Exploring the links between Van Hiele Levels and 3-dimensional geometry

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Abstract

This paper presents some hypotheses about the use of the Van Hiele Model of thinking as a framework for understanding the processes of learning in 3-dimensional Geometry.
In this area it is specially important to differentiate the processes concerning spatial visualization from those which relate to solids (polyhedrons, etc.) and their properties. They are two inter-related but different topics, having each one its own characteristics. Therefore, two different fields of research can be considered (although both are strongly related): The use of the Van Hiele Model to understand and organize the acquisition of abilities of spatial visualization and the use of the Van Hiele Model to understand and organize the learning of 3-dimensional Geometry.

This paper deals with the first field, presenting results of an exploratory experiment aimed at improving the abilities of spatial visualization in sixth-grade students. Possible

descriptors of the Van Hiele levels in this context are stated on the basis of the students' behaviour and answers.

1. Introduction

Since I. Wirszup reported about the Soviet Curriculum in Geometry in the 60's [28], the Van Hiele Model of Geometrical Thinking has become a continuous focus of attention among researchers in Mathematical Education. The Van Hiele Model consists of two parts:

- The first one, the "levels of thinking", is a description of the ways of thinking that can be found in the student's Geometry. The Van Hiele Model states that a student can progress through several levels of reasoning during his/her learning process. The main educational concern for the Van Hiele Model is the progress from a level to the following one; this progress cannot be taught, but it is highly dependent on the kind of teaching.
- The second part of the Van Hiele Model, the "phases of learning", is a suggestion to the teachers on how to organize the teaching of Geometry in order to facilitate and promote the students to pass from their current level of thinking to the following one.

The Van Hiele levels have been described in detail several times[1] (see [4], [7], [11], or [18]).

From the Van Hiele levels, some properties may be apparent:

- Levels are **ordered** and **sequential**: Each level means an improvement over the reasoning abilities of the previous one. Then a student can reach level n ($n \geq 2$) only if he/she has previously reached level n -1.
- Each level has its **own language**: There is a close relationship between levels and language; for example, we have shown three different meanings of the word "proof": Verification (in level 2), infbrmal deduction (in level 3), and formal deduction (in level 4). Then, two persons speaking (therefore reasoning) at different levels cannot understand each other ([10], p. 246).
- Levels are **continuous**: Although the original Van Hiele's statement was in favour of the discreteness of the levels, results ftom further research has currently led to most of the researchers to consider that the movement from a level to the following one is a continuous process, since the acquisition of a thinking level by a student is gradual and it can be observed along the time.

2. The problematical

For the last 14 years much research has focused on the Van Hiele Model as an aim of research by itself, to determine or analyse the characteristics of the levels of thinking ([8], [9], [13]), as a tool, to evaluate the students' achieve me nt in a particular environment ([4], [14], [15], [22], [27]), or as a framework to design experimental teaching units ([S], [1] [18]). Nevertheless, most relevant results obtained from these investigations refer to plane Geometry and, almost always, to polygons and related concepts, where we can find accurate relationships of descriptors for every Van Hiele level, as

[1] Some authors number Van Hiele levels from level 0; in this paper they are numbered as follow: Level 1 (Recognition), Level 2 (Analysis), Level 3 (Informal deduction), Level 4 (Formal deduction).

those in [4] and [11].

At present, the Van Hiele levels are quite accurately identified for some geometrical topics, but we know very little about them in other geometrical areas, such as 3-dimensional Geometry. This lack of research, shown in [6] and [16], seems more paradoxical if we recall that there have been several National Primary and Secondary curricula where geometrical contents and methodology were based on the Van Hiele Model: The Soviet curriculum in the 60's [25], the Dutch curriculum in the 70's [26], and the new North-American Standards [23].

There has been several isolated approaches to 3-dimensional Geometry based on the Van Hiele Model. The first attempt to characterize explicitly the Van Hiele levels in 3-dimensional Geometry appeared in [17], where, several geometrical skills (visual, verbal, drawing, logical and applied skills) were described and short general theoretical descriptions for every skill and Van Hiele levels 1 to 5 were stated; some examples for visual and drawing skills which referred to solids were considered, but even then, it is not possible to derive from this paper a specific conception of ways of thinking in 3-dimensional Geometry based on the Van Hiele levels.

Other research relating the learning of 3-dimensional Geometry and the Van Hiele levels was carried out by D. Lunkenbein, who described the different ways in which his pupils worked with polyhedrons, matching totally the characteristics of Van Hiele levels 1 and 2 [19]. On the other hand, taking the Piagetian concept of "grouping" as the starting point, [20] defined three types of groupings (infralogical, logical partition, and logical inclusion) which would fit Van Hiele levels 1, 2, and 3, respectively. Later on, groupings were used in [21] to analyse the different ways used by the students when solving activities with polyhedrons. This is an interesting proposal, although it only allows to partially characterize the lower Van Hiele levels, as logical classification is not the only thinking activity involved in these levels.

When the teaching or learning of 3-dimensional Geometry is mentioned, spatial visualization arises immediately as an element to bear in mind. It is evident that the spatial abilities, especially visualization, are a basic component in this field, as it has been proved that the students' degree of development in such abilities influences their achievement in attending a course in 3-dimensional Geometry. For instance, a bad visualizer will likely bee in trouble to properly understand the graphical contexts of the textbook and to clearly express his/her own ideas. Nevertheless, both aspects are not to be identified, as the capability of visualizing and any other spatial ability are only some of the various factors influencing the way the students build their network of geometrical concepts, properties, and relationships (in 1, 2, or 3 dimensions). Therefore, when a process of learning 3-dimensional Geometry is to be analyzed, it is necessary to distinguish between:

- Acquiring and using what is usually known as 3-dimensional Geometry: knowledge and classification of the various types of solids, in particular polyhedrons, their structures, elements, geometrical and metric properties, etc., in order to solve problems.
- Acquiring and developing the spatial abilities which are

present in the activities used for the learning and use of 3-dimensional Geometry. According to Ben-Chaim et al. [1], by spatial abilities one has to understand the set of skills of representing, transforming, generating and using non-linguistic information, from which the abilities which conform the spatial visualization, as described by Bishop [2], stand out.

From what has been written above, it can be said that research con the learning processes in 3-dimensional Geometry from the point of view of the Van Hiele Model is needed, and also that it is necessary to divide the problem by considering, on one hand, the acquisition of the spatial abilities and, on the other hand, the understanding of the networks of geometrical concepts. Although it is quite easy to work in spatial visualization avoiding the use of any aspect of 3-dimensional Geometry, it is probably not possible to work in 3-dimensional Geometry without considering some aspects of visualization, so the results of these two investigations could be different. Consequently, it seems that a complete analysis of the learning processes in 3-dimensional Geometry should be subordinated to an analysis of the acquisition of the abilities of spatial visualization.

In this paper we give some ideas about some possible characteristics of the Van Hiele levels in the topic of spatial visualization. This proposals is based con a research in which we have been involved, consisting of the design, and experimentation with sixth-grade students of activities of manipulating solids aimed at developing the abilities of spatial visualization. The final aim of this experiment was to obtain, in a future, operative and detailed characterizations of every Van Hiele level in terms of the student's behaviour in the topic of spatial visualization.

3. The Van Hiele levels and the spatial visualization[2]

The literature presenting results of research in visualization not always refers to the same concept "visualization" since some researchers include in the field of visualization abilities or types of problems which are not taken under consideration by other researchers working in different investigative paradigms. To define the context of our research, I will point out the distinction made by Bishop [3] between objects of visualization and processes of visualization. The objects of visualization identified by Presmeg [24] are concrete imagery, pattern imagery, memory images of formulae, kinetic imagery, and dynamic imagery. The relevant processes of visualization are *interpreting figural information* (IFI) and the *visual processing* (VP) of abstract information identified by Bishop [2].

Some of the objects and processes of visualization stated above were present in the research that we carried out, applied to the use of 3-dimensional figures: The dynamic imagery was present in the students' activity, as they had to move objects physically and mentally. The students also used the kinetic imagery, for instance, when moving their hand to help themselves to determine the rotation which should be applied to a solid on the computer screen. Some IFI abilities, in particular understanding of visual representations used in geometric work, and VP abilities, such as ma-

[2] The results presented on this section are part of a research project funded by the Institución Valenciana de Estudios a Investigación "Alfonso el Magnánimo" de Valencia (Spain).

nipulation and transformation of visual representations and imagery, were also used by the students.

Gaulin [12] presents a complete description of the usual kinds of plane representations of 3-dimensional objects. From this experimentation we chose perspective drawings, orthogonal drawings on isometric dot paper, side views, and side views with numeric information (we named them "numerical views"). They were chosen because they fit well the materials used by our students.

Students usually work in three contexts when they handle 3-dimensional objects or they study space Geometry: Manipulation of real physical objects, manipulation of 3-dimensional representations on a computer screen and reading or drawing plane representations on paper. Each one of these contexts, being important for the learning process and the everyday life, has advantages and disadvantages. The physical objects are the most versatile and easiest to manipulate. Although this easiness hides many of the manipulations involved in working with solids. The students lacks awareness of the links between these physical movements and the mental movements requited in other contexts. Plane representations are the most frequent in our world, the ones which can give the most complete information about the characteristics of the represented solids, but the most difficult to be mentally manipulated. Previous learning of some reading codes is required. Computers are a very valuable aid for teachers, although research about their advantage with respect to other classical contexts shows contradictory results; according to their advantages and disadvantages, computers can be located in between real solids and plane representations, closer to one or the other according to the specific software used.

The microworld that we created integrated the three contexts (physical objects, plane and computer dynamic representations), related to each other through the use of the same geometric solids, so that the students' activity always took place in two of the contexts simultaneously (**Figure 1**).

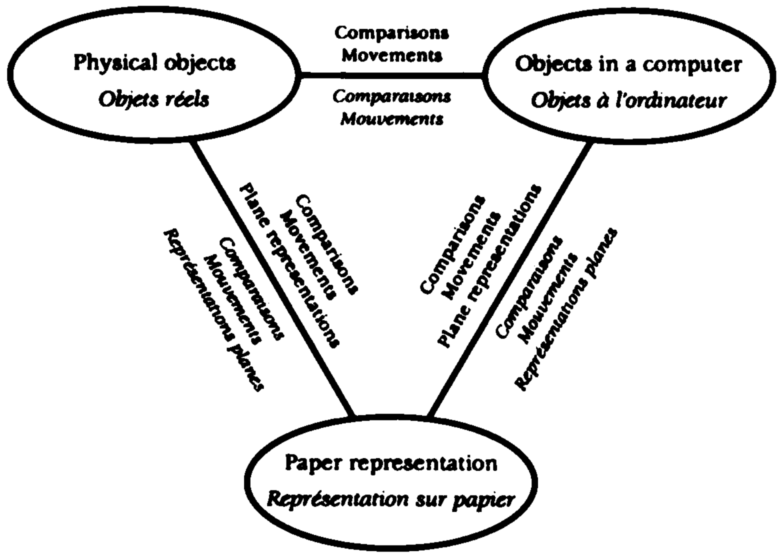

Figure 1

An analysis of the students' behaviour when solving activities of comparing or moving solids is the ground for the following proposal of characteristics of the Van Hiele levels in the field of spatial visualization. It must be noted that the descriptors are closely related to the microworld used and, in particular, to the software.

Level 1 (Recognition). The comparison of solids is based on a global perception of the shapes of the solids or some particular elements (faces, edges, vertices) without paying attention to properties such as angle sizes, edge lengths, parallelism, etc. When some one of these mathematical characteristics appears in a student's answer, it has just a visual role.

Students are unable to visualize both solids or positions of solids that they cannot see, neither their movements. Then, students move (physically, or on a computer screen) a solid by guess, in spite of the fact that they may have a specific objective. First a movement (a rotation in our experiment) is selected. If it does not produce any suitable result, another movement is selected. If a movement does produce an acceptable result, then a more accurate sequence of movements may follow.

At the first level there is a] a lack of coordination between the desired movement (usually figured out by moving the hands) and the one actually selected on the computer screen. It is probably a consequence of the students' lack of ability for visualizing the effect of movements on the screen. For example, the action of moving a hand from left to right, may be interpreted as a positive turn around axis y or a negative turn around z (**Figure 2**).

Level 2 (Analysis). The comparison of solids is based on a global perception of the solids or their elements leading to the examination of differences in isolated mathematical properties (such as angles sizes, edges lengths, parallelism, etc.), apparent from the observation of the solids or known from the solid's name. Observation is the main basis for the students' explanations.

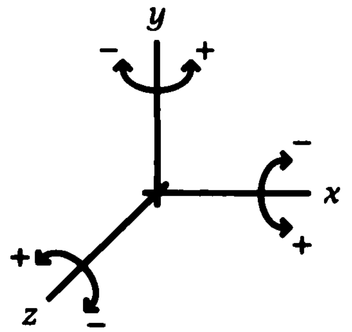

Figure 2

Students are able to visualize simple movements of solids between two concrete positions. Then the movements are not made by guess but determined depending on its current and final positions. There is not a pre-planned strategy, but a "direct mode" activity (in the same way as it happens in Logo): A movement is made; after observing the result, the next movement is selected and made; and so on.

Level 3 (Informal deduction). In order to decide whether two solids may be congruent or not, there is a mathematical analysis of the solids and their elements prior to any (physical or mental) movement. Students' answers include informal justifications based on isolated mathematical properties of the solids; these properties may be observed in the solids' representations or known from their mathematical structure.

The better students' ability of visualization allows them to visualize movements from or to non-visible positions. Then, when a solid has to be moved, students do previously an analysis of the initial and final positions, and a plan is done about which sequence of movements should be made. Justifications for such a decision are usually based on evident mathematical properties of or relationships between one or more parts (faces, edges, vertices) of the solid's original position and the matching parts of the final position.

Level 4 (Formal deduction). Also in this level students analyse the solids prior to any manipulation. Students' reasoning is based on the mathematical structure of the solids or their elements, including properties not seen but formally deduced from definitions or other properties.

Students in this level have a high ability of visualization. They can pre-plan the! movement of solids, and the selection of movements is done on the basis of the solid's mathematical structure and properties. This ability allows students to make an economic and accurate use of movements; for example, rotation angles depend on the angles between particular faces or edges. Students can also transform a non-available movement in a sequence of available ones.

3.1 The teaching unit

The aim of the activities in this teaching unit was to promote the acquisition or the development of the students' visual abilities. The basic material used by the students in the activities are the following:

A) Several solid cubes, the faces of which were decorated with drawings (**Figure 3**). They were used in problems (about changes of position) where only 90° rotations around the axes *x*, *y*, and *z* could be applied (**Figure 7a**).

B) A set of usual polyhedrons (cube, tetrahedron, octahedron, square pyramid and right prism) and a module made of small cubes. They were available as solid polyhedrons, made of cardboard with faces shaded using the same patterns as in the computer or paper representations (**Figure 4**), and as transparent polyhedrons, except the module, made of straws and pipe cleaners (**Figure 5**). These solids were used in problems about matching solids and about changes of position.

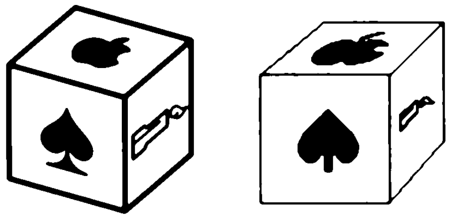

Figure 3

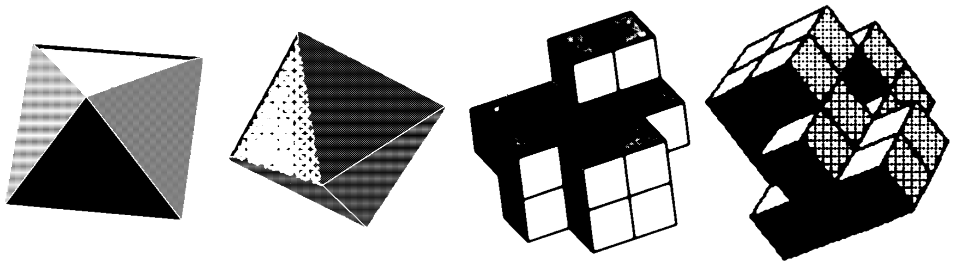

Figure 4

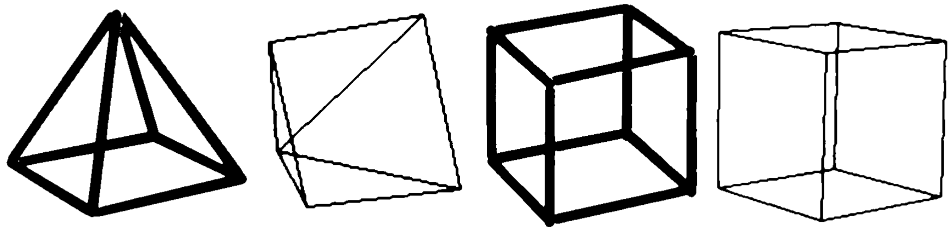

Figure 5

C) A set of Multilink cubes and several modules made of such cubes, used to study the three plane representations: Isometric, side views, and numerical views (**Figure 6**).

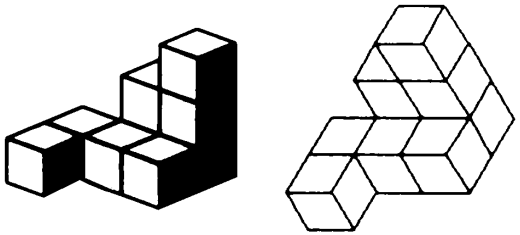

Figure 6

The software used in the teaching unit consisted of three different programs running on Macintosh SE.

D) Several Hypercard stacks (**Figure 7**), used mainly to move the cubes in A) and the modules in C). Another stack was the controller for moving from any part of the software to the others.

E) A pre-release version of the program 3D Images, designed by A. Hoffer and R. Koch, which allows to represent and to rotate 3-dimensional solids with transparent faces (**Figure 8**). In this program, rotations are performed automatically so that, while the user keeps on pressing the button for a specific rotation, the figure keeps on rotating continuously.

F) The program Phœnix 3D allows to represent and rotate 3-dimensional solids with either transparent or opaque faces (**Figure 9**). In this program rotations do not take place automatically, but the user has to select previously the size of the rotation. However, the size of the rotation can be graded intuitively by a small pyramid, visualized in the centre of the screen, which moves continuously in the direction of the selected rotation and shows the rotation's size.

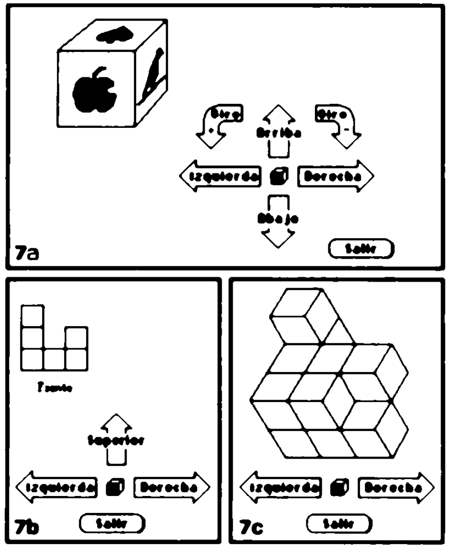

Figure 7

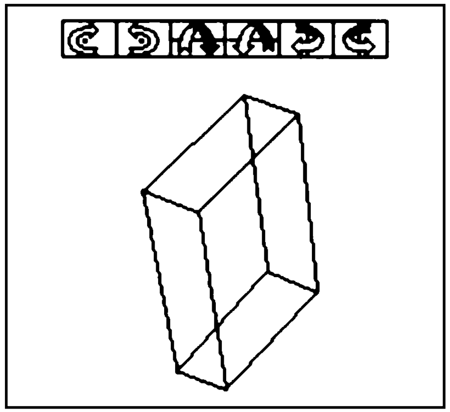

Figure 8

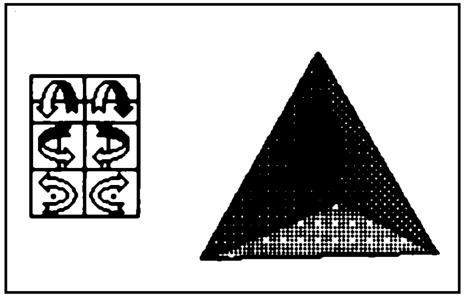

Figure 9

As can be seen in the previous figures, the solids appearing on the computer screen were similar to the corresponding real ones. It is worth pointing out that 3D Images and Phœnix 3D are very different programs since the automatic movement of solids in the first one supports the search by guess by the less able students, while the second program supports an analysis prior to the selection of a movement.

G) A booklet with the statement of the activities, some of which, dealing with plane representations, including hard copies of solids as seen on the computer screen (**Figure 10**).

The teaching unit consisted of three main types of activities.

A) Activities asking to compare a certain polyhedron to a given perspective representation of the same or a similar solid (**Table 1**). Usually, students did these comparisons by trying to move the polyhedron up to the position shown by the paper representation, and deciding whether they were alike or not.

B) Activities asking to move (mentally, physically, or on a computer) a certain polyhedron from its current position to another given one (**Table 2**).

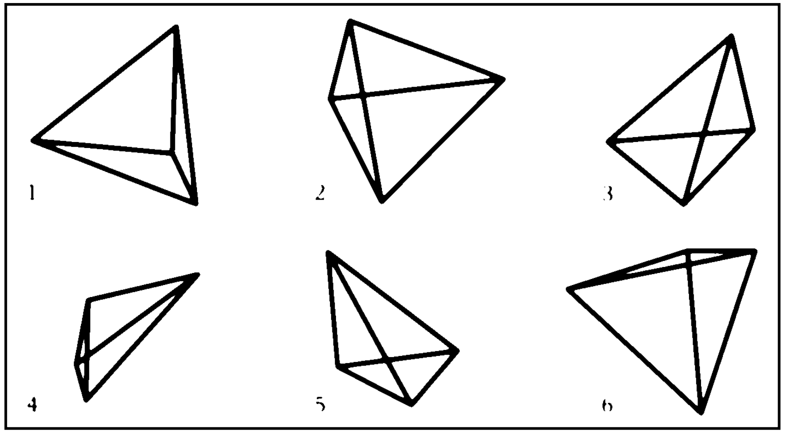

Figure 10

C) Activities asking to build modules made of Multilink cubes from given plane representations (isometric, side views, or numerical views), to draw the plane representations of given modules, or to check whether a certain representation corresponded to a given module.

Compare this solid ...	... to this one	
	Opaque	Transparent
Real	●	●
On computer	●	●

Table 1

This teaching unit was experimented with three sixth-grade children, two girls and one boy, with different ability levels. A girl (María) had a low-average ability, the boy (Enrique) had an average ability, and the other girl (Carmen) had a high ability. The experiment took 9 weeks, with 2 to 3 classes per week, approximately an hour long each.

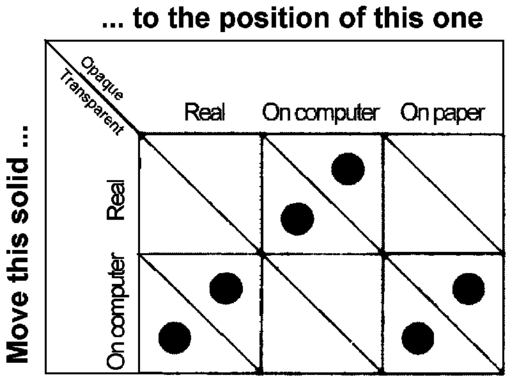

Table 2

This was the third part of a longer experiment (20 weeks in total) in which the children had previously worked on problems of definition and classification of different kinds of polyhedrons and solids with curved surface. All the work the students did in the two first parts was done using real solids. So the first formal contact of the children with the omputer or paper representations of solids was in the part of the experiment we are presenting.

3.2 Analysis of responses

In the next paragraphs we present some examples of the students' behaviour when answering different activities of types (A) and (B). We interpret them in terms of the Van Hiele levels. Although three students were engaged in the experiment, we restrict our comments to the two girls since they show the greatest behaviour contrast.

A typical activity of kind (A) was: *"Open the file "Straw Tetrahedron." Write YES or NO in the following views of solids (**Figure 10**), depending on whether they correspond to the tetrahedron on the screen or not. You can move the tetrahedron if you like."* The mentioned file belongs to the 3D Images program. In other activities like this one, students were provided with the corresponding files or with real straw polyhedrons, and other activities dealt with opaque polyhedrons (real ones or files of Phœnix 3D).

When working on the first view (**Figure 10.1**), María moved the tetrahedron to the position in **Figure 11a**. Then María said:

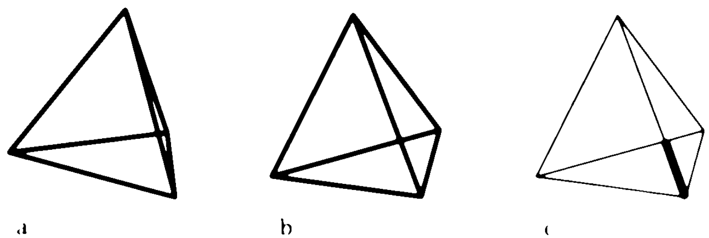

Figure 11

MARÍA: *Ok!*
TEACHER: *Ok? Are they now equal?*

MARÍA: *Yes.*

TEACHER: *I see different things.*

MARÍA: *This is thicker than that, isn't it?* [she pointed at the triangle on the right in **Figure 10.1**].

TEACHER: *Ah! I see them quite different* [she moved the tetrahedron and put it in the position of **Figure 11b**].

MARÍA: *Ok!*

TEACHER: *Do you now see them equal?*

MARÍA: *Yes, I do ...... No, quite wrong.*

TEACHER: *What is different?*

MARÍA: *This line* [she pointed to the segment highlighted in **Figure 11c**]. *That one over there shouldn't be there* [she continued moving the tetrahedron by guess].

In this excerpt, a Van Hiele level 1 of thinking could be noted. The girl just globally watched the whole shape, and she did not pay attention to the elements of the solid (the edges in this case). When she was asked to compare the tetrahedrons carefully, first (paragraph 5 of the dialogue) she only appreciated a global difference, and later (last paragraph) she noted that the tetrahedron on the screen had one segment more than the one on the sheet, but she did not mention the different directions of the edges, nor any mathematical property.

If children decided that a picture did not match the given polyhedron, and they were asked for a reason, the following two answers, clearly associated with Van Hiele level 1, were given several times:

CARMEN: *It cannot be so because I look at it and I wonder "is it a tetrahedron?" and I say "no, this is an oblique pyramid!"*

MARÍA: *It cannot be so because I have tried it many more times* [and I have been unable to match the solids].

The following excerpt corresponded to an activity, similar to the previous one, where the children had to decide whether **Figure 12a** matched a real straw cube. Before making any movement, Carmen said:

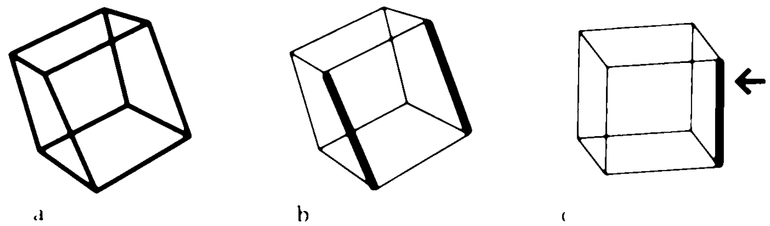

Figure 12

CARMEN: *This cannot be at all.*

TEACHER: *How fast you are!*

CARMEN: *Well, these two edges am slanted* [pointing the edges highlighted in **Figure 12b**] *and they should be perpendicular to this face* [pointing the face on the top of the solid in **Figure 12a**]. *They should be perpendicular* [in order to reinforce her statement, she pointed at the same edges on the real cube].

TEACHER: *But, could not it be that when you see it in perspective, they won't be perpendicular any longer?* [the fact that perspective some times modifies parallelism was discussed in a previous class].

CARMEN: *But it cannot be, because this face does not seem to be equal to this one* [pointing the faces on the top and the bottom, respectively, of the solid in **Figure 12a**]. *It seems as if this cube* [the real one] *was like this* [she put it approximately in the same position as in **Figure 12a**], *but this edge* [**Figure 12c**] *like this, pushed towards the inside* [while pushing the upper end of the edge].

This answer showed a Van Hiele level 2 of thinking, since it was based on the visual perception of the picture and the use of mathematical properties: The girl observed some non parallel edges in the picture and she knew that all the opposite edges in a cube are parallel; then she associated both facts and concluded that the picture could not match a cube. A more elaborated argument (although not completely correct) was provided by Carmen some time later when she and Enrique had to decide whether **Figure 13a** matched a cube on the computer screen. They moved the cube on the screen to the position in **Figure 13b**; the teacher asked whether it was right, and they said that it had to be put upside down. When they began to select the rotation that should be done, the teacher asked them:

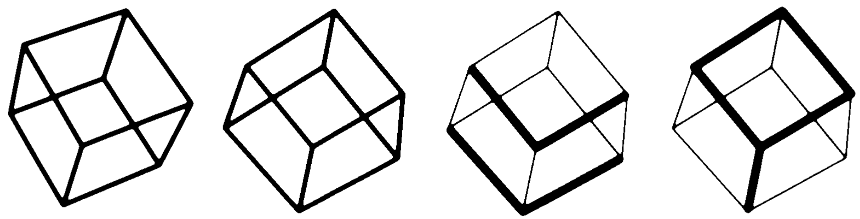

Figure 13

TEACHER: *Wait a moment. Why is it upside down? Why do you* [Enrique] *say that it is upside down?*

ENRIQUE: *Because if I put it this way* [turning the sheet round], *they look the same* [a level 1 answer].

CARMEN: *No. It is because these two edges are the outside ones* [really she pointed to the four edges highlighted in **Figure 13c**; likely she wanted to point at the faces].

TEACHER: *I see it different too. I also think it is upside down.*

CARMEN: *Could not these, these, and these* [**Figure 13d**] *be the outside edges?*

TEACHER: *Yes.*

Then Carmen and Enrique moved the cube to the right position. We can notice that, when comparing the cubes, Carmen used the codes of perspective representation for differentiating which edges were in the front or back of the cube.

The random selection of movements typical of students at Van Hiele level 1 came out once again when Enrique told Carmen: *Turn it that way to see what happens.* Comments like this one appeared often during this experiment.

A typical activity of kind (B) was: *"Open the file "Solid Pyramid." Move the pyramid on the screen to the first view you see below (**Figure 14**). Then, move it to the second view, and so on."* The mentioned file belongs to the Phœnix 3D program. In other activities like this one, students were provided with files of 3D Images (straw polyhedrons) or Hypercard (cubes; **Figure 7a**).

María was trying to move the pyramid to the last view (**Figure 14.6**). She had been making some movements, The following excerpt reflects the kind of work that María did for several classes: She moved the solids randomly until she found some suitable position; then she tried to fit it to the view given in the activity.

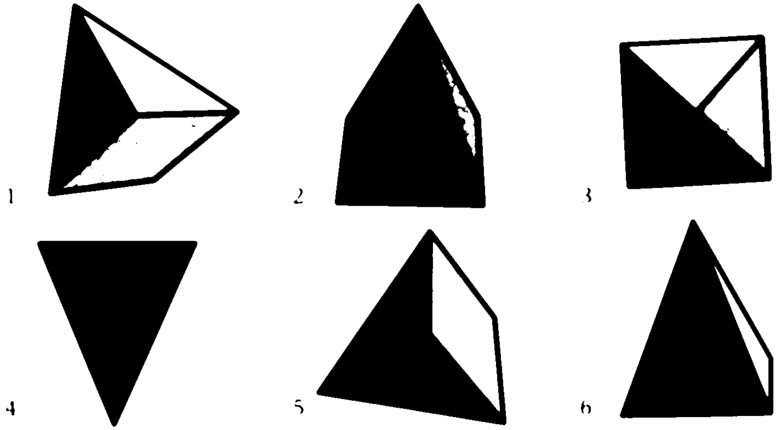

Figure 14

MARÍA: *I move by guess.*
TEACHER: *You move by guess. But, cannot you think a little which movement is useful to you?*
MARÍA: *No.*
TEACHER: *Why? ...* [she did not answer] *... Let us see, what are you looking for?*
MARÍA: *The white face.*
TEACHER: *It is there* [during the talk María had been moving the pyramid, and now it was like in **Figure 15a**]. *What do you want to do?*
MARÍA: *To put it right.*
TEACHER: *But, what is "to put it right"?*
MARÍA: *The base must he downwards.*
TEACHER: *Which movement makes it?*

María selected the rotation around axis *y*, and the teacher guided her (with the help of the real pyramid) to select the right movement. The pyramid was now like in **Figure 15b**.

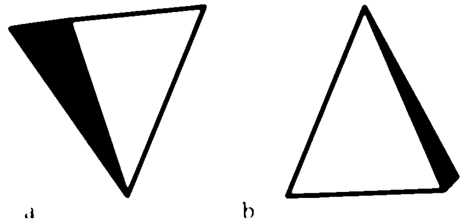

Figure 15

MARÍA: *Now it is ok.*
TEACHER: *Yes, but it is not in the right position. Let us see, which* [movement] *shall we do now?*
MARÍA: *To turn it that way* [she moved her hand from left to right in front of the screen].
TEACHER: *With which one?*

María selected the rotation around axis *z*. Then, the teacher showed her, with the real pyramid, the two turns from left to right (around axes *y* and *z*). Now María selected the right movement and finished the activity.

In another activity, María had to move a cube on the screen from the position in **Figure 16a** to the one in **Figure 16b**. She showed her strategy:

TEACHER: *You are looking for the candle, aren't you? You look for one* [face], *I mean, you make rotations until you find one, don't you?*

MARÍA: *Of course.*

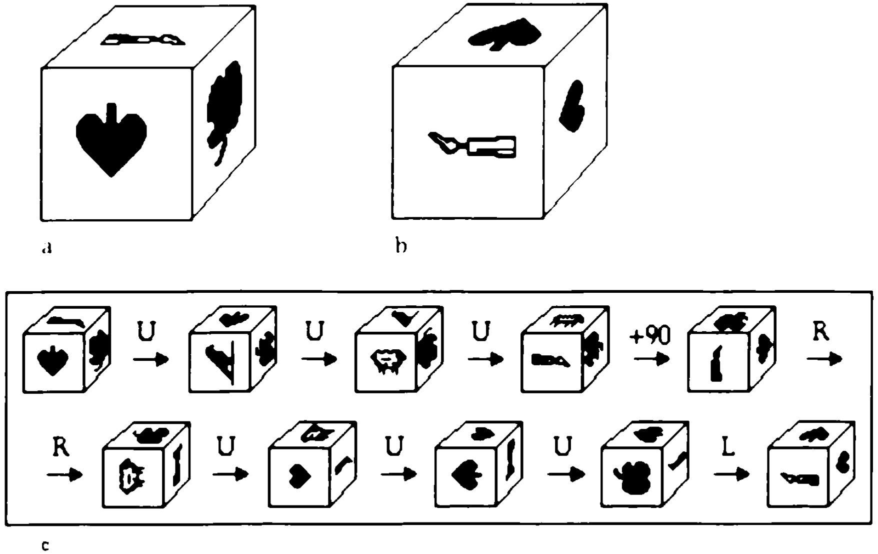

Figure 16

TEACHER: *And what do you do when you get it?*
MARÍA: *Trying to put it in that position* [the one in the sheet]. *As soon as this face is in its position, every one fits.*

However, the cube was rotated 9 times for moving from **Figure 16a** to **Figure 16b**; **Figure 16c** shows the sequence (the movements refer to the front face, like in **Figure 7a**: Up, Down, Left, Right, +90, -90). In the following exercise, María did as many as 21 rotations for reaching the final position, although she also tried to apply her strategy.

Now, let us analyse an example of behaviour corresponding to Van Hiele level 2. Carmen and Enrique had to move the module from the position in **Figure 17a** to the one in **Figure 17c**. Here Carmen showed a level 2 of thinking, since she identified some intermediate objectives (positions of the solid), one after reaching the previous, and can choose the rotations for directly moving the solid to these positions. However, she was not able to think at level 3 since she could not plan a complete sequence of movements from the current position to the final one:

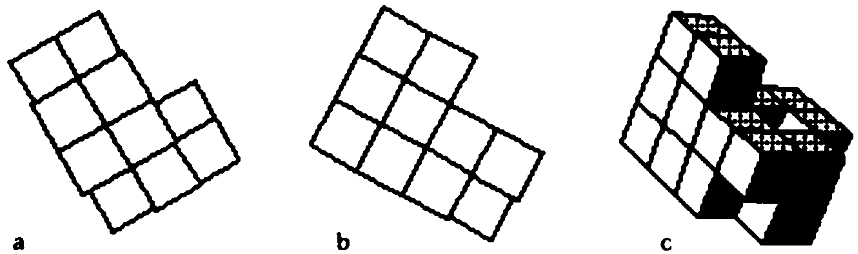

Figure 17

TEACHER: *Tell me, how many movements will you need* [to reach the position in **Figure 17c**]?
CARMEN: *I don't know!*

The students use a negative rotation around axis *z*, trying to slope the solid on the screen like the one in **Figure 17a**.

CARMEN: *I know it is not this way. It is another face, this is not the face. It is another face which is also white.*
TEACHER: *Why do you know this is not the face?*
CARMEN: *Because it does not matter how many turns I make, it cannot be that way. This will be here and that just there* [pointing at two couples of cubes on the screen].

TEACHER: *And if you want to put the other face, what shall you do?*

CARMEN: *We have to move with these* [pointing at the 4 arrows of rotations around axes *x* and *y*. She chose the negative turn around y and gave directions to Enrique].

CARMEN: *Move 180 ... or 90; 90, move 90!*

CARMEN: [after the first rotation of -90°] *This face is black. It is still wrong. Look for a white face. Turn it 90 more.*

CARMEN: [after the second rotation of -90°] *Look, we got the right face* [the white one]. *Don't you see that this face can be the right one? Because we turn it round and we put this upside* [pointing at the two little cubes corresponding to the ones on the top of the module in **Figure 17c**].

Now Carmen and Enrique made some rotations around the axis *z*, and they put the solid like in **Figure 17b**. After that, they chose the negative turn around axis *y*.

ENRIQUE: *It is just to see* [a little piece of level 1 of thinking].

CARMEN: *If it isn't* [correct], *then undo it.*

Finally, after some more rotations like the previous one, they reached the final position.

Lastly, I present an excerpt, from one of the last classes, as an example of Van Hiele level 3 of thinking. Carmen really was not at level 3, but during the last part of the experiment, some short episodes of level 3 could be recognized. Now she had to move the prism from the position in **Figure 18a** to the one in **Figure 18c**; she was trying to understand **Figure 18c**, and she said:

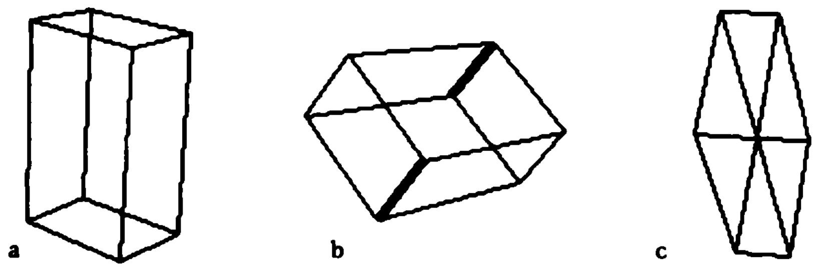

Figure 18

CARMEN: *This one is very difficult ... No! I can see what it is.* [to the teacher] *Look, a lateral side, another lateral side* [pointing at the rhombuses on the left and the right of **Figure 18c**]. *Now I see it. I see how it is.*

Now Carmen moved the prism from the position in **Figure 18a** to the one in **Figure 18b**, and she explained her objective:

CARMEN: *I have made a rhombus here so that these* [the two edges highlighted in **Figure 18b**] *join when it is raised* [the prism].

In this Carmen's work, a pre-planned strategy for moving the prism from the initial position to the final one could be noted. It was not a simple strategy since two intermediate steps were planned (making a rhombus, and putting two edges vertically) and different kinds of rotations had to be used. Therefore it is quite different from the work at Van Hiele level 2, in which students always tried to move the solids from the current to another position, usually the final one, without considering the possibility of a multi-step sequence, but deciding at each moment what to make.

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