Some comments on the fundamentals and the applications of the Shapley value[1]

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1 Summary of the lecture

In this lecture some recent results concerning the Shapley value are presented. To start with, a brief introduction to game theory and to the Shapley value is provided. Then, following Carpente et al. (2004a) and Carpente et al. (2004b), a valuation function for strategic games in which players cooperate is introduced and characterized. This valuation function assigns to every non-empty coalition of players in a strategic game the Shapley value of an associated coalitional game. The interest of this valuation function is also explained and motivated. Finally, some applications of the Shapley value are briefly presented, stressing one related with the reorganization of the railways system in the European Union, which is strongly conected with the results in Fragnelli et al. (2000).

2 Values for strategic games in which players cooperate

This section displays the paper Carpente et al. (2004a).

2.1 Introduction

Classical game theory makes a radical distinction between non-cooperative games and cooperative games. Usually, non-cooperative games are defined

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as games that do not permit enforceable agreements among players. This is contrast to cooperative games, in which enforceable agreements are possible. However, we think that the point of view adopted in Van Damme and Furth (2002) reflects the difference between non-cooperative games and cooperative games more accurately. They write:

> *"The terminology that is used sometimes gives rise to confusion; it is not the case that in non-cooperative games players do not wish to cooperate and that in cooperative games players automatically do so. The difference instead is in the level of detail of the model; non-cooperative models assume that all the possibilities for cooperation have been included as formal moves in the game, while cooperative models are 'incomplete' and allow players to act outside of the detailed rules that have been specified."*

This description is, in fact, more in accordance with the approach in Von Neumann and Morgenstern (1944). Given a non-cooperative game, they formulate a cooperative game that describes for each coalition the benefits that this coalition can secure for its members, independently of the actions taken by the players outside the coalition. Hence, the cooperative-game description abstracts away from the details of the non-cooperative game and collapses those into simple numbers, one for each coalition of players. For a coalition of players to secure the benefits (or worth) of the coalition for its members, these members will most likely have to coordinate their actions and this in itself will generally require them to act outside the detailed rules of the non-cooperative game. To more clearly reflect the interpretations provided above, we prefer to use the terminology strategic game (instead of non-cooperative game) and coalitional game (instead of cooperative game).

The main objective of the current paper is to highlight the connection between strategic games and coalitional games. We do so by providing axiomatic foundations for two procedures that associate a coalitional game with each strategic game. The first procedure we study is that introduced in Von Neumann and Morgenstern (1944), which defines the worth of a coalition of players S to be the value of the mixed extension of a finite two-player zero-sum game between coalition S on the one hand and the coalition $N \setminus S$ consisting of all the other players on the other hand. In this zero-sum game, coalition $N \setminus S$ tries to keep the payoff to coalition S as low as possible, while coalition S tries to maximize its payoff. We also introduce a second procedure, which defines the worth of a coalition S to be the lower value of the finite two-player zero-sum game between coalition S and coalition $N \setminus S$. Both of these procedures can be interpreted as representing the pessimistic point of view where the worth of a coalition S is the minimum that the members of S can guarantee themselves (note that the lower value is equal to the value for the mixed extension of the game). The benefit of considering the lower value of the finite game instead of the (lower) value of its mixed extension is that the former requires no use of mixed strategies, whereas the latter does. Hence, in situations where the use of mixed strategies is not plausible, the procedure using the lower value will provide a more plausible method to determine the benefits that each coalition can secure for its members.

To provide axiomatic foundations for these two procedures we start by looking into axiomatic characterizations of the values of matrix games and of the lower values of finite two-player zero-sum games, using the characterizations of the value by Vilkas (1963) and Tijs (1981) as starting points.[3]

We point out that there are papers that propose procedures to associate coalitional games with strategic games that are quite different from the ones we study in the current paper. Harsanyi (1963), Myerson (1991) and Bergantiños and García-Jurado (1995) are some of these papers. All of these take an approach that involves the Nash equilibrium concept rather than the value or lower value. Myerson (1978) is a paper that is remotely related to the current one. It studies the role of threats in strategic games in which players are assumed to cooperate. Finally, we remark that we do consider only situations where utilities are transferable between the players in a coalition. Procedures to associate NTU-games with strategic games were proposed in Aumann (1961, 1967). Aumann's work was continued in Borm and Tijs (1992).

The structure of this paper is as follows. In section 2, we provide definitions related to finite two-player zero-sum games and we develop axiomatic characterizations of the value of mixed extensions of such games. In section 3, we discuss axiomatic characterizations of the lower value of finite two-player zero-sum games. In section 4, we include definitions related to strategic games and coalitional games and we study methods to associate a coalitional game with each strategic game. In section 5, we provide axiomatic characterizations of the method defined in Von Neumann and Morgenstern (1944) as well as a newly-defined method that is based on the lower value.

2.2 Values of two-player zero-sum games

In this section, we look at two-player zero-sum games. We provide definitions of the lower value of finite two-player zero-sum games and of the value

[3] A contemporary and independent paper, Norde and Voorneveld (2004), contains axiomatic characterizations of the value of matrix games that are very similar to ours but not quite the same.

of mixed extensions of such games. We also develop axiomatic characterizations of the value. These axiomatizations are used in section 5 to construct axiomatic characterizations of methods that associate coalitional games with strategic games.

We consider finite two-player zero-sum games in which the action sets of players 1 and 2 are $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$ respectively, and $A = [a_{ij}]_{i \in M,\ j \in N}$ is an $m \times n$ matrix of real numbers that gives the payoffs to the players. If player 1 chooses action $i \in M$ and player 2 chooses action $j \in N$, then player 2 pays a_{ij} to player 1, so player 1's payoff is a_{ij} and player 2's payoff is $-a_{ij}$. We identify a finite two-player zero-sum game with its payoff matrix A and denote it by A.

Given a payoff matrix A, the matrix game $E(A)$ is the mixed extension of the finite two-player zero-sum game A.[4] It is the two-player zero-sum game in which the strategy set of player 1 is

$$S_M = \{x \in \mathbb{R}^M \mid x_i \geq 0 \ \text{ for all } i \in M, \sum_{i \in M} x_i = 1\},$$

i.e., player 1 chooses a probability distribution on his actions, and the strategy set of player 2 is similarly defined by

$$S_N = \{y \in \mathbb{R}^N \mid y_j \geq 0 \ \text{ for all } j \in N, \sum_{j \in N} y_j = 1\}.$$

If player 1 chooses $x \in S_M$ and player 2 chooses $y \in S_N$, player 1's (expected) payoff is $x^T A y$ (or $\sum_{i \in M} \sum_{j \in N} x_i\, y_j\, a_{ij}$) and player 2's is $-x^T A y$.[5]

The value of a matrix game is defined using the lower and upper values. The lower and upper values of $E(A)$, $\underline{V}(E(A))$ and $\overline{V}(E(A))$, are defined as:

$$\underline{V}(E(A)) := \max_{x \in S_M} \min_{y \in S_N} x^T A y$$

$$\overline{V}(E(A)) := \min_{y \in S_N} \max_{x \in S_M} x^T A y.$$

At the basis of these definitions is that player 1 wants to maximize $x^T A y$ and player 2 wants to minimize it. By choosing an appropriate $x \in S_M$, player 1 can ensure he gets at least the lower value. Similarly, by choosing

[4]Note that our notation deviates from the usual notation, which uses A for the mixed extension. We denote the finite two-player zero-sum game by A and its mixed extension by $E(A)$. We do this because we mainly concentrate on the finite two-player game and would like to use the simplest notation for this game.

[5]x^T denotes the transpose of x.

an appropriate $y \in S_N$, player 2 can ensure he does not have to pay more than the upper value. Note that it follows easily from these definitions that $\underline{V}(E(A)) \leq \overline{V}(E(A))$. The classical minimax theorem (cf. Von Neumann (1928)) asserts that $\underline{V}(E(A)) = \overline{V}(E(A))$ for every matrix game $E(A)$, so that player 1 expects to get exactly the amount $\underline{V}(E(A))$. This leads to the definition of the value of the matrix A, which is the value of the matrix game $E(A)$; $V(A) = V(E(A)) := \underline{V}(E(A)) = \overline{V}(E(A))$. If the players cannot use probability distributions on their actions, we get the lower value

$$\underline{V}(A) := \max_{i \in M} \min_{j \in N} a_{ij}$$

of the finite two-player zero-sum game A and its upper value

$$\overline{V}(A) := \min_{j \in N} \max_{i \in M} a_{ij}.$$

In general, $\underline{V}(A) < \overline{V}(A)$. Note that $\underline{V}(A) \leq V(A) \leq \overline{V}(A)$ for all matrices A.

We denote the set of real matrices by $\mathcal{A}$. The *value function* $V : \mathcal{A} \to \mathbb{R}$ associates with each matrix $A \in \mathcal{A}$ its value $V(A)$. The value function is an example of an *evaluation function*, which we define as a real-valued function $f : \mathcal{A} \to \mathbb{R}$ that assigns to every matrix $A \in \mathcal{A}$ a real number reflecting the evaluation of a game based on the matrix A from the point of view of player 1. The value function was characterized as an evaluation function in Vilkas (1963). His characterization was extended to a broader class of zero-sum games in Tijs (1981). The value function only makes sense in situations where mixed strategies can be used by the players. We are interested in situations where the players can only use (pure) actions and therefore we will consider the lower-value evaluation function. We axiomatically characterize this evaluation function in the next section. Our characterizations of the lower-value evaluation function are inspired by axiomatizations of the value function, which we consider in the remainder of the current section.

We start by recalling Vilkas's (1963) characterization of the value function. He used the following four properties of an evaluation function $f : \mathcal{A} \to \mathbb{R}$.

Objectivity For all $a \in \mathbb{R}$, $f([a]) = a$. [6]

Monotonicity For all $A, B \in \mathcal{A}$, if $A \geq B$, then $f(A) \geq f(B)$.

[6]Here, $[a]$ denotes the 1×1 matrix A with $a_{11} = a$.

Row dominance The i^{th} row of the matrix A, denoted

$$r_i = [a_{i1} \dots a_{in}],$$

is *dominated* if there exists a convex combination x of the other rows[7] of A such that $x_j \geq a_{ij}$ for all $j \in N$. For all $A \in \mathcal{A}$, if row r is dominated, then $f(A) = f(A \setminus r)$, where $A \setminus r$ represents the matrix obtained from A by deleting row r.

Symmetry For all $A \in \mathcal{A}$, $f(-A^T) = -f(A)$.

Objectivity establishes the evaluation for player 1 in a trivial situation where both players have exactly one action available. Monotonicity states that the evaluation for player 1 should not decrease when his payoff weakly increases for every possible choice of actions by both players. Row dominance states that player 1's evaluation should not change if he can no longer choose an action that is worse for him than some combination of other actions. Note that this property makes sense only in a setting where players can randomize over their actions. Symmetry establishes that the roles of players 1 and 2 can be interchanged if the matrix is adapted accordingly. Transposing the matrix interchanges the roles of the players and the minus sign appears because the payoff of player 2 is the opposite of that of player 1.

Theorem 1 (Vilkas (1963)) *The value function V is the unique evaluation function that satisfies objectivity, monotonicity, row dominance, and symmetry.*

The counterpart for player 2 to row dominance would be what we call column dominance, which is related to dominated actions of player 2.

Column dominance The j^{th} column of the matrix A, denoted $c_j = [a_{1j} \dots a_{mj}]^T$, is *dominated* if there exists a convex combination y of the other columns of A such that $y_i \leq a_{ij}$ for all $i \in M$. For all $A \in \mathcal{A}$, if column c is dominated, then $f(A) = f(A \setminus c)$, where $A \setminus c$ represents the matrix obtained from A by deleting column c.

Column dominance states that player 1's evaluation should not change if player 2 can no longer choose an action that is dominated for him by some combination of his other actions. Note that player 2 wants to minimize the payoff of player 1 (thereby maximizing his own payoff), so an action for

[7] x is a convex combination of the other rows if there exist $(\alpha_k)_{k \in M \setminus i}$ such that $\alpha_k \geq 0$ for each $k \in M \setminus i$, $\sum_{k \in M \setminus i} \alpha_k = 1$, and $x = \sum_{k \in M \setminus i} \alpha_k r_k$.

player 2 is dominated if it always gives a (weakly) larger payoff to player 1. Column dominance is the flip side of row dominance and can in fact replace symmetry in Theorem 1.

Theorem 2 *The value function V is the unique evaluation function that satisfies objectivity, monotonicity, row dominance, and column dominance.*

Proof. *Existence.* It follows from Theorem 1 that V satisfies the first three properties listed in the theorem. To see that V satisfies *column dominance*, note that if c is a dominated column in A, then this column becomes a dominated row in the matrix $-A^T$ and can be eliminated by *row dominance*. *Symmetry* of V assures that we can switch from the matrix to the negative of its transpose and back after the elimination of the dominated row to obtain the matrix $A \setminus c$.

Uniqueness. Let $f : \mathcal{A} \to \mathbb{R}$ be an evaluation function satisfying the four axioms listed in the theorem and let $A = [a_{ij}]_{i \in M, j \in N} \in \mathcal{A}$. Because $V(A) = \underline{V}(E(A))$, there exists a mixed strategy of player 1 that guarantees player 1 a payoff of at least $V(A)$. Then, using *row dominance*, we can add to the matrix a dominated row in which all elements equal $V(A)$ without changing the value. Similarly, using that $V(A) = \overline{V}(E(A))$ and *column dominance*, we can add a dominated column in which all elements equal $V(A)$ without changing the value. Hence,

$$f(A) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \\ V(A) & \cdots & V(A) \end{bmatrix}) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}).$$

Now, we use *monotonicity* to make all the elements of the matrix less than or equal to $V(A)$. This makes all the rows but the last one dominated. Hence, we can use *row dominance* (repeatedly) to eliminate one by one all rows but the last one. Then we have a $1 \times (n+1)$ matrix left in which all elements equal $V(A)$. We subsequently use *column dominance* (repeatedly) to eliminate all but one of the columns of the remaining matrix. This leaves us with a 1×1 matrix, to which we can apply *objectivity*. Doing so, we obtain

$$f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}) \geq$$

$$f(\begin{bmatrix} \min\{a_{11}, V(A)\} & \cdots & \min\{a_{1n}, V(A)\} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ \min\{a_{m1}, V(A)\} & \cdots & \min\{a_{mn}, V(A)\} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}) =$$

$$f([V(A) \ldots V(A)]) = f([V(A)]) = V(A).$$

Making all the elements of the matrix greater than or equal to $V(A)$, and using *monotonicity*, *column dominance*, *row dominance*, and *objectivity*, respectively, we derive in a similar manner that

$$f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}) \leq$$

$$f(\begin{bmatrix} \max\{a_{11}, V(A)\} & \cdots & \max\{a_{1n}, V(A)\} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ \max\{a_{m1}, V(A)\} & \cdots & \max\{a_{mn}, V(A)\} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}) =$$

$$f(\begin{bmatrix} V(A) \\ \vdots \\ V(A) \end{bmatrix}) = f([V(A)]) = V(A).$$

Putting together all the (in)equalities that we have derived, we obtain that $f(A) = V(A)$. This shows that the value function V is the only evaluation function that satisfies *objectivity*, *monotonicity*, *row dominance*, and *column dominance*. □

Theorems 1 and 2 show that symmetry and column dominance are equivalent in the presence of objectivity, monotonicity, and row dominance. However, this equivalence does not hold in general. For example, the evaluation function defined by $f(A) = a_{11}$ for all $A \in \mathcal{A}$, satisfies symmetry (as well as objectivity and monotonicity) but does not satisfy column dominance (or row dominance).

Row dominance and column dominance state that the elimination of a dominated action of player 1 or 2 has no effect on the evaluation of player 1. A natural question at this point is what the effect is of the elimination of an arbitrary action, dominated or not. The following two properties deal with this question.

Row elimination For all $A \in \mathcal{A}$ and all rows r of A, $f(A) \geq f(A \setminus r)$.

Column elimination For all $A \in \mathcal{A}$ and all columns c of A, $f(A) \leq f(A \setminus c)$.

Row elimination states that player 1's evaluation should not increase when an action of player 1 is eliminated. Basically, it means that player 1 cannot be better off when his possibilities are further restricted. Column elimination states that the same is true for player 2; as player 2's payoff is the opposite of that of player 1, player 1's evaluation should not decrease when an action of player 2 is eliminated. It is not hard to see that the value function satisfies both row elimination and column elimination. These two properties highlight that the value function satisfies a form of monotonicity with respect to the elimination of actions. The question arises whether monotonicity could be replaced by row elimination and column elimination in the previous results. The answer is affirmative.[8]

Theorem 3 *The value function V is the unique evaluation function that satisfies objectivity, row dominance, column dominance, row elimination, and column elimination.*

Proof. *Existence.* We already established that V satisfies *objectivity, row dominance*, and *column dominance*. To see that it also satisfies *row elimination* and *column elimination*, it suffices to note that taking the maximum over a smaller set leads to a weakly smaller value and that taking the minimum over a smaller set leads to a weakly larger value.

Uniqueness. The proof of uniqueness is analogous to that in Theorem 2. Let $f : \mathcal{A} \to \mathbb{R}$ be an evaluation function that satisfies the five properties and let $A = [a_{ij}]_{i \in M, j \in N} \in \mathcal{A}$. Then, by *row dominance*, and *column dominance* we have

[8]Norde and Voorneveld (2004) provide axiomatic characterizations of the value function that are very similar to the characterization in Theorem 3. Their subgame property is our row elimination, and their strictly dominated action property is a slightly weaker version of our row dominance. Theorems 2.2 and 2.4 in Norde and Voorneveld (2004), the two theorems that are closest to our Theorem 3, however, both use symmetry as well as monotonicity.

$$f(A) = f(\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \\ V(A) & \cdots & V(A) \end{bmatrix}) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}).$$

Using *row elimination*, *column dominance*, and *objectivity*, respectively, we obtain

$$f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}) \geq f\left([V(A), \ldots, V(A)]\right) =$$

$$f\left([V(A)]\right) = V(A).$$

In a similar way, using *column elimination*, *row dominance*, and *objectivity*, respectively, we have

$$f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & V(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & V(A) \\ V(A) & \cdots & V(A) & V(A) \end{bmatrix}) \leq f(\begin{bmatrix} V(A) \\ \vdots \\ V(A) \end{bmatrix}) = f\left([V(A)]\right) = V(A).$$

We conclude that $f(A) = V(A)$. $\square$

Monotonicity is a very different property from column elimination and row elimination, even though it can be replaced by these two properties in characterizing the value function. Monotonicity deals with matrices of equal dimensions in which the elements differ, whereas row and column elimination deal with matrices of changing dimension. The same example as we used before, the evaluation function f with $f(A) = a_{11}$ for all $A \in \mathcal{A}$, satisfies monotonicity but does not satisfy column elimination or row elimination.

2.3 Characterizations of the lower value

This section is devoted to the *lower value function* $\underline{V}\colon \mathcal{A} \to \mathbb{R}$, which assigns to every matrix $A \in \mathcal{A}$ its lower value $\underline{V}(A)$. The main advantage of the lower value function is that one does not have to assume that the players have preferences over lotteries; it only uses pure actions in the finite two-player zero-sum game. For this reason, this evaluation function is more appropriate to apply to wider classes of two-player zero-sum games. The lower value can be interpreted as the absolute minimum payoff player 1 can guarantee himself in the finite two-player zero-sum game and it thereby represents player 1's most pessimistic expectations in the game.

To try and better understand the lower value function, we first ask which ones of the axioms used by Vilkas (1963) are satisfied by this evaluation function. Obviously, the lower value function does satisfy objectivity and monotonicity. However, the following two examples show that it does not satisfy row dominance or symmetry.

Example 1 *Consider the two matrices*

$$A = \begin{pmatrix} 5 & 0 \\ 1 & 4 \end{pmatrix} \qquad \text{and} \qquad A' = \begin{pmatrix} 5 & 0 \\ 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

Note that A' is obtained from A by adding a row that is a convex combination of the other rows, namely $\frac{1}{2}$ times the first row plus $\frac{1}{2}$ times the second one. To find the lower value of A, note that player 1 gets at least 0 if he chooses the first row, whereas he gets at least 1 if he chooses the second row. Hence, $\underline{V}(A) = 1$. Similarly, we derive that $\underline{V}(A') = 2$. Hence, the elimination of the dominated third row from A' changes the lower value and this is a violation of row dominance.

Example 2 *Consider the matrices A and $-A^T$ below.*

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad -A^T = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$$

$\underline{V}(A) = 1$ and $\underline{V}(-A^T) = -2$, which illustrates that $\underline{V}$ does not satisfy symmetry.

The lower value does satisfy the following property, which is a weaker form of row dominance.

> **Weak row dominance** The i^{th} row of the matrix A, denoted r_i, is *strongly dominated* if there exists a row r_k $(k \neq i)$ in the matrix that is weakly larger than row r_i, i.e., $a_{kj} \geq a_{ij}$ for all $j \in N$. For all $A \in \mathcal{A}$, if row r is strongly dominated, then $f(A) = f(A \setminus r)$.

Note that every row that is strongly dominated is also dominated but that the reverse is not necessarily true. Therefore, every row that can be eliminated under weak row dominance can also be eliminated under row dominance but not the other way around. Hence, weak row dominance is a weaker property than row dominance.

In addition to objectivity, monotonicity, and weak row dominance, the lower value function satisfies column dominance. However, taking into account Theorem 2, it is clear that these four properties cannot characterize the lower value function. We need a property that is satisfied by the lower value function but not by the value function. Strong column dominance is such a property.

> **Strong column dominance** The j^{th} column of the matrix A, denoted c_j, is *weakly dominated* if for all $i \in M$ there exists an other column c_k $(k \neq j)$ such that $a_{ik} \leq a_{ij}$. For all $A \in \mathcal{A}$, if column c is weakly dominated, then $f(A) = f(A \setminus c)$.

Weak domination of a column is a fairly weak requirement. It means that for every row, there is another column which has a weakly lower value in that row than the weakly dominated column does. Note that this can be a different column for every row. Because a column that is dominated is also weakly dominated, strong column dominance is a stronger property than column dominance. This stronger property is satisfied by the lower value function, which in fact can be characterized using this property. It is not satisfied by the value function.[9]

Theorem 4 *The lower value function $\underline{V}$ is the unique evaluation function that satisfies objectivity, monotonicity, weak row dominance, and strong column dominance.*

Proof. *Existence.* It is easily seen that $\underline{V}$ satisfies *objectivity* and *monotonicity.* To see that $\underline{V}$ satisfies *weak row dominance*, note that if row r_i is strongly dominated by row r_k in matrix A, then $\min_{j \in N} a_{ij} \leq \min_{j \in N} a_{kj}$. Hence, player 1 does not need row r_i to reach the maximum of these expressions, which equals $\underline{V}(A)$. To see that $\underline{V}$ satisfies *strong column dominance*, note that if column c_j is weakly dominated in matrix A, then $\min_{k \in N} a_{ik} = \min_{k \in N \setminus j} a_{ik}$ for every $i \in M$. Hence, deleting c_j does not change the lower value.

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[9] To see this directly, consider, for example, the matrix $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$, in which the third column is weakly dominated. However, the deletion of this column changes the value from 1 to $1\frac{1}{2}$.

Uniqueness. To prove that there is no other evaluation function that satisfies the four axioms listed in the theorem, let $f : \mathcal{A} \to \mathbb{R}$ be an evaluation function satisfying these properties and take a matrix $A = [a_{ij}]_{i \in M, j \in N} \in \mathcal{A}$. Suppose, without loss of generality, that $\underline{V}(A)$ is the element in the i^{th} row and the j^{th} column. First, we use *monotonicity* to make all the rows strongly dominated by the i^{th} row. Then we apply *weak row dominance* (repeatedly) to delete all the other rows. We are then left with a $1 \times n$ matrix consisting of the i^{th} row of A. Because $\underline{V}(A) = a_{ij}$, we know that $a_{ik} \geq a_{ij}$ for all $k \in N$. Hence, in the $1 \times n$ matrix, all columns different from the j^{th} are weakly dominated and can be eliminated by *strong column dominance.* Then we can apply *objectivity*, and obtain

$$f(A) \geq f(\begin{bmatrix} \min\{a_{11}, a_{i1}\} & \cdots & \min\{a_{1n}, a_{in}\} \\ \vdots & \ddots & \vdots \\ \min\{a_{m1}, a_{i1}\} & \cdots & \min\{a_{mn}, a_{in}\} \end{bmatrix}) =$$

$$f([a_{i1} \dots a_{in}]) = f([a_{ij}]) = a_{ij} = \underline{V}(A).$$

To show that $f(A) \leq \underline{V}(A)$, we first add a column to the matrix A in which all elements are equal to $\underline{V}(A)$. Note that such a column is weakly dominated, so by *strong column dominance*, this addition will not alter the lower value. Then we apply *monotonicity* to make all columns weakly dominated by the newly added one, after which we use *strong column dominance* again (repeatedly) to eliminate all these other columns. We are then left with a $m \times 1$ matrix in which all elements are equal to $\underline{V}(A)$, in which all rows are strongly dominated so that we can eliminate all but one of them by *weak row dominance.* Then we can apply *objectivity*, and obtain

$$f(A) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & \underline{V}(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & \underline{V}(A) \end{bmatrix}) \leq$$

$$f(\begin{bmatrix} \max\{a_{11}, \underline{V}(A)\} & \cdots & \max\{a_{1n}, \underline{V}(A)\} & \underline{V}(A) \\ \vdots & \ddots & \vdots & \vdots \\ \max\{a_{m1}, \underline{V}(A)\} & \cdots & \max\{a_{mn}, \underline{V}(A)\} & \underline{V}(A) \end{bmatrix}) =$$

$$f(\begin{bmatrix} \underline{V}(A) \\ \vdots \\ \underline{V}(A) \end{bmatrix}) = f([\underline{V}(A)]) = \underline{V}(A).$$

We have shown that $f(A) = \underline{V}(A)$, which proves that the lower value $\underline{V}$ is the unique evaluation function that satisfies *objectivity*, *monotonicity*, *weak row dominance*, and *strong column dominance*. $\square$

Like in the characterization of the value function, monotonicity can be substituted by column elimination and row elimination in Theorem 4.

Theorem 5 *The lower value function $\underline{V}$ is the unique evaluation function that satisfies objectivity, row elimination, column elimination, weak row dominance, and strong column dominance.*

Proof. *Existence.* We already established that V satisfies *objectivity*, *weak row dominance*, and *strong column dominance*. To see that it also satisfies *row elimination* and *column elimination*, it suffices to note that taking the maximum over a smaller set leads to a weakly smaller value and that taking the minimum over a smaller set leads to a weakly larger value.

Uniqueness. The proof of uniqueness is analogous to that in Theorem 4. Let $f : \mathcal{A} \to \mathbb{R}$ be an evaluation function that satisfies the five axioms listed in the theorem and let $A = [a_{ij}]_{i\in M, j\in N} \in \mathcal{A}$. Suppose, without loss of generality, that $\underline{V}(A)$ is the element in the i^{th} row and the j^{th} column. Then, applying *row elimination*, *strong column dominance*, and *objectivity*, successively, we obtain

$$f(A) \geq f([a_{i1}, \ldots, a_{in}]) = f([a_{ij}]) = a_{ij} = \underline{V}(A).$$

Using *strong column dominance*, *column elimination*, *weak row dominance*, and *objectivity*, we obtain

$$f(A) = f(\begin{bmatrix} a_{11} & \cdots & a_{1n} & \underline{V}(A) \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & \underline{V}(A) \end{bmatrix}) \leq f(\begin{bmatrix} \underline{V}(A) \\ \vdots \\ \underline{V}(A) \end{bmatrix}) =$$

$$f([\underline{V}(A)]) = \underline{V}(A).$$

This proves that $f(A) = \underline{V}(A)$. $\square$

In the following table we provide an overview of the various properties that we have encountered in Sections 2 and 3, and for each property we indicate whether it is satisfied (+) by the value function and the lower value function or not (−). We also indicate for each property the number(s) of the theorem(s) in which it is used to characterize the (lower) value.

	V	$\underline{V}$
Objectivity	$+_{1,2,3}$	$+_{4,5}$
Monotonicity	$+_{1,2}$	$+_{4}$
Symmetry	$+_{1}$	$-$
Row elimination	$+_{3}$	$+_{5}$
Column elimination	$+_{3}$	$+_{5}$
Weak row dominance	$+$	$+_{4,5}$
Row dominance	$+_{1,2,3}$	$-$
Column dominance	$+_{2,3}$	$+$
Strong column dominance	$-$	$+_{4,5}$

2.4 Characteristic functions associated with strategic games

In this section, we consider methods to associate a characteristic function with each strategic game. We consider the method defined in Von Neumann and Morgenstern (1944), which is based on the value function, and we also introduce a new method based on the lower value function. We remind the reader that the motivation for considering methods to associate a characteristic function with each strategic game is to tackle those situations in which players recognize the implicit cooperation possibilities in a strategic game, whose rules describe only the actions available to each of the players and their payoff functions explicitly. We start by providing the necessary definitions.

A *strategic game* $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N})$ consists of a set of players $N = \{1, \ldots, n\}$, and for every player $i \in N$ a set of actions X_i available to this player, and a payoff function $u_i : \prod_{j\in N} X_j \to \mathbb{R}$. We consider only finite strategic games[10], which are those games in which the action sets $\{X_i\}_{i\in N}$ are all finite. The class of finite strategic games with player set N is denoted by G^N. We denote the class of all finite strategic games by G.

A *coalitional game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function of the game, assigning to each coalition $S \subset N$ its worth $v(S)$. The worth $v(S)$ of a coalition S represents the benefits that this coalition can guarantee its members independently of what the other players (those in $N \setminus S$) do. By convention, $v(\emptyset) = 0$. From now on, we identify a coalitional game (N, v) with its characteristic function v. We denote the class of coalitional games with player set N by Γ^N and we use Γ to denote the class of all coalitional games. A

[10]However, some results we obtain can easily be extended to wider classes of strategic games.

coalitional game $v \in \Gamma^N$ is said to be *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all coalitions $S, T \subset N$ with $S \cap T = \emptyset$.

Von Neumann and Morgenstern (1944) propose the following procedure to associate a coalitional game with every strategic game. Let $g \in G^N$ be a strategic game and $S \subset N$, $S \neq N$, a non-empty coalition. The two-player zero-sum game g_S is defined by

$$g_S = (\{S, N \setminus S\}, \{X_S, X_{N \setminus S}\}, \{u_S, -u_S\}),$$

where, for all $T \subset N$, $X_T = \prod_{i \in T} X_i$ and $u_T = \sum_{i \in T} u_i$. In this game, there are two players, coalition S and coalition $N \setminus S$. The actions available to each of these two coalitions are all the combinations of the actions available to its members in the game g. The payoff to coalition S is the sum of the payoffs of its members for every possible action tuple, and the payoff to coalition $N \setminus S$ is the negative of this. Note that the game g_S is a finite two-player zero-sum game. We denote by A_S the payoff matrix of this game. If the players can randomize over their actions in the strategic game, then it is more appropriate to consider the mixed extension of the game g_S, i.e. the matrix game $E(A_S)$. Now, in the coalitional game $v_g \in \Gamma^N$ associated with the strategic game g, the worth of coalition S is the value of this matrix game;

$$v_g(S) = V(A_S).$$

This is the worth that the coalition S can secure for itself whatever the players in $N \setminus S$ do (even if they cooperate to keep the worth of coalition S as low as possible). The worth of the grand coalition N is simply defined as $v_g(N) = \max_{x \in X_N} u_N(x)$. The interpretation of v_g is that the players in a coalition S assume that all the players who are not in the coalition will coordinate their action choices to keep the payoff to S as small as possible. Note that the value of A_S is the value of the matrix game $E(A_S)$, so that it is implicitly assumed that the players in a coalition cannot only coordinate their pure actions, but can even choose a probability distribution over their coordinated actions. This is a very strong assumption.

The philosophy underlying Von Neumann and Morgenstern's procedure is intimately connected to the characteristic function concept. Since the characteristic function provides the benefits that every coalition can guarantee its members, independently of what the other players do, Von Neumann and Morgenstern's procedure seems to be a sensible one, at least in situations in which coalitions of players have preferences over lotteries and in which their utility functions are linear.

However, in settings in which randomizing over coordinated actions is not possible or reasonable, it is more appropriate to stick to (pure) actions and the lower value of the finite two-player zero-sum game. This leads us to associate with a strategic game $g \in G^N$ the coalitional game $\underline{v}_g \in \Gamma^N$ defined by

$$\underline{v}_g(S) = \underline{V}(A_S)$$

for all non-empty $S \subset N$, $S \neq N$, and $\underline{v}_g(N) = \max_{x \in X_N} u_N(x)$. Note that this coalitional game is more pessimistic than Von Neumann and Morgenstern's, in the sense that $\underline{v}_g(S) \leq v_g(S)$ for all $g \in G^N$ and all $S \subset N$. We illustrate both games in the following example.

Example 3 *Consider the following three-player strategic game g.*[11]

α_3	α_2	β_2
α_1	$(1,1,1)$	$(0,0,0)$
β_1	$(0,0,0)$	$(1,1,1)$

β_3	α_2	β_2
α_1	$(0,0,0)$	$(1,1,1)$
β_1	$(1,1,1)$	$(0,0,0)$

Let us take $S = \{1,2\}$. The matrix of the 2-player zero-sum game associated with S is $A_S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 2 \\ 2 & 0 \end{pmatrix}$, where the columns correspond to the actions α_3 and β_3 (from left to right) of player $3 \in N \setminus S$ and the rows are ordered as follows. The first row corresponds to the actions (α_1, α_2) by the players in S, the second row to (β_1, α_2), the third row to (α_1, β_2), and the fourth row to (β_1, β_2). The value of this matrix is $V(A_S) = 1$ and its lower value equals $\underline{V}(A_S) = 0$. Hence, $v_g(1,2) = 1$ and $\underline{v}_g(1,2) = 0$.[12]

The worths $v_g(S)$ and $\underline{v}_g(S)$ of other coalitions S are found in a similar manner. We list them in the following table.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v_g(S)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	3
$\underline{v}_g(S)$	0	0	0	0	0	0	3

This example illustrates that in general the two coalitional games v_g and $\underline{v}_g$ are different and also that $\underline{v}_g(S) \leq v_g(S)$ for all coalitions $S \subset N$.

[11] As is standard, player 1 is the row player, player 2 the column player, and player 3 chooses the matrix to the left or that to the right.

[12] Following a common practice, we simplify our notation and leave out the brackets $\{$ and $\}$ around the elements of a coalition.

The games v_g and $\underline{v}_g$ in the previous example are both superadditive. This is not a coincidence. This result holds for all coalitional games derived in the described manner from strategic games using the value or the lower value. For the games v_g this was shown in Von Neumann and Morgenstern (1944) and for the games $\underline{v}_g$, it is the content of the following Proposition.

Proposition 1 *For every strategic game $g \in G^N$, the associated coalitional game $\underline{v}_g$ is superadditive.*

Proof. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ be a strategic game and take two non-empty coalitions $S, T \subset N$, such that $S \cap T = \emptyset$. Then

$$\underline{v}_g(S \cup T) = \max_{x_{ST}\in X_{S\cup T}} \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_{S\cup T}(x_{ST}, x_{-ST})$$

$$= \max_{x_S\in X_S} \max_{x_T\in X_T} \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_{S\cup T}(x_S, x_T, x_{-ST}).$$

Now, let $y_S \in X_S$ and $y_T \in X_T$. We derive

$$\underline{v}_g(S \cup T) \geq \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_{S\cup T}(y_S, y_T, x_{-ST})$$

$$\geq \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_S(y_S, y_T, x_{-ST}) + \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_T(y_S, y_T, x_{-ST})$$

$$\geq \min_{x_T\in X_T} \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_S(y_S, x_T, x_{-ST})$$

$$+ \min_{x_S\in X_S} \min_{x_{-ST}\in X_{N\setminus(S\cup T)}} u_T(x_S, y_T, x_{-ST})$$

$$= \min_{x_{-S}\in N\setminus S} u_S(y_S, x_{-S}) + \min_{x_{-T}\in N\setminus T} u_T(y_T, x_{-T}).$$

Since this holds for all $y_S \in X_S$ and $y_T \in X_T$, we can now derive

$$\underline{v}_g(S \cup T) \geq \max_{x_S\in X_S} \min_{x_{-S}\in N\setminus S} u_S(x_S, x_{-S}) + \max_{x_T\in X_T} \min_{x_{-T}\in N\setminus T} u_T(x_T, x_{-T})$$

$$= \underline{v}_g(S) + \underline{v}_g(T).$$

This proves that $\underline{v}_g$ is superadditive. □

The *core* of a coalitional game (N, v) consists of payoff vectors in $\mathbb{R}^N$ that divide the payoff $v(N)$ of the grand coalition in such a way that each coalition $S \subset N$ of players gets at least their worth $v(S)$. The two games v_g and $\underline{v}_g$ in example 3 are not only superadditive, but also have non-empty cores. In example 3 we found $v_g(N) = \underline{v}_g(N) = 3$. If we divide this amount equally among the three players, giving them 1 each, then every single player gets at least his worth (which is $\frac{1}{2}$ in the game v_g and 0 in the game $\underline{v}_g$) and every 2-player coalition gets 2, whereas its worth in v_g equals 1 and that in $\underline{v}_g$ equals 0. Hence, this equal division is in the core of both games. It is not true in general, however, that the games v_g and $\underline{v}_g$ have non-empty cores. This is illustrated by the following example.

Example 4 *Consider the following three-player strategic game g.*

α_3	α_2	β_2
α_1	$(1, 1, 0)$	$(0, 0, 0)$
β_1	$(1, 0, 1)$	$(1, 0, 1)$

β_3	α_2	β_2
α_1	$(1, 1, 0)$	$(0, 1, 1)$
β_1	$(0, 0, 0)$	$(0, 1, 1)$

Following the procedure explained in example 3, we derive that

$$v_g(i) = \underline{v}_g(i) = 0 \text{ for each } i \in N,$$

$$v_g(i, j) = \underline{v}_g(i, j) = 2 \text{ for each pair } i, j \in N,$$

$$\text{and } v_g(N) = \underline{v}_g(N) = 2.$$

Note that in any core-division of the worth $v_g(N) = \underline{v}_g(N) = 2$, each individual player should get no less than 0 and any two players should get at least 2 together. It is clearly impossible to meet all these conditions simultaneously. Hence, the cores of the games (N, v_g) and $(N, \underline{v}_g)$ are empty.

2.5 Characterizations of the methods

In this section we consider more closely the two methods based on the value and the lower value that we discussed in the previous section. We axiomatically characterize these two methods to associate a coalitional game with every strategic game. We define a method in general as a map $\phi : G \to \Gamma$ that associates a coalitional game $\phi(g) \in \Gamma^N$ with every strategic game $g \in G^N$. We denote the Von Neumann and Morgenstern (1944) method by Ψ_V and our method by $\Psi_{\underline{V}}$. Hence, $\Psi_V(g) = v_g$ and $\Psi_{\underline{V}}(g) = \underline{v}_g$ for all $g \in G^N$. To characterize these two methods, we use properties that are derived from the properties we used in sections 2 and 3 to characterize the value function and the lower value function.

Individual objectivity states that if player i gets the same payoff for any possible actions tuple, then in the associated coalitional game the worth of the coalition consisting of player i only is equal to this amount.

Individual objectivity For all $g \in G^N$ and all players $i \in N$, if there exists a $c \in \mathbb{R}$ such that $u_i(x) = c$ for all $x \in X_N$, then $\phi(g)(i) = c$.

Monotonicity states that the worth of player i in the associated coalitional game does not decrease if his payoff in the strategic game weakly increases for all possible action tuples.

Monotonicity If $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ and

$$g' = (N, \{X_i\}_{i\in N}, \{u'_i\}_{i\in N}) \in G^N$$

are two strategic games and player $i \in N$ is such that $u_i \geq u'_i$, then $\phi(g)(i) \geq \phi(g')(i)$.

Irrelevance of dominated actions and irrelevance of strongly dominated actions mean that a player's worth in the coalitional game does not change if in the strategic game he loses the ability to use an action that was weakly worse for him than another one of his (mixed) strategies. Irrelevance of dominated actions and irrelevance of strongly dominated actions are derived from row dominance and weak row dominance, respectively. Correspondingly, irrelevance of dominated actions implies irrelevance of strongly dominated actions but not the other way around.

Irrelevance of dominated actions In a game $g \in G^N$, an action $x_i \in X_i$ of player i is *dominated* if there exists a convex combination y of the other actions of player i, with the property that $u_i(y, x_{N\setminus i}) \geq u_i(x_i, x_{N\setminus i})$ for all $x_{N\setminus i} \in X_{N\setminus i}$.[13] For all $g \in G^N$ and $i \in N$, if the action $x_i \in X_i$ is dominated, then $\phi(g)(i) = \phi(g')(i)$, where $g' \in G^N$ is the game obtained from g by deleting action x_i.

[13] $u_i(y, x_{N\setminus i}) := \sum_{\hat{x}_i \in X_i} y(\hat{x}_i) u_i(\hat{x}_i, x_{N\setminus i})$, where $y = \sum_{\hat{x}_i \in X_i} y(\hat{x}_i)\hat{x}_i$. Note that $y(x_i) = 0$, $y(\hat{x}_i) \geq 0$, for all $\hat{x}_i \in X_i$, and $\sum_{\hat{x}_i \in X_i} y(\hat{x}_i) = 1$. Observe that y is simply a mixed strategy of player i.

Irrelevance of strongly dominated actions In a game $g \in G^N$, an action $x_i \in X_i$ of player i is *strongly dominated* if there exists an action $x_i' \in X_i$, $x_i' \neq x_i$, such that $u_i(x_i', x_{N\setminus i}) \geq u_i(x_i, x_{N\setminus i})$ for all $x_{N\setminus i} \in X_{N\setminus i}$. For all $g \in G^N$ and $i \in N$, if action $x_i \in X_i$ is strongly dominated, then $\phi(g)(i) = \phi(g')(i)$, where $g' \in G^N$ is the game obtained from g by deleting action x_i.

Irrelevance of dominated threats and irrelevance of weakly dominated threats are derived from column dominance and strong column dominance, but they are adapted to be used in games with more than two players. They state that a player i's worth in the associated coalitional game is not affected if another player j is prohibited from using an action whose deletion does not change player i's worst-case scenario. Irrelevance of weakly dominated threats is the stronger property of the two, as every threat that is dominated is also weakly dominated.

Irrelevance of dominated threats In a game $g \in G^N$, an action $x_j \in X_j$ of a player j is a *dominated threat* to player $i \neq j$ if there exists a convex combination y of the other actions of player j, with the property that $u_i(y, x_{N\setminus j}) \leq u_i(x_j, x_{N\setminus j})$ for all $x_{N\setminus j} \in X_{N\setminus j}$. For all $g \in G^N$ and players $i, j \in N$, $i \neq j$, if action $x_j \in X_j$ is a dominated threat to player i, then $\phi(g)(i) = \phi(g')(i)$, where $g' \in G^N$ is the game obtained from g by deleting action x_j.

Irrelevance of weakly dominated threats In a game $g \in G^N$, an action $x_j \in X_j$ of a player j is a *weakly dominated threat* to player $i \neq j$ if for every $x_{N\setminus j} \in X_{N\setminus j}$ there exists an action $x_j' \in X_j$, $x_j' \neq x_j$, such that $u_i(x_j', x_{N\setminus\ j}) \leq u_i(x_j, x_{N\setminus\ j})$. For all $g \in G^N$ and players $i, j \in N$, $i \neq j$, if action $x_j \in X_j$ is a weakly dominated threat to player i, then $\phi(g)(i) = \phi(g')(i)$, where $g' \in G^N$ is the game obtained from g by deleting action x_j.

Both irrelevance of dominated threats and irrelevance of dominated actions make sense only in an environment where it is reasonable to assume that players can use mixed strategies, whereas irrelevance of strongly dominated actions and irrelevance of weakly dominated threats are more adequate in situations in which players can only use (pure) actions. To understand the relevance of these four properties, note that the worth of a player in the game $\phi(g)$ is interpreted as the payoff that this player can guarantee himself.

We need one additional property that has no equivalent in our characterizations of the (lower) value. It appears because we have to consider coalitions consisting of more than one player in the setting of the current section. We need some additional notation to introduce this property. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ and $S \subset N$, $S \neq \emptyset$. To study the opportunities of the members of S as a group, we introduce a new player $p(S)$ with set of actions $X_{p(S)} := X_S$ and utility function $u_{p(S)} : \prod_{j\in(N\setminus S)\cup\{p(S)\}} X_j \to \mathbb{R}$ defined by $u_{p(S)}(x_{N\setminus S}, x_{p(S)}) = u_S(x_{N\setminus S}, x_S)$ for all $x_{N\setminus S} \in X_{N\setminus S}$ and all $x_{p(S)} = x_S \in X_S = X_{p(S)}$. Denote $N(S) := (N \setminus S) \cup \{p(S)\}$. The game $g(S) \in G^{N(S)}$ is defined by $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$.[14] The property merge invariance states that the worth of coalition S in the original strategic game g is the same as that of player $p(S)$ in the game $g(S)$. Its interpretation is that a coalition of players cannot influence its worth by merging and acting as one player. Its validity derives from the very interpretation of a coalition in a coalitional game as a group of players acting in the best interests of the group.

> **Merge invariance** Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ and $S \subset N$, $S \neq \emptyset$. Then $\phi(g)(S) = \phi(g(S))(p(S))$, where $g(S)$ is the strategic game that is obtained from g by considering the coalition S as a single player.

The properties introduced above can be used to axiomatically characterize the two methods Ψ_V and $\Psi_{\underline{V}}$. We only provide a proof of one of the following theorems, as their structure is similar to the proofs of Theorems 2 and 4, respectively, and providing one proof suffices to illustrate the role of the extra property merge invariance.

Theorem 6 *The method Ψ_V is the unique method satisfying individual objectivity, monotonicity, irrelevance of dominated actions, irrelevance of dominated threats, and merge invariance.*

Theorem 7 *The method $\Psi_{\underline{V}}$ is the unique method satisfying individual objectivity, monotonicity, irrelevance of strongly dominated actions, irrelevance of weakly dominated threats, and merge invariance.*

Proof. *Existence.* First, we show that $\Psi_{\underline{V}}$ satisfies the five properties. Let $g \in G^N$, $i \in N$, and $c \in \mathbb{R}$ be such that $u_i(x) = c$, for all $x \in X_N$. Then, in the matrix A_i of the game g_i, all entries are equal to c. The lower value of this matrix equals c. Hence, $\Psi_{\underline{V}}(g)(i) = \underline{v}_g(i) = c$, which shows that $\Psi_{\underline{V}}$ satisfies *individual objectivity.*

[14] Note the difference with the game $g_S = (\{S, N\setminus S\}, \{X_S, X_{N\setminus S}\}, \{u_S, -u_S\})$, in which not only coalition S is viewed as one player, but coalition $N \setminus S$ as well. Also, in the game g_S, the objective of the players in $N \setminus S$ is to keep the payoffs to S as low as possible, rather than to maximize their own, as is the case in $g(S)$.

Now, let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ and $g' = (N, \{X_i\}_{i\in N}, \{u'_i\}_{i\in N}) \in G^N$ be two strategic games such that $u_i \geq u'_i$ for player $i \in N$. Then, $A_i \geq A'_i$, where A_i denotes the matrix of the game g_i and A'_i denotes the matrix of the game g'_i. It now follows from monotonicity of the lower value function that $\Psi_{\underline{V}}(g)(i) = \underline{v}_g(i) \geq \underline{v}_{g'}(i) = \Psi_{\underline{V}}(g')(i)$. This proves that $\Psi_{\underline{V}}$ satisfies *monotonicity*.

To see that $\Psi_{\underline{V}}$ satisfies *irrelevance of strongly dominated actions*, note that if action x_i for player i is strongly dominated in the game $g \in G^N$, then it corresponds to a strongly dominated row in the matrix A_i of the game g_i. Hence, by weak row dominance of the lower value function, $\Psi_{\underline{V}}(g)(i) = \underline{v}_g(i) = \underline{v}_{g'}(i) = \Psi_{\underline{V}}(g')(i)$, where $g' \in G^N$ is the game that is obtained from g by deleting action x_i.

To see that $\Psi_{\underline{V}}$ satisfies *irrelevance of weakly dominated threats*, note that if action $x_j \in X_j$ of a player j is a weakly dominated threat to player $i \neq j$ in the game $g \in G^N$, then for all $x_{N\backslash i,j} \in X_{N\backslash i,j}$ action $(x_j, x_{N\backslash i,j})$ corresponds to a weakly dominated column in the matrix A_i of the game g_i. Hence, using strong column dominance of the lower value function (repeatedly), we can eliminate the action (column) $(x_j, x_{N\backslash i,j})$ for each $x_{N\backslash i,j} \in X_{N\backslash i,j}$, without changing the lower value. The matrix that we have left is that corresponding to the game $g' \in G^N$ that is obtained from g by deleting action x_j. For this game, we then have $\Psi_{\underline{V}}(g)(i) = \underline{v}_g(i) = \underline{v}_{g'}(i) = \Psi_{\underline{V}}(g')(i)$.

It follows easily that $\Psi_{\underline{V}}$ satisfies *merge invariance* by noting that the matrix A_S of the strategic game g_S derived from g and the matrix $A_{p(S)}$ of the strategic game $g(S)_{p(S)}$ derived from $g(S)$ are, in fact, the same.

Uniqueness. We now proceed to show that any method satisfying the five properties coincides with $\Psi_{\underline{V}}$. Let $\phi : G \to \Gamma$ be a method satisfying *individual objectivity*, *monotonicity*, *irrelevance of strongly dominated actions*, *irrelevance of weakly dominated threats*, and *merge invariance*. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ be a finite strategic game and $S \subset N$ a non-empty coalition. If $S = N$, then *merge invariance*, *irrelevance of strongly dominated actions*, and *individual objectivity* clearly imply that $\phi(g)(N) = \underline{v}_g(N) = \Psi_{\underline{V}}(g)(N)$. Assume now that $S \neq N$. We will prove that $\phi(g)(S) = \underline{v}_g(S) = \Psi_{\underline{V}}(g)(S)$ holds. The proof is divided into two parts.

Part I. First, we prove that $\phi(g)(S) \geq \Psi_{\underline{V}}(g)(S)$.

Consider the game $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$, which is obtained from g by considering the coalition S as a single player $p(S)$. Because ϕ satisfies *merge invariance*, we know that

$$\phi(g)(S) = \phi(g(S))(p(S)). \tag{1}$$

We know from the definition of $g(S)$ that the matrix A_S of the strategic game g_S derived from g and the matrix $A_{p(S)}$ of the strategic game $g(S)_{p(S)}$ derived from $g(S)$ are the same. From this we conclude that $\underline{v}_g(S) = \underline{V}(A_S) = \underline{V}(A_{p(S)}) = \underline{v}_{g(S)}(p(S))$. Now, let $\bar{x} = (\bar{x}_i)_{i \in N} \in \prod_{i \in N} X_i$ be an action such that the lower value of A_S is obtained in the row corresponding to action $\bar{x}_S$ for coalition S and the column corresponding to action $\bar{x}_{N \setminus S}$ for coalition $N \setminus S$. Then the lower value of $A_{p(S)}$ is obtained in the row corresponding to action $\bar{x}_{p(S)} = \bar{x}_S \in X_{p(S)}$ for player $p(S)$ and the column corresponding to action $\bar{x}_{N \setminus S}$ for coalition $N \setminus S$.

Let g_1 be the game that is obtained from the game $g(S)$ by bounding the utility of player $p(S)$ from above by $\underline{v}_g(S)$, i.e.,

$$g_1 = (N(S), \{X_i\}_{i \in N(S)}, \{u_i\}_{i \in N \setminus S}, u'_{p(S)}),$$

where

$$u'_{p(S)}(x_{N \setminus S}, x_{p(S)}) = \min\{u_S(x_{N \setminus S}, x_S), \underline{v}_g(S)\}$$

for all $x_{N \setminus S} \in X_{N \setminus S}$ and all $x_{p(S)} = x_S \in X_S = X_{p(S)}$. Because ϕ satisfies *monotonicity*,

$$\phi(g(S))(p(S)) \geq \phi(g_1)(p(S)). \tag{2}$$

Now, note that

$$\underline{v}_g(S) = \underline{v}_{g(S)}(p(S)) = \max_{x_{p(S)} \in X_{p(S)}} \min_{x_{N \setminus S} \in X_{N \setminus S}} u_{p(S)}(x_{p(S)}, x_{N \setminus S})$$

is obtained at $(\bar{x}_{p(S)}, \bar{x}_{N \setminus S})$, so that $\min_{x_{N \setminus S} \in X_{N \setminus S}} u_{p(S)}(\bar{x}_{p(S)}, x_{N \setminus S}) = \underline{v}_g(S)$ and $u_{p(S)}(\bar{x}_{p(S)}, x_{N \setminus S}) \geq \underline{v}_g(S)$ for all $x_{N \setminus S} \in X_{N \setminus S}$. Hence,

$$u'_{p(S)}(\bar{x}_{p(S)}, x_{N \setminus S}) = \underline{v}_g(S) \tag{3}$$

for all $x_{N \setminus S} \in X_{N \setminus S}$. Moreover, $u'_{p(S)}(x_{p(S)}, x_{N \setminus S}) \leq \underline{v}_g(S)$ for all $x_{N \setminus S} \in X_{N \setminus S}$ and all $x_{p(S)} \in X_{p(S)}$. Hence, every action $x_{p(S)} \in X_{p(S)}$, $x_{p(S)} \neq \bar{x}_{p(S)}$, is strongly dominated by action $\bar{x}_{p(S)}$. Because ϕ satisfies *irrelevance of strongly dominated actions*, we can eliminate all the strongly dominated actions of player $p(S)$ without changing the worth of $p(S)$ in the image of the game under ϕ. Hence,

$$\phi(g_1)(p(S)) = \phi(g_2)(p(S)), \tag{4}$$

where g_2 is the game that is obtained from g_1 by deleting all actions of player $p(S)$ except action $\bar{x}_{p(S)}$.

In the game g_2, for every player $j \neq p(S)$ every action $x_j \in X_j \setminus \bar{x}_j$ is a weakly dominated threat to player $p(S)$, because $u'_{p(S)}(\bar{x}_{p(S)}, \bar{x}_{N \setminus S})$

$= \min_{x_{N\setminus S} \in X_{N\setminus S}} u'_{p(S)}(\bar{x}_{p(S)}, x_{N\setminus S})$.[15] Because ϕ satisfies *irrelevance of weakly dominated threats*, we can eliminate all the weakly dominated threats to player $p(S)$ without changing the worth of $p(S)$ in the image of the game under ϕ. Hence,

$$\phi(g_2)(p(S)) = \phi(g_3)(p(S)), \tag{5}$$

where g_3 is the game that is obtained from g_2 by deleting all actions $x_j \in X_j \setminus \bar{x}_j$ for every player $j \in N \setminus S$.

In the game g_3 every player j has exactly one action, $\bar{x}_j$. Hence, for this game we can use *individual objectivity* of ϕ to derive that

$$\phi(g_3)(p(S)) = u'_{p(S)}(\bar{x}). \tag{6}$$

Putting (in)equalities (1), (2), (4), (5), (6) and (3) together, we have proved that $\phi(g)(S) = \phi(g(S))(p(S)) \geq \phi(g_1)(p(S)) = \phi(g_2)(p(S)) = \phi(g_3)(p(S)) = u'_{p(S)}(\bar{x}) = \underline{v}_g(S) = \Psi_{\underline{V}}(g)(S)$.

Part II. Now, we prove that $\phi(g)(S) \leq \Psi_{\underline{V}}(g)(S)$.

Consider again the game $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$ that is obtained from g by considering the coalition S as a single player $p(S)$. We have already seen that $\phi(g(S))(p(S)) = \phi(g)(S)$ and $\underline{v}_{g(S)}(p(S)) = \underline{v}_g(S)$.

We define a new game g_4 by adding an action $x_i^* \notin X_i$ for each player $i \in N \setminus S$. The actions x_i^* are introduced as additional threats to player $p(S)$. We add these actions one by one. Without loss of generality, we assume that $N \setminus S = \{1, 2, \ldots, k\}$, where k denotes the number of players in $N \setminus S$.

We first define the game g_1^*, by adding action x_1^* for player 1. The payoff to player $p(S)$ in the game g_1^* is as in the game $g(S)$ when player 1 plays an action $x_1 \in X_1$. When player 1 plays his action x_1^*, then the payoff to player $p(S)$ is defined by

$$u^1_{p(S)}(x_{p(S)}, x_1^*, (x_i)_{i\in\{2,3,\ldots,k\}}) = \min_{x_1 \in X_1} \{u_{p(S)}(x_{p(S)}, x_1, (x_i)_{i\in\{2,3,\ldots,k\}})\},$$

where $x_i \in X_i$ for all $i \in \{2, 3, \ldots, k\}$ and $x_{p(S)} \in X_{p(S)}$. In the game g_1^*, action x_1^* is a weakly dominated threat to player $p(S)$. Because ϕ satisfies *irrelevance of weakly dominated threats*, we can eliminate this weakly dominated threat to player $p(S)$ from g_1^* without changing the worth of $p(S)$ in the image of the game under ϕ. This shows that $\phi(g_1^*)(p(S)) = \phi(g(S))(p(S))$.

Now, let $2 \leq j \leq k$ and suppose that we have added an action x_i^* for each player $i = 1, 2, \ldots j-1$ and defined the corresponding games g_i^* with payoff

[15] In fact, $u'_{p(S)}(\bar{x}_{p(S)}, \bar{x}_{N\setminus S}) = u'_{p(S)}(\bar{x}_{p(S)}, x_{N\setminus S}) = \underline{v}_g(S)$ for all $x_{N\setminus S} \in X_{N\setminus S}$.

functions $u^i_{p(S)}$ for player $p(S)$ so that in each game g^*_i action x^*_i is a weakly dominated threat to player $p(S)$ and $\phi(g^*_i)(p(S)) = \phi(g(S))(p(S))$. To obtain the game g^*_j, we add an action x^*_j for player j and define the payoff to player $p(S)$ to be as in the game g^*_{j-1} when player j plays an action $x_j \in X_j$, and when player j plays his action x^*_j it is

$$u^j_{p(S)}(x_{p(S)}, (y_i)_{i\in\{1,\dots,j-1\}}, x^*_j, (x_i)_{i\in\{j+1,\dots,k\}}) =$$

$$\min_{x_j \in X_j} \{u^{j-1}_{p(S)}(x_{p(S)}, (y_i)_{i\in\{1,\dots,j-1\}}, x_j, (x_i)_{i\in\{j+1,\dots,k\}})\},$$

where $y_i \in X_i \cup \{x^*_i\}$ for all $i \in \{1,\dots,j-1\}$, $x_i \in X_i$ for all $i \in \{j+1,\dots,k\}$, and $x_{p(S)} \in X_{p(S)}$. In the game g^*_j, action x^*_j is a weakly dominated threat to player $p(S)$. Because ϕ satisfies *irrelevance of weakly dominated threats*, we can eliminate this weakly dominated threat to player $p(S)$ from g^*_j without changing the worth of $p(S)$ in the image of the game under ϕ. This shows that $\phi(g^*_j)(p(S)) = \phi(g^*_{j-1})(p(S)) = \phi(g(S))(p(S))$.

The game g_4 is the game g^*_k which emerges from the procedure described above after an action x^*_i has been added for each player $i \in N\setminus S$. The payoff function of player $p(S)$ in the game g_4 is denoted by $u'_{p(S)} := u^k_{p(S)}$. We have that

$$\phi(g_4)(p(S)) = \phi(g(S))(p(S)). \tag{7}$$

Note that

$$u'_{p(S)}(x_{p(S)}, (x^*_i)_{i\in N\setminus S}) =$$

$$u^k_{p(S)}(x_{p(S)}, (x^*_i)_{i\in N\setminus S}) = \min_{x_k\in X_k} \{u^{k-1}_{p(S)}(x_{p(S)}, (x^*_i)_{i\in\{1,\dots,k-1\}}, x_k)\} =$$

$$\min_{x_k\in X_k} \min_{x_{k-1}\in X_{k-1}} \{u^{k-2}_{p(S)}(x_{p(S)}, (x^*_i)_{i\in\{1,\dots,k-2\}}, x_{k-1}, x_k)\} =$$

$$\dots = \min_{x_{N\setminus S}\in X_{N\setminus S}} u_{p(S)}(x_{p(S)}, x_{N\setminus S}) \leq \underline{v}_{g(S)}(p(S)) \tag{8}$$

for all $x_{p(S)} \in X_{p(S)}$.

Let g_5 be the game that is obtained from the game g_4 by bounding the utility of player $p(S)$ from below by $\underline{v}_g(S)$, i.e., the payoff function of player $p(S)$ is

$$u''_{p(S)}(x_{p(S)}, y_{N\setminus S}) = \max\{u'_{p(S)}(x_{p(S)}, y_{N\setminus S}), \underline{v}_g(S)\}$$

for all $x_{p(S)} \in X_{p(S)}$ and all $y_{N\setminus S} \in \prod_{i\in N\setminus S} X_i \cup \{x^*_i\}$. Because ϕ satisfies *monotonicity*,

$$\phi(g_5)(p(S)) \geq \phi(g_4)(p(S)). \tag{9}$$

Note that the game g_5 has been constructed in such a way that

$$u''_{p(S)}(x_{p(S)}, x_i^*, y_{N\setminus(S\cup i)}) = \min_{x_i \in X_i} u''_{p(S)}(x_{p(S)}, x_i, y_{N\setminus(S\cup i)})$$

for all $i \in N\setminus S$, all $x_{p(S)} \in X_{p(S)}$, and all $y_{N\setminus(S\cup i)} \in \prod_{j\in N\setminus(S\cup i)} X_j \cup \{x_j^*\}$. Hence, every action $x_i \in X_i$ is a weakly dominated threat to player $p(S)$ for every player $i \neq p(S)$. Because ϕ satisfies *irrelevance of weakly dominated threats*, we can eliminate all the weakly dominated threats to player $p(S)$ without changing the worth of $p(S)$ in the image of the game under ϕ. Hence,

$$\phi(g_5)(p(S)) = \phi(g_6)(p(S)), \tag{10}$$

where g_6 is the game that is obtained from g_5 by deleting all actions $x_i \in X_i$ for every player $i \in N \setminus S$.

In the game g_6 all players $i \neq p(S)$ have only one action, action x_i^*, and as by (8) $u'_{p(S)}(x_{p(S)}, (x_i^*)_{i\in N\setminus S}) \leq \underline{v}_{g(S)}(p(S)) = \underline{v}_g(S)$,

$$u''_{p(S)}(x_{p(S)}, (x_i^*)_{i\in N\setminus S}) = \max\{u'_{p(S)}(x_{p(S)}, (x_i^*)_{i\in N\setminus S}), \underline{v}_g(S)\} = \underline{v}_g(S)$$

for every $x_{p(S)} \in X_{p(S)}$. Hence, the conditions for *individual objectivity* are satisfied and we can conclude that

$$\phi(g_6)(p(S)) = \underline{v}_g(S). \tag{11}$$

Putting (in)equalities (1), (7), (9), (10), and (11), together, we have proved that $\phi(g)(S) = \phi(g(S))(p(S)) = \phi(g_4)(p(S)) \leq \phi(g_5)(p(S)) = \phi(g_6)(p(S)) = \underline{v}_g(S) = \Psi_{\underline{V}}(g)(S)$.

This finishes the proof of the Theorem. □

We already pointed out that irrelevance of dominated actions implies irrelevance of strongly dominated actions and that irrelevance of weakly dominated threats implies irrelevance of dominated threats. Hence, Theorems 6 and 7 allow us to conclude that Ψ_V does not satisfy irrelevance of weakly dominated threats and that $\Psi_{\underline{V}}$ does not satisfy irrelevance of dominated actions.

In Sections 2 and 3 we showed that monotonicity can be replaced by row elimination and column elimination in the characterizations of the value function and the lower value function (see Theorems 3 and 5). Analogously, we can replace the monotonicity property in Theorems 6 and 7 by the two following properties, which are derived from row elimination and column elimination.

Elimination of own actions For all strategic games $g \in G^N$, all players $i \in N$, and all actions $x_i \in X_i$, $\phi(g)(i) \geq \phi(g')(i)$, where $g' \in G^N$ is that game that is obtained from g by deleting action x_i.

Elimination of others' actions For all strategic games $g \in G^N$, all players $i, j \in N$, $i \neq j$, and all actions $x_j \in X_j$, $\phi(g)(i) \leq \phi(g')(i)$, where $g' \in G^N$ is that game that is obtained from g by deleting action x_j.

3 The Shapley valuation function for strategic games in which players cooperate

This section displays the paper Carpente et al. (2004b).

3.1 Introduction

In this note we consider valuation functions. A valuation function associates with every non-empty coalition of players in a strategic game a vector of payoffs for the members of the coalition that provides these players' valuations of cooperating in the coalition. We formulate axioms for such a valuation function and prove that there exists a unique valuation function that satisfies these axioms. This valuation function is found by applying Shapley values (cf. Shapley (1953)) to coalitional games that are obtained by applying the lower-value based method to associate a coalitional game with every strategic game. The lower-value based method was introduced in Carpente et al. (2004a).

Von Neumann and Morgenstern (1944) explicitly consider the possibility that coalitions of players cooperate in strategic games. To formalize this, they associate with every strategic game a coalitional game in which the worth of a coalition of players represents the worth that these players can jointly obtain when they coordinate their actions. This worth is defined to be the value of the mixed extension of the zero-sum game that the coalition plays with the complementary coalition consisting of all other players. Carpente et al. (2004a) axiomatically characterize the method of associating a coalitional game with every strategic game that was proposed by Von Neumann and Morgenstern. In addition, they formulate and axiomatically characterize a variation of Von Neumann and Morgenstern's method by considering lower values of the zero-sum games between coalitions and their complementary coalitions. They argue that the method based on lower values rather than values is more appropriate in situations where it is not reasonable to assume that coalitions of players can mix coordinated actions.

We introduce our framework and axioms in Section 2, in which we also prove that the so-called Shapley valuation function, which is based on the Carpente et al. (2004a) method to associate coalitional games with strategic games, is the unique valuation function satisfying all the axioms. We conclude in Section 3 by pointing out how the axioms need to be adapted to axiomatically characterize a valuation function based on Shapley values and the Von Neumann and Morgenstern (1944) method to associate coalitional games with strategic games. In this section, we also point out how one of the axioms, which deals with monotonicity in payoffs, can be replaced by two other axioms that deal with monotonicity when deleting actions.

3.2 A valuation function for strategic games in which players cooperate

A *strategic game* $g = (N, \{X_i\}_{i \in N}, \{u_i\}_{i \in N})$ consists of a set of players $N = \{1, \cdots, n\}$ and, for every player $i \in N$, a set of actions X_i available to this player, as well as a payoff function $u_i : \prod_{j \in N} X_j \to \mathbb{R}$. In this note, we consider only finite strategic games, i.e. games with finitely many players in which the action set X_i is finite for each player i. The class of finite strategic games with player set N is denoted by G^N.

We assume that in a strategic game $g \in G^N$ a coalition of players $S \subset N$ is able to coordinate and play any action profile $x_S \in X_S := \prod_{i \in S} X_i$. We associate with each game $g \in G^N$ and coalition $S \subset N$ the payoffs attainable in game g by the players in S if they decide to cooperate, independent of which actions are played by the other players. A *valuation function* is a map φ that associates a payoff vector $\varphi(S, g) \in \mathbb{R}^S$ with every game $g \in G^N$ and non-empty coalition $S \subset N$, where $\varphi_i(S, g)$ provides a valuation for player i of cooperating in coalition S in game g, for each $i \in S$.

We consider the following properties for a valuation function. Most of these properties are inspired by analogous ones introduced in Carpente et al. (2004a).

Individual objectivity states that if a player gets the same payoff for any possible action tuple in a game, then the valuation for this player of forming a singleton coalition is equal to this amount.

Individual objectivity. For all $g \in G^N$ and all players $i \in N$, if $c \in \mathbb{R}$ is such that $u_i(x) = c$ for all $x \in X_N$, then

$$\varphi_i(\{i\}, g) = c.$$

Monotonicity states that the valuation for a player of forming a singleton coalition does not decrease if his payoff in the strategic game weakly increases for all possible action tuples.

Monotonicity. For all strategic games $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N})$ and $g' = (N, \{X_i\}_{i\in N}, \{u'_i\}_{i\in N})$, and player $i \in N$ such that $u_i(x) \geq u'_i(x)$ for all $x \in X_N$,

$$\varphi_i(\{i\}, g) \geq \varphi_i(\{i\}, g').$$

Irrelevance of strongly dominated actions states that the valuation for a player of forming a singleton coalition does not change if in the strategic game he loses the ability to use an action that is weakly worse for him than another of his actions, no matter what actions the other players choose. To understand the relevance of this property (as well as of the next one, irrelevance of weakly dominated threats), note that the valuation for a player of forming a singleton coalition is interpreted as the payoff that this player can guarantee himself independent of what actions are played by the other players.

Irrelevance of strongly dominated actions. In a game $g \in G^N$, an action $x_i \in X_i$ of player $i \in N$ is *strongly dominated* if there exists an action $x'_i \in X_i$, $x'_i \neq x_i$, such that $u_i(x'_i, x_{N\setminus i}) \geq u_i(x_i, x_{N\setminus i})$ for all $x_{N\setminus i} \in X_{N\setminus i}$. For all $g \in G^N$ and $i \in N$, if action $x_i \in X_i$ is strongly dominated, then $\varphi_i(\{i\}, g) = \varphi_i(\{i\}, g')$, where $g' \in G^N$ is the game obtained from g by deleting action x_i.

Irrelevance of weakly dominated threats states that the valuation for a player i of forming a singleton coalition is not affected if another player j is prohibited from using an action whose deletion does not change player i's worst-case scenario.

Irrelevance of weakly dominated threats. In a game $g \in G^N$, an action $x_j \in X_j$ of a player $j \in N$ is a *weakly dominated threat* to player $i \in N$, $i \neq j$, if for every $x_{N\setminus j} \in X_{N\setminus j}$ there exists an action $x'_j \in X_j$, $x'_j \neq x_j$, such that $u_i(x'_j, x_{N\setminus j}) \leq u_i(x_j, x_{N\setminus j})$. For all $g \in G^N$ and players $i, j \in N$, $i \neq j$, if action $x_j \in X_j$ is a weakly dominated threat to player i, then $\varphi_i(\{i\}, g) = \varphi_i(\{i\}, g')$, where $g' \in G^N$ is the game obtained from g by deleting action x_j.

We now introduce some additional notation in order to be able to formulate the next property. Let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$ and $S \subset N$,

$S \neq \emptyset$. Suppose that the members of coalition S decide to merge and act as one player. In order to study the opportunities of S as a coalition, we introduce a new player $i(S)$ with action set $X_{i(S)} := X_S$ and utility function $u_{i(S)} : \Pi_{j \in (N \backslash S) \cup \{i(S)\}} X_j \to \mathbb{R}$ defined by $u_{i(S)}(x_{N\backslash S}, x_{i(S)}) = \sum_{j \in S} u_j(x_{N\backslash S}, x_S)$ for all $x_{N\backslash S} \in X_{N\backslash S}$ and all $x_{i(S)} = x_S \in X_S = X_{i(S)}$. Denote $N(S) := (N\backslash S) \cup \{i(S)\}$. The strategic game $g(S) \in G^{N(S)}$ is defined by $g(S) = (N(S), \{X_i\}_{i\in N(S)}, \{u_i\}_{i\in N(S)})$. The property merge invariance states that the total valuation for the players in S of forming coalition S in the game g is the same as the valuation for player $i(S)$ of forming a singleton coalition in the game $g(S)$. The interpretation of this property is that a coalition of players cannot influence their joint valuation by merging and acting as one player. Such a requirement seems natural as a coalition of players who decide to cooperate is supposed to act in the best interest of the group.

Merge invariance. For all $g \in G^N$ and non-empty $S \subset N$,

$$\sum_{i \in S} \varphi_i(S, g) = \varphi_{i(S)}(\{i(S)\}, g(S)).$$

In the literature on the Shapley value and other solution concepts for coalitional games, a principle of reciprocity between the players is often used. We use balanced contributions, as introduced in Myerson (1980). The principle of balanced contributions asserts that for any two players the gains or losses that they can inflict on each other by leaving the game should be equal. Myerson used this principle to extend the Shapley value to a setting of coalitional games without transferable utility with conferences structures.

Our aim is to apply the principle of balanced contributions to valuation functions. In this setting, rather than considering that players leave the game, we assert that players leave the coalition of cooperating players and consider the losses or gains that this inflicts on other players in the cooperating coalition.

Balanced contributions. For all $g \in G^N$ and non-empty $S \subset N$, and all $i, j \in S$,

$$\varphi_i(S, g) - \varphi_i(S\backslash\{j\}, g) = \varphi_j(S, g) - \varphi_j(S\backslash\{i\}, g).$$

As we pointed out above, in coalitional games the property balanced contributions is intimately connected with the Shapley value. This is still true in the context of valuation functions, in which we associate payoff vectors with coalitions of cooperating players in strategic games. We show in Theorem 8 below that there exists a unique valuation function that satisfies the properties that we have stated above. This valuation function is derived from the Shapley value in the following manner.

Take a strategic game $g \in G^N$. With this game we associate the coalitional game $(N, \underline{v}_g)$ as defined in Carpente et al. (2004a). A *coalitional game* is a pair (N, v) consisting of a player set $N = \{1, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ that assigns to each coalition $S \subset N$ its worth $v(S)$ representing the benefits that this coalition can guarantee its members independently of what the other players (those in $N \backslash S$) do (by convention, $v(\emptyset) = 0$). In the game $(N, \underline{v}_g)$, the worth $\underline{v}_g(S)$ of a coalition of players $S \subset N$ is the lower value of a finite two-person zero-sum game between coalition S on the one hand and the coalition $N \backslash S$ on the other hand. For any non-empty coalition $S \subset N$, $S \neq N$, the two-player zero-sum game g_S is defined by

$$g_S = (\{S, N \setminus S\}, \{X_S, X_{N\setminus S}\}, \{u_S, -u_S\}),$$

where, for all $T \subset N$, $X_T = \prod_{i \in T} X_i$ and $u_T = \sum_{i \in T} u_i$. This game has two players, coalitions S and $N \backslash S$. The actions available to each of these two coalitions are all the combinations of the actions available to its members in the game g and for every possible action tuple the payoff to coalition S is the sum of the payoffs of its members while the payoff to coalition $N \backslash S$ is the opposite of this. The lower value of this game is $\underline{v}_g(S) = \max\limits_{x_S \in X_S} \min\limits_{x_{N\setminus S} \in X_{N\setminus S}} u_S(x_S, x_{N\setminus S})$. This is the worth that the players in coalition S can secure for themselves by coordinating their actions even if the players in $N \backslash S$ cooperate to keep the worth of coalition S as low as possible. The worth of the grand coalition N is simply defined as $\underline{v}_g(N) = \max_{x \in X_N} u_N(x)$.

The *Shapley valuation function* $\phi_{\underline{V}}$ assigns to each strategic game $g \in G^N$ and non-empty coalition $S \subset N$ the Shapley value $\phi(S, \underline{v}_g)$ of the coalitional game $(S, \underline{v}_g)$ associated with g and S, which is defined by

$$\phi_i(S, \underline{v}_g) = \sum_{T \subset S, i \in T} \frac{(|T| - 1)!\,(|S| - |T|)!}{|S|!} \left(\underline{v}_g(T) - \underline{v}_g(T \backslash \{i\})\right)$$

for all $i \in S$.

Theorem 8 *The Shapley valuation function $\phi_{\underline{V}}$ is the unique valuation function satisfying individual objectivity, monotonicity, irrelevance of strongly dominated actions, irrelevance of weakly dominated threats, merge invariance, and balanced contributions.*

Proof. *Existence.* The proof that $\phi_{\underline{V}}$ satisfies the five properties individual objectivity, monotonicity, irrelevance of strongly dominated actions, irrelevance of weakly dominated threats, and merge invariance, uses the results in Carpente et al. (2004a). In that paper it is proved that the lower value method, which associates with each strategic game $g \in G^N$ the coalitional game $(N, \underline{v}_g)$, is the unique method of associating a coalitional game with each strategic game that satisfies the appropriate equivalents of these five properties.

To check that $\phi_{\underline{V}}$ satisfies *individual objectivity*, let $g \in G^N$, $i \in N$, and $c \in \mathbb{R}$ be such that $u_i(x) = c$ for all $x \in X_N$. Then $(\phi_{\underline{V}})_i(\{i\}, g) = \phi_i(\{i\}, \underline{v}_g) = \underline{v}_g(\{i\}) = c$, where the first equality simply uses the definition of the Shapley valuation function, the second one follows from efficiency of the Shapley value, and the third one follows from individual objectivity of the lower value method.

To check that $\phi_{\underline{V}}$ satisfies *monotonicity*, let $g = (N, \{X_i\}_{i\in N}, \{u_i\}_{i\in N}) \in G^N$, $g' = (N, \{X_i\}_{i\in N}, \{u'_i\}_{i\in N}) \in G^N$, and $i \in N$ be such that $u_i(x) \geq u'_i(x)$ for all $x \in X_N$. Then $(\phi_{\underline{V}})_i(\{i\}, g) = \phi_i(\{i\}, \underline{v}_g) = \underline{v}_g(\{i\}) \geq \underline{v}_{g'}(\{i\}) = \phi_i(\{i\}, \underline{v}_{g'}) = (\phi_{\underline{V}})_i(\{i\}, g')$, where the inequality follows from monotonicity of the lower value method.

To check that $\phi_{\underline{V}}$ satisfies *irrelevance of strongly dominated actions*, we use that the equivalent of this property for the lower value method implies that $\underline{v}_g(\{i\}) = \underline{v}_{g'}(\{i\})$ for any two strategic games $g, g' \in G^N$ where the game g' is the game obtained from g by deleting a strongly dominated action $x_i \in X_i$ of player $i \in N$. *Irrelevance of weakly dominated threats* of $\phi_{\underline{V}}$ follows in a similar manner from the equivalent property for the lower value method.

To check that $\phi_{\underline{V}}$ satisfies *merge invariance*, let $g \in G^N$ and $S \subset N$, $S \neq \emptyset$. Then $\sum_{i\in S}(\phi_{\underline{V}})_i(S, g) = \sum_{i\in S} \phi_i(S, \underline{v}_g) = \underline{v}_g(S) = \underline{v}_{g(S)}(i(S)) = \phi_{i(S)}(\{i(S)\}, \underline{v}_{g(S)}) = (\phi_{\underline{V}})_{i(S)}(\{i(S)\}, g(S))$, where the second and fourth equalities use efficiency of the Shapley value and the third one follows from merge invariance of the lower value method.

To check that $\phi_{\underline{V}}$ satisfies *balanced contributions*, let $g \in G^N$, $S \subset N$, and $i, j \in S$. Then $(\phi_{\underline{V}})_i(S, g) - (\phi_{\underline{V}})_i(S\backslash\{j\}, g) = \phi_i(S, \underline{v}_g) - \phi_i(S\backslash\{j\}, \underline{v}_g) = \phi_j(S, \underline{v}_g) - \phi_j(S\backslash\{i\}, \underline{v}_g) = (\phi_{\underline{V}})_j(S, g) - (\phi_{\underline{V}})_j(S\backslash\{i\}, g)$, where the second equality follows from balanced contributions of the Shapley value for the coalitional game $(S, \underline{v}_g)$ and players $i, j \in S$.

Uniqueness. Let φ be a valuation function that satisfies the six properties in the statement of the theorem. We start by proving that $\sum_{i\in S} \varphi_i(S, g) = \underline{v}_g(S)$ for all $S \subset N$. To do so, we consider the function Φ that associates with every strategic game $g \in G^N$ a coalitional game $(N, \Phi(g))$ defined by $\Phi(g)(S) = \sum_{i\in S} \varphi_i(S, g)$ for all $S \subset N$. Then *individual objectivity*, *monotonicity*, *irrelevance of strongly dominated actions*, *irrelevance of weakly dom-*

inated threats and *merge invariance* of φ imply that Φ satisfies the appropriate equivalents of these five properties for methods of associating a coalitional game with each strategic game. Carpente et al. (2004a) proved that the lower value method, which associates with each strategic game $g \in G^N$ the coalitional game $(N, \underline{v}_g)$, is the unique method of associating a coalitional game with each strategic game that satisfies these five properties. Hence, we know $\Phi(g)(S) = \underline{v}_g(S)$ for all $g \in G^N$and for all $S \subset N$, which of course proves that $\sum_{i \in S} \varphi_i(S, g) = \underline{v}_g(S)$ for all $g \in G^N$ and for all $S \subset N$.

Suppose that φ^1 and φ^2 are two valuation functions satisfying the six properties in the statement of the theorem. Let $g \in G^N$. We prove that $\varphi^1(S, g) = \varphi^2(S, g)$ for all $S \subset N$ by induction to the size of S.

If $S = \{i\}$, then $\varphi^1_i(S, g) = \sum_{j \in S} \varphi^1_j(S, g) = \underline{v}_g(S) = \sum_{j \in S} \varphi^2_j(S, g) = \varphi^2_i(S, g)$. Now, suppose we have proved that $\varphi^1(S, g) = \varphi^2(S, g)$ for all $S \subset N$ with $|S| \leq t$, where $1 \leq t < n$. Let $S \subset N$ with $|S| = t + 1$. Using balanced contributions of φ^l, $l = 1, 2$, we derive that for all $i, j \in S$, $i \neq j$, $\varphi^1_j(S, g) - \varphi^1_i(S, g) = \varphi^1_j(S \backslash \{i\}, g) - \varphi^1_i(S \backslash \{j\}, g) = \varphi^2_j(S \backslash \{i\}, g) - \varphi^2_i(S \backslash \{j\}, g) = \varphi^2_j(S, g) - \varphi^2_i(S, g)$, where the second equality uses the induction hypothesis. Together with $\underline{v}_g(S) = \sum_{i \in S} \varphi^1_i(S, g) = \sum_{i \in S} \varphi^2_i(S, g)$, this implies that $\varphi^1_i(S, g) = \varphi^2_i(S, g)$ for all $i \in S$. This finishes the proof of the theorem. $\square$

3.3 Concluding remarks

In Section 2 we defined and axiomatically characterized the Shapley valuation function, which associates with each strategic game and cooperating coalition of players a payoff vector that provides a valuation for each of the members of the coalition. The Shapley valuation function is defined using the lower-value based method to associate a coalitional game with each strategic game that was introduced in Carpente et al. (2004a). As argued in that paper, the use of the lower value is appropriate in settings in which mixing coordinated actions is not possible or reasonable. In situations where the use of such strategies is possible, however, the value based method introduced in Von Neumann and Morgenstern (1944) can be used to construct a valuation function ϕ_V by considering Shapley values of coalitional games (S, v_g) that are defined using this value based method. Given a strategic game $g \in G^N$, this method defines the worth $v_g(S)$ of a coalition of players $S \subset N$ as the value of the mixed extension of the two-person zero-sum game g_S between coalitions S and $N \backslash S$. The axiomatic characterization of this method in Carpente et al. (2004a) can be used to find an axiomatic characterization of the valuation function ϕ_V. In addition to some of the properties that we have already encountered, this axiomatization uses the following two properties.

Irrelevance of dominated actions. In a game $g \in G^N$, an action $x_i \in X_i$ of player $i \in N$ is *dominated* if there exists a convex combination y of the other actions of player i, with the property that $u_i(y, x_{N\setminus i}) \geq u_i(x_i, x_{N\setminus i})$ for all $x_{N\setminus i} \in X_{N\setminus i}$.[16] For all $g \in G^N$ and $i \in N$, if action $x_i \in X_i$ is dominated, then $\varphi_i(\{i\}, g) = \varphi_i(\{i\}, g')$, where $g' \in G^N$ is the game obtained from g by deleting action x_i.

Irrelevance of dominated threats. In a game $g \in G^N$, an action $x_j \in X_j$ of a player $j \in N$ is a *dominated threat* to player $i \in N$, $i \neq j$, if there exists a convex combination y of the other actions of player j with the property that $u_i(y, x_{N\setminus j}) \leq u_i(x_j, x_{N\setminus j})$ for all $x_{N\setminus j} \in X_{N\setminus j}$. For all $g \in G^N$ and players $i, j \in N$, $i \neq j$, if action $x_j \in X_j$ is a dominated threat to player i, then $\varphi_i(\{i\}, g) = \varphi_i(\{i\}, g')$, where $g' \in G^N$ is the game obtained from g by deleting action x_j.

Irrelevance of dominated actions states that the valuation for a player of forming a singleton coalition does not change if in the strategic game he loses the ability to use an action that is weakly worse for him than a mix of his other actions, no matter what actions the other players choose. Note that irrelevance of dominated actions is a stronger property than irrelevance of strongly dominated actions. Irrelevance of dominated threats has a similar interpretation to irrelevance of weakly dominated threats but is weaker than that property as every threat that is dominated is also weakly dominated.

Theorem 9 *The valuation function ϕ_V is the unique valuation function satisfying individual objectivity, monotonicity, irrelevance of dominated actions, irrelevance of dominated threats, merge invariance, and balanced contributions.*

Theorem 9 can be proved in a manner similar to the proof of Theorem 8 by using Carpente et al.'s axiomatic characterization of Von Neumann and Morgenstern's value based method.

We conclude this note by pointing out that in the axiomatic characterizations in Theorems 8 and 9 monotonicity can be replaced by two properties called elimination of own actions and elimination of others' actions. These two properties are inspired by the properties irrelevance of (strongly) dominated actions and irrelevance of (weakly) dominated threats and address the elimination of arbitrary actions, dominated or not. Elimination of own actions states that the elimination of an (arbitrary) action of player i does not

[16] $u_i(y, x_{N\setminus i}) := \sum_{\hat{x}_i \in X_i} y(\hat{x}_i) u_i(\hat{x}_i, x_{N\setminus i})$, where $y = \sum_{\hat{x}_i \in X_i} y(\hat{x}_i)\hat{x}_i$. Note that $y(x_i) = 0$, $y(\hat{x}_i) \geq 0$, for all $\hat{x}_i \in X_i$, and $\sum_{\hat{x}_i \in X_i} y(\hat{x}_i) = 1$. Observe that y is simply a mixed strategy of player i.

increase this player's valuation of forming a singleton coalition, and elimination of others' actions states that the valuation for a player i of forming a singleton coalition does not decrease when an (arbitrary) action of another player j is eliminated. Hence, these properties highlight a form of monotonicity with respect to the elimination of actions. It is shown in Carpente et al. (2004a) that these properties can replace monotonicity in the axiomatic characterizations of both the value based and the lower-value based methods to associate a coalitional game with each strategic game and this result can be adapted to the valuation function setting of this note. These properties have also been used in Norde and Voorneveld (2004) to characterize the value of the mixed extension of a matrix game.

4 How to Share Railways Infrastructure Costs?

This section displays the paper Fragnelli et al. (2000).

4.1 Introduction

In this paper we deal with a cost allocation problem arising from the reorganization of the railway sector in Europe, after the application of the EEC directive 440/91 and the EC directives 18/95 and 19/95, which involve the separation between infrastructure management and transport operations. In this situation two main economic problems arise. One is to allocate the track capacity among the various operators. This issue has been treated, for instance, in Nilsson (1996), Brewer and Plott (1996) and Bassanini and Nastasi (1997). The second problem is to determine the access tariff that the railway transport operators must pay to the firm in charge of the infrastructure management for a particular journey. This tariff should take into account several aspects such as the a priori profitability and social utility of the journey, congestion issues, the number of passengers and/or goods transported, the services required by the operator, infrastructure costs, etc. The tariff is conceived in an additive way, i.e. as the sum of various tariffs corresponding to the various aspects to be considered.

The main motivation of this paper is a practical one. We were approached by *Ferrovie dello Stato*[17] (the italian national railway company) to study how the infrastructure costs should be allocated to the operators through a fair *infrastructure access tariff* (i.e. we were asked to define one part of

[17]Ferrovie dello Stato is the coordinator of the EuROPE-TRIP research project, sponsored by the European Community. Formally, our research has been requested and financed by the European Community.

the additive access tariff: that corresponding to the infrastructure costs). In this work we treat this problem from a game theoretical point of view, making use of the Shapley value. The Shapley value is a very important solution concept for TU-games, which has excellent properties and has been applied successfully in cost allocation problems (see Shapley (1953), Tijs and Driessen (1986), Young (1994) and Moulin and Shenker (2001)). Moreover, in our particular problem, it is especially appropriate because of the following two reasons.

1. It is well-known that the Shapley value is an additive solution. This feature fits well with the "additive nature" of the access tariff, as commented above.

2. In this paper we will show that the infrastructure access tariff based on the Shapley value can be computed very easily (using, once more, the additivity of the Shapley value). In a practical environment this is certainly an important property. Take into account that a very big amount of fees will have to be computed by the infrastructure manager every new season, so computational issues become highly relevant.

Let us now describe informally the problem we are facing. Consider a railway path (for instance, Milano-Roma), that is used by different types of trains belonging to several operators, and consider the problem of dividing among these trains the infrastructure costs. Clearly it is a problem of joint cost allocation. To settle the question, one can see the infrastructure as consisting of some kinds of "facilities" (track, signalling system, stations, etc.). Different groups of trains need these facilities at different levels: for example, fast trains need a more sophisticated track and signalling system, compared to local trains, for which instead station services are more important (particularly in small stations).

So, a straightforward approach can be that of viewing the infrastructure as a "sum" of different facilities, each of them required by the trains at a different level of cost.

Furthermore, infrastructure costs can be seen as the sum of "building" costs and "maintenance" costs (for a better understanding of the distinction between these two types of costs, we refer to the example in section 4). If we consider only building costs, especially in the case of a single facility, we are facing a problem similar to the so-called "airport game" (see, for instance, Littlechild and Owen (1973) and Dubey (1982)). For what concerns maintenance costs, it seems to be a reasonable first order approximation to assume that they are proportional both to the building costs and to the number of trains that use the facility.

Similar considerations extend to related problems: for example the costs for a bridge, to be used by small and big cars. There are building costs, that are different in the case of a bridge for small or big cars, and maintenance costs, that can be assumed to be proportional to the number of vehicles using the bridge, and to the kind of bridge needed.

In this paper we analyze these infrastructure cost games (sums for various facilities of a building cost game and a maintenance cost game) from the point of view of the Shapley value. In section 2 we introduce and briefly study the infrastructure cost games. In section 3 we provide a simple expression of the Shapley value for this class of games. In section 4 we elaborate an example where we apply the models and results presented in sections 2 and 3.

4.2 Infrastructure Cost Games

For simplicity, we concentrate first on infrastructure cost games when we are dealing with the building and maintenance of one facility. To begin with, we recall the definition of an "airport game".

Definition 1 *Consider k groups of players $g_1, \dots, g_k$ with $n_1, \dots, n_k$ players respectively and k non-negative numbers $b_1, \dots, b_k$. The* airport game *corresponding to $g_1, \dots, g_k$ and $b_1, \dots, b_k$ is the cooperative (cost) game $< N, c >$ with $N = \cup_{i=1}^{k} g_i$ and cost function c defined by*

$$c(S) = b_1 + \dots + b_{j(S)}$$

for every $S \subset N$, where $j(S) = \max\{j \,:\, S \cap g_j \neq \emptyset\}$.

Airport games are cost games for the building of one facility (for instance, a landing strip) where the wishes of the coalitions are linearly ordered. Coalitions desiring a more sophisticated facility (a larger landing strip) have to pay at least as much as coalitions desiring a less sophisticated facility (a smaller landing strip). Every b_i represents the extra building cost that should be made in order that a facility that can be used by players in groups $g_1, ...g_{i-1}$ can also be used by the more sophisticated players in group g_i. Airport games are known to be concave. Consequently, the Shapley value of such a game provides a core element. Sometimes we will refer to an airport game as a *building cost game.* Denote by $B(g_1, ..., g_k)$ the set of all building cost games with groups of players $g_1, ..., g_k$.

In airport games costs for the building of one facility are modeled. Now we consider the maintenance costs of this facility, which lead to the class of "maintenance cost games". Basic assumptions are that maintenance costs are increasing with the degree of sophistication of the facility and that maintenance costs are proportional to the number of users.

Definition 2 *Consider k groups of players $g_1, \dots, g_k$ with $n_1, \dots, n_k$ players respectively and $\frac{k(k+1)}{2}$ non-negative numbers $\{\alpha_{ij}\}_{i,j\in\{1,\dots,k\},j\geq i}$. The* maintenance cost game *corresponding to $g_1, \dots, g_k$ and $\{\alpha_{ij}\}_{i,j\in\{1,\dots,k\},j\geq i}$ is the cooperative (cost) game $< N, c >$ with $N = \cup_{i=1}^{k} g_i$ and cost function c defined by*

$$c(S) = \sum_{i=1}^{j(S)} |S \cap g_i| A_{ij(S)} \tag{12}$$

for every $S \subset N$, where $A_{ij} = \alpha_{ii} + ... + \alpha_{ij}$ for all $i, j \in \{1, ..., k\}$ with $j \geq i$.

The interpretation of the numbers α_{ij} and A_{ij} is the following. Suppose that one player in g_i has used the facility. In order to restore the facility up to level i (the level of sophistication desired by this player) the maintenance costs are $A_{ii} = \alpha_{ii}$. If, however, the facility is going to be restored up to level $i + 1$, then extra maintenance costs α_{ii+1} will be made. So, in order to restore the facility up to level j (with $j \geq i$) the maintenance costs are $A_{ij} = \alpha_{ii} + \alpha_{ii+1} + ... + \alpha_{ij}$. Hence, $c(S)$ represents the maintenance costs corresponding to the facility up to the level $j(S)$ (so that all the players in S can use it), after all players in S have used it. Observe that, for every $i \leq j$, the more sophisticated the facility is (the larger j is), the higher the maintenance costs produced by a player in g_i are. In Section 4 we provide an example which illustrates the above definition of a maintenance cost game.

We denote by $M(g_1, ..., g_k)$ the set of all maintenance cost games with groups of players $g_1, ..., g_k$. Obviously, to characterize a game $< N, c > \in M(g_1, ..., g_k)$ it is equivalent to give either the set of parameters $\{\alpha_{ij}\}_{i,j\in\{1,\dots,k\},j\geq i}$ or the set of parameters $\{A_{ij}\}_{i,j\in\{1,\dots,k\},j\geq i}$.

The following decomposition of a maintenance cost game $< N, c > \in M(g_1, ..., g_k)$ will be useful. For every $S \subset N$,

$$\begin{aligned} c(S) &= \sum_{i=1}^{j(S)} |S \cap g_i| A_{ij(S)} = \\ &= \sum_{i=1}^{j(S)} |S \cap g_i| (\alpha_{ii} + ... + \alpha_{ij(S)}) = \sum_{i=1}^{k} \sum_{j=i}^{k} \alpha_{ij} c^{ij}(S), \end{aligned}$$

where

$$c^{ij}(S) = \begin{cases} |S \cap g_i| & \text{if } j \leq j(S) \\ 0 & \text{if } j > j(S) \end{cases}$$

for all $i, j \in \{1, ..., k\}$ with $j \geq i$.

We know that building cost games are concave. The following result shows that this is not true for maintenance cost games. Moreover, it shows that maintenance cost games are essentially neither concave nor balanced.

Theorem 10 *Let* $< N, c >$ *be the maintenance cost game corresponding to* $g_1, \dots, g_k$ *and* $\{\alpha_{ij}\}_{i,j \in \{1,\dots,k\}, j \geq i}$. *Then the following four statements are equivalent:*

(1) $< N, c >$ *is concave*

(2) $< N, c >$ *is balanced*

(3) $\sum_{i \in N} c(i) \geq c(N)$

(4) $\alpha_{ij} = 0$ *for every* $j > i$.

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear. For the implication (3) $\Rightarrow$ (4) suppose that (3) holds. Then

$$\sum_{i=1}^{k} \sum_{j=i}^{k} \alpha_{ij} n_i = c(N) \leq \sum_{i \in N} c(i) = \sum_{i=1}^{k} \alpha_{ii} n_i$$

which implies that $\alpha_{ij} = 0$ for every $j > i$. For the implication (4) $\Rightarrow$ (1) suppose that (4) holds. Note that c^{ii} defined as above is an additive characteristic function for every $i \in \{1, ..., k\}$. Hence, c can be expressed as a non-negative combination of additive characteristic functions. Thus, $< N, c >$ is concave.$\square$

Now we can introduce the class of infrastructure cost games.

Definition 3 *A* one facility infrastructure cost game *with groups of players* $g_1, \dots, g_k$ *is the cooperative (cost) game* $< N, c >$ *with* $N = \cup_{i=1}^{k} g_i$ *and cost function* $c = c_b + c_m$ *such that* $< N, c_b > \in B(g_1, \dots, g_k)$ *and* $< N, c_m > \in M(g_1, \dots, g_k)$. *An* infrastructure cost game *with groups of players* $g_1, \dots, g_k$ *is the cooperative (cost) game* $< N, c >$ *with* $N = \cup_{i=1}^{k} g_i$ *and cost function* $c = c^1 + ... + c^l$ *such that, for every* $r \in \{1, ..., l\}$, $< N, c^r >$ *is a one facility infrastructure cost game with groups of players* $g_{\pi^r(1)}, \dots, g_{\pi^r(k)}$, *where* π^r *is a permutation of* $\{1, ..., k\}$.

From the definition above we see that a one facility infrastructure cost game is the sum of a building cost game plus a maintenance cost game with the same groups of players ordered in the same way. An infrastructure cost game is the sum of a finite set of one facility infrastructure cost games with the same groups of players but, perhaps, ordered in a different way. This means that group i can require a higher level of sophistication than group j for facility r, whereas group j requires a higher level of sophistication than group i for facility s. Because of this reason, it is not true that every infrastructure cost game is a one facility infrastructure cost game. An interesting consequence of Theorem 10, the concavity of airport games and the additivity of the Shapley value is the following. Since an infrastructure cost game is the sum of building cost games and maintenance cost games, then its Shapley value is the sum of allocations, which are moreover core allocations for those such games having a non-empty core.

The class of infrastructure cost games is the model we designed to solve the practical problem which motivates this work: how to allocate in a fair way the infrastructure costs to the users of a certain railway path. A game in our class describes the infrastructure costs imputable to every possible collection of users. Now we have to choose an allocation rule which allocates the total cost to the users. As we announced in the introduction of this paper, we chose the Shapley value because of the two reasons already discussed. The access tariff we propose for a certain path in a certain time period is simply the Shapley value of the infrastructure cost game corresponding to this path and time period.

Note that an infrastructure cost game is the sum of a finite collection of airport games and maintenance cost games. It is well known that there is a simple expression of the Shapley value for airport games (see Littlechild and Owen, 1973). In the next section we obtain a simple expression of the Shapley value for maintenance cost games. Hence, since the Shapley value is additive, we can compute easily the Shapley value of an infrastructure cost game even when the number of players is large, which will be the case in practice: take into account that the players here are the trains using the path in a certain period. Thus, we are proposing an access tariff system which is at the same time reasonable (based on a general theory of fairness) and computable in an efficient way.

4.3 The Shapley Value of a Maintenance Cost Game

This section contains a theorem providing a simple expression of the Shapley value of a maintenance cost game.

Theorem 11 *Let $< N, c >$ be the maintenance cost game corresponding to the groups $g_1, ..., g_k$ (with $n_1, ..., n_k$ players respectively) and to*

$\{\alpha_{lm}\}_{l,m\in\{1,...,k\},m\geq l}$. *Then, for every* $i \in N$*,*

$$\begin{aligned}\varphi_i(c) &= \alpha_{j(i)j(i)} + \sum_{m=j(i)+1}^{k} \alpha_{j(i)m}\frac{n_m+...+n_k}{n_m+...+n_k+1} \\ &\quad + \sum_{m=2}^{j(i)}\sum_{l=1}^{m-1}\alpha_{lm}\frac{n_l}{(n_m+...+n_k)(n_m+...+n_k+1)},\end{aligned}$$

where $\varphi_i(c)$ *denotes the* i*-th component of the Shapley value of the game* $< N, c >$ *and* $j(i)$ *is the group to which* i *belongs (i.e.* $i \in g_{j(i)}$*).*

Proof. Recall that $c = \sum_{l=1}^{k}\sum_{m=l}^{k}\alpha_{lm}c^{lm}$ where

$$c^{lm}(S) = \begin{cases} |S\cap g_l| & \text{if } m \leq j(S) \\ 0 & \text{if } m > j(S). \end{cases}$$

Then, since the Shapley value is linear,

$$\varphi_i(c) = \sum_{l=1}^{k}\sum_{m=l}^{k}\alpha_{lm}\varphi_i(c^{lm})$$

for all $i \in N$. It is clear that, for every $l \in \{1,...,k\}$, c^{ll} is an additive characteristic function and that

$$\varphi_i(c^{ll}) = \begin{cases} 1 & \text{if } i \in g_l \\ 0 & \text{in any other case.} \end{cases} \tag{13}$$

Suppose now that $l < m$. In this case only players in $g_l \cup (\cup_{r=m}^{k} g_r)$ are not null players. By symmetry we may put $\varphi_i(c^{lm}) = a$ for every $i \in g_l$ and $\varphi_i(c^{lm}) = b$ for every $i \in \cup_{r=m}^{k} g_r$. In order to compute a take $i \in g_l$ and note that for every $S \subset N\backslash\{i\}$ we have

$$c^{lm}(S\cup\{i\}) - c^{lm}(S) = \begin{cases} 0 & \text{if } j(S) < m \\ 1 & \text{else.} \end{cases}$$

So, if the players of N are ordered at random, a is the probability that player i has at least one predecessor in $\cup_{r=m}^{k} g_r$. Equivalently, if the players of N are ordered at random, a is the probability that player i is not the first player of the players in $\{i\} \cup (\cup_{r=m}^{k} g_r)$. Consequently,

$$a = \frac{n_m+...+n_k}{n_m+...+n_k+1}. \tag{14}$$

Thus, by symmetry and efficiency,

$$b = \frac{n_l - n_l a}{n_m + ... + n_k} = \frac{n_l}{(n_m + ... + n_k)(n_m + ... + n_k + 1)}. \qquad (15)$$

Now, in view of (13), (14) and (15) the proof is concluded.□

As we mentioned above, the Shapley value of the corresponding infrastructure game is our proposal to share railways infrastructure costs. It is clear that, using Theorem 11 and the formula for the Shapley value of an airport game, the computations that should be made are not difficult; however, the potentially very large amount of data that will have to be handled to compute a very large collection of fees makes necessary to have a good computer program to do it. For this purpose, we have prepared a software package that will be delivered to *Ferrovie dello Stato*, the coordinator of EuROPE-TRIP. The name of this package is ShRInC (Sharing Railways Infrastructure Costs). It has been created with the collaboration of Luisa Carpente and Claudia Viale.

Obviously, from a game theoretical point of view, there are many interesting questions concerning infrastructure cost games that have not been treated here. The main motivation of this paper is to report the practical solution we proposed for the real problem of allocating railways infrastructure costs. In Norde et al (2002), we study other game theoretical properties of infrastructure cost games.

4.4 An Example

In this section we illustrate our solution with an example. We shall elaborate it on data taken from Baumgartner (1997). The aim of that paper is to provide "order of magnitude" of costs concerning the railway system: we shall exploit it to analyze a rough but realistic example. In practical models, making a realistic example uses to be an enlightening exercise. Here, for instance, the example we are proposing shows that our building or maintenance cost games do not necessarily correspond to real building or maintenance costs. Actually, the costs for one facility can be decomposed into:

- a fixed part (in the sense that it does not depend on the number of players), that corresponds to the building cost game associated with this facility, and
- a variable part (in the sense that it is proportional to the number of players), that corresponds to the maintenance cost game part.

For simplicity, we shall concentrate on a single element (the track), even if Baumgartner (1997) provides data also for other elements (line, catenary, signaling and security system, etc.), that can be analyzed in a similar fashion. If we consider one kilometer of track, from Baumgartner (1997) we get two kind of costs[18], that depend on the type of train (slow/fast) and on the number of trains running. More precisely, we have both *renewal costs* and *repairing costs*. According to this division of costs we will divide the track into two facilities: "track renewal" and "track repairing".

Renewal costs can be approximated by the following formula:

$$RWC = 0.001125X + 11,250$$

where RWC are the renewal costs per kilometer and per year (expressed in swiss francs) and X measures the "number" of trains, expressed in yearly TGCK (TGCK means Tons Gross and Complete per Kilometer).

So, if we assume for ease of exposition that all of the trains running are of the same weight, the facility "track renewal" has a fixed component (to be included in our building costs), and a part which is proportional to the number of trains running (to be included in our maintenance costs). If the assumption of equal weight cannot be sustained, our model still fits: simply divide trains into groups of similar weight. In such a case each group will have different unitary maintenance costs.

Similarly, for the facility "track repairing", costs can be given by analogous formulas:

$$RPC_s = 0.001X + 10,000$$

$$RPC_f = 0.00125X + 12,500.$$

RPC_s denotes the repairing costs (in swiss francs) per kilometer and per year of a track prepared only for slow trains, whereas RPC_f denotes the repairing costs (in swiss francs) per kilometer and per year of a track prepared for all trains. X denotes the same as before.

So, consider one kilometer of line, which will be used this year by a total weight of 10^7 TGCK (corresponding to 20,000 trains, assuming a weight per train of approximately 500 tons). Assume that 5,000 trains are fast and that the remaining are slow. The infrastructure cost game that can be used to allocate the costs is $< N, c >$ given by:

- $N = g_1 \cup g_2$, g_1 being the set of slow trains ($n_1 =$ 15,000) and g_2 being the set of fast trains ($n_2 =$ 5,000).

[18] We assumed the weight of 50Kg for a meter of rail and made a linear approximation of the costs given in table 2 of Baumgartner (1997).

- $c = c^1 + c^2$, c^1 and c^2 being one facility infrastructure cost games both having the same groups of players and ordered in the same way: g_1, g_2.

Now, c^1 and c^2 are characterized by the following parameters.

- $c^1 : b_1^1 = 11,250$; $b_2^1 = 0$; $\alpha_{11}^1 = 0.5625$; $\alpha_{12}^1 = 0$; $\alpha_{22}^1 = 0.5625$.
- $c^2 : b_1^2 = 10,000$; $b_2^2 = 2,500$; $\alpha_{11}^2 = 0.5$; $\alpha_{12}^2 = 0.125$; $\alpha_{22}^2 = 0.625$.

Hence, making use of Theorem 11 and the formula for the Shapley value of an airport game, it is easy to check that, if $\varphi_s(c)$ and $\varphi_f(c)$ denote the Shapley value of a slow and a fast train respectively, then:

- $\varphi_s(c) = \frac{b_1^1}{n_1+n_2} + \alpha_{11}^1 + \frac{b_1^2}{n_1+n_2} + \alpha_{11}^2 + \alpha_{12}^2 \frac{n_2}{n_2+1} = 2.25$
- $\varphi_f(c) = \frac{b_1^1}{n_1+n_2} + \alpha_{22}^1 + \frac{b_1^2}{n_1+n_2} + \frac{b_2^2}{n_2} + \alpha_{22}^2 + \alpha_{12}^2 \frac{n_1}{n_2(n_2+1)} = 2.75$.

These are the fees, in swiss francs, that every slow and fast train (respectively) should pay per kilometer of track used, according to our solution. Clearly, in front of a specific allocation problem regarding a specific line, with specific transport operators and trains, appropriate data should be collected. Here, we only presented an illustrative approximation to a real example.

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