From Fresnel patterns to fractional Fourier transform through geometrical optics

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Abstract. A convenient relation between fractional Fourier transform patterns and diffraction patterns is obtained by applying the Gauss equation of geometrical optics. Thus, Fresnel and fractional domains cannot be considered as independent domains, since one is just a geometrical image of the other, providing a physical and direct connection. © 2000 *Society of Photo-Optical Instrumentation Engineers.* [S0091-3286(00)02006-7]

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In 1980, the fractional Fourier transform (FRT) was introduced by Namias[1] with applications to quantum mechanics. This transform remained unknown to many scientific communities until Ozaktas and Mendlovic,[2,3] and Lohmann[4] introduced it into optics. From that moment, many publications have successfully applied this operation in the area of signal processing, and have extended the concept of fractionalization to other mathematical transformations in optics (see Ref. 5 for an extensive review of these operations and their applications).

The FRT is a generalization of the ordinary Fourier transform depending on a parameter p ($p=1$ for the ordinary Fourier transform) and can be interpreted as a rotation of the original distribution in a space-frequency domain by an angle $p\pi/2$. From a physical point of view, the FRT describes the diffraction patterns of a light distribution when it propagates through a graded-refractive-index (GRIN) optical fiber. From this perspective, since both FRT and Fresnel integral describe light propagation, it is clear that they must be related.

The advantages that can be obtained from this relation are clear: the FRT provides a compact and coherent way of describing the scalar diffraction problem and introduces a new perspective into signal processing in domains different than object or Fourier domains.[6,7] Moreover, unlike the Fresnel integral, the FRT can be accurately calculated through fast numerical algorithms.[8,9] On the other hand, the Fresnel diffraction integral provides a more intuitive description. Thus, it is interesting to develop an easy connection between both formulations, in order to exploit the advantages of both points of view.

The relation between the fractional Fourier transform and the free-space propagation integral has been stated in many papers through different approaches. In Ref. 10, it is shown, through a formal solution of the wave equation, that the field distribution at any distance from the object can be represented as an FRT of the input distribution corrected by a scaling function depending on the distance and an additional phase factor. The same conclusion can be reached through the ABCD matrix formalism (see Refs. 11 and 12 for instance). In those papers, it is shown that FRT and Fresnel integrals are particular cases of a generic ABCD matrix. Another approach to the problem can be found in Ref. 13, where the FRT is adapted to the expression for Fresnel diffraction in the same way that the standard Fourier transform is adapted to Fraunhofer diffraction.

The above-cited methods provide the same relation between the two integrals, but always from a mathematical point of view, and thus, an optical or physical interpretation of that relation must be given *a posteriori*. Here we present a connection between the FRT and Fresnel patterns from a new point of view. The approach is from a novel perspective, closer to that of the experimentalist. In what follows, we demonstrate that FRT and Fresnel patterns are related through a simple imaging relation. Application of the Gauss equation to a modified Lohmann's Type II setup (see Figs. 1 and 2) will determine the relation between a Fresnel distribution (which will be considered as the object) and a FRT distribution (which will be considered the image). The method here developed admits an easy physical interpretation: since object and image are not isolated but complementary domains, it makes no sense to speak about pro-

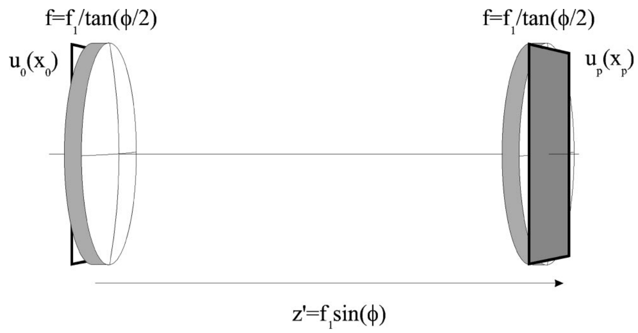

Fig. 1 Lohmann's Type II optical setup for obtaining the FRT pattern of order $p=2\phi/\pi$ of the input object $u_0(x_0)$.

cessing into Fresnel or FRT domains as separate issues, and any operation in an FRT domain will have its counterpart in a Fresnel domain and vice versa.

The Fresnel pattern produced by an object u_0 at a distance z when it is illuminated by a monochromatic plane wave whose wavelength is λ can be expressed as

$$u_z(x_z)=\exp\left(i\frac{\pi}{\lambda z}x_z^2\right)\int_{-\infty}^{+\infty}\left[u_0(x_0)\exp\left(i\frac{\pi}{\lambda z}x_0^2\right)\right]\times\exp\left(-i\frac{2\pi}{\lambda z}x_z x_0\right)\,\mathrm{d}x_0, \tag{1}$$

where constant factors have been dropped. In this expression, and in the remainder of the paper, we will use a 1-D formulation. Extension to 2-D is straightforward.

The FRT of order $0<p<2$ of an input function $u_0(x_0)$ provided by a Lohmann Type II system[4] (see Fig. 1) can be expressed as

$$u_p(x_p)=\exp\left(\frac{i\pi}{\lambda f_1\tan\phi}x_p^2\right)\int_{-\infty}^{\infty}u(x_0)\exp\left(\frac{i\pi}{\lambda f_1\tan\phi}x_0^2\right)\times\exp\left(-\frac{i2\pi}{\lambda f_1\sin\phi}x_0x_p\right)\,\mathrm{d}x_0, \tag{2}$$

where $\phi=p\pi/2$, λ is the illuminating-light wavelength, and f_1 is an arbitrary fixed length.

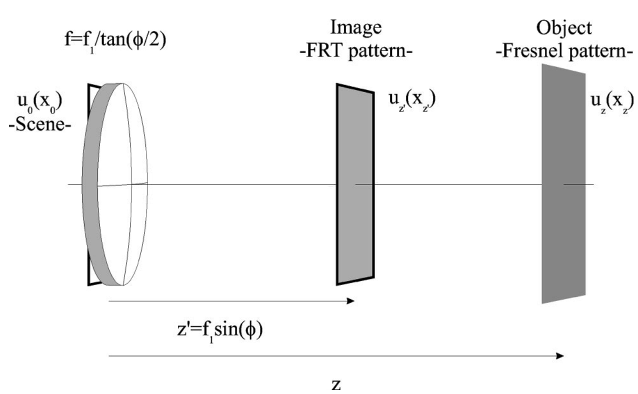

Fig. 2 Modified Type II system, considered as an imaging system.

Comparing the Fresnel diffraction expression in Eq. (1) and the FRT expression in Eq. (2), one can notice that they are quite similar. In fact, the only difference is that the scaling factors affecting the exponential functions are different. In the Fresnel-integral case, these factors are the same for the outer factor and for the ones inside the integral. This equality allows writing the Fresnel integral as a convolution, while the FRT cannot be written in this manner. Thus, converting Eq. (2) into (1) will require, at least, scaling of the input function, and then applying an additional quadratic phase factor. As was said at the beginning, these corrections can be found just by algebraic manipulation of the integrals or through an optical setup that performs such conversion.

Let us consider the Type II Lohmann system depicted in Fig. 1. Note that the second lens in front of the output plane can be removed, and the final FRT pattern in Eq. (2) will be modified only by a quadratic phase factor.[14] This factor is the complex conjugate of the phase factor that would be introduced by the lens. We will call this system the modified Type II system; except for this well-defined quadratic phase factor, it will provide the same output as the Type II system.

In order to obtain Fresnel diffraction patterns, the modified Type II system can be considered as an imaging system. For this purpose, the output plane must be considered as the image of the Fresnel pattern rendered by the lens that would be obtained without it, i.e., an FRT pattern can be considered as a geometrical image of a certain Fresnel pattern.

The connection between the FRT distribution at the output of the Type II system and its corresponding Fresnel pattern can be made through the Gauss equation of geometrical optics:

$$-\frac{1}{z}+\frac{1}{z'}=\frac{1}{f}, \tag{3}$$

where z and z' are the distances of the object (Fresnel pattern) and the image (FRT pattern) from the lens, respectively (see Fig. 2). The focal length f is determined by the Type II setup. Note that, since the input scene is attached to a thin lens, all the distances can be referred to the object itself. Note also that, like every imaging system, this one performs a magnification over the output, given by $\beta=z'/z$. Summarizing, and bearing in mind Fig. 2, the physical magnitudes determining this system are:

$$\begin{aligned}&\text{object-lens distance} && z=f_1\tan\phi,\\ &\text{lens-image distance} && z'=f_1\sin\phi,\\ &\text{lens focal length} && f=f_1/\tan(\phi/2),\\ &\text{magnification} && \beta=\cos\phi,\end{aligned} \tag{4}$$

where the object must be identified with a Fresnel pattern, and the image with a quasi-FRT pattern.

Let us now consider carefully the modified FRT-imaging system depicted in Fig. 2. From Ref. 15 it can be derived that such a system provides an impulse response:

$$h(x_z, x_{z'}) = \exp\left[\frac{i\pi}{\lambda}\left(\frac{x_{z'}^2}{z'} - \frac{x_z^2}{z}\right)\right]\delta\left(\frac{x_{z'}}{z'} - \frac{x_z}{z}\right). \tag{5}$$

Taking the Fresnel pattern of the scene $u_z(x_z)$ as the input object, the image provided by the system is given by

$$u_{z'}(x_{z'}) = u_z\left(\frac{x_{z'}}{\beta}\right)\exp\left[i\frac{\pi}{\lambda z'}x_{z'}^2\left(1 - \frac{1}{\beta}\right)\right]. \tag{6}$$

As we pointed out before, $u_{z'}(x_{z'})$ must be identified with an FRT pattern affected by two quadratic phase factors: one coming from the absence of the second lens in the quasi-FRT system, and the other coming from the impulse response of the system.

Since we are interested in the Fresnel distribution, and not in the image output of the system, it is more convenient to write Eq. (6) in the form

$$\begin{aligned} u_z\left(\frac{x_{z'}}{\beta}\right) &= u_{z'}(x_{z'})\exp\left[\frac{i\pi}{\lambda f_1}\left(\frac{1-\cos\phi}{\sin\phi\cos\phi}\right)x_{z'}^2\right] \\ &= \exp\left[\frac{i\pi}{\lambda f_1}\frac{x_{z'}^2}{\sin\phi\cos\phi}\right]\int_{-\infty}^{\infty}u_0(x_0) \\ &\quad \times\exp\left(\frac{i\pi}{\lambda f_1\tan\phi}x_0^2\right)\exp\left(-\frac{i2\pi}{\lambda f_1\sin\phi}x_0x_{z'}\right)\,\mathrm{d}x_0 \\ &= \exp\left(i\pi x_{z'}^2\frac{\tan\phi}{\lambda f_1}\right)F^p[u_0(x_0)], \end{aligned} \tag{7}$$

which gives the final conversion formula from FRT to Fresnel patterns. As was expected, a scaled Fresnel pattern is equivalent to an FRT pattern multiplied by a correcting quadratic phase factor.

The expression just obtained with the ''geometrical method'' coincides with the relations obtained in Refs. 10, 11 through other methods. What has been presented here is an optical connection between the two patterns. It has been proved that any FRT process can be viewed as a process in some diffraction plane, or vice versa. Therefore this approach may provide a new point of view when implementing FRT or Fresnel-based setups, since the two concepts are easily interchangeables. An application of this conversion formula can be found in Ref. 16. In that paper, several fast Fresnel calculation algorithms are presented. Computer simulations are described there that demonstrate the feasibility of the Fresnel-through-FRT approach.

Acknowledgments

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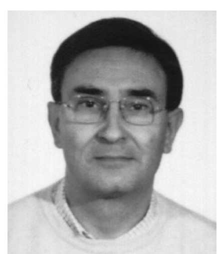

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