COMPUTATIONS ON THREE-DIMENSIONAL SUBMANIFOLDS IN $\mathbb{R}^{3+n}$

ANGEL MONTESINOS-AMILIBIA

1. Preliminary concepts

Let A and B be real vector spaces of finite dimensions a and b. We denote by $S(A)$ the subspace of $A \otimes A$ of symmetric tensors, that is $S(A)$ is generated by the elements as $v \otimes v,\ v \in A$. Note that if w is another element of A, then $(v+w) \otimes (v+w) - v \otimes v - w \otimes w = v \otimes w + w \otimes v$. In the following, suppose that A and B are Euclidean vector spaces with inner product denoted by a dot. If $(u_1, \ldots, u_a)$ is an orthonormal basis of A and $\beta, \gamma \in A^*$, then we can define the inner product of β and γ by

$$\beta \cdot \gamma = \sum_{i=1}^{a} \beta(u_i)\gamma(u_i).$$

In fact, as it is easily proved, the result does not depend on the chosen orthonormal basis. This may be generalized for defining the inner product of elements of $\bigotimes A^*$. In fact, if for example $g, h \in A^* \otimes A^*$ is a bilinear form on A, we define $g \cdot h = \sum_{i,j=1}^{a} g(u_i, u_j)h(u_i, u_j)$, and as before this does not depend on the orthonormal basis. Let $(u^1, \ldots, u^a)$ be the dual basis of $(u_1, \ldots, u_a)$. Then the elements $u^i \otimes u^j,\ i, j = 1, \ldots, a$, are an orthonormal basis of $A^* \otimes A^*$. In fact,

$$(u^i \otimes u^j) \cdot (u^k \otimes u^l) = \sum_{p,q=1}^{a} (u^i \otimes u^j)(u_p, u_q)(u^k \otimes u^l)(u_p, u_q) = \delta_{ki}\delta_{lj},$$

as required. In the same manner we may define the inner product in, say $\bigotimes_s^r A$ by declaring that the elements

$$u^{i_1} \otimes \cdots \otimes u^{i_r} \otimes u_{j_1} \otimes \cdots \otimes u_{j_s}, \quad i_1, \ldots, i_r, j_1, \ldots, j_s = 1, \ldots, a$$

are an orthonormal basis of $\bigotimes_s^r A$. With that definition it is easy to prove that if for instance we have the tensors $m = \beta \otimes v,\ n = \gamma \otimes w \in A^* \otimes A$, then $m \cdot n = (\beta \cdot \gamma)(v \cdot w)$.

Let us show that the elements $s_{ii} = u_i \otimes u_i,\ i = 1, \ldots, a$, and the elements $s_{ij} = \frac{1}{\sqrt{2}}(u_i \otimes u_j + u_j \otimes u_i)$, when $1 \le i < j \le a$, are an orthonormal basis of $S(T)$.

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It is clear that they are a basis. Now,

$$\begin{aligned}
s_{ii}\cdot s_{jj} &= (u_i\cdot u_j)^2 = \delta_{ij}^2 = \delta_{ij},\\
s_{ii}\cdot s_{jk} &= \frac{1}{\sqrt{2}}((u_i\cdot u_j)(u_i\cdot u_k)+(u_i\cdot u_k)(u_i\cdot u_j)) = \sqrt{2}\delta_{ij}\delta_{ik}=0,\\
s_{jk}\cdot s_{pq} &= \frac{1}{2}(2\delta_{jp}\delta_{kq}+2\delta_{jq}\delta_{kp}) = \delta_{jp}\delta_{kq},
\end{aligned}$$

because the second term is zero since $j<k$ and $p<q$. Hence, the affirmation is true. If, as before, $g,h\in A^*\otimes A^*$ are symmetric we will have

$$\begin{aligned}
g\cdot h = \sum_{i,j=1}^{a} g(u_i,u_j)h(u_i,u_j) &= \sum_{i=1}^{a} g(u_i,u_i)h(u_i,u_i)\\
&+\sum_{i<j}^{a}\big(g(u_i,u_j)h(u_i,u_j)+g(u_j,u_i)h(u_j,u_i)\big)\\
&=\sum_{i=1}^{a} g(s_{ii})h(s_{ii})+\sum_{i<j}^{a} g(s_{ij})h(s_{ij}) = \sum_{1\le i\le j\le a}^{a} g(s_{ij})h(s_{ij}).
\end{aligned}$$

We introduce another notation. If A is an Euclidean vector space we put $S^A=\{u\in A\ :\ u\cdot u=1\}$ to denote the sphere in A.

Let now $h: A\to B$ be a homomorphism. Then, we can define the *pull-back* of the inner product of B as the symmetric bilinear form $h_2: A\times A\to\mathbb{R}$ given by $h_2(u,v)=h(u)\cdot h(v)$. It is well known that there is an orthonormal basis $(u_1,\dots,u_a)$ of A and unique real numbers $\mu_1\ge\cdots\ge\mu_a\ge 0$ such that $h_2(u_i,u_j)=0$ if $i\ne j$ and $h_2(u_i,u_i)=\mu_i$. Let $c\le \mathrm{Min}(a,b)$ be the number of non zero elements μ_i. If we call $\lambda_i=\sqrt{\mu_i}$ for $1\le i\le a$, and $w_i=h(u_i)/\lambda_i$ for $1\le i\le c$, we will have $w_i\cdot w_i=h_2(u_i,u_i)=1$, and $w_i\cdot w_j=0$, $i\ne j$. Also, if $\mu_i=0$, then $h_2(u_i,u_i)=h(u_i)\cdot h(u_i)=0$, whence $h(u_i)=0$. Therefore the kernel of h is generated by the vectors $u_1,\dots,u_c$. We will assume that we have completed the vectors w_i to form an orthonormal basis of B and that (w^i) denotes the dual basis. It is clear that $h(u_i)=\lambda_i w_i$, for $1\le i\le a$. Using the Einstein convention, we will have

$$h = w^i(h(u_j))w_i\otimes u^j = (w_i\cdot\lambda_j w_j)w_i\otimes u^j = \sum_{j=1}^{c}\lambda_j w_j\otimes u^j.$$

Since h is lineal, $h(S^A)$ must be a compact quadric, that is an ellipsoid that may be degenerate. In other words, it consists in the intersection of a solid ellipsoid centered at the origin with a subspace of B. Its axes are determined by the vectors $v\in S^A$ such that the function $f(u)=h(u)\cdot h(u)$, $u\in A$, when restricted to S^A, is extremal at v. This means that the 1-form df_v annihilates the subspace orthogonal to v. So, df_v must be a multiple of dr_v^2, say $df_v=\mu dr_v^2$, where $r^2: A\to\mathbb{R}$ is given by $r^2(u)=u\cdot u$. If $t\mapsto\gamma(t)\in A$ is a smooth curve such that $\gamma(0)=v$ and we put $u=\gamma'(0)$, then

$$(df_v-dr_v^2)(u) = \big(f(\gamma,\gamma)-\mu r^2(\gamma,\gamma)\big)'(0) = 2(h(v)\cdot h(u)-\mu v\cdot u)=0$$

Suppose that $v=u_i$, $\mu=\mu_i$ and $u=u_j$. Then

$$h(v)\cdot h(u)-\lambda v\cdot u = h_2(u_i,u_j)-\mu_i\delta_{ij}=0.$$

Therefore, $w_i = h(u_i)$ determines an axis of the ellipsoid and $\sqrt{\mu_i}$ is the half-axis corresponding to it.

We may define the *adjoint* $h^* : B \to A$ of h by saying that $h^*(w)$ is the unique element of A such that $h^*(w) \cdot u = w \cdot h(u)$, for any u. We will have $h^* = u^j(h^*(w_i))u_j \otimes w^i$, where we use the Einstein convention, and where (w^i) is the basis dual to (w_i). Then

$$h^* = u_j \cdot h^*(w_i)u_j \otimes w^i = (h(u_j) \cdot w_i)u_j \otimes w^i = (\lambda_j w_j \cdot w_i)u_j \otimes w^i = \sum_{j=1}^{c} \lambda_j u_j \otimes w^j.$$

Hence, the half-axes of the corresponding ellipsoid in A are the same as those of the ellipsoid in B, and the principal directions corresponding to non-zero half-axes are images of each other.

Three-dimensional submanifolds: notation and inventory of invariants

In the following, α will denote the value of the second fundamental form of a three-dimensional submanifold S in $\mathbb{R}^{3+n}$ at some point p. The tangent space to S at that point will be denoted T and its orthogonal complement by N. Thus, $\dim N = n$, and $T \oplus N = \mathbb{R}^{3+n}$. The inner product will be denoted by a dot and the first fundamental form at p will be denoted by g, so that $g(X, Y) = X \cdot Y$. The map from the unit sphere in T to N defined by α will be denoted by η; thus $\eta(t) = \alpha(t, t)$ for $t \in T$ satisfying $t \cdot t = 1$. Some of our computations will be realized by using a chart $\mathbf{x} : U \subset \mathbb{R}^3 \to S$, and we will use the following notation:

$$\mathbf{x}_i = \partial_i \mathbf{x}, \quad \mathbf{x}_{ij} = \partial_j \partial_i \mathbf{x}, \qquad g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j, \quad i, j = 1, \ldots, 3.$$

Also, for any vector $z \in T$ let us put $z = \sum_{i=1}^{3} z_i \mathbf{x}_i$. We shall denote by (t_1, t_2, t_3) the orthonormal basis of T built as follows

$$t_1 = \frac{\mathbf{x}_1}{|\mathbf{x}_1|}, \quad t_2 = \frac{g_{11}\mathbf{x}_2 - g_{12}\mathbf{x}_1}{|g_{11}\mathbf{x}_2 - g_{12}\mathbf{x}_1|}, \quad t_3 := \mathbf{x}_3 - (t_1 \cdot \mathbf{x}_3)t_1 - (t_2 \cdot \mathbf{x}_3)t_2, \quad t_3 := \frac{t_3}{|t_3|}.$$

In this manner, we will have

$$\alpha(z, z) = \sum_{i,j=1}^{3} z_i z_j \alpha(\mathbf{x}_i, \mathbf{x}_j) = \sum_{i,j=1}^{3} z_i z_j (D_{\mathbf{x}_i} \mathbf{x}_j)^{\perp}$$

and

$$(D_{\mathbf{x}_i} \mathbf{x}_j)^{\perp} = \mathbf{x}_{ij} - \sum_{k=1}^{3} (\mathbf{x}_{ij} \cdot t_k) t_k.$$

If $A : V \to V^*$ is a symmetric bilinear form in any Euclidean n-dimensional vector space (V,g), we will call *principal directions* of A the non-vanishing vectors $v \in V$ such that $(A - \lambda g)(v) = 0$ and the corresponding values $\lambda \in \mathbb{R}$ will be called the *eigenvalues* of A; a unit vector v that defines a principal direction will be called an *eigenvector* of A. This may also be expressed equivalently as follows. Since $g : V \to V^*$ is an isomorphism, we can consider its inverse $g^{-1} : V^* \to V$, which is also a bilinear symmetric form on V^*. Then $\tilde{A} := g^{-1} \circ A \in \mathrm{End}(V)$, and the eigenvalues and eigenvectors of A are the eigenvalues and eigenvectors of $\tilde{A}$ in the usual sense. The trace or determinant of A will be defined as the trace and determinant of $\tilde{A}$. The characteristic polynomial of $\tilde{A}$ may be written as

$(-1)^n\big(\lambda^n - \operatorname{tr}(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} - \cdots + (-1)^{n-1}c_{n-1}(A)\lambda + (-1)^n \det(A)\big)$, and it is clear that the $c_i(A)$ are invariants of A.

The geometric interpretation of some formulas will be related to the interpretation of the second fundamental form. Let $u \in N$ be a unit vector. Then, we can orthogonally project S, in a neighborhood of p, to the 4-space $\mathbb{R}u \oplus T$. Thus we obtain a hypersurface in a Euclidean 4-space, whose second fundamental form at p is given by $u \cdot \alpha$. Thus, we will say that $u \cdot \alpha$ is the u-second fundamental form of S at p, or that $u \cdot \alpha(t,t)$, $t \in S^T$, is the u-normal curvature of S at p in the direction t, etc.

Now we describe some *concomitants* of α. By a concomitant we understand here some mathematical object obtained by means of α using only the properties of α and the Euclidean structure of T and N, including the use of orthonormal bases, provided that the obtained object does not depend on the choice of those bases. A concomitant that is a real number will be called an *invariant.* Let us put $b_i = \alpha(t_i,t_i)$ and $b_{ij} = \alpha(t_i,t_j)$ for $i,j = 1,\dots,3,\ i \neq j$. We have the following concomitants:

Mean curvature vector:

$$H = \frac{1}{3}\sum_{a=1}^{3} \alpha(t_a,t_a) = \frac{1}{3}(b_1 + b_2 + b_3).$$

Interpretation: If $u \in S^N$, $u \cdot H$ is the u-mean curvature of S at p.

Gauss curvature form: the symmetric trilinear form $K : N \times N \times N \to \mathbb{R}$ defined as the determinant of α, that is

$$\begin{aligned}K(u,u,u) &= \det\big(u\cdot\alpha(t_a,t_b)\big)\\ &= (u\cdot b_1)\big((u\cdot b_2)(u\cdot b_3) - (u\cdot b_{23})^2\big)\\ &\quad + (u\cdot b_{12})\big((u\cdot b_{31})(u\cdot b_{23}) - (u\cdot b_{12})(u\cdot b_3)\big)\\ &\quad + (u\cdot b_{31})\big((u\cdot b_{12})(u\cdot b_{23}) - (u\cdot b_{31})(u\cdot b_2).\end{aligned}$$

Interpretation: If $u \in S^N$, $K(u,u,u)$ is the u-Gauss curvature of S.

The ellipsoid: Let $S(T)$ be the subspace of symmetric elements of $T \otimes T$. As we know, the elements $(s_{ij}$ of $S(T)$ are an orthonormal basis.

Let $S^{S(T)} = \{s \in S(T)\ :\ s\cdot s = 1\}$. The *ellipsoid of curvature* is $\alpha(S^{S(T)}) \subset N$. It is an ellipsoid that could be degenerate. Its axes are the images of the extremal points of the real function in $S^{S(T)}$ given by $U \mapsto \alpha_2(U,U) = \alpha(U)\cdot\alpha(U)$, $U \in S^{S(T)}$. That function is the restriction to $S^{S(T)}$ of the quadratic form defined by the bilinear form α_2 in $S(T)$ given by $\alpha_2(U,V) = \alpha(U)\cdot\alpha(V)$. Then, the non-zero vector U defines a principal direction or an axis iff there is a real number λ such that $\alpha_2(U,V) = \lambda U\cdot V$, $\forall V \in S(T)$. The half-axis corresponding to that principal direction is $\sqrt{\lambda}$. The matrix of α_2 in the preceding orthonormal basis $s_{ij},\ i<j$, of $S(T)$ is easily computed.

$$\begin{aligned}\alpha_2(s_{ii},s_{jj}) &= \alpha(s_{ii})\cdot\alpha(s_{jj}) = b_i\cdot b_j,\\ \alpha_2(s_{ii},s_{jk}) &= \alpha(s_{ii})\cdot\alpha(s_{jk}) = \sqrt{2}\,b_i\cdot b_{jk},\\ \alpha_2(s_{hi},s_{jk}) &= \alpha(s_{hi})\cdot\alpha(s_{jk}) = 2b_{hi}\cdot b_{jk}.\end{aligned}$$

It is easy to show that the determinant of α_2 is a multiple of the squared length of $b_1 \wedge b_2 \wedge b_3 \wedge b_{12} \wedge b_{23} \wedge b_{31}$. Therefore, the determinant of α_2 vanishes iff the rank of $\alpha : S(T) \to N$ is less than 6, and then the ellipsoid is degenerate.

Curvature energy form: The symmetric bilinear form given by

$$E(u,u) = (u{\cdot}\alpha){\cdot}(u{\cdot}\alpha) = (u{\cdot}b_1)^2 + (u{\cdot}b_2)^2 + (u{\cdot}b_3)^2 + 2(u{\cdot}b_{12})^2 + 2(u{\cdot}b_{23})^2 + 2(u{\cdot}b_{31})^2,$$

Interpretation: If $u \in S^N$, we could have chosen (t_1, t_2, t_3) to be an orthonormal basis of eigenvectors of the u-second fundamental form, so that $u \cdot b_i = u \cdot \alpha(t_i, t_i) = k_i(u)$, $i = 1, \dots, 3$, and $u \cdot b_{ij} = 0$, $i < j \leq 3$, where the $k_i(u)$ would be the u-principal normal curvatures. Thus, $E(u,u) = \sum_{i=1}^{3} k_i(u)^2$, and this explains the adopted name.

There is another interpretation of E. We can consider α as a linear map $\alpha : N \to S(T)^*$, defined by $u \in N \mapsto u \cdot \alpha \in S(T)^*$. Then, E is the pull-back by α of the inner product in $S(T)^*$.

Let us consider the map $\alpha^* : N \to S(T)$, adjoint to $\alpha : S(T) \to N$. Thus, if $u \in N$ and $s \in S(T)$ we will have $\alpha^*(u) \cdot s = u \cdot \alpha(s)$. The unit sphere in N applies now to some ellipsoid in $S(T)$ and we will have also its corresponding axes. Let us compute the action of $\alpha_2^* : N \times N \to \mathbb{R}$. We will have

$$\begin{aligned}\alpha_2^*(u,u) = \alpha^*(u) \cdot \alpha^*(u) &= \sum_{i \leq j} (\alpha^*(u) \cdot s_{ij})(\alpha^*(u) \cdot s_{ij}) \\ &= \sum_{i=1}^{3} (u \cdot \alpha(s_{ij}))(u \cdot \alpha(s_{ij})) = (u \cdot \alpha) \cdot (u \cdot \alpha) \\ &= E(u,u),\end{aligned}$$

that is we have $E = \alpha_2^*$. Therefore, the ellipsoid and E have the same information about α.

The trace of E will give another invariant:

Mean curvature energy:

$$\begin{aligned}E_M &= \frac{1}{n} \sum_{i=1}^{n} E(u_i, u_i) \\ &= \frac{1}{n} \sum_{i=1}^{n} \left((u_i \cdot b_1)^2 + (u_i \cdot b_2)^2 + (u_i \cdot b_3)^2 + 2(u_i \cdot b_{12})^2 + 2(u_i \cdot b_{13})^2 + 2(u_i \cdot b_{23})^2 \right) \\ &= \frac{1}{n} (b_1 \cdot b_1 + b_2 \cdot b_2 + b_3 \cdot b_3 + 2b_{12} \cdot b_{12} + 2b_{23} \cdot b_{23} + 2b_{31} \cdot b_{31}).\end{aligned}$$

Third fundamental form W in T given by

$$W(t,t) = \alpha(t,.) \cdot \alpha(t,.) = \sum_{i=1}^{3} \alpha(t, t_i) \cdot \alpha(t, t_i).$$

Interpretation: For a fixed $t \in T$, let $h := \alpha(t,.) : T \to N$. For each $s \in S^T$, we have that $h(s)$ is the projection on N of the covariant derivative of (any extension of) s with respect to t. Thus, it measures the extrinsic twist, along t, of the tangent direction s of S. Hence, $h(s) \cdot h(s)$ is the squared length (called also energy) of that twist, and as a consequence, up to a constant factor, $W(t,t)$ is the average twisting energy along t.

As for its trace, we have

$$\begin{aligned}\sum_{i=1}^{3} W(t_i,t_i) &= \sum_{i,j=1}^{3}\sum_{a=1}^{n} \big(u_a\cdot\alpha(t_i,t_j)\big)\big(u_a\cdot\alpha(t_i,t_j)\big)\\ &= \sum_{a=1}^{n}(u_a\cdot\alpha)\cdot(u_a\cdot\alpha) = \sum_{a=1}^{n} E(u_a,u_a) = nE_M.\end{aligned}$$

that is a constant multiple of E_M.

Riemannnian curvature of S: The standard formula for the Riemann tensor field is the following

$$R^T(X,Y,Z,W) = \alpha(X,W)\cdot\alpha(Y,Z) - \alpha(X,Z)\cdot\alpha(Y,W).$$

Therefore, the Ricci tensor field is given by

$$\begin{aligned}\mathrm{Ricci}(X,Z) &= \sum_{i=1}^{3}\big(\alpha(X,t_i)\cdot\alpha(t_i,Z) - \alpha(X,Z)\cdot\alpha(t_i,t_i)\big)\\ &= W(X,Z) - 3H\cdot\alpha(X,Z).\end{aligned}$$

Now we compute the scalar curvature ρ.

$$\rho = \sum_{j=1}^{3}\mathrm{Ricci}(t_j,t_j) = \sum_{j=1}^{3} W(t_j,t_j) - 9H\cdot H = nE_M - 9H\cdot H.$$

Curvature of the normal bundle: Let us denote by $A : N\times T \to T$ the map that satisfies $Y\cdot A_u(X) = -u\cdot\alpha(X,Y)$ for $u\in N$ and $X,Y\in T$. Then the curvature R^N of the normal bundle at p satisfies

$$v\cdot R^N(X,Y)u = A_u(Y)\cdot A_v(X) - A_u(X)\cdot A_v(Y),\quad u,v\in N_pS,\ X,Y\in T_pS.$$

Thus, we have

$$\begin{aligned}v\cdot R^N(X,Y)u &= \sum_{k=1}^{3}\big((t_k\cdot A_u(X))(t_k\cdot A_v(Y) - (t_k\cdot A_u(Y))(t_k\cdot A_v(X)\big)\\ &= \sum_{k=1}^{3}\big((u\cdot\alpha(t_k,Y))(v\cdot\alpha(t_k,X)) - (u\cdot\alpha(t_k,X))(v\cdot\alpha(t_k,Y)\big).\end{aligned}$$

So, if we denote by $w^\sharp \in N_pS$ the 1-form given by $w^\sharp(u) = w\cdot u$, we can put:

$$\hat{R}^N(X,Y)(u,v) = v\cdot R^N(X,Y)u = \Big(\sum_{k=1}^{3}\alpha(t_k,Y)^\sharp\wedge\alpha(t_k,X)^\sharp\Big)(u,v).$$

Then we have

$$\hat{R}^N(X,Y) = \sum_{k=1}^{3}\alpha(t_k,Y)^\sharp\wedge\alpha(t_k,X)^\sharp$$

Some properties relative to the second fundamental form

The second fundamental form defines a map from the projective space PT to N, that we will call the *Veronese of curvature.* It is given by

$$\eta([t]) = \eta(t) = \frac{\alpha(t,t)}{t \cdot t}.$$

Its image will be the same as the image of the map $t \mapsto \eta(t)$, for $t \in S^T$. Let us put $V = \eta(S^T)$. We begin with the following property that does not depend on the dimension of the immersed manifold S.

Proposition 1.1. *With the preceding notation let us assume that the dimension of S is $a \geq 2$. Then V is contained in the affine subspace of N generated by the vectors $b_{ii} - b_{11}$, $i = 2, \ldots, a$ and the vectors b_{ij}, $1 \leq i < j \leq a$, that contains H. It has dimension less or equal to $\frac{1}{2}(a+2)(a-1)$.*

Proof. Suppose that $(t_i)_{i=1}^a$ is an orthonormal basis of T. Then $H = \frac{1}{a}\sum_{i=1}^a \alpha(t_i, t_i)$. Now, we consider the $\frac{1}{2}a(a+1)$-dimensional vector subspace $S(T)$ of symmetric elements of $T \otimes T$ and denote by $\tilde{\alpha} : S(T) \to N$ the natural linear map defined by α. Any element of V is the image by $\tilde{\alpha}$ of some element $v \otimes v \in S(A)$ with $v \in S^T$. Let us consider the orthonormal basis $(s_{ij})_{1\leq i \leq j}^a$ of $S(T)$ defined as in section 1. Let us define $\tilde{H} \in S(T)$ as

$$\tilde{H} = \frac{1}{a} g^{-1} = \frac{1}{a}(t_1 \otimes t_1 + \cdots + t_a \otimes t_a) = \frac{1}{a}\sum_{i=1}^{a} s_{ii},$$

where g^{-1} defines the inner product in N^*. Thus, $\tilde{H}$ does not depend on the chosen orthonormal basis (t_i) of N. We consider the map $\mathrm{tr}_1 : S(T) \to \mathbb{R}$ given by $\mathrm{tr}_1(h) = (h - \tilde{H}) \cdot \tilde{H}$. The equation $\mathrm{tr}_1(h) = 0$ is the equation of an affine hyperplane C in $S(T)$, whose dimension is

$$\frac{1}{2}a(a+1) - 1 = \frac{1}{2}(a+2)(a-1).$$

Now, if $h = t \otimes t$ for $t \in S^A$ we can suppose that t is the first vector of an orthonormal basis (t_i) of T. Thus

$$\mathrm{tr}_1(h) = (s_{11} - \tilde{H}) \cdot \tilde{H} = \frac{1}{a^2}\big((a-1)s_{11} - s_{22} - \cdots - s_{aa}\big) \cdot (s_{11} + \cdots + s_{aa}) = 0.$$

Hence it is clear that $V \subset \tilde{\alpha}(C)$. Since $\tilde{\alpha}$ is linear, the s_{ii} belong to C and $\tilde{H} \cdot s_{ij} = 0$ if $i < j$ the claim follows immediately. ☐

The Veronese V lies in an affine hyperplane of N iff there is a unit vector $k \in N$ such that $k \cdot \alpha = \mu g$, where $\mu \in \mathbb{R}$. In fact, if $t \in T$ is a unit vector then $k \cdot \eta(t) = \mu g(t,t) = \mu$, that is $\eta(t)$ belongs to the affine hyperplane in N with unit normal k, at a distance μ of the origin. By polarization we obtain the converse. In the same manner, iff there are orthonormal vectors $k_1, \ldots, k_r \in N$ such that $k_i \cdot \alpha = \mu_i g$, $i = 1, \ldots, r$, then, $\eta(S^T)$ lies in an affine subspace of N of codimension r, etc.

Proposition 1.2. *Let $n = 3$ and $k \in S^T$ be such that $k \cdot \alpha = \mu g$ for some $\mu \in \mathbb{R}$. Then, $k \cdot R^N = 0$ and $R^N(X,Y)k = 0$ for any $X, Y \in T$. On the contrary, let us suppose that $k \cdot R^N = 0$. Then either V lies in an affine plane or in a cone.*

Proof. If $k\cdot\alpha=\mu g$ then $\hat{R}^N(X,Y)(k,\cdot)=k\cdot R^N(X,Y)=0$ for any $X,Y\in T$. In fact, we have

$$\sum_{i=1}^{3}(k\cdot\alpha(t_i,Y))(v\cdot\alpha(t_i,X))=\mu\sum_{i=1}^{3}g(t_i,Y)(v\cdot\alpha(t_i,X))=\mu v\cdot\alpha(Y,X),$$

and the claim follows from the symmetry of α and the antisymmetry of $\hat{R}^N$. Suppose now that $k\cdot R^N=0$. Let (t_1,t_2,t_3) an orthonormal principal basis of the bilinear form $k\cdot\alpha$. Then there are real numbers μ_i such that $k\cdot\alpha(t_i,t_j)=\mu_i\delta_{ij}=-t_j\cdot A_k(t_i)$ and this implies $A_k(t_i)=-\mu_i t_i$ If $\mu_1=\mu_2=\mu_3:=\mu$, then we have $A_k=-\mu\,\mathrm{id}_T$, that is $k\cdot\alpha=\mu g$, so that V lies in a hyperplane. If not all the μ_i are equal and $1\le i<j\le 3$ we have

$$\begin{aligned}&k\cdot R^N(t_i,t_j)v=A_k(t_i)\cdot A_v(t_j)-A_k(t_j)\cdot A_v(t_i)=\mu_i v\cdot\alpha(t_i,t_j)-\mu_j v\cdot\alpha(t_j,t_i)\\&=(\mu_i-\mu_j)v\cdot\alpha(t_i,t_j)=0.\end{aligned}$$

Suppose first that the μ_i are different from each other. Then, $v\cdot\alpha(t_i,t_j)=0$ for any $v\in N$, whence $b_{ij}=0,\ 1\le i<j\le 3$. Thus, if $X=\sum_{i=1}^3 x_i t_i$ is a unit vector, we will have $\eta(X)=x_1^2b_{11}+x_2^2b_{22}+x_3^2b_{33}$, so that the sum of the components of X in the system of vectors b_{11},b_{22},b_{33} is equal to one and this entails that V lies in an affine subspace of N whose dimension is equal to the rank of that system minus one. Finally, let us suppose that $\mu_1=\mu_2\neq\mu_3$. Then, $b_{13}=b_{23}=0$ and since $x_3^2=1-x_1^2-x_2^2$ we will have

$$\eta(X)=x_1^2b_{11}+x_2^2b_{22}+x_3^2b_{33}+2x_1x_2b_{23}=b_{33}+x_1^2(b_{11}-b_{33})+x_2^2(b_{22}-b_{33})+2x_1x_2b_{23}.$$

Then, the components x,y,z of $\eta(X)-b_{33}$ in the system of vectors $(b_{11}-b_{33},b_{22}-b_{33},b_{23})$ satisfy $4xy-z^2=0$. That is, if those vectors are linearly independent then V lies in a cone of vertex b_{33}, else V lies in an affine plane or in an affine line of N. □

We can identify $\hat{R}^N$ with a linear map from T to N as follows. Let us suppose that T and N are oriented and that $X\times Y$ denotes the cross product defined by g and the orientation of T and that $u\times v$ denotes the same in N. Then we define the map $\tilde{R}^N:T\to N$ by

$$\tilde{R}^N(Z)=\sum_{k=1}^{3}\alpha(t_k,X)^\sharp\times\alpha(t_k,Y)^\sharp$$

if $Z=X\times Y$. In fact, $X\times Y=X'\times Y'$ iff $X\wedge Y=X'\wedge Y'$, as it is easy to verify.

This map allows us to define special directions in T and N at p. In fact, as we have shown in section 1, there are orthonormal bases $(t_i)_{i=1}^3$ of T and $(n_i)_{i=1}^3$ of N and real numbers ν_i such that $\tilde{R}^N(t_i)=\nu_i n_i$. From these bases is easy to find vectors as in the hypotheses of the preceding proposition.

In the same manner we may define other special directions in T by the map $\tilde{R}^T:T\to T$ obtained in the same manner from the Riemann curvature tensor field.

Computation of the Veronese of curvature

We may parameterize S^T by putting

$$\Psi(\theta,\phi)=\sin\theta(\cos\phi\,t_1+\sin\phi\,t_2)+\cos\theta\,t_3.$$

Then we have the following parameterization of $\eta(S^T)$:

$$\begin{aligned}\eta(\theta,\phi) := \eta\big(\Psi(\theta,\phi)\big) &= \sin^2\theta(\cos^2\phi\, b_1 + \sin^2\phi\, b_2 + \sin 2\phi\, b_{12}) + \cos^2\theta\, b_3 \\ &\quad + \sin 2\theta(\cos\phi\, b_{13} + \sin\phi\, b_{23}) \\ &= H + \frac{1}{12}(1+3\cos 2\theta)(2b_3 - b_1 - b_2) + \frac{1}{2}\cos 2\phi \sin^2\theta(b_1 - b_2) \\ &\quad + \sin 2\phi \sin^2\theta\, b_{12} + \sin 2\theta(\cos\phi\, b_{13} + \sin\phi\, b_{23}).\end{aligned}$$

The last expression, obtained with the use of Mathematica®, says that the Veronese of curvature lies in an affine subspace of N of dimension less or equal to five.

Since

$$b_i = \sum_{j,k=1}^{3} t_{ij} t_{ik} \alpha(\mathbf{x}_j, \mathbf{x}_k), \quad b_{ij} = \sum_{h,k=1}^{3} t_{ih} t_{jk} \alpha(\mathbf{x}_h, \mathbf{x}_k),$$

the computation of the Veronese is straightforward.

For drawing the Veronese we will need the normals to it at the vertexes of the triangulation. They will be computed through the partial derivatives η_ϕ and η_θ of η with respect to the variables ϕ and θ. Now, we have

$$\begin{aligned}\eta_\theta &= 2\cos 2\theta(\cos\phi\, b_{13} + \sin\phi\, b_{13}) + \sin 2\theta(\cos^2\phi\, b_1 + \sin^2\phi\, b_2 - 2b_3), \\ \eta_\phi &= \sin^2\theta(2\cos 2\phi\, b_{12} + \sin 2\phi(b_2 - b_1)) + \sin 2\theta(\cos\phi\, b_{23} - \sin\phi\, b_{13}).\end{aligned}$$

The normal is parallel to the cross product of these two vectors.

Fot the record, let us show the computations when we use a "flat" chart:

$$(u,v) \mapsto \frac{(u,v,1)}{\sqrt{u^2+v^2+1}}, \qquad u,v \in \mathbb{R}.$$

Then, the point in the Veronese corresponding to (u,v) will be

$$\eta(u,v) = \frac{1}{u^2+v^2+1}(u^2\, b_1 + v^2\, b_2 + b_3 + 2uv\, b_{12} + 2u\, b_{13} + 2v\, b_{23}).$$

The partial derivatives of $\eta(u,v)$ will be given by

$$\begin{aligned}\eta_u(u,v) &= \frac{2(u\, b_1 + v\, b_{12} + b_{13} - u\,\eta(u,v))}{u^2+v^2+1}, \\ \eta_v(u,v) &= \frac{2(v\, b_2 + u\, b_{12} + b_{23} - v\,\eta(u,v))}{u^2+v^2+1}.\end{aligned}$$

Now, if we use a stereographic chart for the projective tangent space at p, we will have the map

$$(u,v) \mapsto \frac{(2u, 2v, 1-u^2-v^2)}{u^2+v^2+1}, \qquad u^2+v^2 \le 1.$$

Then, the point in the Veronese corresponding to (u,v) will be

$$\eta(u,v) = \frac{4u^2\, b_1 + 4v^2\, b_2 + 8uv\, b_{12} + (1-u^2-v^2)((1-u^2-v^2)b_3 + 4u\, b_{13} + 4v\, b_{23})}{(u^2+v^2+1)^2}.$$

The partial derivatives of $\eta(u,v)$ will be given by

$$\frac{1}{4}\eta_u(u,v)$$
$$= \frac{2vb_{12} + (1-3u^2-v^2)b_{13} + u(2b_1 - 2vb_{23} + (u^2+v^2-1)b_3 - (1+u^2+v^2)\eta(u,v))}{(u^2+v^2+1)^2},$$
$$\frac{1}{4}\eta_v(u,v)$$
$$= \frac{2ub_{12} + (1-3v^2-u^2)b_{23} + v(2b_2 - 2ub_{13} + (u^2+v^2-1)b_3 - (1+u^2+v^2)\eta(u,v))}{(u^2+v^2+1)^2}.$$

Computation of the focal set

It may be interesting to compute the focal locus in the normal space at a point of S. The focal locus is given as the following set:

$$\mathcal{F}(S) = \{u \in NS \ : \ \det(g_p - u\cdot\alpha_p) = 0, \text{where } p = \pi_N(u)\},$$

where $\pi_N : NS \to S$ is the normal bundle over S and g is the first fundamental form of S. Let $\mathcal{F}_p(S) = \mathcal{F}(S) \cap N_pS$.

Suppose that we have orthonormal bases $(t_i)_{i=1}^3$ and $(n_i)_{i=1}^3$ of T and N respectively and suppose that $u = \sum_{i=1}^3 u_i n_i$. Then $u\cdot\alpha(t_i,t_j) = \sum_{k=1}^3 u_k n_k \cdot b_{ij}$. Let us put $b_{k,ij} = n_k \cdot b_{ij}$. Then

$$(g - u\cdot\alpha)(t_i,t_j) = \delta_{ij} - \sum_{k=1}^3 u_k b_{k,ij}.$$

Therefore $\det(g_p - u\cdot\alpha_p)$ is a polynomial of degree three in the components u_k of u.

Departamento de Geometría y Topología, Universidad de Valencia, c/. Vicente Andrés Estellés, Burjasot (Valencia), Spain

E-mail address: montesin@uv.es