ON THE COMPLETENESS THEOREM OF MANY-SORTED EQUATIONAL LOGIC AND THE EQUIVALENCE BETWEEN HALL ALGEBRAS AND BÉNABOU THEORIES

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ABSTRACT. The completeness theorem of equational logic of Birkhoff asserts the coincidence of the model-theoretic and proof-theoretic consequence relations. Goguen and Meseguer, giving a sound and adequate system of inference rules for many-sorted deduction, founded ultimately on the congruences on Hall algebras, generalized the completeness theorem of Birkhoff to the completeness theorem of many-sorted equational logic. In this paper, after simplifying the specification of Hall algebras as given by Goguen-Meseguer, we obtain another many-sorted equational calculus from which we prove that the inference rules of abstraction and concretion due to Goguen-Meseguer are derived rules. Finally, after defining the Bénabou algebras for a set of sorts S we prove that the category of Bénabou algebras for S is *equivalent* to the category of Hall algebras for S and *isomorphic* to the category of Bénabou theories for S, i.e., the many-sorted counterpart of the category of Lawvere theories, hence that Hall algebras and Bénabou theories are equivalent.

1. INTRODUCTION.

The completeness theorem of many-sorted equational logic of Goguen-Meseguer (in [4]), under which the classical completeness theorem of equational logic of Birkhoff (in [2]) falls, asserts, for a set of sorts S and an S-sorted signature Σ, the coincidence of two consequence relations defined between subfamilies of the many-sorted set $\mathrm{Eq_H}(\Sigma)$, of finitary Σ-equations, and elements of such a many-sorted set, for an S-sorted signature Σ and an S-sorted set of variables $V = (V_s)_{s\in S}$ where, for every sort s in S, $V_s = \{ v^s_n \mid n \in \mathbb{N} \}$ is a standard infinite countable set of variables of type s.

Concretely, the above completeness theorem affirms that the consequence relations $\models^{\Sigma}$ and $\vdash^{\Sigma}$ are identical, where $\models^{\Sigma} = (\models^{\Sigma}_{w,s})_{(w,s)\in S^\star\times S}$, with $S^\star$ the underlying set of the free monoid on S, the so-called semantical consequence relation, is obtained from the contravariant Galois connection between the ordered set $\mathbf{Sub}(\mathbf{Alg}(\Sigma))$, of subsets of $\mathbf{Alg}(\Sigma)$, the category of Σ-algebras (identified in this case to its underlying set of objects), and the ordered set $\mathbf{Sub}(\mathrm{Eq_H}(\Sigma))$, of subfamilies of $\mathrm{Eq_H}(\Sigma)$; while $\vdash^{\Sigma} = (\vdash^{\Sigma}_{w,s})_{(w,s)\in S^\star\times S}$, the so-called entailment relation, or syntactical consequence relation, can be obtained, for instance, as has been pointed out in [4], as the operator $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}$, of generated congruence, on the Hall algebra $\mathbf{HTer}_S(\Sigma)$ that has as underlying $S^\star \times S$-sorted set $(\mathrm{T}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star\times S}$ where, for a word $w \in S^\star$, $\downarrow w$ is the S-sorted set that has, for $s \in S$, as s-th coordinate the subset of V_s defined as $(\downarrow w)_s = \{ v^s_i \in V_s \mid w_i = s \}$, while $\mathrm{T}_\Sigma(\downarrow w)$ is the underlying S-sorted set of $\mathbf{T}_\Sigma(\downarrow w)$, the free Σ-algebra on the S-sorted set $\downarrow w$.

Date: September 14, 2005.

2000 *Mathematics Subject Classification.* Primary: 03C05, 08A68, 08B05, 18A32, 18C15, 18C20; Secondary: 06B23.

Key words and phrases. Many-sorted algebra, term, equation, many-sorted equational completeness, Hall algebra, Bénabou algebra, Bénabou theory.

In the second section of this paper, once defined the variety of Hall algebras for a set of sorts S, through a many-sorted specification slightly different from that presented in [4], and after reproving the completeness theorem of many-sorted equational logic, we obtain another many-sorted equational calculus from which we prove that the inference rules of abstraction and concretion in [4] are derived rules, thus providing a system of sound and adequate inference rules somewhat less redundant than that presented by Goguen-Meseguer in [4].

In the third and last section, after defining the variety of Bénabou algebras for a set of sorts S, through a many-sorted specification, we prove, on the one hand, that the category of Bénabou theories for S, defined in [1], has the form of the category of models for a convenient many-sorted specification because it is *isomorphic* to the category of Bénabou algebras for S and, on the other hand, that the category of Hall algebras for S, used by Goguen-Meseguer in their proof of the completeness theorem of many-sorted equational logic, is *equivalent* to that of Bénabou algebras for S, hence that Hall algebras and Bénabou theories are equivalent. Finally, we prove that the algebraic lattice $\mathbf{Cgr}(\mathbf{BTer}_S(\Sigma))$ associated to the Bénabou algebra $\mathbf{BTer}_S(\Sigma)$ is isomorphic to the algebraic lattice of fixed points of the operator Cn_Σ, canonically associated to the semantical consequence relation $\models^\Sigma$.

We point out that the category of Bénabou algebras for a set of sorts S is not only interesting because it is isomorphic to the category of Bénabou theories for S and equivalent to the category of Hall algebras for S, but also because in [3] the Bénabou algebras have been used, among other things, to define what we have called morphisms of Fujiwara from a many-sorted signature into another, as well as morphisms from a many-sorted specification into another, from which we have proved, in a convenient 2-category of many-sorted specifications, the equivalence between the many-sorted specifications of Hall and Bénabou, and also, as a direct consequence of the existence of a certain pseudo-functor from such a 2-category into the 2-category of categories, the equivalence between the associated varieties.

In what follows we use standard concepts from many-sorted algebra, see e.g., [4]. Sometimes, to avoid any confusion, we will denote the family of structural operations of a given Σ-algebra $\mathbf{A}$ by $F^{\mathbf{A}}$ and the components of $F^{\mathbf{A}}$ corresponding to the different formal operations σ, τ, ..., as $F^{\mathbf{A}}_\sigma$, $F^{\mathbf{A}}_\tau$, ..., respectively. Moreover, every set we consider will be an element or subset of a Grothendieck universe $\boldsymbol{\mathcal{U}}$, fixed once and for all.

2. Hall algebras, the many-sorted completeness theorem of Goguen-Meseguer, and some derived inference rules.

Hall algebras, as reflected by the defining axioms stated below, are a species of algebraic construct in which the essential properties of the concepts of substitution, for the many-sorted terms in the free many-sorted algebras, and of generalized composition, for the many-sorted operations on sorted sets, are embodied. And this is precisely one of the reasons why Hall algebras are a powerful and fundamental instrument to investigate many-sorted algebras. To this we add that Hall algebras are not only worth of study because of its source in the above mentioned procedures. Besides that, Hall algebras are interesting in themselves since they furnish important examples of equationally defined many-sorted algebras, and also because they have been used, as we have said in the introduction, by Goguen and Meseguer in [4] to prove the completeness theorem of finitary many-sorted equational logic (that generalizes the classical completeness theorem of finitary equational logic of Birkhoff), providing in this way, a full algebraization of many-sorted equational deduction.

In this section after defining, for a set of sorts S, $\mathbf{Alg}(\mathrm{H}_S)$, the category of Hall algebras for S, through a many-sorted specification H_S slightly different from that presented in [4], we prove the existence, for every S-sorted signature Σ, of an isomorphism between $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, the free Hall algebra on Σ, and $\mathbf{HTer}_S(\Sigma)$, the Hall algebra for (S,Σ) which, we advance, formalizes the concept of substitution and has as underlying $S^\star \times S$-sorted set precisely $(\mathrm{T}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star\times S}$, i.e., the different sets of finitary many-sorted Σ-terms. We point out that this isomorphism, which allows us to replace everywhere $\mathbf{HTer}_S(\Sigma)$ for $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, together with the adjunction $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$ from $\mathbf{Set}^{S^\star\times S}$ to $\mathbf{Alg}(\mathrm{H}_S)$, will be specially useful to state some results in a more concrete and tractable way. Then, once reproved the completeness theorem of many-sorted equational logic, we obtain from it a many-sorted equational calculus from which we prove that the rules of abstraction and concretion in [4] are derived rules, hence providing a somewhat less redundant set of sound and adequate inference rules than those in [4].

But before we begin to realize what has been announced we consider, for a set of sorts S and an S-sorted signature Σ, the concepts of finitary Σ-term, finitary Σ-equation and the relation of validation between finitary Σ-equations and Σ-algebras. From these concepts we obtain, as is well known, a contravariant Galois connection between the ordered set of families of finitary Σ-equations and the ordered set of families of Σ-algebras and, in particular, the closure operator of semantical consequence on the set of finitary Σ-equations.

Definition 1. Let Σ be an S-sorted signature, $w \in S^\star$, and $s \in S$.

(1) A *finitary Σ-term of type* (w,s) is an element P of $\mathrm{T}_\Sigma(\downarrow w)_s$.
(2) A *finitary Σ-equation of type* (w,s) is an element (P,Q) of $\mathrm{T}_\Sigma(\downarrow w)_s^2$, i.e., a pair of finitary Σ-terms of type (w,s).

From now on we agree that $\mathrm{HTer}_S(\Sigma)$ denotes $(\mathrm{T}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star\times S}$, the $S^\star\times S$-sorted set of finitary Σ-terms, and that $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ denotes $(\mathrm{T}_\Sigma(\downarrow w)_s^2)_{(w,s)\in S^\star\times S}$, the $S^\star\times S$-sorted set of finitary Σ-equations.

Next we define for an S-sorted signature Σ, on the one hand, the realization of the finitary Σ-terms in the Σ-algebras and, on the other, the concept of validation of a finitary Σ-equation in a Σ-algebra.

Definition 2. Let Σ be an S-sorted signature, $w \in S^\star$, $s \in S$, $\mathbf{A}$ a Σ-algebra, and $P \in \mathrm{T}_\Sigma(\downarrow w)_s$ a finitary Σ-term of type (w,s). Then

(1) The **Σ**-algebra of the *many-sorted w-ary operations on* $\mathbf{A}$ is $\mathbf{A}^{A_w}$, i.e., the direct A_w-power of $\mathbf{A}$, where A_w is $\prod_{i\in|w|} A_{w_i}$, with $|w|$ the length of the word w, or, since, for every $s\in S$, the sets $(\downarrow w)_s = \{\, v_i^s \in V_s \mid w_i = s \,\}$ and $\{\, i \in |w| \mid w_i = s \,\}$ are isomorphic, $\mathbf{A}^{A_{\downarrow w}}$, i.e., the direct $A_{\downarrow w}$-power of $\mathbf{A}$, where $A_{\downarrow w}$ is $\mathrm{Hom}(\downarrow w, A)$, the set of all S-sorted mappings from $\downarrow w$ to A. From now on, to shorten terminology, we will speak of *w-ary operations on* $\mathbf{A}$ instead of *many-sorted w-ary operations on* $\mathbf{A}$.
(2) We denote by $\mathrm{Tr}^{\downarrow w,\mathbf{A}}$ the unique homomorphism from $\mathbf{T}_{\boldsymbol{\Sigma}}(\downarrow w)$ to $\mathbf{A}^{A_w}$ such that $\mathrm{pr}^A_{\downarrow w} = \mathrm{Tr}^{\downarrow w,\mathbf{A}} \circ \eta_{\downarrow w}$, where $\mathrm{pr}^A_{\downarrow w}$ is the S-sorted mapping $(\mathrm{pr}^A_{\downarrow w,s})_{s\in S}$ from $\downarrow w$ to A^{A_w} defined, for $s\in S$, as $\mathrm{pr}^A_{\downarrow w,s} = (\mathrm{pr}^A_{\downarrow w,s,x})_{x\in(\downarrow w)_s}$, and $\eta_{\downarrow w}$ the canonical embedding of $\downarrow w$ into $\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow w)$, the underlying S-sorted set of $\mathbf{T}_{\boldsymbol{\Sigma}}(\downarrow w)$. Furthermore, $P^{\mathbf{A}}$ denotes the image of P under $\mathrm{Tr}_s^{\downarrow w,\mathbf{A}}$, and we call the mapping $P^{\mathbf{A}}$ from $A_{\downarrow w}$ to A_s, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$.

Definition 3. Let $\mathbf{A}$ be a Σ-algebra and (P,Q) a finitary Σ-equation of type (w,s). We say that (P,Q) is *valid* in $\mathbf{A}$, denoted by $\mathbf{A} \models^\Sigma_{w,s} (P,Q)$, if $P^{\mathbf{A}} = Q^{\mathbf{A}}$.

If $\mathcal{K} \subseteq \mathbf{Alg}(\Sigma)$, then we agree that $\mathcal{K} \models^{\Sigma}_{w,s} (P, Q)$ means that, for every $\mathbf{A} \in \mathcal{K}$, $\mathbf{A} \models^{\Sigma}_{w,s} (P, Q)$.

From the concept of validation we obtain, as is well-known, the following contravariant Galois connection.

Definition 4. Let Σ be an S-sorted signature.

(1) If $\mathcal{K} \subseteq \mathbf{Alg}(\Sigma)$, then $\mathrm{Th}_{\Sigma}(\mathcal{K}) = (\mathrm{Th}_{\Sigma}(\mathcal{K})_{w,s})_{(w,s)\in S^{\star}\times S}$, the *finitary Σ-equational theory determined by* $\mathcal{K}$, is the sub-$(S^{\star} \times S)$-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ whose (w, s)-th coordinate $\mathrm{Th}_{\Sigma}(\mathcal{K})_{w,s}$, for $(w, s) \in S^{\star} \times S$, has as elements those finitary Σ-equations (P, Q) of type (w, s) such that $\mathcal{K} \models^{\Sigma}_{w,s} (P, Q)$, therefore

$$\mathrm{Th}_{\Sigma}(\mathcal{K}) = \left(\left\{(P, Q) \in \mathrm{Eq}_{\mathrm{H}}(\Sigma)_{w,s} \mid \forall\, \mathbf{A} \in \mathcal{K}\, (\mathbf{A} \models^{\Sigma}_{w,s} (P, Q))\right\}\right)_{(w,s)\in S^{\star}\times S}.$$

(2) If $\mathcal{E} \subseteq \mathrm{Eq}_{\mathrm{H}}(\Sigma)$, then $\mathrm{Mod}_{\Sigma}(\mathcal{E})$, the *finitary Σ-equational class determined by* $\mathcal{E}$, has as elements the Σ-algebras $\mathbf{A}$ that validate each equation of $\mathcal{E}$, i.e.,

$$\mathrm{Mod}_{\Sigma}(\mathcal{E}) = \left\{\mathbf{A} \in \mathbf{Alg}(\Sigma) \;\middle|\; \begin{array}{c} \forall (w, s) \in S^{\star} \times S,\ \forall (P, Q) \in \mathcal{E}_{w,s}, \\ \mathbf{A} \models^{\Sigma}_{w,s} (P, Q) \end{array}\right\}.$$

Proposition 1. *Let Σ be an S-sorted signature, $\mathcal{E}$, $\mathcal{E}'$ two families of finitary Σ-equations and $\mathcal{K}$, $\mathcal{K}'$ two sets of Σ-algebras. Then the following holds:*

(1) *If $\mathcal{E} \subseteq \mathcal{E}'$, then* $\mathrm{Mod}_{\Sigma}(\mathcal{E}') \subseteq \mathrm{Mod}_{\Sigma}(\mathcal{E})$.
(2) *If $\mathcal{K} \subseteq \mathcal{K}'$, then* $\mathrm{Th}_{\Sigma}(\mathcal{K}') \subseteq \mathrm{Th}_{\Sigma}(\mathcal{K})$.
(3) $\mathcal{E} \subseteq \mathrm{Th}_{\Sigma}(\mathrm{Mod}_{\Sigma}(\mathcal{E}))$ *and* $\mathcal{K} \subseteq \mathrm{Mod}_{\Sigma}(\mathrm{Th}_{\Sigma}(\mathcal{K}))$.

Therefore the pair of mappings Th_{Σ} *and* Mod_{Σ} *is a contravariant Galois connection.*

The categories associated to the lattices of sets of Σ-algebras and families of finitary Σ-equations are related by the adjunction $\mathrm{Mod}_{\Sigma} \dashv \mathrm{Th}_{\Sigma}$, i.e., for every set $\mathcal{K}$ of Σ-algebras and every family $\mathcal{E}$ of finitary Σ-equations, we have that $\mathcal{K} \subseteq \mathrm{Mod}_{\Sigma}(\mathcal{E})$ iff $\mathcal{E} \subseteq \mathrm{Th}_{\Sigma}(\mathcal{K})$, because of the contravariance.

Definition 5. We denote by Cn_{Σ} the closure operator $\mathrm{Th}_{\Sigma} \circ \mathrm{Mod}_{\Sigma}$ on $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and we call the Cn_{Σ}-closed sets *Σ-equational theories*. If $\mathcal{E}$ is a family of finitary Σ-equations and (P, Q) a finitary Σ-equation of type (w, s), then we say that (P, Q) is a *semantical consequence* of $\mathcal{E}$ if $\mathrm{Mod}_{\Sigma}(\mathcal{E}) \subseteq \mathrm{Mod}_{\Sigma}(P, Q)$, i.e., if $(P, Q) \in \mathrm{Cn}_{\Sigma}(\mathcal{E})_{w,s} = \mathrm{Th}_{\Sigma}(\mathrm{Mod}_{\Sigma}(\mathcal{E}))_{w,s}$, which we denote also by $\mathcal{E} \models^{\Sigma}_{w,s} (P, Q)$.

Before we define the Hall algebras, through an appropriate many-sorted specification, we agree that for a set of sorts U, a word $x \in U^{\star}$ and a standard U-sorted set of variables $V^{U} = (\{\, v^{u}_{n} \mid n \in \mathbb{N} \,\})_{u\in U}$, $\downarrow\! x$ is the U-sorted subset of V^{U} defined, for every $u \in U$ as $(\downarrow\! x)_{u} = \{\, v^{u}_{i} \mid i \in x^{-1}[u] \,\}$, this will apply, in particular, when the set of sorts U is $S^{\star} \times S$ or $S^{\star} \times S^{\star}$.

Definition 6. Let S be a set of sorts and V^{H_S} the $S^{\star} \times S$-sorted set of variables $(V_{w,s})_{(w,s)\in S^{\star}\times S}$ where, for every $(w, s) \in S^{\star} \times S$, $V_{w,s} = \{\, v^{w,s}_{n} \mid n \in \mathbb{N} \,\}$. A *Hall algebra for* S is a $\mathrm{H}_{S} = (S^{\star} \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S})$-algebra, where Σ^{H_S} is the $S^{\star} \times S$-sorted signature, i.e., the $(S^{\star} \times S)^{\star} \times (S^{\star} \times S)$-sorted set, defined as follows:

HS_1. For every $w \in S^{\star}$ and $i \in |w|$,

$$\pi^{w}_{i} \colon \lambda \longrightarrow (w, w_{i}),$$

where $|w|$ is the *length* of the word w and λ the *empty word* in $(S^{\star} \times S)^{\star}$.

HS_2. For every u, $w \in S^{\star}$ and $s \in S$,

$$\xi_{u,w,s} \colon ((w, s), (u, w_{0}), \ldots, (u, w_{|w|-1})) \longrightarrow (u, s);$$

while $\mathcal{E}^{\mathrm{H}_S}$ is the sub-$(S^\star \times S)^\star \times (S^\star \times S)$-sorted set of $\mathrm{Eq}(\Sigma^{\mathrm{H}_S})$, where

$$\mathrm{Eq}(\Sigma^{\mathrm{H}_S}) = (\mathrm{T}_{\Sigma^{\mathrm{H}_S}}(\downarrow\overline{w})^2_{(u,s)})_{(\overline{w},(u,s))\in(S^\star\times S)^\star\times(S^\star\times S)},$$

defined as follows:

H_1. *Projection.* For every u, $w \in S^\star$ and $i \in |w|$, the equation

$$\xi_{u,w,w_i}(\pi^w_i, v^{u,w_0}_0, \dots, v^{u,w_{|w|-1}}_{|w|-1}) = v^{u,w_i}_i$$

of type $(((u,w_0),\dots,(u,w_{|w|-1})),(u,w_i))$.

H_2. *Identity.* For every $u \in S^\star$ and $j \in |u|$, the equation

$$\xi_{u,u,u_j}(v^{u,u_j}_j, \pi^u_0, \dots, \pi^u_{|u|-1}) = v^{u,u_j}_j$$

of type $(((u,u_j)),(u,u_j))$.

H_3. *Associativity.* For every u, v, $w \in S^\star$ and $s \in S$, the equation

$$\begin{aligned}\xi_{u,v,s}(\xi_{v,w,s}(v^{w,s}_0, v^{v,w_0}_1, \dots, v^{v,w_{|w|-1}}_{|w|}), v^{u,v_0}_{|w|+1}, \dots, v^{u,v_{|v|-1}}_{|w|+|v|}) = \\ \xi_{u,w,s}(v^{w,s}_0, \xi_{u,v,w_0}(v^{v,w_0}_1, v^{u,v_0}_{|w|+1}, \dots, v^{u,v_{|v|-1}}_{|w|+|v|}), \dots, \\ \xi_{u,v,w_{|w|-1}}(v^{v,w_{|w|-1}}_{|w|}, v^{u,v_0}_{|w|+1}, \dots, v^{u,v_{|v|-1}}_{|w|+|v|}))\end{aligned}$$

of type $(((w,s),(v,w_0),\dots,(v,w_{|w|-1}),(u,v_0),\dots,(u,v_{|v|-1})),(u,s))$.

Remark. From H_3, for $w = \lambda$, the empty word on S, we get the invariance of constant functions axiom in [4]: For every u, $v \in S^\star$ and $s \in S$, we have the equation

$$\xi_{u,v,s}(\xi_{v,\lambda,s}(v^{\lambda,s}_0), v^{u,v_0}_1, \dots, v^{u,v_{|v|-1}}_{|v|}) = \xi_{u,\lambda,s}(v^{\lambda,s}_0)$$

of type $(((\lambda,s),(u,v_0),\dots,(u,v_{|v|-1})),(u,s))$.

We call the formal constants π^w_i *projections*, and the formal operations $\xi_{u,w,s}$ *substitution operators*. Furthermore, we denote by $\mathbf{Alg}(\mathrm{H}_S)$ the category of Hall algebras for S and homomorphisms between Hall algebras. Since $\mathbf{Alg}(\mathrm{H}_S)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{H}_S}$ from $\mathbf{Alg}(\mathrm{H}_S)$ to $\mathbf{Set}^{S^\star\times S}$ has a left adjoint $\mathbf{T}_{\mathrm{H}_S}$, situation denoted by $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$, or diagrammatically by

$$\mathbf{Alg}(\mathrm{H}_S) \underset{\mathbf{T}_{\mathrm{H}_S}}{\overset{\mathrm{G}_{\mathrm{H}_S}}{\rightleftarrows}} \mathbf{Set}^{S^\star\times S} \quad (\top)$$

which assigns to an $S^\star \times S$-sorted set Σ the corresponding free Hall algebra $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$.

For every S-sorted set A, $\mathrm{HOp}_S(A) = (\mathrm{Hom}(A_w, A_s))_{(w,s)\in S^\star\times S}$, the $S^\star \times S$-sorted set of operation for A, is naturally endowed with a structure of Hall algebra, as stated in the following proposition, if we realize the projections as the true projections and the substitution operators as the generalized composition of mappings.

Proposition 2. *Let A be an S-sorted set and $\mathbf{HOp}_S(A)$ the Σ^{H_S}-algebra with underlying many-sorted set $\mathrm{HOp}_S(A)$ and algebraic structure defined as follows*

(1) *For every $w \in S^\star$ and $i \in |w|$, $(\pi^w_i)^{\mathbf{HOp}_S(A)} = \mathrm{pr}^A_{w,i} \colon A_w \longrightarrow A_{w_i}$.*

(2) *For every $u, w \in S^\star$ and $s \in S$, $\xi^{\mathbf{HOp}_S(A)}_{u,w,s}$ is defined, for every $f \in A_s^{A_w}$ and $g \in A_w^{A_u}$, as $\xi^{\mathbf{HOp}_S(A)}_{u,w,s}(f, g_0, \dots, g_{|w|-1}) = f \circ \langle g_i\rangle_{i\in|w|}$, where $\langle g_i\rangle_{i\in|w|}$ is the unique mapping from A_u to A_w such that, for every $i \in |w|$, we have that*

$$\mathrm{pr}^A_{w,i} \circ \langle g_i\rangle_{i\in|w|} = g_i.$$

Then $\mathbf{HOp}_S(A)$ is a Hall algebra, the Hall algebra for (S, A).

Remark. The closed sets of the Hall algebra $\mathbf{HOp}_S(A)$ for (S, A) are precisely the clones of (many-sorted) operations on the S-sorted set A.

We agree that, for every Σ-algebra $\mathbf{A}$, $\mathbf{HOp}_S(\mathbf{A})$ is $\mathbf{HOp}_S(A)$, where A is the underlying S-sorted set of $\mathbf{A}$. Thus, under this convention, every Σ-algebra $\mathbf{A}$ has associated a Hall algebra.

For every S-sorted signature Σ, $\mathrm{HTer}_S(\Sigma) = (\mathrm{T}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star\times S}$ is also endowed with a structure of Hall algebra that formalizes the concept of substitution as stated in the following

Proposition 3. *Let Σ be an S-sorted signature and $\mathbf{HTer}_S(\Sigma)$ the Σ^{H_S}-algebra with underlying many-sorted set $\mathrm{HTer}_S(\Sigma)$ and algebraic structure defined as follows*

1. *For every $w \in S^\star$ and $i \in |w|$, $(\pi^w_i)^{\mathbf{HTer}_S(\Sigma)}$ is the image under $\eta_{\downarrow w,w_i}$ of the variable $v^{w_i}_i$, where $\eta_{\downarrow w} = (\eta_{\downarrow w,s})_{s\in S}$ is the canonical embedding of $\downarrow w$ into $\mathrm{T}_\Sigma(\downarrow w)$. Sometimes, to abbreviate, we will write π^w_i instead of $(\pi^w_i)^{\mathbf{HTer}_S(\Sigma)}$.*
2. *For every $u, w \in S^\star$ and $s \in S$, $\xi^{\mathbf{HTer}_S(\Sigma)}_{u,w,s}$ is the mapping*
$$\xi^{\mathbf{HTer}_S(\Sigma)}_{u,w,s}\begin{cases}\mathrm{T}_\Sigma(\downarrow w)_s \times \mathrm{T}_\Sigma(\downarrow u)_{w_0} \times \cdots \times \mathrm{T}_\Sigma(\downarrow u)_{w_{|w|-1}} \longrightarrow \mathrm{T}_\Sigma(\downarrow u)_s\\ \qquad (P, (Q_i)_{i\in|w|}) \longmapsto \mathcal{Q}^\sharp_s(P)\end{cases}$$
where, for $\mathcal{Q}$ the S-sorted mapping from $\downarrow w$ to $\mathrm{T}_\Sigma(\downarrow u)$ canonically associated to the family $(Q_i)_{i\in|w|}$, $\mathcal{Q}^\sharp$ is the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow w)$ into $\mathbf{T}_\Sigma(\downarrow u)$ such that $\mathcal{Q}^\sharp \circ \eta_{\downarrow w} = \mathcal{Q}$. Sometimes, to abbreviate, we will write $\xi_{u,w,s}$ instead of $\xi^{\mathbf{HTer}_S(\Sigma)}_{u,w,s}$.

Then $\mathbf{HTer}_S(\Sigma)$ is a Hall algebra, the Hall algebra for (S, Σ).

Our next goal is to prove that, for every $S^\star \times S$-sorted set Σ, $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, the free Hall algebra on Σ, is isomorphic to $\mathbf{HTer}_S(\Sigma)$. We remark that the existence of this isomorphism is interesting because it enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, and, on the other hand, as we will show afterwards, to state, for every Σ-algebra $\mathbf{A}$, taking into account the adjunction $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$, the existence of a homomorphism of Hall algebras $\mathrm{Tr}^{\mathbf{A}}$ from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HOp}_S(\mathbf{A}) = \mathbf{HOp}_S(A)$ such that $\mathrm{Th}_\Sigma(\mathbf{A})$, the finitary Σ-equational theory determined by $\mathbf{A}$, is precisely $\mathrm{Ker}(\mathrm{Tr}^{\mathbf{A}})$, the kernel of the homomorphism $\mathrm{Tr}^{\mathbf{A}}$.

To attain the goal just stated we begin by defining, for a Hall algebra $\mathbf{A}$, an S-sorted signature Σ, an $S^\star \times S$-mapping $f\colon \Sigma \longrightarrow A$, and a word $u \in S^\star$, the concept of derived Σ-algebra of $\mathbf{A}$ for (f, u), since it will be used afterwards in the proof of the isomorphism between $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$ and $\mathbf{HTer}_S(\Sigma)$.

Definition 7. Let $\mathbf{A}$ be a Hall algebra and Σ an S-sorted signature. Then, for every $f\colon \Sigma \longrightarrow A$ and $u \in S^\star$, $\mathbf{A}^{f,u}$, the *derived Σ-algebra of $\mathbf{A}$ for (f, u)*, is the Σ-algebra with underlying S-sorted set $A^{f,u} = (A_{u,s})_{s\in S}$ and algebraic structure $F^{f,u}$, defined, for every $(w, s) \in S^\star \times S$, as
$$F^{f,u}_{w,s}\begin{cases}\Sigma_{w,s} \longrightarrow \mathrm{HOp}_w(A^{f,u})_s\\ \sigma \longmapsto \begin{cases}\prod_{i\in|w|} A_{u,w_i} \longrightarrow A_{u,s}\\ (a_0, \ldots, a_{|w|-1}) \longmapsto \xi^{\mathbf{A}}_{u,w,s}(f_{(w,s)}(\sigma), a_0, \ldots, a_{|w|-1})\end{cases}\end{cases}$$
where $\mathrm{HOp}_w(A^{f,u})_s = A_{u,s}^{\prod_{i\in|w|} A_{u,w_i}}$.

Furthermore, we denote by p^u the S-sorted mapping from $\downarrow u$ to $A^{f,u}$ defined, for every $s \in S$ and $i \in |u|$, as $p^u_s(v^s_i) = (\pi^u_i)^{\mathbf{A}}$, and by $(p^u)^\sharp$ the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow u)$ to $\mathbf{A}^{f,u}$ such that $(p^u)^\sharp \circ \eta_{\downarrow u} = p^u$.

Remark. For a Σ-algebra $\mathbf{B} = (B, G)$, we have that $G\colon \Sigma \longrightarrow \mathrm{HOp}_S(B)$ and $\mathbf{B} \cong \mathbf{HOp}_S(B)^{G,\lambda}$, where λ is the empty word on S. Besides, for every $u \in S^\star$, we have that $\mathbf{B}^{B_u}$, the direct B_u-power of $\mathbf{B}$, is isomorphic to $\mathbf{HOp}_S(B)^{G,u}$.

Lemma 1. *Let Σ be an S-sorted signature, $\mathbf{A}$ a Hall algebra, $f\colon \Sigma \longrightarrow A$ and $u \in S^{\star}$. Then, for every $(w,s) \in S^{\star} \times S$, $P \in \mathrm{T}_{\Sigma}(\downarrow w)_s$ and $a \in \prod_{i\in|w|} A_{u,w_i}$, we have that*

$$P^{\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1}) = \xi^{\mathbf{A}}_{u,w,s}((p^w)^{\sharp}_s(P), a_0,\ldots,a_{|w|-1}).$$

Proof. By algebraic induction on the complexity of P. If P is a variable v^s_i, with $i \in |w|$, then

$$\begin{aligned} v_i^{s,\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1}) &= a^{\sharp}_{w_i}(v^s_i) \\ &= a_i \\ &= \xi^{\mathbf{A}}_{u,w,s}((\pi^w_i)^{\mathbf{A}}, a_0,\ldots,a_{|w|-1}) \quad \text{(by H}_1\text{)} \\ &= \xi^{\mathbf{A}}_{u,w,s}((p^w)^{\sharp}_s(v^s_i), a_0,\ldots,a_{|w|-1}). \end{aligned}$$

Let us assume that $P = \sigma(Q_0,\ldots,Q_{|x|-1})$, with $\sigma\colon x \longrightarrow s$ and that, for every $j \in |x|$, $Q_j \in \mathrm{T}_{\Sigma}(\downarrow w)_{x_j}$ fulfills the induction hypothesis. Then we have that

$$\begin{aligned} &(\sigma(Q_0,\ldots,Q_{|x|-1}))^{\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1}) \\ &= \sigma^{\mathbf{A}^{f,u}}(Q_0^{\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1}),\ldots,Q_{|x|-1}^{\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1})) \\ &= \xi^{\mathbf{A}}_{u,x,s}(f(\sigma), Q_0^{\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1}),\ldots,Q_{|x|-1}^{\mathbf{A}^{f,u}}(a_0,\ldots,a_{|w|-1})) \\ &= \xi^{\mathbf{A}}_{u,x,s}(f(\sigma), \xi^{\mathbf{A}}_{u,w,x_0}((p^w)^{\sharp}_{x_0}(Q_0), a_0,\ldots,a_{|w|-1}),\ldots, \\ &\qquad \xi^{\mathbf{A}}_{u,w,x_{|x|-1}}((p^w)^{\sharp}_{x_{|x|-1}}(Q_{|x|-1}), a_0,\ldots,a_{|w|-1})) \quad \text{(by Ind. Hypothesis)} \\ &= \xi^{\mathbf{A}}_{u,w,s}(\xi^{\mathbf{A}}_{w,x,s}(f(\sigma),(p^w)^{\sharp}_{x_0}(Q_0),\ldots,(p^w)^{\sharp}_{x_{|x|-1}}(Q_{|x|-1})), a_0,\ldots,a_{|w|-1}) \text{(by H}_3\text{)} \\ &= \xi^{\mathbf{A}}_{u,w,s}(\sigma^{\mathbf{A}_w}((p^w)^{\sharp}_{x_0}(Q_0),\ldots,(p^w)^{\sharp}_{x_{|x|-1}}(Q_{|x|-1})), a_0,\ldots,a_{|w|-1}) \\ &= \xi^{\mathbf{A}}_{u,w,s}((p^w)^{\sharp}_s(\sigma, Q_0,\ldots,Q_{|x|-1}), a_0,\ldots,a_{|w|-1}) \\ &= \xi^{\mathbf{A}}_{u,w,s}((p^w)^{\sharp}_s(P), a_0,\ldots,a_{|w|-1}). \end{aligned}$$

□

Next we prove that, for every $S^{\star} \times S$-sorted set Σ, the Hall algebra for (S,Σ) is isomorphic to the free Hall algebra on Σ.

Proposition 4. *Let Σ be an S-sorted signature, i.e., an $S^{\star} \times S$-sorted set. Then the Hall algebra $\mathbf{HTer}_S(\Sigma)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$.*

Proof. It is enough to prove that $\mathbf{HTer}_S(\Sigma)$ has the universal property of the free Hall algebra on Σ. Therefore we have to specify an $S^{\star} \times S$-sorted mapping h from Σ to $\mathrm{HTer}_S(\Sigma)$ such that, for every Hall algebra $\mathbf{A}$ and $S^{\star} \times S$-sorted mapping f from Σ to A, there is a unique homomorphism $\widehat{f}$ from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{A}$ such that $\widehat{f} \circ h = f$. Let h be the $S^{\star} \times S$-sorted mapping defined, for every $(w,s) \in S^{\star} \times S$, as

$$h_{w,s}\begin{cases} \Sigma_{w,s} & \longrightarrow \mathrm{T}_{\Sigma}(\downarrow w)_s \\ \quad\sigma & \longmapsto \sigma(v^s_0,\ldots,v^s_{|w|-1}) \end{cases}$$

Let $\mathbf{A}$ be a Hall algebra, $f\colon \Sigma \longrightarrow A$ an $S^{\star} \times S$-sorted mapping and $\widehat{f}$ the $S^{\star} \times S$-sorted mapping from $\mathrm{HTer}_S(\Sigma)$ to A defined, for every $(w,s) \in S^{\star} \times S$, as $\widehat{f}_{w,s} = (p^w)^{\sharp}_s$, where, we recall, $(p^w)^{\sharp}$ is the unique homomorphism from $\mathbf{T}_{\Sigma}(\downarrow w)$ to $\mathbf{A}^{f,w}$ such that $(p^w)^{\sharp} \circ \eta_{\downarrow w} = p^w$. Then $\widehat{f}$ is a homomorphism of Hall algebras, because, on the one hand, for $w \in S^{\star}$ and $i \in |w|$ we have that

$$\begin{aligned} \widehat{f}_{w,w_i}((\pi^w_i)^{\mathbf{HTer}_S(\Sigma)}) &= \widehat{f}_{w,w_i}(v^s_i) \\ &= p^w_{w_i}(v^s_i) \\ &= (\pi^w_i)^{\mathbf{A}}, \end{aligned}$$

and, on the other hand, for $P \in \mathrm{T}_\Sigma(\downarrow w)_s$ and $(Q_i \mid i \in |w|) \in \mathrm{T}_\Sigma(\downarrow u)_w$ we have that

$$\begin{aligned}
&\widehat{f}_{u,s}(\xi^{\mathbf{HTer}_S(\Sigma)}_{u,w,s}(P, Q_0, \dots, Q_{|w|-1}))\\
&= (p^u)^\sharp_s(\mathcal{Q}^\sharp_s(P))\\
&= ((p^u)^\sharp \circ \mathcal{Q})^\sharp_s(P) \qquad (\text{because } (p^u)^\sharp \circ \mathcal{Q}^\sharp = ((p^u)^\sharp \circ \mathcal{Q})^\sharp)\\
&= P^{\mathbf{A}^{f,u}}((p^u)^\sharp_{w_0}(Q_0), \dots, (p^u)^\sharp_{w_{|w|-1}}(Q_{|w|-1}))\\
&= \xi^{\mathbf{A}}_{u,w,s}((p^w)^\sharp_s(P), (p^u)^\sharp_{w_0}(Q_0), \dots, (p^u)^\sharp_{w_{|w|-1}}(Q_{|w|-1})) \qquad (\text{by Lemma 1})\\
&= \xi^{\mathbf{A}}_{u,w,s}(\widehat{f}_{w,s}(P), \widehat{f}_{u,w_0}(Q_0), \dots, \widehat{f}_{u,w_{|w|-1}}(Q_{|w|-1})).
\end{aligned}$$

Therefore the $S^\star \times S$-sorted mapping $\widehat{f}$ is a homomorphism. Furthermore, $\widehat{f} \circ h = f$, because, for every $w \in S^\star$, $s \in S$, and $\sigma \in \Sigma_{w,s}$, we have that

$$\begin{aligned}
\widehat{f}_{w,s}(h_{w,s}(\sigma)) &= (p^w)^\sharp_s(\sigma(v^s_0, \dots, v^s_{|w|-1}))\\
&= \sigma^{\mathbf{A}_w}(p^w_{w_0}(v^s_0), \dots, p^w_{w_{|w|-1}}(v^s_{|w|-1}))\\
&= \xi^{\mathbf{A}}_{w,w,s}(f_{(w,s)}(\sigma), (\pi^w_0)^{\mathbf{A}}, \dots, (\pi^w_{|w|-1})^{\mathbf{A}})\\
&= f_{w,s}(\sigma) \quad (\text{by } \mathrm{H}_2).
\end{aligned}$$

It is obvious that $\widehat{f}$ is the unique homomorphism such that $\widehat{f} \circ h = f$. Henceforth $\mathbf{HTer}_S(\Sigma)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$. □

As was announced above, this isomorphism together with the adjunction $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$ has as an immediate consequence that, for every S-sorted set A and every S-sorted signature Σ, the sets $\mathrm{Hom}(\Sigma, \mathrm{HOp}_S(A))$, in the category $\mathbf{Set}^{S^\star \times S}$, and $\mathrm{Hom}(\mathbf{HTer}_S(\Sigma), \mathbf{HOp}_S(A))$, in the category $\mathbf{Alg}(\mathrm{H}_S)$, are naturally isomorphic.

Actually, the isomorphism assigns, for an S-sorted set A, as we will prove immediately below for the case in which A is the underlying S-sorted set of a Σ-algebra $\mathbf{A}$, to a structure of Σ-algebra F on A (i.e., an $S^\star \times S$-sorted mapping F from Σ to $\mathrm{HOp}_S(A)$) the homomorphism of Hall algebras $\mathrm{Tr}^{(A,F)} = (\mathrm{Tr}^{\downarrow w,(A,F)}_s)_{(w,s)\in S^\star \times S}$ from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HOp}_S(A)$, where, for every $w \in S^\star$, the subfamily $\mathrm{Tr}^{\downarrow w,(A,F)} = (\mathrm{Tr}^{\downarrow w,(A,F)}_s)_{s\in S}$ of $\mathrm{Tr}^{(A,F)}$ is the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow w)$ to $(A,F)^{A_w}$, the direct A_w-power of (A,F), such that $\mathrm{Tr}^{\downarrow w,(A,F)} \circ \eta_{\downarrow w} = \mathrm{p}^A_{\downarrow w}$, where $\mathrm{p}^A_{\downarrow w}$ is the S-sorted mapping from $\downarrow w$ to A^{A_w} defined, for every $s \in S$ and $v^s_i \in (\downarrow w)_s$, as $\mathrm{p}^A_{\downarrow w,s}(v^s_i) = \mathrm{pr}^A_{w,i}$; while the inverse isomorphism sends an homomorphism h from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HOp}_S(A)$ to, essentially, the algebraic structure $\mathrm{G}_{\mathrm{H}_S}(h) \circ \eta_\Sigma$ on A, where η_Σ is the canonical embedding of Σ into $\mathrm{T}_{\mathrm{H}_S}(\Sigma)$.

After having stated, for an S-sorted set A and a structure of Σ-algebra F on A, the definition of the $S^\star \times S$-sorted mapping $\mathrm{Tr}^{(A,F)}$, we prove in the following proposition, among others, that, for a Σ-algebra $\mathbf{A} = (A,F)$, it is in fact an homomorphism of Hall algebras from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HOp}_S(\mathbf{A}) = \mathbf{HOp}_S(A)$.

Proposition 5. *Let $\mathbf{A} = (A,F)$ be a Σ-algebra. Then $\mathrm{Tr}^{\mathbf{A}} = \mathrm{Tr}^{(A,F)}$ is a homomorphism of Hall algebras from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HOp}_S(\mathbf{A}) = \mathbf{HOp}_S(A)$. Moreover, $\mathrm{Ker}(\mathrm{Tr}^{\mathbf{A}}) = \mathrm{Th}_\Sigma(\mathbf{A})$, the Σ-equational theory determined by $\mathbf{A}$.*

Proof. Let $w \in S^\star$ be and $i \in |w|$. Then we have that

$$\mathrm{Tr}^{\downarrow w,\mathbf{A}}_s((\pi^w_i)^{\mathbf{HTer}_S(\Sigma)}) = \mathrm{pr}^A_{w,i} = (\pi^w_i)^{\mathbf{HOp}_S(A)}.$$

Next given $u, w \in S^{\star}$, $s \in S$, $P \in \mathrm{T}_{\Sigma}(\downarrow w)_s$, and $(Q_i)_{i\in|w|} \in \mathrm{T}_{\Sigma}(\downarrow u)_w$, we have to prove that

$$\begin{aligned}&\mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}(P,(Q_i)_{i\in|w|})) =\\ &\xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_s^{\downarrow w,\mathbf{A}}(P),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}).\end{aligned}$$

Let $\mathcal{X}^{u,w}$ be the S-sorted set whose s-th coordinate, for $s \in S$, is the set of all terms $P \in \mathrm{T}_{\Sigma}(\downarrow w)_s$ which, for every $(Q_i)_{i\in|w|} \in \mathrm{T}_{\Sigma}(\downarrow u)_w$, satisfy the above equation. We prove that $\mathcal{X}^{u,w} = \mathrm{T}_{\Sigma}(\downarrow w)$ by algebraic induction.

For every $v_i^s \in (\downarrow w)_s$, we have that v_i^s, identified to $\eta_{\downarrow w,s}(v_i^s) = (\pi_i^w)^{\mathbf{HTer}_S(\Sigma)}$ belongs to $\mathcal{X}_s^{u,w}$ since

$$\begin{aligned}&\mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}(v_i^s,(Q_i)_{i\in|w|}))\\ &= \mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}((\pi_i^w)^{\mathbf{HTer}_S(\Sigma)},(Q_i)_{i\in|w|}))\\ &= \mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(Q_i) \quad (\text{by } \mathrm{H}_1)\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}((\pi_i^w)^{\mathbf{HOp}_S(A)},(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|})\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\mathrm{pr}_{w,i}^{A},(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|})\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_s^{\downarrow w,\mathbf{A}}(v_i^s),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}).\end{aligned}$$

For every $\sigma \in \Sigma$, with $\sigma\colon x \longrightarrow s$, and every $(R_j)_{j\in|x|} \in \mathcal{X}_x$, $\sigma((R_j)_{j\in|x|}) \in \mathcal{X}_s^{u,w}$ since

$$\begin{aligned}&\mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}(\sigma((R_j)_{j\in|x|}),(Q_i)_{i\in|w|}))\\ &= \mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}(\xi_{w,x,s}^{\mathbf{HTer}_S(\Sigma)}(\sigma((v_j)_{j\in|x|}),(R_j)_{j\in|x|}),(Q_i)_{i\in|w|}))\\ &= \mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\xi_{u,x,s}^{\mathbf{HTer}_S(\Sigma)}(\sigma((v_j)_{j\in|x|}),\xi_{u,w,x_0}^{\mathbf{HTer}_S(\Sigma)}(R_0,(Q_i)_{i\in|w|}),\dots,\\ &\qquad\qquad \xi_{u,w,x_{|x|-1}}^{\mathbf{HTer}_S(\Sigma)}(R_{|x|-1},(Q_i)_{i\in|w|}))) \quad (\text{by } \mathrm{H}_3)\\ &= \mathrm{Tr}_s^{\downarrow u,\mathbf{A}}(\sigma((\xi_{u,w,x_j}^{\mathbf{HTer}_S(\Sigma)}(R_j,(Q_i)_{i\in|w|}))_{j\in|x|}))\\ &= F_\sigma^{\mathbf{A}^{A_u}}(\mathrm{Tr}_{x_0}^{\downarrow u,\mathbf{A}}(\xi_{u,w,x_0}^{\mathbf{HTer}_S(\Sigma)}(R_0,(Q_i)_{i\in|w|})),\dots,\\ &\qquad \mathrm{Tr}_{x_{|x|-1}}^{\downarrow u,\mathbf{A}}(\xi_{u,w,x_{|x|-1}}^{\mathbf{HTer}_S(\Sigma)}(R_{|x|-1},(Q_i)_{i\in|w|})))\\ &= F_\sigma^{\mathbf{A}^{A_u}}(\xi_{u,w,x_0}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_{x_0}^{\downarrow w,\mathbf{A}}(R_0),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}),\dots,\\ &\qquad \xi_{u,w,x_{|x|-1}}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_{x_{|x|-1}}^{\downarrow w,\mathbf{A}}(R_{|x|-1}),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|})) \quad (\text{by Ind. Hypothesis})\\ &= \xi_{u,x,s}^{\mathbf{HOp}_S(A)}(F_\sigma^{\mathbf{A}},\xi_{u,w,x_0}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_{x_0}^{\downarrow w,\mathbf{A}}(R_0),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}),\dots,\\ &\qquad\qquad \xi_{u,w,x_{|x|-1}}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_{x_{|x|-1}}^{\downarrow w,\mathbf{A}}(R_{|x|-1}),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}))\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\xi_{w,x,s}^{\mathbf{HOp}_S(A)}(F_\sigma^{\mathbf{A}},(\mathrm{Tr}_{x_j}^{\downarrow w,\mathbf{A}}(R_j))_{j\in|x|}),\\ &\qquad (\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}) \quad (\text{by } \mathrm{H}_3)\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\xi_{w,x,s}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_s^{\downarrow x,\mathbf{A}}(\sigma((v_j)_{j\in|x|})),(\mathrm{Tr}_{x_j}^{\downarrow w,\mathbf{A}}(R_j))_{j\in|x|},\\ &\qquad (\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}))\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_s^{\downarrow w,\mathbf{A}}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}(\sigma((v_j)_{j\in|x|}),(R_j)_{j\in|x|})),\\ &\qquad (\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|})\\ &= \xi_{u,w,s}^{\mathbf{HOp}_S(A)}(\mathrm{Tr}_s^{\downarrow w,\mathbf{A}}(\sigma((R_j)_{j\in|x|})),(\mathrm{Tr}_{w_i}^{\downarrow u,\mathbf{A}}(Q_i))_{i\in|w|}).\end{aligned}$$

Finally, $\mathrm{Th}_{\Sigma}(\mathbf{A})$, the Σ-equational theory determined by $\mathbf{A}$, is, by definition $(\mathrm{Ker}(\mathrm{Tr}^{\downarrow w,\mathbf{A}})_s)_{(w,s)\in S^{\star}\times S}$, which is precisely the kernel of $\mathrm{Tr}^{\mathbf{A}}$ and, therefore, it is a congruence on $\mathbf{HTer}_S(\Sigma)$. □

The last part of the proposition just stated can be extended to sets of Σ-algebras and, in particular, to the models of a family $\mathcal{E}$ of finitary Σ-equations. From this it will follow that the operator $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}$ is sound relative to the operator of semantical consequence Cn_Σ.

Proposition 6. *Let $\mathcal{K}$ a set of Σ-algebras. Then $\mathrm{Th}_\Sigma(\mathcal{K})$ is a congruence on* $\mathrm{HTer}_S(\Sigma)$.

Proof. Because $\mathrm{Th}_\Sigma(\mathcal{K})$ is $\bigcap_{\mathbf{A}\in\mathcal{K}} \mathrm{Ker}(\mathrm{Tr}^{\mathbf{A}}) \in \mathrm{Cgr}(\mathbf{HTer}_S(\Sigma))$. □

Corollary 1 (Soundness Theorem). *Let Σ be an S-sorted signature. Then we have that* $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)} \leq \mathrm{Cn}_\Sigma$.

Proof. Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$. By definition $\mathrm{Cn}_\Sigma(\mathcal{E}) = \mathrm{Th}_\Sigma(\mathrm{Mod}_\Sigma(\mathcal{E}))$. But $\mathrm{Th}_\Sigma(\mathrm{Mod}_\Sigma(\mathcal{E}))$ is a congruence on $\mathbf{HTer}_S(\Sigma)$ and contains $\mathcal{E}$. Therefore $\mathrm{Cn}_\Sigma(\mathcal{E})$ contains $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}(\mathcal{E})$. □

The congruence generated in $\mathbf{HTer}_S(\Sigma)$ by a family of finitary Σ-equations $\mathcal{E}$ can be characterized as follows.

Proposition 7. *Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$. Then $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}(\mathcal{E})$ is the smallest sub-sorted set $\overline{\mathcal{E}}$ of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ that contains $\mathcal{E}$ and is such that, for every $u, w \in S^\star$ and $s \in S$, satisfies the following conditions:*

(1) Reflexivity. *For every $P \in \mathrm{HTer}_S(\Sigma)_{w,s}$, $(P, P) \in \overline{\mathcal{E}}_{w,s}$.*
(2) Symmetry. *For every P, $Q \in \mathrm{HTer}_S(\Sigma)_{w,s}$, if $(P, Q) \in \overline{\mathcal{E}}_{w,s}$, then $(Q, P) \in \overline{\mathcal{E}}_{w,s}$.*
(3) Transitivity. *For every P, Q, $R \in \mathrm{HTer}_S(\Sigma)_{w,s}$, if (P, Q), $(Q, R) \in \overline{\mathcal{E}}_{w,s}$, then $(P, R) \in \overline{\mathcal{E}}_{w,s}$.*
(4) Substitutivity. *For every $(M_i)_{i\in|w|}, (N_i)_{i\in|w|} \in \prod_{i\in|w|} \mathrm{HTer}_S(\Sigma)_{u,w_i}$ and every $(P, Q) \in \overline{\mathcal{E}}_{w,s}$, if, for every $i \in |w|$, it happens that $(M_i, N_i) \in \overline{\mathcal{E}}_{u,w_i}$, then*

$$(\xi_{u,w,s}(P, M_0, \ldots, M_{|w|-1}), \xi_{u,w,s}(Q, N_0, \ldots, N_{|w|-1})) \in \overline{\mathcal{E}}_{u,s}.$$

□

Let us remark that in the proposition just stated, the substitutivity condition for $w = \lambda$, the empty word on S, demands that if $(P, Q) \in \overline{\mathcal{E}}_{\lambda,s}$ then, for every $u \in S^\star$, $(P, Q) \in \overline{\mathcal{E}}_{u,s}$.

Proposition 8. *Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $\sigma \in \Sigma_{w,s}$. If, for every $i \in |w|$, we have that $(P_i, Q_i) \in \overline{\mathcal{E}}_{w,w_i}$, then $(\sigma(P_0, \ldots, P_{|w|-1}), \sigma(Q_0, \ldots, Q_{|w|-1})) \in \overline{\mathcal{E}}_{w,s}$.*

Proof. By reflexivity $(\sigma(v_0, \ldots, v_{|w|-1}), \sigma(v_0, \ldots, v_{|w|-1})) \in \overline{\mathcal{E}}_{w,s}$ hence, by substitutivity, $(\sigma(P_0, \ldots, P_{|w|-1}), \sigma(Q_0, \ldots, Q_{|w|-1})) \in \overline{\mathcal{E}}_{w,s}$. □

Proposition 9. *Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $(w, s) \in S^\star \times S$. If $(P, Q) \in \overline{\mathcal{E}}_{w,s}$ and f is an endomorphism of $\mathbf{T}_\Sigma(\downarrow w)$, then $(f_s(P), f_s(Q)) \in \overline{\mathcal{E}}_{w,s}$.*

Proof. For every $i \in |w|$, the equation $(f_{w_i}(v_i), f_{w_i}(v_i))$ is in $\overline{\mathcal{E}}_{w,w_i}$. By substitutivity, we have that

$$(\xi_{w,w,s}(P, f_{w_0}(v_0), \ldots, f_{w_{|w|-1}}(v_{|w|-1})), \xi_{w,w,s}(Q, f_{w_0}(v_0), \ldots, f_{w_{|w|-1}}(v_{|w|-1})))$$

is in $\overline{\mathcal{E}}_{w,s}$, hence $(f_s(P), f_s(Q)) \in \overline{\mathcal{E}}_{w,s}$. □

Corollary 2. *Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $w \in S^\star$. Then $\overline{\mathcal{E}}_w = (\overline{\mathcal{E}}_{w,s})_{s\in S}$ is a fully invariant congruence on $\mathbf{T}_\Sigma(\downarrow w)$.*

Proof. By definition, $\overline{\mathcal{E}}_w$ is an equivalence on $\mathrm{T}_\Sigma(\downarrow w)$, by Proposition 8 is compatible with the operations in Σ and by Proposition 9 is closed under endomorphisms. □

We remark that the congruence $\overline{\mathcal{E}}_w$ contains $\mathrm{Cg}^{\mathrm{fi}}_{\mathbf{T}_\Sigma(\downarrow w)}(\mathcal{E}_w)$, the fully invariant congruence generated by $\mathcal{E}_w = (\mathcal{E}_{w,s})_{s\in S}$ and, in general, the containment is strict, because $\mathrm{Cg}^{\mathrm{fi}}_{\mathbf{T}_\Sigma(\downarrow w)}(\mathcal{E}_w)$ contains only the consequences of the subfamily of $\mathcal{E}$ which has the equations in $\mathcal{E}$ with variables in $\downarrow w$, whereas $\overline{\mathcal{E}}_w$ contains the equations with variables in $\downarrow w$ that are consequence of all equations in $\mathcal{E}$.

Proposition 10. *Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ and $w \in S^\star$. Then $\mathbf{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w$ is a model of $\mathcal{E}$.*

Proof. Let $(P,Q) \in \mathcal{E}_{u,s}$ be and $R\colon \downarrow u \longrightarrow \mathrm{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w$ a valuation. Then

$$R^\sharp(P) = [P(R_0,\ldots,R_{|u|-1})] = [Q(R_0,\ldots,R_{|u|-1})] = R^\sharp(Q).$$

□

Proposition 11 (Adequacy Theorem)**.** *Let Σ be an S-sorted signature. Then we have that* $\mathrm{Cn}_\Sigma \leq \mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}$.

Proof. Let $\mathcal{E}$ be a sub-sorted set of $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$. If $(P,Q) \in \mathrm{Cn}_\Sigma(\mathcal{E})_{w,s}$, then, because $\mathbf{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w$ is a model of $\mathcal{E}$, $P^{\mathbf{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w} = Q^{\mathbf{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w}$. Hence

$$\begin{aligned}
[P] &= [\xi_{w,w,s}(P,\pi^w_0,\ldots,\pi^w_{|w|-1})]\\
&= [P^{\mathbf{T}_\Sigma(\downarrow w)}(v_0,\ldots,v_{|w|-1})]\\
&= P^{\mathbf{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w}([v_0],\ldots,[v_{|w|-1}])\\
&= Q^{\mathbf{T}_\Sigma(\downarrow w)/\overline{\mathcal{E}}_w}([v_0],\ldots,[v_{|w|-1}])\\
&= [Q^{\mathbf{T}_\Sigma(\downarrow w)}(v_0,\ldots,v_{|w|-1})]\\
&= [\xi_{w,w,s}(Q,\pi^w_0,\ldots,\pi^w_{|w|-1})]\\
&= [Q],
\end{aligned}$$

and $(P,Q) \in \mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}(\mathcal{E})_{w,s}$. □

Corollary 3 (Completeness theorem of Goguen-Meseguer)**.** *Let Σ be an S-sorted signature. Then we have that $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)} = \mathrm{Cn}_\Sigma$, or, what is equivalent, the algebraic lattice of all Σ-equational theories is isomorphic to the algebraic lattice of all congruences on the Hall algebra $\mathbf{HTer}_S(\Sigma)$.*

The completeness theorem of Goguen-Meseguer allows us to obtain a calculus of finitary Σ-equations, i.e., a calculus on sets of variables of the form $\downarrow w$, for $w \in S^\star$, or, what amounts to the same, on finite sub-S-sorted sets X of the S-sorted set $V = (V_s)_{s\in S}$. Before we state the finitary Σ-equational inference rules we agree that $(P,Q) : (X,s)$ means that the finitary Σ-equation (P,Q) is of type (X,s), i.e., that $P, Q \in \mathrm{T}_\Sigma(X)_s$, in addition if $P \in \mathrm{T}_\Sigma(X)_s$ and $\mathcal{P} = (P_s)_{s\in S}\colon X \longrightarrow \mathrm{T}_\Sigma(Y)$, then $P(x/P_{s,x})_{s\in S,\, x\in X_s}$ is $\mathcal{P}^\sharp_s(P)$.

Proposition 12 (Inference Rules)**.** *The following finitary Σ-equational inference rules determine a closure operator on $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$ that is identical to the closure operator Cn_Σ.*

(R1) Reflexivity. *For all $P \in \mathrm{T}_\Sigma(X)_s$, $(P,P) \in \overline{\mathcal{E}}_{X,s}$, or diagrammatically*

$$\frac{}{(P,P) : (X,s)}\quad P \in \mathrm{T}_\Sigma(X)_s.$$

(R2) Symmetry. *For all* $P, Q \in \mathrm{T}_\Sigma(X)_s$*, if* $(P,Q) \in \overline{\mathcal{E}}_{X,s}$*, then* $(Q,P) \in \overline{\mathcal{E}}_{X,s}$*, or diagrammatically*

$$\frac{(P,Q) : (X,s)}{(Q,P) : (X,s)}.$$

(R3) Transitivity. *For all* $P, Q, R \in \mathrm{T}_\Sigma(X)_s$*, if* $(P,Q) \in \overline{\mathcal{E}}_{X,s}$ *and* $(Q,R) \in \overline{\mathcal{E}}_{X,s}$*, then* $(P,R) \in \overline{\mathcal{E}}_{X,s}$ *or diagrammatically*

$$\frac{(P,Q) : (X,s) \qquad (Q,R) : (X,s)}{(P,R) : (X,s)}.$$

(R4) Generalized substitutivity. *For all* $(P,Q) \in \overline{\mathcal{E}}_{X,s}$ *and* $\mathcal{P}, \mathcal{Q} \colon X \longrightarrow \mathrm{T}_\Sigma(Y)$ *such that, for every* $s \in S$*,* $x \in X_s$*,* $(P_{s,x}, Q_{s,x}) \in \overline{\mathcal{E}}_{Y,s}$*,*

$$(\xi_{Y,X,s}(P, (P_{s,x})_{s\in S,\, x\in X_s}), \xi_{Y,X,s}(Q, (Q_{s,x})_{s\in S,\, x\in X_s})) \in \overline{\mathcal{E}}_{Y,s},$$

or diagrammatically

$$\frac{(P,Q) : (X,s) \qquad ((P_{s,x}, Q_{s,x}) : (Y,s))_{s\in S,\, x\in X_s}}{(P(x/P_{s,x})_{s\in S,\, x\in X_s}, Q(x/Q_{s,x})_{s\in S,\, x\in X_s}) : (Y,s)}.$$

Proof. Because the finitary Σ-equational inference rules are the translation of the conditions in Proposition 7. □

Proposition 13. *The inference rule R4 is equivalent, assuming R1, to the following inference rule*

(R4′) Substitutivity.

$$\frac{(P,Q) : (X,s) \qquad (P',Q') : (Y,t)}{(P(x/P'), Q(x/Q')) : ((X - \delta^{t,x}) \cup Y, s)} \quad x \in X_t \ [\delta^{t,x}_t = \{x\},\ \delta^{t,x}_s = \varnothing,\ \textit{if } s \neq t].$$

Proof. We begin by proving that R4 implies R4′. If $(P,Q) : (X,s)$ and $(P',Q') : (Y,t)$ are deducible and $x \in X_t$, then also, by reflexivity, the finitary Σ-equations in the family $((P''_{s,x}, Q''_{s,x}) : ((X - \delta^{t,x}) \cup Y, s))_{s\in S,\, x\in X_s}$, where $P''_{t,x} = P'$, $Q''_{t,x} = Q'$, and otherwise $P''_{s,y} = Q''_{s,y} = y$, are deducible. Then, by generalized substitutivity, $(P(x/P'), Q(x/Q')) : ((X - \delta^{t,x}) \cup Y, s)$ is deducible, because $P(x/P') = (P(x/P''_{s,x})_{s\in S,\, x\in X_s}$ and $Q(x/Q') = Q(x/P''_{s,x})_{s\in S,\, x\in X_s}$.

Reciprocally, R4′ implies R4, by reiterating the application of R4′ $\mathrm{card}(\coprod X)$-times, where $\coprod X$ is the coproduct of the S-sorted set X. □

In some presentations of many-sorted equational logic, e.g., in [4], two additional inference rules that allow the adjunction and suppression of variables, under some conditions, are introduced. But as we will prove below both rules are derived rules, relative to the system of rules R1 to R4.

Definition 8 (Abstraction and concretion)**.**

(R5) *Abstraction.*

$$\frac{(P,Q) : (X,s)}{(P,Q) : (X \cup \delta^{t,x}, s)} \quad x \in V_t - X_t.$$

(R6) *Concretion.*

$$\frac{(P,Q) : (X,s)}{(P,Q) : (X - \delta^{t,x}, s)} \quad x \in X_t,\ x \notin \mathrm{var}(P,Q),\ \mathrm{T}_\Sigma((\varnothing)_{s\in S})_t \neq \varnothing.$$

Proposition 14. *The abstraction and concretion rules are derived rules.*

Proof. Abstraction is a derived rule. Let $y \in V_s$ be such that $y \notin X_s$. Then, by reflexivity, the finitary Σ-equation $(y, y) : (\delta^{s,y} \cup \delta^{t,x}, s)$ is deducible. Hence, by substitutivity, the finitary Σ-equation

$$(y(y/P), y(y/Q)) : (((\delta^{s,y} \cup \delta^{t,x}) - \delta^{s,y}) \cup X, s)$$

that is identical to $(P, Q) : (X \cup \delta^{t,x}, s)$, is also deducible. As a particular case we have that if $(P, Q) : ((\varnothing)_{s\in S}, s)$ is deducible, then $(P, Q) : (\delta^{t,x}, s)$ is also deducible.

Concretion is a derived rule. Since $\mathrm{T}_\Sigma((\varnothing)_{s\in S})_t \neq \varnothing$ let us choose an $R \in \mathrm{T}_\Sigma((\varnothing)_{s\in S})_t$. Then, by reflexivity, the finitary Σ-equation $(R, R) : ((\varnothing)_{s\in S}, t)$ is deducible. Hence, by substitutivity, $(P(x/R), Q(x/R)) : ((X - \delta^{t,x}) \cup (\varnothing)_{s\in S}, s)$ is also deducible and, because $x \notin \mathrm{var}(P, Q)$, $(P, Q) : (X - \delta^{t,x}, s)$ is deducible. □

Definition 9 (Replacement rule)**.**

(R7) *Replacement.*

$$\frac{(P^i, Q^i) : (X, w_i)}{(\sigma(P_0, \dots, P_{|w|-1}), \sigma(Q_0, \dots, Q_{|w|-1})) : (X, s)} \quad \sigma \in \Sigma_{w,s}.$$

Proposition 15. *The replacement rule is a derived rule.*

Proof. By reflexivity, $(\sigma(v_0, \dots, v_{|w|-1}), \sigma(v_0, \dots, v_{|w|-1})) : (\downarrow w, s)$ is deducible. Now, by reiterating substitutivity $|w|$-times, we obtain the desired finitary Σ-equation. □

Everything we have done until now can be extended to the case of S-finitary Σ-equations, where, for $X \in \mathrm{Sub}_{S-\mathrm{f}}(V) = \{ X \subseteq V \mid \forall s \in S\,(\mathrm{card}(X_s) < \aleph_0) \}$, the set of *$S$-finite* sub-$S$-sorted sets of V, and $s \in S$, an *S-finitary Σ-equation of type* (X, s) is a pair of coterminal parallel S-sorted mappings from the S-sorted set $\delta^s = (\delta^s_t)_{t\in S}$, the *delta of Kronecker in* s, such that $\delta^s_t = \varnothing$ if $s \neq t$ and $\delta^s_s = 1$, to $\mathrm{T}_\Sigma(X)$. In this respect we only have to change the (finitary) structural operations of the Hall algebras to S-finitary operations. Moreover, the equational calculus has the same inference rules R1–R4, but generalized to S-sorted sets of variables which are S-finite. However, the rule of substitution is no longer equivalent to the generalized rule of substitution. Finally, the rules of abstraction and concretion for this case are the following.

Definition 10.

(R5′) *Generalized abstraction.*

$$\frac{(P, Q) : (X, s)}{(P, Q) : (X \cup Y, s)}\,.$$

(R6′) *Generalized concretion.*

$$\frac{(P, Q) : (X, s)}{(P, Q) : (X - Y, s)} \quad Y \cap \mathrm{var}(P, Q) = \varnothing,\ \mathrm{supp}(Y) \subseteq \mathrm{supp}(\mathrm{T}_\Sigma((\varnothing)_{s\in S})),$$

where, for an S-sorted set Z, we agree that $\mathrm{supp}(Z)$, the *support* of Z, is precisely $\mathrm{supp}(Z) = \{ s \in S \mid Z_s \neq \varnothing \}$.

3. The equivalence between Hall algebras and Bénabou theories.

Another approximation to the study of many-sorted algebras has been proposed by Bénabou in [1], by making use of the finitary many-sorted algebraic theories (categories with objects the words on a set of sorts S such that, for every word $w = (w_i)_{i\in n}$, there exists a family of morphisms $(p^w_i)_{i\in n}$, where, for $i \in n$, p^w_i is a morphism from w to (w_i), the word of length one associated to the letter w_i, such that $(w, (p^w_i)_{i\in n})$ is a product of the family $((w_i))_{i\in n})$, that are the generalization to the many-sorted case of the finitary single-sorted algebraic theories of Lawvere, see [5].

The equational presentation of the finitary many-sorted algebraic theories of Bénabou gives rise to what we call Bénabou algebras. And the Bénabou algebras, even having a many-sorted specification different from that of the Hall algebras, are also models of the essential properties of the clones for the many-sorted operations. This is so since, as we will prove below, for an arbitrary but fixed set of sorts S, the Bénabou algebras for S are equivalent to the Hall algebras for S, i.e., there exists an equivalence between the category $\mathbf{Alg}(\mathrm{H}_S)$, of Hall algebras for S, and the category $\mathbf{Alg}(\mathrm{B}_S)$, of Bénabou algebras for S.

Moreover, the Bénabou algebras for S, as we will show below, are more strongly linked to the finitary many-sorted theories algebraic theories than are the Hall algebras, because, as we will prove afterwards, there exists an isomorphism between the category $\mathbf{Alg}(\mathrm{B}_S)$ and the category $\mathbf{BTh}_{\mathrm{f}}(S)$, of finitary many-sorted algebraic theories for S.

In order to accomplish what has been announced we begin by defining the Bénabou algebras as those that satisfy the laws of a convenient many-sorted specification.

Definition 11. Let S be a set of sorts and V^{B_S} the $(S^\star)^2$-sorted set of variables $(V_{u,w})_{(u,w)\in(S^\star)^2}$ where, for every $(u,w)\in(S^\star)^2$, $V_{u,w}=\{\, v_n^{u,w} \mid n\in\mathbb{N}\,\}$. A *Bénabou algebra for* S is a $\mathrm{B}_S=((S^\star)^2,\Sigma^{\mathrm{B}_S},\mathcal{E}^{\mathrm{B}_S})$-algebra, where Σ^{B_S} is the $(S^\star)^2$-sorted signature defined as follows:

BS_1. For the empty word $\lambda\in S^\star$, every $w\in S^\star$ and $i\in|w|$, where $|w|$ is the domain of the word w, the formal operation of *projection*:
$$\pi_i^w\colon \lambda \longrightarrow (w,(w_i)).$$

BS_2. For every u, $w\in S^\star$, the formal operation of *finite tupling*:
$$\langle\,\rangle_{u,w}\colon ((u,(w_0)),\dots,(u,(w_{|w|-1}))) \longrightarrow (u,w).$$

BS_3. For every u, x, $w\in S^\star$, the formal operation of *substitution*:
$$\circ_{u,x,w}\colon ((u,x),(x,w)) \longrightarrow (u,w);$$

while $\mathcal{E}^{\mathrm{B}_S}$ is the sub-$((S^\star)^2)^\star\times(S^\star)^2$-sorted set of $\mathrm{Eq}(\Sigma^{\mathrm{B}_S})$, where
$$\mathrm{Eq}(\Sigma^{\mathrm{B}_S}) = (\mathrm{T}_{\Sigma^{\mathrm{B}_S}}(\downarrow\overline{w})^2_{(u,x)})_{(\overline{w},(u,x))\in((S^\star)^2)^\star\times(S^\star)^2},$$

defined as follows:

B_1. For every u, $w\in S^\star$ and $i\in|w|$, the equation:
$$\pi_i^w \circ_{u,w,(w_i)} \langle v_0^{u,(w_0)},\dots,v_{|w|-1}^{u,(w_{|w|-1})}\rangle_{u,w} = v_i^{u,(w_i)},$$
of type $(((u,(w_0)),\dots,(u,(w_{|w|-1}))),(u,(w_i)))$.

B_2. For every u, $w\in S^\star$, the equation:
$$v_0^{u,w} \circ_{u,u,w} \langle \pi_0^u,\dots,\pi_{|u|-1}^u\rangle_{u,u} = v_0^{u,w},$$
of type $(((u,w)),(u,w))$.

B_3. For every u, $w\in S^\star$, the equation:
$$\langle \pi_0^w \circ_{u,w,w_0} v_0^{u,w},\dots,\pi_{|w|-1}^w \circ_{u,w,w_{|w|-1}} v_0^{u,w}\rangle_{u,w} = v_0^{u,w},$$
of type $(((u,w)),(u,w))$.

B_4. For every $w\in S^\star$, the equation:
$$\langle \pi_0^w\rangle_{w,(w_0)} = \pi_0^w,$$
of type $(((w,(w_0))),(w,(w_0)))$.

B_5. For every u, x, w, $y \in S^\star$, the equation:

$$v_0^{w,y} \circ_{u,w,y} (v_1^{x,w} \circ_{u,x,w} v_2^{u,x}) = (v_0^{w,y} \circ_{x,w,y} v_1^{x,w}) \circ_{u,x,y} v_2^{u,x},$$

of type $(((w,y),(x,w),(u,x)),(u,y))$,

where $v_n^{u,w}$ is the n-th variable of type (u,w), $Q \circ_{u,x,w} P$ is $\circ_{u,x,w}(P,Q)$, and $\langle P_0,\ldots,P_{|w|-1}\rangle_{u,w}$ is $\langle\,\rangle_{u,w}(P_0,\ldots,P_{|w|-1})$.

Since $\mathbf{Alg}(\mathrm{B}_S)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{B}_S}$ from $\mathbf{Alg}(\mathrm{B}_S)$ to $\mathbf{Set}^{S^\star\times S^\star}$ has a left adjoint $\mathbf{T}_{\mathrm{B}_S}$

$$\mathbf{Alg}(\mathrm{B}_S) \underset{\mathbf{T}_{\mathrm{B}_S}}{\overset{\mathrm{G}_{\mathrm{B}_S}}{\rightleftarrows}} \mathbf{Set}^{S^\star\times S^\star} \quad (\top)$$

which assigns to an $S^\star\times S^\star$-sorted set the corresponding free Bénabou algebra.

For every S-sorted set A, $\mathrm{BOp}_S(A) = (\mathrm{Hom}(A_w, A_u))_{(w,u)\in S^\star\times S^\star}$ is endowed with a structure of Bénabou algebra as stated in the following

Proposition 16. *Let A be an S-sorted set and $\mathbf{BOp}_S(A)$ the Σ^{B_S}-algebra with underlying many-sorted set $\mathrm{BOp}_S(A)$ and algebraic structure defined as follows*

1. *For every $w \in S^\star$ and $i \in |w|$, $(\pi_i^w)^{\mathbf{BOp}_S(A)} = \mathrm{pr}_{w,i}^A\colon A_w \longrightarrow A_{(w_i)}$.*
2. *For every $u,w \in S^\star$, $\langle\,\rangle_{u,w}^{\mathbf{BOp}_S(A)}$ is defined, for every $(f_0,\ldots,f_{|w|-1})$ in $\prod_{i\in|w|}\mathrm{Hom}(A_w, A_{(w_i)})$, as $\langle\,\rangle_{u,w}^{\mathbf{BOp}_S(A)}(f_0,\ldots,f_{|w|-1}) = \langle f_i\rangle_{i\in|w|}$, where $\langle f_i\rangle_{i\in|w|}$ is the unique mapping from A_u to A_w such that, for every $i\in|w|$, $\mathrm{pr}_{w,i}^A \circ \langle f_i\rangle_{i\in|w|} = f_i$.*
3. *For every u, x, $w \in S^\star$, $\circ_{u,x,w}^{\mathbf{BOp}_S(A)}$ is defined as the composition of mappings.*

Then $\mathbf{BOp}_S(A)$ is a Bénabou algebra, the Bénabou algebra for (S,A).

For every S-sorted signature Σ, $\mathrm{BTer}_S(\Sigma) = (\mathrm{T}_{\mathbf{\Sigma}}(\downarrow w)_u)_{(w,u)\in S^\star\times S^\star}$, that is naturally isomorphic to $(\mathrm{Hom}(\downarrow u, \mathrm{T}_{\mathbf{\Sigma}}(\downarrow w)))_{(w,u)\in S^\star\times S^\star}$, is endowed with a structure of Bénabou algebra as stated in the following

Proposition 17. *Let Σ be an S-sorted signature and $\mathbf{BTer}_S(\Sigma)$ the Σ^{B_S}-algebra with underlying many-sorted set $\mathrm{BTer}_S(\Sigma)$ and algebraic structure that obtained, by transport of structure, from the algebraic structure defined on the $S^\star\times S^\star$-sorted set $(\mathrm{Hom}(\downarrow u, \mathrm{T}_{\mathbf{\Sigma}}(\downarrow w)))_{(w,u)\in S^\star\times S^\star}$ as follows*

1. *For every $w \in S^\star$ and $i\in|w|$, $(\pi_i^w)^{\mathbf{BTer}_S(\Sigma)}$ is the composition of the canonical embedding from $\downarrow(w_i)$ to $\downarrow w$ and the canonical embedding from $\downarrow w$ to $\mathrm{T}_\Sigma(\downarrow w)$.*
2. *For every $u,w\in S^\star$, $\langle\,\rangle_{u,w}^{\mathbf{BTer}_S(\Sigma)}$ is the canonical isomorphism from the cartesian product $\prod_{i\in|w|}\mathrm{Hom}(\downarrow(w_i), \mathrm{T}_{\mathbf{\Sigma}}(\downarrow u))$ to $\mathrm{Hom}(\downarrow w, \mathrm{T}_{\mathbf{\Sigma}}(\downarrow u))$.*
3. *For every u, x, $w\in S^\star$, $\circ_{u,x,w}^{\mathbf{BTer}_S(A)}$ is defined as the mapping which sends a pair $\mathcal{P}\in\mathrm{Hom}(\downarrow x, \mathrm{T}_{\mathbf{\Sigma}}(\downarrow u))$ and $\mathcal{Q}\in\mathrm{Hom}(\downarrow w, \mathrm{T}_{\mathbf{\Sigma}}(\downarrow x))$ to $\mathcal{P}^\sharp\circ\mathcal{Q}$.*

Then $\mathbf{BTer}_S(\Sigma)$ is a Bénabou algebra, the Bénabou algebra for (S,Σ).

Next, after defining the category $\mathbf{BTh}_{\mathrm{f}}(S)$, of finitary many-sorted algebraic theories of Bénabou (defined for the first time in [1]), that generalize the finitary single-sorted algebraic theories of Lawvere, we prove that there exists an isomorphism between the category $\mathbf{BTh}_{\mathrm{f}}(S)$ and the category $\mathbf{Alg}(\mathrm{B}_S)$.

Definition 12. We denote by $\mathbf{BTh}_{\mathrm{f}}(S)$ the category with objects pairs $\boldsymbol{\mathcal{B}} = (\mathbf{B}, p)$, where $\mathbf{B}$ is a category that has as objects the words on S and p a family $(p^w)_{w\in S^\star}$ such that, for every word $w\in S^\star$, p^w is a family $(p_i^w\colon w\longrightarrow (w_i))_{i\in|w|}$ of morphisms in $\mathbf{B}$, the *projections* for w, where (w_i) is the word of length 1 on S whose only letter is w_i, such that (w, p^w) is a product in $\mathbf{B}$ of the family of words $((w_i))_{i\in|w|}$, and as morphisms from $\boldsymbol{\mathcal{B}}$ to $\boldsymbol{\mathcal{B}}'$ functors F from $\mathbf{B}$ to $\mathbf{B}'$ such that the object mapping of F is the identity and the morphism mapping of F preserves the projections, i.e., for every $w \in S^\star$ and $i \in |w|$, $F(p_i^{w,\boldsymbol{\mathcal{B}}}) = p_i^{w,\boldsymbol{\mathcal{B}}'}$.

Proposition 18. *There exists an isomorphism from the category* $\mathbf{Alg}(\mathrm{B}_S)$ *to the category* $\mathbf{BTh}_{\mathrm{f}}(S)$.

Proof. The isomorphism from $\mathbf{Alg}(\mathrm{B}_S)$ to $\mathbf{BTh}_{\mathrm{f}}(S)$ is the functor $B_{a,t}$ which to a Bénabou algebra $\mathbf{B}$ assigns the Bénabou theory $B_{a,t}(\mathbf{B})$ which has as underlying category that given by the following data

(1) The set of objects is $S^\star$ and, for $u, w \in S^\star$, $\mathrm{Hom}(u, w) = B_{u,w}$,
(2) For every $w \in S^\star$, $\mathrm{id}_w = \langle (\pi_i^w)^{\mathbf{B}} \mid i \in |w| \rangle_{w,w}$,
(3) If $P\colon u \longrightarrow x$, $Q\colon x \longrightarrow w$, then the composition of P and Q is $\circ_{u,x,w}^{\mathbf{B}}(P, Q)$,

and as underlying family of projections that given, for every $w \in S^\star$, as $\pi^w = ((\pi_i^w)^{\mathbf{B}})_{i\in|w|}$; and which to a morphism of Bénabou algebras $f\colon \mathbf{B} \longrightarrow \mathbf{B}'$ assigns the morphism of Bénabou theories $B_{a,t}(f)$ that to $P\colon w \longrightarrow u$ associates $f_{w,u}(P)\colon w \longrightarrow u$.

The inverse of $B_{a,t}$ is the functor $B_{t,a}$ which to a Bénabou theory $\boldsymbol{\mathcal{B}} = (\mathbf{B}, p)$ assigns the Bénabou algebra $B_{t,a}(\boldsymbol{\mathcal{B}})$ that has

(1) As underlying $(S^\star)^2$-sorted set the family $(\mathrm{Hom}_{\mathbf{B}}(w, u))_{(w,u)\in(S^\star)^2}$, and
(2) As structure of Bénabou algebra on $(\mathrm{Hom}_{\mathbf{B}}(w, u))_{(w,u)\in(S^\star)^2}$ that obtained by interpreting, for every $w \in S^\star$ and $i \in |w|$, π_i^w as p_i^w, for every $u, w \in S^\star$, $\langle\,\rangle_{u,w}$ as the canonical mapping from $\prod_{i\in|w|} \mathrm{Hom}_{\mathbf{B}}(u, (w_i))$ to $\mathrm{Hom}_{\mathbf{B}}(u, w)$ obtained by the universal property of the product for w, and, for every $u, x, w \in S^\star$, $\circ_{u,x,w}$ as the composition in $\mathbf{B}$;

and which to a morphism of Bénabou theories $F\colon \boldsymbol{\mathcal{B}} \longrightarrow \boldsymbol{\mathcal{B}}'$ assigns the morphism of Bénabou algebras $B_{t,a}(F)$, that for every $u, w \in S^\star$, is the bi-restriction of F to the corresponding hom-sets $\mathrm{Hom}(u, w)$ and $\mathrm{Hom}(u, w)$. □

Remark. The isomorphism between $\mathbf{BTh}_{\mathrm{f}}(S)$ and $\mathbf{Alg}(\mathrm{B}_S)$ can be interpreted as meaning, and this can be algebraically reassuring, that the category of finitary many-sorted algebraic theories of Bénabou, a purely formal entity, has the form of a category of models for a finitary many-sorted equational presentation, a semantical, or substantial, entity, therefore confirming, once more, that apparently *form is substance*. Moreover, the isomorphism shows that the Bénabou algebras are more closely related to the finitary many-sorted algebraic theories of Bénabou than are the Hall algebras.

Next we prove that the categories $\mathbf{Alg}(\mathrm{H}_S)$ and $\mathbf{Alg}(\mathrm{B}_S)$ of Hall and Bénabou algebras, respectively, are equivalent.

Proposition 19. *For every set of sorts* S*, the categories* $\mathbf{Alg}(\mathrm{H}_S)$ *and* $\mathbf{Alg}(\mathrm{B}_S)$ *are equivalent.*

Proof. The equivalence from $\mathbf{Alg}(\mathrm{H}_S)$ to $\mathbf{Alg}(\mathrm{B}_S)$ is the functor $F_{h,b}$ which to a Hall algebra $\mathbf{A}$ assigns the Bénabou algebra $F_{h,b}(\mathbf{A})$ that has

(1) As underlying $(S^\star)^2$-sorted set $((A_w)_u)_{(w,u)\in(S^\star)^2}$ where $A_w = (A_{w,s})_{s\in S}$ and $(A_w)_u = \prod_{i\in|u|} A_{w,u_i}$, and

(2) As structure of Bénabou algebra on $((A_w)_u)_{(w,u)\in(S^\star)^2}$ that defined as

$$
\begin{aligned}
(\pi^w_i)^{F_{h,b}(\mathbf{A})} &= ((\pi^w_i)^{\mathbf{A}}),\\
\langle (a_0),\dots,(a_{|w|-1})\rangle^{F_{h,b}(\mathbf{A})}_{u,w} &= (\xi^{\mathbf{A}}_{u,w,w_0}(\pi^w_0,a_0,\dots,a_{|w|-1}),\dots,\\
&\qquad \xi^{\mathbf{A}}_{u,w,w_{|w|-1}}(\pi^w_{|w|-1},a_0,\dots,a_{|w|-1})),\\
\circ^{F_{h,b}(\mathbf{A})}_{u,x,w}(a,b) &= (\xi^{\mathbf{A}}_{u,x,w_0}(b_0,a_0,\dots,a_{|x|-1}),\dots,\\
&\qquad \xi^{\mathbf{A}}_{u,x,w_{|w|-1}}(b_{|w|-1},a_0,\dots,a_{|x|-1}));
\end{aligned}
$$

and which to a morphism $f\colon \mathbf{A}\longrightarrow \mathbf{B}$ of Hall algebras assigns the morphism $F_{h,b}(f) = ((f_w)_u)_{(w,u)\in(S^\star)^2}$ from $F_{h,b}(\mathbf{A})$ to $F_{h,b}(\mathbf{B})$ defined, for $(a_0,\dots,a_{|u|-1})$ in $(A_w)_u$, as

$$(a_0,\dots,a_{|u|-1}) \longmapsto (f_{w,u_0}(a_0),\dots,f_{w,u_{|u|-1}}(a_{|u|-1}))).$$

The quasi-inverse equivalence from $\mathbf{Alg}(\mathrm{B}_S)$ to $\mathbf{Alg}(\mathrm{H}_S)$ is the functor $F_{b,h}$ which to a Bénabou algebra $\mathbf{A}$ assigns the Hall algebra $F_{b,h}(\mathbf{A})$ that has

(1) As underlying $S^\star\times S$-sorted set $(A_{w,(s)})_{(w,s)\in S^\star\times S}$, and
(2) As structure of Hall algebra on $(A_{w,(s)})_{(w,s)\in S^\star\times S}$ that defined as

$$
\begin{aligned}
(\pi^w_i)^{F_{b,h}(\mathbf{A})} &= (\pi^w_i)^{\mathbf{A}},\\
\xi^{F_{b,h}(\mathbf{A})}_{u,w,s}(a,a_0,\dots,a_{|w|-1}) &= a\circ_{u,w,s}\langle a_0,\dots,a_{|w|-1}\rangle_{u,w};
\end{aligned}
$$

and which to a homomorphism $f\colon \mathbf{A}\longrightarrow \mathbf{B}$ of Bénabou algebras assigns the birestriction of f to $F_{b,h}(\mathbf{A})$ and $F_{b,h}(\mathbf{B})$.

Next, for a Bénabou algebra $\mathbf{A}$, we prove that $\mathbf{A}$ and $F_{h,b}(F_{b,h}(\mathbf{A}))$ are isomorphic. Let $f\colon \mathbf{A}\longrightarrow F_{h,b}(F_{b,h}(\mathbf{A}))$ be the $S^\star\times S^\star$-sorted mapping defined, for $(u,w)\in S^\star\times S^\star$ and $a\in A_{u,w}$, as

$$a\mapsto ((\pi^w_0)^{\mathbf{A}}\circ a,\dots,(\pi^w_{|w|-1})^{\mathbf{A}}\circ a).$$

The definition is sound because, for $a\in A_{u,w}$, we have that $(\pi^w_i)^{\mathbf{A}}\circ a\in F_{b,h}(A)_{u,w_i}$, hence $((\pi^w_0)^{\mathbf{A}}\circ a,\dots,(\pi^w_{|w|-1})^{\mathbf{A}}\circ a)\in F_{h,b}(F_{b,h}(A))_{u,w}$. Thus defined f is a homomorphism, since we have, on the one hand, that

$$
\begin{aligned}
f((\pi^w_i)^{\mathbf{A}}) &= (\pi^{(w_i)}_0\circ\pi^w_i)\\
&= (\langle\pi^{(w_i)}_0\rangle_{(w_i),(w_i)}\circ\pi^w_i) && \text{(by B}_4\text{)}\\
&= (\langle\pi^{(w_i)}_0\circ(\langle\pi^{(w_i)}_0\rangle\circ\pi^w_i)\rangle_{w,(w_i)}) && \text{(by B}_3\text{)}\\
&= (\langle\pi^{(w_i)}_0\circ\pi^w_i\rangle_{w,(w_i)}) && \text{(by B}_2\text{ and B}_5\text{)}\\
&= (\pi^w_i) && \text{(by B}_3\text{)}\\
&= (\pi^w_i)^{F_{h,b}(F_{b,h}(\mathbf{A}))},
\end{aligned}
$$

on the other hand, that

$$
\begin{aligned}
f(\langle a_0,\dots,a_{|w|-1}\rangle^{\mathbf{A}}_{u,w}) &= ((\pi^w_0)^{\mathbf{A}}\circ\langle a_0,\dots,a_{|w|-1}\rangle^{\mathbf{A}}_{u,w},\dots,\\
&\qquad (\pi^w_{|w|-1})^{\mathbf{A}}\circ\langle a_0,\dots,a_{|w|-1}\rangle^{\mathbf{A}}_{u,w})\\
&= (\xi^{F_{b,h}(\mathbf{A})}((\pi^w_0)^{F_{b,h}(\mathbf{A})},a_0,\dots,a_{|w|-1}),\dots,\\
&\qquad \xi^{F_{b,h}(\mathbf{A})}((\pi^w_{|w|-1})^{F_{b,h}(\mathbf{A})},a_0,\dots,a_{|w|-1}))\\
&= \langle (a_0),\dots,(a_{|w|-1})\rangle^{F_{h,b}(F_{b,h}(\mathbf{A}))}_{u,w}\\
&= \langle f(a_0),\dots,f(a_{|w|-1})\rangle^{F_{h,b}(F_{b,h}(\mathbf{A}))}_{u,w},
\end{aligned}
$$

and, lastly, that

$$\begin{aligned} f(b \circ^{\mathbf{A}} a) &= ((\pi^w_0)^{\mathbf{A}} \circ (b \circ a), \dots, (\pi^w_{|w|-1})^{\mathbf{A}} \circ (b \circ a)) \\ &= ((\pi^w_0)^{\mathbf{A}} \circ b \circ \langle a_0, \dots, a_{|w|-1}\rangle, \dots, \\ &\qquad (\pi^w_{|w|-1})^{\mathbf{A}} \circ b \circ \langle a_0, \dots, a_{|w|-1}\rangle) \\ &= (f(b_0) \circ \langle a_0, \dots, a_{|w|-1}\rangle, \dots, \\ &\qquad b(b_{|w|-1}) \circ \langle a_0, \dots, a_{|w|-1}\rangle) \\ &= (\xi^{F_{b,h}(\mathbf{A})}(f(b_0), f(a_0), \dots, f(a_{|w|-1})), \dots, \\ &\qquad \xi^{F_{b,h}(\mathbf{A})}(f(b_{|w|-1}), f(a_0), \dots, f(a_{|w|-1}))) \\ &= \ f(b) \circ^{F_{h,b}(F_{b,h}(\mathbf{A}))} f(a). \end{aligned}$$

Reciprocally, let $g\colon F_{h,b}(F_{b,h}(\mathbf{A})) \longrightarrow \mathbf{A}$ be the $S^\star \times S^\star$-sorted mapping defined, for $(u, w) \in S^\star \times S^\star$ and $b \in F_{h,b}(F_{b,h}(A))$, as

$$b \mapsto \langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}.$$

The definition is sound because, for $b = (b_0, \dots, b_{|w|-1}) \in F_{h,b}(F_{b,h}(A))$, we have that $b_i \in F_{b,h}(A)_{u,w_i}$, hence $b_i \in A_{u,(w_i)}$, therefore $\langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}} \in A_{u,w}$. Thus defined it is easy to prove that g is a homomorphism.

Now we prove that the homomorphisms f and g are such that $g \circ f = \mathrm{id}_{\mathbf{A}}$ and $f \circ g = \mathrm{id}_{F_{h,b}(F_{b,h}(\mathbf{A}))}$. On the one hand, if $a \in A_{u,w}$, then, by B_3, we have that

$$\langle (\pi^w_0)^{\mathbf{A}} \circ a, \dots, (\pi^w_{|w|-1})^{\mathbf{A}} \circ a\rangle = a,$$

hence $g \circ f = \mathrm{id}_{\mathbf{A}}$. On the other hand, if $b \in F_{h,b}(F_{b,h}(A))$, then $g_{u,w}$ sends b to $\langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}$, and $f_{u,w}$ sends $\langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}$ to

$$((\pi^w_0)^{F_{h,b}(F_{b,h}(\mathbf{A}))} \circ \langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}, \dots, (\pi^w_{|w|-1})^{F_{h,b}(F_{b,h}(\mathbf{A}))} \circ \langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}),$$

but this last coincides with

$$((\pi^w_0)^{F_{h,b}(\mathbf{A})} \circ \langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}, \dots, (\pi^w_{|w|-1})^{F_{h,b}(\mathbf{A})} \circ \langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w}),$$

thus, by the axiom B_1, we have that this, in its turn, coincides with

$$\langle b_0, \dots, b_{|w|-1}\rangle^{\mathbf{A}}_{u,w},$$

therefore $f_{u,w} \circ g_{u,w}(b) = b$. From which we can assert that $f \circ g = \mathrm{id}_{F_{h,b}(F_{b,h}(\mathbf{A}))}$.

Finally, for a Hall algebra $\mathbf{A}$ we have that $\mathbf{A}$ and $F_{b,h}(F_{h,b}(\mathbf{A}))$ are identical, because $a \in A_{w,s}$ iff $a \in F_{h,b}(A)_{w,(s)}$ iff $a \in F_{b,h}F_{h,b}(A)_{w,s}$. □

Corollary 4. *There exists an equivalence between the category* $\mathbf{Alg}(\mathrm{H}_S)$ *and the category* $\mathbf{BTh}_{\mathrm{f}}(S)$.

In the following proposition, for a set of sorts S, we state some relations among the equivalence between the categories $\mathbf{Alg}(\mathrm{H}_S)$ and $\mathbf{Alg}(\mathrm{B}_S)$, the adjunctions $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$ and $\mathbf{T}_{\mathrm{B}_S} \dashv \mathrm{G}_{\mathrm{B}_S}$, and the adjunction $\coprod_{1\times \wr_S} \dashv \Delta_{1\times \wr_S}$ determined by the mapping $1\times \wr_S$ from $S^\star \times S$ to $S^\star \times S^\star$ which sends a pair (w, s) in $S^\star \times S$ to the pair $(w, (s))$ in $S^\star \times S^\star$. From these relations we will get as an easy, but interesting, corollary, that, for every $S^\star \times S$-sorted set Σ, $\mathbf{T}_{\mathrm{B}_S}(\coprod_{1\times \wr_S} \Sigma)$, the free Bénabou algebra on $\coprod_{1\times \wr_S} \Sigma$, is isomorphic to $\mathbf{BTer}_S(\Sigma)$.

Proposition 20. *Let S be a set of sorts. Then for the diagram*

$$
\begin{array}{ccc}
\mathbf{Set}^{S^\star\times S} & \underset{\mathbf{T}_{\mathrm{H}_S}}{\overset{\mathrm{G}_{\mathrm{H}_S}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{Alg}(\mathrm{H}_S) \\
\coprod_{1\times\between_S}\Big\downarrow \dashv \Big\uparrow \Delta_{1\times\between_S} & & F_{h,b}\Big\downarrow \equiv \Big\uparrow F_{b,h} \\
\mathbf{Set}^{S^\star\times S^\star} & \underset{\mathbf{T}_{\mathrm{B}_S}}{\overset{\mathrm{G}_{\mathrm{B}_S}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} & \mathbf{Alg}(\mathrm{B}_S)
\end{array}
$$

we have that $\Delta_{1\times\between_S}\circ \mathrm{G}_{\mathrm{B}_S} = \mathrm{G}_{\mathrm{H}_S}\circ F_{b,h}$ *and* $\mathbf{T}_{\mathrm{B}_S}\circ\coprod_{1\times\between_S}\cong F_{h,b}\circ\mathbf{T}_{\mathrm{H}_S}$.

Proof. The equality $\Delta_{1\times\between_S}\circ \mathrm{G}_{\mathrm{B}_S} = \mathrm{G}_{\mathrm{H}_S}\circ F_{b,h}$ follows from the definitions of the functors involved. Then, being $\mathbf{T}_{\mathrm{B}_S}\circ\coprod_{1\times\between_S}$ and $F_{h,b}\circ\mathbf{T}_{\mathrm{H}_S}$ left adjoints to the same functor, we can assert that $\mathbf{T}_{\mathrm{B}_S}\circ\coprod_{1\times\between_S}\cong F_{h,b}\circ\mathbf{T}_{\mathrm{H}_S}$. □

Corollary 5. *Let Σ be an S-sorted signature. Then the free Bénabou algebra* $\mathbf{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)$ *on* $\coprod_{1\times\between_S}\Sigma$ *is isomorphic to the Bénabou algebra* $\mathbf{BTer}_S(\Sigma)$ *for* (S,Σ).

Proof. It follows after $\mathbf{BTer}_S(\Sigma) = F_{h,b}(\mathbf{HTer}_S(\Sigma))$. □

If we agree that $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$ denotes $\mathrm{BTer}_S(\Sigma)^2$, then the congruence generated in $\mathbf{BTer}_S(\Sigma)$ by a subfamily $\mathcal{E}$ of $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$ can be characterized as follows.

Proposition 21. *Let $\mathcal{E}$ be a sub-sorted set of* $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$. *Then* $\mathrm{Cg}_{\mathbf{BTer}_S(\Sigma)}(\mathcal{E})$ *is the smallest subfamily $\overline{\mathcal{E}}$ of* $\mathrm{BTer}_S(\Sigma)$ *that contains $\mathcal{E}$ and is such that, for every $u,w,x\in S^\star$ satisfies the following conditions:*

(1) Reflexivity. *For every $\mathcal{P}\in\mathrm{BTer}_S(\Sigma)_{w,u}$, $(\mathcal{P},\mathcal{P})\in\overline{\mathcal{E}}_{w,u}$.*
(2) Symmetry. *For every $\mathcal{P}$, $\mathcal{Q}\in\mathrm{BTer}_S(\Sigma)_{w,u}$, if $(\mathcal{P},\mathcal{Q})\in\overline{\mathcal{E}}_{w,u}$, then $(\mathcal{Q},\mathcal{P})\in\overline{\mathcal{E}}_{w,u}$.*
(3) Transitivity. *For every $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}\in\mathrm{BTer}_S(\Sigma)_{w,u}$, if $(\mathcal{P},\mathcal{Q})$, $(\mathcal{Q},\mathcal{R})\in\overline{\mathcal{E}}_{w,u}$, then $(\mathcal{P},\mathcal{R})\in\overline{\mathcal{E}}_{w,s}$.*
(4) Product compatibility. *For every $\mathcal{P}$, $\mathcal{Q}\in\mathrm{BTer}_S(\Sigma)_{u,w}$, if, for every $i\in|w|$, $(P_i,Q_i)\in\overline{\mathcal{E}}_{u,(w_i)}$, then $(\langle P_0,\ldots,P_{|w|-1}\rangle,\langle Q_0,\ldots,Q_{|w|-1}\rangle)\in\overline{\mathcal{E}}_{u,w}$.*
(5) Substitutivity. *For every $\mathcal{P}$, $\mathcal{Q}\in\mathrm{BTer}_S(\Sigma)_{u,x}$ and $\mathcal{M},\mathcal{N}\in\mathrm{BTer}_S(\Sigma)_{x,w}$, if $(\mathcal{P},\mathcal{Q})\in\overline{\mathcal{E}}_{u,x}$ and $(\mathcal{M},\mathcal{N})\in\overline{\mathcal{E}}_{x,w}$, then it happens that $(\mathcal{M}\circ\mathcal{P},\mathcal{N}\circ\mathcal{Q}) = (\mathcal{P}^\sharp\circ\mathcal{M},\mathcal{Q}^\sharp\circ\mathcal{N})\in\overline{\mathcal{E}}_{u,w}$.*

Next we define two pairs of order preserving mappings, in opposite directions, between the ordered sets $\mathbf{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ and $\mathbf{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ that will allow us to determine the exact relation that there exists between the category $\mathbf{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ and the category $\mathbf{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ in the category $\mathbf{Adj}$ of categories and adjunctions.

Proposition 22. *Let Σ be an S-sorted signature. Then the mappings H, D from* $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ *into* $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ *defined, for every sub-sorted set $\mathcal{E}$ of* $\mathrm{Eq}_{\mathrm{B}}(\Sigma)$, *respectively, as*

$$
H(\mathcal{E}) = (\{(P,Q)\in\mathrm{Eq}_{\mathrm{H}}(\Sigma)_{w,s}\mid (P,Q)\in\mathcal{E}_{w,(s)}\})_{(w,s)\in S^\star\times S},
$$

$$
D(\mathcal{E}) = \left(\left\{(P,Q)\in\mathrm{Eq}_{\mathrm{H}}(\Sigma)_{w,s}\;\middle|\; \begin{array}{c}\exists(\mathcal{R},\mathcal{S})\in\mathcal{E}_{w,u},\ \exists i\in u^{-1}[s],\\ (P,Q) = (R_i,S_i)\end{array}\right\}\right)_{(w,s)\in S^\star\times S},
$$

and the mappings I, B from $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ *into* $\mathrm{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ *defined, for every sub-sorted set* $\mathcal{E}'$ *of* $\mathrm{Eq}_{\mathrm{H}}(\Sigma)$*, respectively, as*

$$I(\mathcal{E}') = (\{\, (P,Q) \in \mathrm{Eq}_{\mathrm{B}}(\Sigma)_{w,u} \mid \exists s \in S\,(u = (s)\ \&\ (P,Q) \in \mathcal{E}'_{w,s})\,\})_{(w,u)\in S^\star\times S^\star},$$
$$B(\mathcal{E}') = (\{\, (\mathcal{P},\mathcal{Q}) \in \mathrm{Eq}_{\mathrm{B}}(\Sigma)_{w,u} \mid \forall i \in |u|\,((P_i,Q_i) \in \mathcal{E}'_{w,u_i})\,\})_{(w,u)\in S^\star\times S^\star},$$

are order preserving. Moreover, $H\circ I = D\circ I = H\circ B = D\circ B = \mathrm{id}_{\mathrm{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))}$ *and, for every* $\mathcal{E}\subseteq \mathrm{Eq}_{\mathrm{H}}(\Sigma)$ *and* $\mathcal{E}'\subseteq\mathrm{Eq}_{\mathrm{B}}(\Sigma)$*, we have that* $D(\mathcal{E})\subseteq\mathcal{E}'$ *iff* $\mathcal{E}\subseteq B(\mathcal{E}')$ *and* $I(\mathcal{E}')\subseteq\mathcal{E}$ *iff* $\mathcal{E}'\subseteq H(\mathcal{E})$*, hence* $D\dashv B$ *and* $I\dashv H$*. Finally, because the composite adjunction* $D\circ I\dashv H\circ B$ *is the identity adjunction, we conclude that* $\mathbf{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))$ *is a retract of* $\mathbf{Sub}(\mathrm{Eq}_{\mathrm{B}}(\Sigma))$ *in the category* **Adj** *of categories and adjunctions.*

After this we prove, for an S-sorted signature Σ, that there is an isomorphism between the lattices $\mathbf{Cgr}(\mathbf{HTer}_S(\Sigma))$ and $\mathbf{Cgr}(\mathbf{BTer}_S(\Sigma))$.

Proposition 23. *Let* Σ *be an* S*-sorted signature. Then the congruence lattices* $\mathbf{Cgr}(\mathbf{HTer}_S(\Sigma))$ *and* $\mathbf{Cgr}(\mathbf{BTer}_S(\Sigma))$ *are isomorphic.*

Proof. If $\mathcal{E}$ is a congruence on $\mathbf{HTer}_S(\Sigma)$, then we have that $\mathrm{Cg}_{\mathbf{BTer}_S(\Sigma)}(B(\mathcal{E})) = B(\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}(\mathcal{E}))$ is included in $B(\mathcal{E})$ and $B(\mathcal{E})\in\mathrm{Cgr}(\mathbf{BTer}_S(\Sigma))$.

Reciprocally, if $\mathcal{E}$ is a congruence on $\mathbf{BTer}_S(\Sigma)$, then $\mathrm{Cg}_{\mathbf{HTer}_S(\Sigma)}(H(\mathcal{E}))$ is included in $H(\mathrm{Cg}_{\mathbf{BTer}_S(\Sigma)}(\mathcal{E}))$, which in its turn is included in $H(\mathcal{E})$, and $H(\mathcal{E})$ is a congruence on $\mathbf{HTer}_S(\Sigma)$. But, because $H\circ B = \mathrm{id}_{\mathbf{Sub}(\mathrm{Eq}_{\mathrm{H}}(\Sigma))}$, we only have to verify that, for every congruence $\mathcal{E}$ on $\mathbf{BTer}_S(\Sigma)$, $B(H(\mathcal{E})) = \mathcal{E}$.

If $(\mathcal{P},\mathcal{Q}) \in B(H(\mathcal{E}))_{u,w}$, then, for every $i\in|w|$, $(P_i,Q_i)\in H(\mathcal{E})_{u,w_i}$, hence $(P_i,Q_i)\in\mathcal{E}_{u,(w_i)}$ and $(\mathcal{P},\mathcal{Q})\in\mathcal{E}_{u,w}$, thus $B(H(\mathcal{E}))\subseteq\mathcal{E}$.

If $(\mathcal{P},\mathcal{Q})\in\mathcal{E}_{u,w}$, then, for every $i\in|w|$, $(P_i,Q_i)\in\mathcal{E}_{u,(w_i)}$, hence $(P_i,Q_i)\in H(\mathcal{E})_{u,w_i}$ and $(\mathcal{P},\mathcal{Q})\in B(H(\mathcal{E}))_{u,w}$, thus $\mathcal{E}\subseteq B(H(\mathcal{E}))$. □

From this it follows immediately the following

Corollary 6. *Let* Σ *be an* S*-sorted signature. Then the algebraic congruence lattice* $\mathbf{Cgr}(\mathbf{BTer}_S(\Sigma))$ *is isomorphic to the algebraic lattice of fixed points of* Cn_Σ*, i.e., the algebraic lattice of the finitary equational theories for* S *is isomorphic to the algebraic lattice of the congruences on the Bénabou algebra* $\mathbf{BTer}_{\mathrm{B}}(\Sigma)$*.*

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