# COMPUTATIONS ON SURFACES

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## 1. Preliminary concepts

Let $A$ and $B$ be real vector spaces of finite dimensions $a$ and $b$. We denote by $S(A)$ the subspace of $A \otimes A$ of symmetric tensors, that is $S(A)$ is generated by the elements as $v \otimes v,\ v \in A$. Note that if $w$ is another element of $A$, then $(v+w) \otimes (v+w) - v \otimes v - w \otimes w = v \otimes w + w \otimes v$. In the following, suppose that $A$ and $B$ are Euclidean vector spaces with inner product denoted by a dot. If $(u_1, \dots, u_a)$ is an orthonormal basis of $A$ and $\beta, \gamma \in A^*$, then we can define the inner product of $\beta$ and $\gamma$ by

$$\beta \cdot \gamma = \sum_{i=1}^{a} \beta(u_i)\gamma(u_i).$$

In fact, as it is easily proved, the result does not depend on the chosen orthonormal basis. This may be generalized for generalizing the inner product to elements of $\bigotimes A^*$. In fact, if for example $g, h \in A^* \otimes A^*$ is a bilinear form on $A$, we define $g \cdot h = \sum_{i,j=1}^{a} g(u_i, u_j)h(u_i, u_j)$, and as before this does not depend on the orthonormal basis. Let $(u^1, \dots, u^a)$ be the dual basis of $(u_1, \dots, u_a)$. Then the elements $u^i \otimes u^j,\ i, j = 1, \dots, a$, are an orthonormal basis of $A^* \otimes A^*$. In fact,

$$(u^i \otimes u^j) \cdot (u^k \otimes u^l) = \sum_{p,q=1}^{a} (u^i \otimes u^j)(u_p, u_q)(u^k \otimes u^l)(u_p, u_q) = \delta_{ki}\delta_{lj},$$

as required. In the same manner we may define the inner product in, say $\bigotimes_s^r A$ by declaring the the elements

$$u^{i_1} \otimes \cdots \otimes u^{i_r} \otimes u_{j_1} \otimes \cdots \otimes u_{j_s}, \quad i_1, \dots, i_r, j_1, \dots, j_s = 1, \dots, a$$

are an orthonormal basis of $\bigotimes_s^r A$. With that definition it is easy to prove that if for instance we have the tensors $m = \beta \otimes v,\ n = \gamma \otimes w \in A^* \otimes A$, then $m \cdot n = (\beta \cdot \gamma)(v \cdot w)$.

Let us show that the elements $s_i = u_i \otimes u_i,\ i = 1, \dots, a$, and the elements $s_{ij} = \frac{1}{\sqrt{2}}(u_i \otimes u_j + u_j \otimes u_i),\ 1 \le i < j \le a$, are an orthonormal basis of $S(T)$. It is clear that they are a basis. Now,

$$\begin{aligned}
s_i \cdot s_j &= (u_i \cdot u_j)^2 = \delta_{ij}^2 = \delta_{ij},\\
s_i \cdot s_{jk} &= \frac{1}{\sqrt{2}}((u_i \cdot u_j)(u_i \cdot u_k) + (u_i \cdot u_k)(u_i \cdot u_j)) = \sqrt{2}\delta_{ij}\delta_{ik} = 0,\\
s_{jk} \cdot s_{pq} &= \frac{1}{2}(2\delta_{jp}\delta_{kq} + 2\delta_{jq}\delta_{kp}) = \delta_{jp}\delta_{kq},
\end{aligned}$$

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*Date*: January 16, 2009.

Partially supported by the Ministerio Español de Ciencia y Tecnología, project number BFM2000-1110], and by the grant MTM 2006-06027 from the Spanish Ministerio Español de Educación y Ciencia .

because the second term is zero since $j < k$ and $p < q$. Hence, the affirmation is true.

We introduce another notation. If $A$ is an Euclidean vector space we put $S^A = \{u \in A \ : \ u \cdot u = 1\}$ to denote the sphere in $A$.

Let now $h : A \to B$ be a homomorphism. Then, we can define the *pull-back* of the inner product of $B$ as the symmetric bilinear form $h_2 : A \times A \to \mathbb{R}$ given by $h_2(u, v) = h(u) \cdot h(v)$. It is well known that there is an orthonormal basis $(u_1, \ldots, u_a)$ of $A$ and unique real numbers $\mu_1 \geq \cdots \geq \mu_a \geq 0$ such that $h_2(u_i, u_j) = 0$ if $i \neq j$ and $h_2(u_i, u_i) = \mu_i$. Let $c \leq \operatorname{Min}(a, b)$ be the number of non zero elements $\mu_i$. If we call $\lambda_i = \sqrt{\mu_i}$ for $1 \leq i \leq a$, and $w_i = h(u_i)/\lambda_,$ for $1 \leq i \leq c$, we will have $w_i \cdot w_i = h_2(u_i, u_i) = 1$, and $w_i \cdot w_j = 0, \ i \neq j$. Also, if $\mu_i = 0$, then $h_2(u_i, u_i) = h(u_i) \cdot h(u_i) = 0$, whence $h(u_i) = 0$. Therefore the kernel of $h$ is generated by the vectors $u_1, \ldots, u_c$. We will assume that we are completed the vectors $w_i$ to form an orthonormal basis of $B$. and that $(w^i)$ denotes the dual basis. It is clear that $h(u_i) = \lambda_i w_i$, for $1 \leq i \leq a$. Using the Einstein convention, we will have

$$h = w^i(h(u_j))w_i \otimes u^j = (w_i \cdot \lambda_j w_j)w_i \otimes u^j = \sum_{j=1}^{c} \lambda_j w_j \otimes u^j.$$

Since $h$ is lineal, $h(S^A)$ must be a compact quadric, that is an ellipsoid that may be degenerate. In other words, it consists in the intersection of a solid ellipsoid centered at the origin with a subspace of $B$. Its axes are determined by the vectors $v \in S^A$ such that the function $f(u) = h(u) \cdot h(u), \ u \in A$, when restricted to $S^A$, is extremal at $v$. This means that the 1-form $df_v$ annihilates the subspace orthogonal to $v$. So, $df_v$ must be a multiple of $dr_v^2$, say $df_v = \mu dr_v^2$, where $r^2 : A \to \mathbb{R}$ is given by $r^2(u) = u \cdot u$. If $t \mapsto \gamma(t) \in A$ is a smooth curve such that $\gamma(0) = v$ and we put $u = \gamma'(0)$, then

$$(df_v - dr_v^2)(u) = \big(f(\gamma, \gamma) - \mu r^2(\gamma, \gamma)\big)'(0) = 2(h(v) \cdot h(u) - \mu v \cdot u) = 0$$

Suppose that $v = u_i, \ \mu = \mu_i$ and $u = u_j$. Then

$$h(v) \cdot h(u) - \lambda v \cdot u = h_2(u_i, u_j) - \mu_i \delta_{ij} = 0.$$

Therefore, $w_i = h(u_i)$ determines an axis of the ellipsoid and $\sqrt{\mu_i}$ is the half-axis corresponding to it.

We may define the *adjoint* $h^* : B \to A$ of $h$ by saying that $h^*(w)$ is the unique element of $A$ such that $h^*(w) \cdot u = w \cdot h(u)$, for any $u$. We will have $h^* = u^j(h^*(w_i))u_j \otimes w^i$, where we use the Einstein convention, and where $(w^i)$ is the basis dual to $(w_i)$. Then

$$h^* = u_j \cdot h^*(w_i)u_j \otimes w^i = (h(u_j) \cdot w_i)u_j \otimes w^i = (\lambda_j w_j \cdot w_i)u_j \otimes w^i = \sum_{j=1}^{c} \lambda_j u_j \otimes w^j.$$

Hence, the half-axes of the corresponding ellipsoid in $A$ are the same as those of the ellipsoid in $B$, and the principal directions corresponding to non-zero half-axes are images of each other.

## Surfaces: notation and inventory of invariants

In the following, $\alpha$ will denote the value of the second fundamental form of a surface $S$ in $\mathbb{R}^{2+n}$ at some point $p$. The tangent space to the surface at that point will be denoted $T$ and its orthogonal complement by $N$. Thus, $\dim N = n$, and

$T \oplus N = \mathbb{R}^{2+n}$. The inner product will be denoted by a dot. We will denote by $(t_1, t_2)$ an orthonormal basis of $T$ and by $(u_1, \ldots, u_n)$ an orthonormal basis of $N$. We shall put $b_1 = \alpha(t_1, t_1),\ b_2 = \alpha(t_2, t_2),\ b_3 = \alpha(t_1, t_2)$.

Our computations will be realized by using a chart $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, so that, if $(u, v)$ are the canonical coordinates in $U$, we will put

$$t_1 = \frac{\mathbf{x}_u}{|\mathbf{x}_u|}, \qquad t_2 = \frac{g_{uu}\mathbf{x}_v - g_{uv}\mathbf{x}_u}{|g_{uu}\mathbf{x}_v - g_{uv}\mathbf{x}_u|},$$

where se have used the following notation: $g_{uu} = \mathbf{x}_u \cdot \mathbf{x}_u,\ g_{uv} = \mathbf{x}_u \cdot \mathbf{x}_v$ and $g_{vv} = \mathbf{x}_v \cdot \mathbf{x}_v$.

If $A : V \to V^*$ is a symmetric bilinear form in any Euclidean $n$-dimensional vector space (V,g), we will call *principal directions* of $A$ the non-vanishing vectors $v \in V$ such that $(A - \lambda g)(v) = 0$ and the corresponding values $\lambda \in \mathbb{R}$ will be called the *eigenvalues* of $A$; a unit vector $v$ that defines a principal direction will be called an *eigenvector* of $A$. This may also be expressed equivalently as follows. Since $g : V \to V^*$ is an isomorphism, we can consider its inverse $g^{-1} : V^* \to V$, which is also a bilinear symmetric form on $V^*$. Then $\tilde{A} := g^{-1} \circ A \in \mathrm{End}(V)$, and the eigenvalues and eigenvectors of $A$ are the eigenvalues and eigenvectors of $\tilde{A}$ in the usual sense. The trace or determinant of $A$ will be defined as the trace and determinant of $\tilde{A}$. The characteristic polynomial of $\tilde{A}$ may be written as $(-1)^n\big(\lambda^n - \mathrm{tr}(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} - \cdots + (-1)^{n-1}c_{n-1}(A)\lambda + (-1)^n \det(A)\big)$, and it is clear that the $c_i(A)$ are invariants of $A$.

The geometric interpretation of some formulas will be related to the interpretation of the second fundamental form. Let $u \in N$ be a unit vector. Then, we can orthogonally project the surface, in a neighborhood of $p$, to the 3-space $\mathbb{R}u \oplus T$. Thus we obtain a surface in a Euclidean 3-space, whose second fundamental form at $p$ is given by $u \cdot \alpha$. Thus, we will say that $u \cdot \alpha$ is the $u$-second fundamental form of the surface at $p$, or that $u \cdot \alpha(t, t),\ t \in S^T$, is the $u$-normal curvature of the surface at $p$ in the direction $t$, etc.

Now we describe some *concomitants* of $\alpha$. By a concomitant we understand here some mathematical object obtained by means of $\alpha$ using only the properties of $\alpha$ and the Euclidean structure of $T$ and $N$, including the use of orthonormal bases, provided that the obtained object does not depend on the choice of those bases. A concomitant that is a real number will be called an *invariant.* We have the following concomitants:

**Mean curvature vector**:

$$H = \frac{1}{2}(\alpha(t_1, t_1) + \alpha(t_2, t_2)) = \frac{1}{2}(b_1 + b_2).$$

*Interpretation*: If $u \in S^N$, $u \cdot H$ is the $u$-mean curvature of the surface at $p$.

**Gauss curvature form**: the symmetric bilinear form $K : N \times N \to \mathbb{R}$ defined as the determinant of $\alpha$, that is

$$K(u, u) = (u \cdot \alpha(t_1, t_1))(u \cdot \alpha(t_2, t_2)) - (u \cdot \alpha(t_1, t_2))^2 = (u \cdot b_1)(u \cdot b_2) - (u \cdot b_3)^2.$$

*Interpretation*: If $u \in S^N$, $K(u, u)$ is the $u$-Gauss curvature of the surface. We can immediately obtain an invariant from K, its mean value obtained through its trace:

**Mean Gauss curvature**:

$$K_M = \frac{1}{n_0}\sum_{i=1}^{n} K(u_i, u_i),$$

where $n_0 = \mathrm{Min}(n, 3)$. A standard calculation shows that $K_M$ is proportional to the sectional curvature of the surface (as a Riemannian manifold) at $p$.

**The ellipsoid**: Let $S(T)$ be the subspace of symmetric elements of $T \otimes T$. As we know, the following elements of $S(T)$ are an orthonormal basis:

$$s_1 = t_1 \otimes t_1, \quad s_2 = t_2 \otimes t_2, \quad s_3 = \frac{1}{\sqrt{2}}(t_1 \otimes t_2 + t_2 \otimes t_1).$$

Let $S^{S(T)} = \{s \in S(T) \ : \ s \cdot s = 1\}$. The *ellipsoid of curvature* is $\alpha(S^{S(T)}) \subset N$. It is an ellipsoid that could be degenerate. Its axes are the images of the extremal points of the real function in $S^{S(T)}$ given by $U \mapsto \alpha_2(U, U) = \alpha(U) \cdot \alpha(U)$, $U \in S^{S(T)}$. That function is the restriction to $S^{S(T)}$ of the quadratic form defined by the bilinear form $\alpha_2$ in $S(T)$ given by $\alpha_2(U, V) = \alpha(U) \cdot \alpha(V)$. Then, the non-zero vector $U$ defines a principal direction or an axis iff there is a real number $\lambda$ such that $\alpha_2(U, V) = \lambda U \cdot V$, $\forall V \in S(T)$. The half-axis corresponding to that principal direction is $\sqrt{\lambda}$. The matrix of $\alpha_2$ in the orthonormal basis $(s_1, s_2, s_3)$ of $S(T)$ is easily computed.

$$\begin{aligned}
&\alpha_2(s_1, s_1) = \alpha(s_1) \cdot \alpha(s_1) = b_1 \cdot b_1,\\
&\alpha_2(s_1, s_2) = \alpha_2(s_2, s_1) = \alpha(s_1) \cdot \alpha(s_2) = b_1 \cdot b_2,\\
&\alpha_2(s_1, s_3) = \alpha_2(s_3, s_1) = \alpha(s_1) \cdot \alpha(s_3) = \sqrt{2} b_1 \cdot b_3,\\
&\alpha_2(s_2, s_2) = b_2 \cdot b_2,\\
&\alpha_2(s_2, s_3) = \sqrt{2} b_2 \cdot b_3,\\
&\alpha_2(s_3, s_3) = 2 b_3 \cdot b_3,
\end{aligned}$$

It is easy to show that the determinant of $\alpha_2$ is a multiple of the squared length of $\sqrt{2} b_1 \wedge b_2 \wedge b_3$. Therefore, the determinant of $\alpha_2$ vanishes iff the rank of $\alpha : S(T) \to N$ is less than 3.

**Curvature energy form**: The symmetric bilinear form given by

$$E(u, u) = (u \cdot \alpha) \cdot (u \cdot \alpha) = (u \cdot b_1)^2 + (u \cdot b_2)^2 + 2(u \cdot b_3)^2,$$

*Interpretation*: If $u \in S^N$, we could have chosen $(t_1, t_2)$ to be an orthonormal basis of eigenvectors of the $u$-second fundamental form, so that $u \cdot b_1 = u \cdot \alpha(t_1, t_1) = k_1(u)$, $u \cdot b_2 = k_2(u)$, $u \cdot b_3 = 0$, where $k_1(u), k_2(u)$ would be the $u$-principal normal curvatures. Thus, $E(u, u) = k_1(u)^2 + k_2(u)^2$, and this explains the adopted name.

There is another interpretation of $E$. We can consider $\alpha$ as a linear map $\alpha : N \to S(T)^*$, defined by $u \in N \mapsto u \cdot \alpha \in S(T)^*$. Then, $E$ is the pull-back by $\alpha$ of the inner product in $S(T)^*$.

Let us consider the map $\alpha^* : N \to S(T)$, adjoint to $\alpha : S(T) \to N$. Thus, if $u \in N$ and $s \in S(T)$ we will have $\alpha^*(u) \cdot s = u \cdot \alpha(s)$. The unit sphere in $N$ applies now to some ellipsoid in $S(T)$ and we will have also its corresponding axes. Let us compute the action of $\alpha_2^* : N \times N \to \mathbb{R}$. We will have

$$\begin{aligned}\alpha_2^*(u,u) = \alpha^*(u)\cdot\alpha^*(u) &= \sum_{i,j=1}^{3}(\alpha^*(u)\cdot s_i)(\alpha^*(u)\cdot s_j)s_i\cdot s_j\\ &=\sum_{i=1}^{3}(u\cdot\alpha(s_i))(u\cdot\alpha(s_i)) = (u\cdot\alpha)\cdot(u\cdot\alpha)\\ &=E(u,u),\end{aligned}$$

that is we have $E = \alpha_2^*$. Therefore, the ellipsoid and $E$ have the same information about $\alpha$.

The trace of $E$ will give another invariant:

**Mean curvature energy**:

$$E_M = \frac{1}{n_0}\sum_{i=1}^{n} E(u_i,u_i) = \frac{1}{n_0}(b_1\cdot b_1 + b_2\cdot b_2 + 2b_3\cdot b_3)$$

**Third fundamental form** $W$ in $T$ given by

$$W(t,t) = \alpha(t,.)\cdot\alpha(t,.) = \alpha(t,t_1)\cdot\alpha(t,t_1) + \alpha(t,t_2)\cdot\alpha(t,t_2).$$

*Interpretations*: For a fixed $t \in T$, let $h := \alpha(t,.) : T \to N$. For each $s \in S^T$, we have that $h(s)$ is the projection on $N$ of the covariant derivative of (any extension of) $s$ with respect to $t$. Thus, it measures the extrinsic twist, along $t$, of the tangent direction $s$ of the surface. Hence, $h(s)\cdot h(s)$ is the squared length (called also energy) of that twist, and as a consequence, up to a constant factor $\frac{1}{2\pi}$, $h(t_1)\cdot h(t_1) + h(t_2)\cdot h(t_2)$ is the average twisting energy along $t$.

We have also the classical interpretation as the form induced by pull-back of the Riemannian meric of the Grassmannian $G_{2,2+n}$ by the map that sends each point of the surface to its tangent plane considered as a point of $G_{2,c}$.

Its trace:

**Mean twist energy:**

$$W_M = \frac{1}{2}\sum_{i=1}^{2} W(t_i,t_i).$$

## Computation of the ellipse of curvature

It should be desirable to compare these invariants with the ones obtained through the use of the *ellipse of curvature.* Let $t(\phi) = t_1\cos\phi + t_2\sin\phi \in S^T$. Then

$$\begin{aligned}\kappa(\phi) := \alpha(t(\phi/2),t(\phi/2)) &= b_1\cos^2\frac{\phi}{2} + b_2\sin^2\frac{\phi}{2} + b_3\sin\phi\\ &= \frac{b_1}{2}(1+\cos\phi) + \frac{b_2}{2}(1-\cos\phi) + b_3\sin\phi\\ &= \frac{1}{2}(b_1+b_2) + \frac{1}{2}(b_1-b_2)\cos\phi + b_3\sin\phi\\ &= H + B\cos\phi + C\sin\phi,\end{aligned}$$

where $H$ is the mean curvature vector, $B := (b_1 - b_2)/2$ and $C := b_3$. We have thus $b_1 = H + B,\ b_2 = H - B$. The curve $\phi \mapsto \kappa(\phi), \phi \in [0, 2\pi)$, is called the ellipse of curvature; its center is the mean curvature vector. That ellipse may degenerate to a segment or to a point. Let $t \in S^T$. Then $t \otimes t \in S^{S(T)}$ and $\alpha(t,t) = \alpha(t \otimes t)$ belongs to the curvature ellipse. Hence, the curvature ellipse is a subset of the curvature ellipsoid.

There are some obvious invariants of $\alpha$ obtained from the geometry of the ellipse. For instance, the direction of its axes, its half-axes, the angle that $H$ makes with the plane of the ellipse (that is the plane generated by $B$ and $C$), and the angle that the major axis of the ellipse makes with $H$. We shall compute some of them. In the following, we will put

$$hh = H \cdot H,\ bb = B \cdot B,\ cc = C \cdot C,\ hb = H \cdot B,\ hc = H \cdot C,\ bc = B \cdot C.$$

Let us see how we can do the computation of these vectors and inner products. Let $\nabla : \mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S)$ be the Levi-Civita connection of $S$, so that if $X, Y \in \mathfrak{X}(S)$, we will have $\nabla_X Y = (D_X Y)^\top$, where the superscript $^\top$ denotes orthogonal projection on the tangent space to $S$ and $D$ denotes the directional derivative. Let us put $G_{ij} = D_{t_i} t_j,\ i = 1, 2$. Since $(t_1, t_2)$ is orthonormal we have

$$\begin{aligned} &G_{11} \cdot t_1 = G_{21} \cdot t_1 = G_{12} \cdot t_2 = G_{22} \cdot t_2 = 0, \\ &G_{11} \cdot t_2 = -G_{12} \cdot t_1, \quad G_{21} \cdot t_2 = -G_{22} \cdot t_1. \end{aligned}$$

Let us put $m_1 := G_{11} \cdot t_2$ and $m_2 := G_{22} \cdot t_1$. Now

$$m_1 = G_{11} \cdot t_2 = (D_{t_i} t_1) \cdot t_2 = \frac{1}{\sqrt{g_{uu}}} \left( D_{\mathbf{x}_u} \frac{\mathbf{x}_u}{\sqrt{g_{uu}}} \right) \cdot t_2 = \frac{\mathbf{x}_{uu} \cdot t_2}{g_{uu}}.$$

Also

$$\begin{aligned} m_2 = -G_{21} \cdot t_2 = -(D_{t_2} t_1) \cdot t_2 &= -\frac{g_{uu} D_{\mathbf{x}_v} \mathbf{x}_u - g_{uv} D_{\mathbf{x}_u} \mathbf{x}_u}{|g_{uu} \mathbf{x}_v - g_{uv} \mathbf{x}_u| \sqrt{g_{uu}}} \cdot t_2 \\ &= \frac{g_{uv} \mathbf{x}_{uu} - g_{uu} \mathbf{x}_{uv}}{|g_{uu} \mathbf{x}_v - g_{uv} \mathbf{x}_u| \sqrt{g_{uu}}} \cdot t_2. \end{aligned}$$

Now,

$$\begin{aligned} b_1 &= \alpha(t_1, t_1) = (D_{t_1} t_1)^\perp = G_{11} - G_{11}^\top = G_{11} - (G_{11} \cdot t_2) t_2 = G_{11} - m_1 t_2, \\ b_2 &= \alpha(t_2, t_2) = G_{22} - G_{22}^\top = G_{22} - (G_{22} \cdot t_1) t_1 = G_{22} - m_2 t_1, \\ b_3 &= \alpha(t_1, t_2) = G_{12} - G_{12}^\top = G_{12} - (G_{12} \cdot t_1) t_1 = G_{12} + m_1 t_1 = G_{21} + m_2 t_2. \end{aligned}$$

Therefore

$$\begin{aligned} H &= \frac{1}{2}(G_{11} + G_{22} - m_1 t_2 - m_2 t_1), \\ B &= \frac{1}{2}(G_{11} - G_{22} - m_1 t_2 + m_2 t_1), \\ C &= G_{12} + m_1 t_1. \end{aligned}$$

From this, we obtain immediately all the inner products $hh$, $hb$, etc.

## Geometric properties of the ellipse of curvature

The ellipse axes will be given by the values of $\phi$ where $(\kappa(\phi) - H) \cdot \kappa'(\phi) = 0$. This is

$$\begin{aligned} 0 &= (B \cos\phi + C \sin\phi) \cdot (-B \sin\phi + C \cos\phi) \\ &= (cc - bb) \sin\phi \cos\phi + bc(\cos^2\phi - \sin^2\phi) \\ &= bc \cos 2\phi + \frac{1}{2}(cc - bb) \sin 2\phi \end{aligned}$$

that is

$$\cos 2\phi = \pm \frac{bb - cc}{\sqrt{(bb - cc)^2 + 4bc^2}}, \qquad \sin 2\phi = \pm \frac{2bc}{\sqrt{(bb - cc)^2 + 4bc^2}}.$$

where we must choose between the upper or the lower signs. Since $(\kappa(\phi) - H) \cdot (\kappa(\phi) - H) = \frac{1}{2}(bb + cc) + \frac{1}{2}(bb - cc)\cos 2\phi + bc \sin 2\phi$, the square of the half-axes will be:

$$\frac{1}{2}\left( bb + cc \pm \frac{(bb - cc)^2 + 4bc^2}{\sqrt{(bb - cc)^2 + 4bc^2}} \right) = \frac{1}{2}\big( bb + cc \pm \sqrt{(bb - cc)^2 + 4bc^2} \,\big).$$

Therefore:

$$\begin{aligned} \text{major half-axis} &= \sqrt{\frac{1}{2}\big(bb + cc + \sqrt{(bb - cc)^2 + 4bc^2}\,\big)}, \\ \text{minor half-axis} &= \sqrt{\frac{1}{2}\big(bb + cc - \sqrt{(bb - cc)^2 + 4bc^2}\,\big)}. \end{aligned}$$

$\pi$ times the product of the half-axes gives the ellipse area. Thus we have the invariant

$$\text{ellipse area} = \pi \sqrt{bb\, cc - bc^2}.$$

Let us compute the cosine of the angle $\nu$ that $H$ makes with the least affine subspace of $N$ that contains the ellipse. That cosine is the length of the orthogonal projection $H'$ of $H$ upon that subspace divided by the length of $H$. Let us write $H' = xB + yC$, where $x, y \in \mathbb{R}$. We will have $H' \cdot B = H \cdot B$ and $H' \cdot C = H \cdot C$, that is we have the linear system

$$bb\, x + bc\, y = hb, \qquad bc\, x + cc\, y = hc$$

If $B$ and $C$ are linearly independent, that is the curvature ellipse is not degenerate, then that system has the unique solution

$$x = \frac{hb\, cc - hc\, bc}{bb\, cc - bc^2}, \qquad y = \frac{hc\, bb - hb\, bc}{bb\, cc - bc^2}.$$

Now

$$\begin{aligned} H' \cdot H' &= x^2\, bb + y^2\, cc + 2x\, y\, bc \\ &= \frac{(hb\, cc - hc\, bc)^2 bb + (hc\, bb - hb\, bc)^2 cc + 2(hb\, cc - hc\, bc)(hc\, bb - hb\, bc) bc}{(bb\, cc - bc^2)^2} \\ &= \frac{cc\, hb^2 - 2bc\, hb\, hc + bb\, hc^2}{bb\, cc - bc^2}, \end{aligned}$$

as one may verify by direct computation. Hence

$$\cos^2\nu = \frac{H' \cdot H'}{H \cdot H} = \frac{cc\, hb^2 - 2bc\, hb\, hc + bb\, hc^2}{hh(bb\, cc - bc^2)},$$

and from this we will have

$$\sin^2\nu = \frac{hh\,bb\,cc + 2bc\,hb\,hc - hh\,bc^2 - cc\,hb^2 - bb\,hc^2}{hh(bb\,cc - bc^2)}.$$

Since we know that the denominator of these expression is an invariant, so are their numerators. If the ellipse collapses to a segment, not to a point, then we will have that $B = mC$, or that $C = mB$ for some real number $m$. Thus we will have

$$\cos^2\nu = \frac{hc^2}{hh\,cc}, \qquad \cos^2\nu = \frac{hb^2}{hh\,bb},$$

respectively. We can dispense the look for the vector that does not vanish by using the following formula, that reduces to the previous ones in each case, as it is easily verified:

$$\cos^2\nu = \frac{hb^2 + hc^2}{hh(bb + cc)}$$

The linear independency of the vectors $H, B, C$ can be computed by means of the following equality:

$$(H\wedge B\wedge C)\cdot(H\wedge B\wedge C) = \begin{vmatrix} hh & hb & hc \\ hb & bb & bc \\ hc & bc & cc \end{vmatrix} = hh\,bb\,cc + 2hb\,hc\,bc - hc^2 bb - hb^2 cc - hh\,bc^2,$$

which is precisely the numerator of the expression above for $\sin^2\nu$.

We can define the "depth" of $H'$ in the ellipse, or better the depth of the orthogonal projection of the origin of $N$ on the ellipse plane. We can write $H' = x\,B + y\,C = r(B\cos\beta + C\sin\beta)$, where $\beta$ is such that $x = r\cos\beta,\ y = r\sin\beta,\ r = \sqrt{x^2+y^2}$. Since $B\cos\beta + C\sin\beta$ gives all points of the ellipse translated to the origin, we see that the projection of the origin in inside the ellipse iff $r < 1$, outside if $r > 1$ and on the ellipse if $r = 1$. So we can define the function

$$\text{depth of } H' = 1 - \sqrt{x^2+y^2} = 1 - \frac{\sqrt{(hb\,cc - hc\,bc)^2 + (hc\,bb - hb\,bc)^2}}{bb\,cc - bc^2}.$$

Another measure of depth that will have no "division by zero" problems consists of multiplying by $bb\,cc - bc^2$ the above equation, that is to use

$$\text{depth of } H' = bb\,cc - bc^2 - \sqrt{(hb\,cc - hc\,bc)^2 + (hc\,bb - hb\,bc)^2}$$

instead.

Another property that may be of interest is the angle that $H'$ forms with the major axis of the ellipse. We have seen that the major axis of the ellipse is parallel to the vector $\bar{E} = B\cos\phi_M + C\sin\phi_M$, where

$$\cos 2\phi_M = \frac{bb - cc}{\sqrt{(bb-cc)^2 + 4bc^2}}, \qquad \sin 2\phi_M = \frac{2bc}{\sqrt{(bb-cc)^2 + 4bc^2}}.$$

Thus,

$$\cos\phi_M = \sqrt{\frac{1}{2}(1 + \cos 2\phi_M)}, \quad \sin\phi_M = \mathrm{sg}(bc)\sqrt{\frac{1}{2}(1 - \cos 2\phi_M)}.$$

Then, the cosine of the angle that $H'$ forms with the major axis is

$$\frac{|\bar{E}\cdot H'|}{|\mathcal{E}||H'|} = \frac{|\bar{E}\cdot H|}{|\mathcal{E}||H'|}$$

## Invariants in terms of the ellipse parameters

The Gauss curvature and the energy forms will be given by

$$\begin{aligned}K(u,u) &= (u\cdot(H+B))(u\cdot(H-B)) - (u\cdot C)^2 = (u\cdot H)^2 - (u\cdot B)^2 - (u\cdot C)^2,\\ E(u,u) &= ((u\cdot H)+(u\cdot B))^2 + ((u\cdot H)-(u\cdot B))^2 + 2(u\cdot C)^2\\ &= 2\big((u\cdot H)^2 + (u\cdot B)^2 + (u\cdot C)^2\big).\end{aligned}$$

Thus, if $g$ is the inner product on $N$, we have $K = g(H)\otimes g(H) - g(B)\otimes g(B) - g(C)\otimes g(C)$, so that $\tilde{K} = H\otimes g(H) - B\otimes g(B) - C\otimes g(C)$. Therefore, $\tilde{K}$ is an endomorphism of the subspace generated by $H, B, C$. Its matrix in the basis $H, B, C$ will be obviously:

$$\begin{pmatrix} hh & hb & hc\\ -hb & -bb & -bc\\ -hc & -bc & -cc\end{pmatrix}.$$

The matrix of $\frac{1}{2}\tilde{E}$ in the same basis is

$$\begin{pmatrix} hh & hb & hc\\ hb & bb & bc\\ hc & bc & cc\end{pmatrix}.$$

Thus

$$\begin{aligned}\operatorname{tr}(K) &= hh - bb - cc,\\ c_2(K) &= bb\,cc - bc^2 - hh\,cc + hc^2 - hh\,bb + hb^2,\\ \det(K) &= hh\,bb\,cc + 2bc\,hb\,hc - bb\,hc^2 - cc\,hb^2 - hh\,bc^2.\end{aligned}$$

In the same manner:

$$\begin{aligned}\operatorname{tr}(E) &= 2(hh+bb+cc),\\ c_2(E) &= 4(bb\,cc - bc^2 + hh\,cc - hc^2 + hh\,bb - hb^2),\\ \det(E) &= 8\det(K)\end{aligned}$$

If we sum and subtract both traces we get the invariants $hh$, $bb{+}cc$. If we sum and subtract both coefficients $c_2$ we have another two invariants $bb\,cc - bc^2$, $-hh(bb+cc) + hb^2 + hc^2$, but this last plus the product of the first two gives the invariant $hb^2 + hc^2$.

As for the third fundamental form form $W$, if $t = at_1 + bt_2$ we will have:

$$\begin{aligned}W(t,t) &= (ab_1 + bb_3)\cdot(ab_1 + bb_3) + (ab_3 + bb_2)\cdot(ab_3 + bb_2)\\ &= (b_1\cdot b_1 + b_3\cdot b_3)a^2 + (b_2\cdot b_2 + b_3\cdot b_3)b^2 + b_3\cdot(b_1+b_2)2\,a\,b\\ &= (hh+bb+cc+2hb)a^2 + (hh+bb+cc-2hb)b^2 + 4\,hc\,a\,b\\ &= \frac{1}{2}\operatorname{tr}(E)g_T(t,t) + 2(hb\,a^2 - hb\,b^2 + 2\,hc\,a\,b),\end{aligned}$$

where $g_T$ is the Euclidean product in $T$. Thus $W$ minus its trivial part $\frac{1}{2}\operatorname{tr}(E)g_T$ will have zero trace and determinant equal to $-4(hb^2 + hc^2)$. Since $W$ is a bilinear form on $T$ it defines its principal directions. However, they will be the same as the mean curvature directions that we will define below.

We observe that $\det(K)$ is the numerator of $\sin^2(\nu)$. Hence, we have the following five invariants, that perhaps are the simplest: $hh$, the squared norm of $H$; $bb+cc$, the sum of the squared half-axes of the curvature ellipse; $bb\ cc - bc^2$, which is a multiple of the squared ellipse area; $hb^2 + hc^2$ and $\det(K)$, which define the relative position of $H$ with respect to the ellipse.

Note that the first four invariants are clearly independent. The fifth has to do with the question whether $H$ belongs to the plane $\langle B, C\rangle$, whereas the first four do not depend on it. Thus, they are independent. Now, the minimum number of a complete set of invariants in this case (for $n \geq 3$) is 5, so that we have completed our search.

Another concomitant is the symmetric bilinear form on $T$ given by $F = H \cdot \alpha$. Since $b_1 = H + B,\ b_2 = H - B$ and $b_3 = C$, the matrix of $F$ in the basis $(t_1, t_2)$ is

$$\begin{pmatrix} hh + hb & hc \\ hc & hh - hb \end{pmatrix}.$$

Therefore $F = (hh - \frac{1}{4}\operatorname{tr}(E))g_T + \frac{1}{2}W$. It defines its principal directions that have merits enough to be called *mean curvature directions* of $\alpha$. These directions and those of $W$ will be the same. Since the basis $t_i$ is orthonormal, we see that the matrix of $F - \frac{1}{2}\operatorname{tr}(F)\, g_T$ will be

$$\begin{pmatrix} hb & hc \\ hc & -hb \end{pmatrix}.$$

In the same manner, for each concomitant $u \in N$ we have the symmetric bilinear form $u \cdot \alpha$ on $T$ with its eigenvalues and eigenvectors.

Therefore, one has here a lot of possibilities. Each of the bilinear forms $K, E, \frac{1}{2}E - K$ will have their eigenvectors in $N$ and for each one of them, one can get principal directions on $T$ via $\alpha^*$ as we have done with $H$. In addition, the form $K$ is not definite so that it has an isotropic cone $\{u \in N \ : \ K(u,u) = 0\}$, whose meaning may also be interesting. The easiest case is given by $\frac{1}{2}E - K = 2(g(B) \otimes g(B) + g(C) \otimes g(C))$. In fact, it is easy to verify that one can choose the orthonormal basis $(t_1, t_2)$ of $T$ so that $B \cdot C = 0$. Then it is clear that the eigenvectors of $\frac{1}{2}E - K$ have directions parallel to the axes of the ellipse, and that they determine directions in $T$ given by pull-back of the vertexes of the ellipse.

Note also that $F$ may have null directions.

It will be interesting to obtain a neat relation between the ellipsoid and the ellipse of curvature. The first remark to do is that the ellipse contains all of the information hidden in the second fundamental form (five invariants), whereas the ellipsoid contains only (in general) the values of its half-axes (three invariant numbers). Therefore, we must go from the ellipse to the ellipsoid and not the other way.

The ellipsoid is the image by the linear map $\alpha : S(T) \to N$ of the unit sphere of $S(T)$, that is $S^{S(T)} = \{as_1 + bs_2 + cs_3 \in S(T) \,:\, a^2 + b^2 + c^2 = 1\}$. The ellipse is the image by $\alpha$ of the elements $s$ of $S(T)$ that may be written as $v \otimes v$, with $v \in S^T$. Since $s \cdot s = (v \otimes v) \cdot (v \otimes v) = (v \cdot v)^2 = 1$, we see that $s \in S^{S(T)}$.

Then, we must find a characterization of such decomposable elements among those of $S^{S(T)}$. A such element $s$, as any other symmetric 2-tensor in an Euclidean space, may be characterized by its eigenvalues; it is clear that in our case these are $0, 1$. The appearance of the zero eigenvalue is equivalent to the vanishing of the determinant of $s$; once this has been verified, the appearance of the eigenvalue 1 is equivalent to the condition $s \cdot s = 1$ together with the condition that $s$ be definite non-negative.

So, let $s = as_1 + bs_2 + cs_3$. In the basis $(t_1, t_2)$, $s$ has the matrix:

$$a\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + c\begin{pmatrix}0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0\end{pmatrix} = \begin{pmatrix}a & \frac{c}{\sqrt{2}}\\ \frac{c}{\sqrt{2}} & b\end{pmatrix}.$$

Therefore, $(a, b, c)$ must satisfy $s \cdot s = a^2 + b^2 + c^2 = 1$ and $c^2 = 2ab$. Carrying this to the first equation we see that $(a + b)^2 = 1$. Then $a + b = \pm 1$. Since $s$ must be non-negative we conclude that $a, b \geq 0$, so that $a + b = 1$. Therefore $s \in S(T)$ is a decomposable element as $v \otimes v$ with $v \in S^T$ iff $a^2 + b^2 + c^2 = 1$ and $a + b = 1$.

But this is the intersection of a sphere with a plane, that is a circle, in the Euclidean three-dimensional space $S(T)$. That circle has as center the point $\tilde{H} = \frac{1}{2}(s_1 + s_2)$ of coordinates $(\frac{1}{2}, \frac{1}{2}, 0)$, and if we put $\tilde{B} = \frac{1}{2}(s_1 - s_2)$, which has coordinates $(\frac{1}{2}, -\frac{1}{2}, 0)$, and $\tilde{C} = \frac{1}{\sqrt{2}}s_3$, whose coordinates are $(0, 0, \frac{1}{\sqrt{2}})$, then that circle can be parameterized as $\phi \mapsto \tilde{H} + \tilde{B}\cos\phi + \tilde{C}\sin\phi$. Now, we can "reconstruct" the whole unit sphere $S^{S(T)}$ from that circle. To do this, it is enough to make the union of the circles parallel to that one and centered on the line $\mathbb{R}\tilde{H}$. Those circles can be parameterized as $r\tilde{H} + t(\tilde{B}\cos\phi + \tilde{C}\sin\phi)$. But in order that those circles lie in the unit sphere, we need only to guarantee that the vector $r\tilde{H} + t\tilde{C}$ is a unit vector. Hence $\frac{1}{2}r^2 + \frac{1}{2}t^2 = 1$. Therefore we can put $r = \sqrt{2}\cos\psi$, $t = \sqrt{2}\sin\psi$. Thus, we reconstruct the unit sphere from the circle by the parametrization $(\phi, \psi) \mapsto \sqrt{2}\big(\tilde{H}\cos\psi + (\tilde{B}\cos\phi + \tilde{C}\sin\phi)\sin\psi\big)$. By applying $\alpha$ to this and taking account that $\alpha(\tilde{H}) = H$, etc., we obtain finally the following parametrization of the ellipsoid:

$$Y(\phi, \psi) = \sqrt{2}\big(H\cos\psi + (B\cos\phi + C\sin\phi)\sin\psi\big), \quad \phi \in [0, 2\pi), \quad \psi \in [0, \pi].$$

In this sense we can describe the ellipsoid by means of the vectors $H, B$ and $C$ that determine the curvature ellipse. Note that the map $\phi \mapsto Y(\phi, \pi/4)$ parameterizes the ellipse of curvature.

## Some computations

Assume that we are given a symmetric bilinear form $L$ whose matrix in the orthonormal basis $t_i$ is given by

$$\begin{pmatrix}p & r\\ r & q\end{pmatrix},$$

and we need to compute its principal and null directions, that is the angles that those directions make with the first basis vector $t_1$.

Let $v = t_1\cos\phi + t_2\sin\phi$. Then

$$\begin{aligned}L(v, v) &= p\cos^2\phi + r\sin 2\phi + q\sin^2\phi\\ &= \frac{1}{2}\left((1 + \cos 2\phi)p + 2r\sin 2\phi + (1 - \cos 2\phi)q\right)\\ &= \frac{1}{2}(p + q) + \frac{1}{2}\big((p - q)\cos 2\phi + 2r\sin 2\phi\big)\end{aligned}$$

We see that if $(p - q)^2 + (2r)^2 = 0$ then $L(v, v)$ is constant. In fact, $L$ is a multiple of the metric tensor. In this case, there are not preferred principal or null directions. If it is not the case, let $\alpha \in (-\pi, \pi]$ be the unique angle such that

$$\cos\alpha = \frac{p - q}{\sqrt{(p - q)^2 + (2r)^2}}, \quad \sin\alpha = \frac{2r}{\sqrt{(p - q)^2 + (2r)^2}}.$$

Then

$$L(v,v) = \frac{\sqrt{(p-q)^2+(2r)^2}}{2}\left(\frac{p+q}{\sqrt{(p-q)^2+(2r)^2}} + \cos(2\phi-\alpha)\right).$$

Thus, $L(v,v)=0$ iff $|p+q| \leq \sqrt{(p-q)^2+(2r)^2}$ and

$$\phi = \frac{\alpha}{2} + \frac{1}{2}\arccos\left(-\frac{p+q}{\sqrt{(p-q)^2+(2r)^2}}\right)$$

Let $\beta \in [0,\pi]$ be the unique angle such that

$$\cos\beta = -\frac{p+q}{\sqrt{(p-q)^2+(2r)^2}}.$$

Then $\phi = \frac{1}{2}(\alpha \pm \beta)$ are the basic solutions; one can add to them any integer multiple of $\pi$. Thus, we have the signed directions given by

$$\frac{1}{2}(\alpha+\beta), \quad \frac{1}{2}(\alpha+\beta)+\pi, \quad \frac{1}{2}(\alpha-\beta), \quad \frac{1}{2}(\alpha-\beta)+\pi.$$

As for the principal directions of $L$, we write down the equation for the extremal points of $L(v,v)$. We have:

$$\sin(2\phi-\alpha)=0.$$

Therefore we shall have the solutions:

$$\phi = \frac{\alpha}{2}, \quad \phi = \frac{\alpha}{2}+\frac{\pi}{2}, \quad \phi = \frac{\alpha}{2}+\pi, \quad \phi = \frac{\alpha}{2}+\frac{3\pi}{2}.$$

## Drawing the ellipsoid and the ellipse

In the following we will assume that $\dim N \geq 3$. Let us denote by $N_1 := \alpha(S(T))$, the sometimes called first-normal space, and put $n_1 = \dim N_1$. We will have $n_1 \leq 3$.

Let us begin with the case $n_1 = 3$. Then we can obtain an orthonormal basis of $N_1$ by the Gram-Schmidt algorithm based on $H, B, C$ more easily than from $b_1, b_2, b_3$ because the inner products of those vectors will be already known. Thus, we will put

$$u_1 = \frac{H}{|H|} = \frac{h}{nh},$$

where $nh = \sqrt{hh}$. Then we put

$$u_2 = a\big(B - (B\cdot u_1)u_1\big) = a\left(B - \frac{hb}{nh}u_1\right).$$

We require $1 = u_2 \cdot u_2 = a^2\left(bb - \frac{hb^2}{hh}\right)$. Therefore

$$a = \frac{nh}{\sqrt{hh\,bb - hb^2}}, \qquad u_2 = a\left(B - \frac{hb}{nh}u_1\right).$$

Now we put provisionally

$$u_3 = C - (C\cdot u_1)u_1 - (C\cdot u_2)u_2,$$

and then normalize it.

Suppose that we project now $N_1$ upon the plane orthogonal to some unit vector $p_n \in N_1$. We extend $p_n$ to a orthonormal basis $(p_h, p_v, p_n)$ of $N_1$ in the same manner that we use for instance in the program *Superficies*. This projection gives us a linear map $p : N_1 \to \mathbb{R}^2$. Then, the drawn curvature ellipsoid is the composition of two linear maps, namely $p \circ \alpha : S^{S(T)} \to \mathbb{R}^2$. From that image we will draw only the singular locus, that is its contour. Let $i : S^2 \approx S^{S(T)} \to S(T)$ be the inclusion of the unit sphere. We look for the singular locus of the map $p \circ \alpha \circ i$. A point $x \in S^2$ is singular if there is some non-zero vector $X \in T_xS^2$ such that $d(\alpha \circ i)(X) \in \ker dp|_{\alpha(x)} = \mathbb{R}\, p_n$. Therefore, there is some non-zero real number $r$ such that $X = r(d\alpha)^{-1}(p_n) = r\,\alpha^{-1}(p_n)$. We see thus that the condition for $x$ being singular is $x \cdot \alpha^{-1}(p_n) = 0$. Therefore the singular locus is the image by $p \circ \alpha \circ i$ of the circle $t \mapsto w_1 \cos t + w_2 \sin t$, where $w_1, w_2$ are orthonormal vectors orthogonal to $\alpha^{-1}(p_n)$.

Thus, *the contour of the projected ellipsoid is the curve*

$$t \mapsto (p \circ \alpha)(w_1 \cos t + w_2 \sin t), \quad \text{where}$$
$$|w_1| = |w_2| = 1, \quad w_1 \cdot w_2 = w_1 \cdot \alpha^{-1}(p_n) = w_2 \cdot \alpha^{-1}(p_n) = 0.$$

Let

$$e_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \quad e_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right), \quad e_2 = \left(0, 0, \frac{1}{\sqrt{2}}\right),$$

where the components refer to the orthonormal basis $s_1, s_2, s_3$. Then

$$t \mapsto (p \circ \alpha)(e_0 + e_1 \cos t + e_2 \sin t)$$

*is the curvature ellipse projected on* $\mathbb{R}^2$. The set of points of $S^{S(T)}$ whose images are in the contour of the ellipsoid consists of the intersection of $S^{S(T)}$ with the plane of $S(T)$ orthogonal to $\alpha^{-1}(p_n)$. Therefore, the sign of the height function $f(x) := x \cdot \alpha^{-1}(p_n)$ tells us whether a point in the ellipsoid is in the near or rear side of the ellipsoid as seen from the viewpoint.

We see that for drawing the ellipsoid and the ellipse it would be enough to know the matrix of $p \circ \alpha$ in the bases $(s_1, s_2, s_3)$ of $S(T)$ and $(p_h, p_v)$ of $\mathbb{R}^2$. Since we need to allow for changes in the viewpoint, we shall compute once and for all the matrices of $\alpha$ and $\alpha^{-1}$ and then, in each case, that of $p$ in the bases $(u_1, u_2, u_3)$ of $N_1$ and $(p_h, p_v)$ of $\mathbb{R}^2$. Let us see this computation in detail. If a vector $x \in S(T)$ with components $(x_1, x_2, x_3)$ in the basis $(s_1, s_2, s_3)$ is given, its image by $\alpha$ will have as components in the basis $(u_1, u_2, u_3)$ the following

$$\alpha(x)_i = \alpha(x) \cdot u_i = \sum_{j=1}^{3} u_i \cdot \alpha(x_j s_j) = \sum_{j=1}^{3} u_i \cdot \alpha(s_j) x_j.$$

Thus we write $\alpha_{ij} = u_i \cdot \alpha(s_j)$, so that $\alpha(x) = \sum_{i,j=1}^{3} \alpha_{ij} x_j u_i$.

Note that if we put $\alpha_i = \sum_{j=1}^{3} \alpha_{ij} s_j$, then $\alpha(x) = \sum_{j=1}^{3} (\alpha_i \cdot x) u_i$. The inverse $\alpha^{-1}$ of $\alpha$ will act on a vector $n = \sum_{j=1}^{3} n_j u_j \in N_1$ and give

$$\alpha^{-1}(n) = \sum_{j=1}^{3} \big(s_i \cdot \alpha^{-1}(n)\big) s_i = \sum_{i,j=1}^{3} \big(s_i \cdot \alpha^{-1}(u_j)\big) n_j s_i.$$

So, we shall put $\alpha^{-1}_{ij} = s_i \cdot \alpha^{-1}(u_j)$, and it is easy to prove that $\sum_{j=1}^{3} \alpha^{-1}_{ij} \alpha_{jk} = \delta_{jk}$.

Let us see now the matrix of the composition $p \circ \alpha$. If as before $x = \sum_{i=1}^{3} x_i s_i$, we will have for $a = u, v$ :

$$(p \circ \alpha)(x)_a := p_a \cdot \alpha(x) = \sum_{i=1}^{3} p_{a\,i} \alpha_{ij} x_j.$$

Thus, we will define $q_{aj} := \sum_{i=1}^3 p_{ai}\alpha_{ij},\ a = u, v;\ j = 1, 2, 3$. Therefore we will have for $a = u, v$ :

$$(p \circ \alpha)(x) = \big(\sum_{i=1}^3 q_{ui}x_i, \sum_{i=1}^3 q_{vi}x_i\big).$$

Now we treat the case of $n_1 = \dim N_1 \leq 2$. We may build an orthonormal set of three vectors of $N$ such that the $n_1$ first ones generate $N_1$ and as a consequence we can define a "camera" as before. Now we don't need to compute a contour because there is a natural contour given by the image of the singular locus $\mathcal{L}$ of the map $\alpha$ restricted to $S^{S(T)}$. Therefore, we only need to find $\mathcal{L}$. It is given by those $x \in S^{S(T)}$ that admit some $0 \neq X \in T_xS^{S(T)}$ such that $d\alpha(X) = \alpha(X) = 0$. Let $K = \ker \alpha \subset S(T)$. Then, $x \in S^{S(T)}$ is singular iff there is $0 \neq X \in K$ such that $x \cdot X = 0$.

If $\dim K = 1$, then $x \in \mathcal{L}$ iff $x$ is orthogonal to $K$. Therefore, it is enough to draw the image of the circle of points of $S^{S(T)}$ orthogonal to $K$.

If $\dim K = 2$, then all points of the sphere are singular and the ellipsoid degenerates to a segment. There are two points $\pm x$ that are 2-singular in the sense that all the tangent vectors to the sphere at them belong to $K$. These points are he unit vectors orthogonal to $K$. Its images are the ends of the segment into which the ellipsoid degenerates. Thus it is enough to draw that segment.

If $\dim K = 3$, there is nothing to draw because then $\alpha = 0$.

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