935, p. 210), introduced a conception of the operations of multiplying and dividing fractions that did not implement the conceptual change.

The multiplication of fractions was presented, on the basis of an interpretation of fractions as an operator, as a double operation of multiplying and dividing by a natural number:

The so-called problems of multiplication of fractions are, strictly speaking, problems of combined multiplication and division.

The division of fractions was presented as an inverse multiplication operation.

Showing the operation $\frac{\boldsymbol{2}}{\boldsymbol{9}} \div \frac{\boldsymbol{3}}{\boldsymbol{7}}$ *means: Is there a fraction which when multiplied by* $\frac{\boldsymbol{3}}{\boldsymbol{7}}$ *gives* $\frac{\boldsymbol{2}}{\boldsymbol{9}}$ *?*

2.5 Key implications for teaching and learning

In the process of generalizing the division of whole numbers to the division of fractions, according to the textbooks there are two options; in one, attention is paid to the conceptual change, in the other it is ignored.

1. When opting for conceptual change, in which the multiplication and division of fractions is conceived through the proportional model, the quaternary relationship is made to appear (described by Vergnaud 1983) which allows direct operations of multiplying or dividing to be identified as "missing-value proportional problems (in which three numbers are given, one of them is unity, and a fourth is asked for).

In this way, word problems of direct operations that involve fractions can be solved using a general method: the rule of three, which implicitly involves the idea of linearity[4].

However, the rule of three is often learnt routinely, giving priority to procedural knowledge (the rules that prescribe how to organise and operate the data), above conceptual knowledge, which is what is needed to exercise control of "linearity" and to limit pupils' tendency to over–use the proportional model. This is a tendency that increases the more pupils acquire linear reasoning skills through practising and solving typical linear problems (Van Dooren, De Bock & Verschaffel, 2006, p. 120) and which is stimulated by various factors, among them the numerical structure of the data involved in the problem (Van Dooren, De Bock. Evers & Verschaffel (this issue).

2. When choosing to drop the conceptual change, the focus is on the approach based on interpretations associated with fractions, and models with natural numbers.

If we look at textbooks we can see that the application of this approach to problem-solving is based on 'analytical' methods (reducing a problem to a simpler case that one already knows how to solve) that are based on a change of unity, through unitary fractions or through reduction to a common denominator.

This can be seen, for example, in the way of solving the following problems taken from the text by Rey Pastor & Puig Adam (ob. cit., pgs. 209-211):

Example. 1. *Each metre of cloth costs 3/5 euros. How much do 7/4 m cost?*

Example 2. $\frac{3}{7}$ *of pizza weighs* $\frac{2}{9}$ *kilos. How much does the pizza weigh?*

Example 3. *If each pizza weighs* $\frac{3}{7}$ *kg, what portion of pizza will I have with* $\frac{2}{9}$ *kg?*

These three problems are direct operation problems, but in this operation it is not easily recognisable.

To explain that they correspond to the multiplication $\frac{3}{5}\cdot\frac{7}{4}$ and the division $\frac{2}{9}\div\frac{3}{7}$, the author of the text resorts to 'analysis' of the problem statement to reduce the problem to a "recognisable" form.

Example 1

[4] This idea can be treated from different perspectives: through proportionality, through first-degree equations, and through linear function. Equations and functions are not formally a part of arithmetic. De Bork (this issue) uses the term linearity through the perspective of arithmetic, so he refers it to the 'proportional model'.

We will indicate the operation thus: $\frac{3}{5}\cdot\frac{7}{4}$*; and we will say: If each m costs* $\frac{3}{5}$ *euros, a fourth of a m, that is to say* $\frac{1}{4}$ *m, will cost* $\frac{3}{5}\div 4=\frac{3}{5\cdot 4}$ *euros, and* $\frac{7}{4}$ *will cost seven times more, that is to say* $\frac{3\cdot 7}{5\cdot 4}=\frac{21}{20}$ *euros.*

Here, the author connects with the interpretation of fractions as "operators".

Example 2

*If **3** sevenths of pizza weigh* $\frac{2}{9}$ *kilos,*

***1** seventh of pizza will weigh* $\frac{2}{9}\div 3 = \frac{2}{9\cdot 3}$

*and the whole pizza = **7** sevenths of pizza, will weigh* $\frac{2}{9\cdot 3}\times 7=\frac{2\cdot 7}{9\cdot 3}$

One notices in this last paragraph that on taking “the sevenths” as a new unity, the problem has been reduced to a known multiplication model with natural numbers: “if 1 unity weighs x, how much will 7 units cost?”.

Example 3

Reducing the weights to the same proportional part of kgs, we are posing the question in this different way: If each pizza weighs $\frac{3\cdot 9}{7\cdot 9}$ *kilograms, how much will I have for* $\frac{2\cdot 7}{9\cdot 7}$*?*

Hence, taking $\frac{1}{7\cdot 9}$ *kg for a new unit, the pizza weights 3·9 units, so with 2·7 units I will have a portion of pizza equal to* $\frac{2\cdot 7}{3\cdot 7}$

This abstract quotient has the same expression as before; but now it represents a fraction of cake, whereas before it was of a kilo.

Just as before, taking the common denominator as a new unity, the problem is reduced to a known division (measurement) problem with natural numbers.

3. Conclusions

Research has suggested that teachers and textbook authors should revise the notions of multiplication and division of fractions.

With respect to this matter, the immediate question is how to approach the process of generalising the multiplication and division of natural numbers to fractions:

a) To help students through the approach of models, it will be necessary to help them create the connections hidden among the natural number models and fraction models.

b) However, if multiplication and division of fractions are notions resistant to models, as happens with multiplying negative numbers, then it will be necessary to drop the model approach or else resort to an ‘analysis’ of the problem in order to reduce it to a simpler one that the student already knows how to solve. Otherwise, there is a third way which would direct the conceptual change towards a more formal mathematical conception.

In any case, in order to answer the proposed question with a sound base, it seems there is still the need for much more research to be done.

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