NOTES ON THE FOCAL SET OF A $k-$DIMENSIONAL IMMERSION IN $\mathbb{R}^{k+n}$

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1. The Veronese of curvature

In the following, M will be a smooth $k-$dimensional manifold immersed in $\mathbb{R}^{k+n}$. However, since all our study will be local, we shall suppose without loss of generality that M is a regular submanifold of $\mathbb{R}^{k+n}$. Over M we have the tangent bundle $\pi : TM \to M$, and the *normal bundle* given by

$$NM = \cup_{p\in M}(T_pM)^{\perp}, \quad \pi_N : NM \to M,$$

where $(T_pM)^{\perp}$ denotes the subspace of $T_p\mathbb{R}^{k+n}$ orthogonal to T_pM. Its fiber over $p \in M$ will be denoted by $N_pM = (T_pM)^{\perp}$. The usual inner product will be denoted by a dot. If $X \in T_p\mathbb{R}^{k+n}$, we will have the decomposition $X = X^{\top} + X^{\perp}$, with $X^{\top} \in T_pM,\ X^{\perp} \in N_pM$. For $X_p \in T_p\mathbb{R}^{k+n}$ we shall denote by $D_{X_p} : C^{\infty}(\mathbb{R}^{k+n}, \mathbb{R}^m)$ (with arbitrary $m \in \mathbb{Z}^+$) the ordinary directional derivative. For this matter, all vectors of T_pM and of N_pM are considered as elements of $T_p\mathbb{R}^{k+n}$.

$\mathfrak{X}(M)$ will denote the Lie algebra of smooth vector fields on the manifold M, and if E is the total space of a vector bundle over M we shall denote by ΓE the $C^{\infty}(M)$-module of its smooth sections.

Let $p \in M,\ X_p, Y_p \in T_pM$ and $u_p \in N_pM$. Let $u \in \Gamma NM$ and $Y \in \mathfrak{X}(M)$ be extensions of u_p and Y_p, respectively. Then we have

$$u_p \cdot D_{X_p}Y = D_{X_p}(u \cdot Y) - (D_{X_p}u) \cdot Y_p = -(D_{X_p}u) \cdot Y_p.$$

The left-hand side says that the expression does not depend on the chosen extension u and the right-hand one that it does not depend on the extension Y. It is clearly linear in the three arguments u_p, X_p and Y_p. This leads to define the *second fundamental form* $\alpha_p : T_pM \times T_pM \to N_pM$ by

$$\alpha_p(X_p, Y_p) = (D_{X_p}Y)^{\perp}$$

and also to the map $A : N_pM \times T_pM \to T_pM$ given by $A_{u_p}(X_p) = (D_{X_p}u)^{\top}$, so that $u_p \cdot \alpha(X_p, Y_p) = -A_{u_p}(X_p) \cdot Y_p$. The map $\alpha : p \mapsto \alpha_p$ defines an element of $\Gamma(NM \otimes T^*M \otimes T^*M)$. Then, for any sections $u \in \Gamma NM$ and $X, Y \in \mathfrak{X}(M)$ we have $u \cdot \alpha(X, Y) = u \cdot D_XY = -(D_Xu) \cdot Y$.

Let PT_pM be the projective space of directions in T_pM. The second fundamental form defines a map $\eta_p : PT_pM \to N_pM$, which we call the *Veronese of curvature*, by

$$\eta_p([t]) = \eta_p(t) = \frac{\alpha_p(t,t)}{t \cdot t}, \quad t \in T_pM\backslash\{0\}.$$

I shall describe in this note the relations between the Veronese of curvature and the focal set of the immersion M.

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2. The focal set

A useful definition of the focal set of M goes as follows. Let $\zeta : NM \to \mathbb{R}^{k+n}$ be the map given by $\zeta(u) = p + u$ if $u \in N_pM$, that is $\zeta(u) = \pi_N(u) + u$. The *focal set of* M, denoted here by $\mathcal{F}(M)$, is defined as the set of critical points of ζ. That is by the equation $\det(d\zeta) = 0$ whose solutions in N_pM can be obtained using arbitrary bases of T_uNM and of $T_{p+u}\mathbb{R}^{k+n}$ for expressing $\det(d\zeta_p) = 0$. Let $w_i,\ i = 1, \ldots, n$, be a local orthonormal frame of NM in a neighborhood U of $p \in M$ and let $t_i,\ i = 1, \cdots, k$, be an orthonormal frame of TM in U. By means of the first of those frames we can work with a trivialization of NM on U given by $u \approx (\pi_N(u), x_1, \ldots, x_n)$, where the x_i are such that $u = \sum_i x_i w_{ip}$, being $p = \pi_N(u)$. Thus the map ζ can be expressed as $\zeta(p, x_1, \ldots, x_n) = p + \sum_i x_i w_{ip}$.

Let $X \in T_pM$. Then

$$d\zeta(X, 0) = X + \sum x_i dw_i(X) = X + \sum x_i D_X w_i,$$

and $d\zeta(0, \partial_{x_i}) = w_i$. Since the w_i are orthonormal, the vanishing of $\det(d\zeta)$ is equivalent ot the vanishing of the determinant of the projection of $d\zeta\big|_{T_pM}$ into T_pM, that is to the vanishing of the determinant of the endomorphism of T_pM given by

$$X \to (d\zeta(X, 0))^\top = X + \sum x_i (D_X w_i)^\top.$$

The component in t_{j_p} of $(d\zeta(t_{ip}, 0))^\top$ is thus

$$t_{ip} \cdot t_{j_p} + \sum_b x_b (D_{t_{i_p}} w_b) \cdot t_{j_p} = t_{ip} \cdot t_{j_p} - u_p \cdot \alpha_p(t_{ip}, t_{j_p}),$$

where $u = \sum_b x_b w_{bp}$. That is, the condition is equivalent to the vanishing at p of the matrix with coefficients $\delta_{ij} - u \cdot \alpha(t_i, t_j) = (g - u \cdot \alpha)(t_i, t_j)$.

We have proved thus:

Proposition 2.1. *The focal set M is given by*

$$\mathcal{F}(M) = \{u \in NM \ :\ \det(g_{\pi_N(u)} - u \cdot \alpha_{\pi_N(u)}) = 0\},$$

where g is the first fundamental form of M and $u \cdot \alpha_p$ is the $u-$second fundamental form of M at $p \in M$.

We shall put $\mathcal{F}_p(M) = \mathcal{F}_p(M) \cap N_pM$. The next properties will be useful:

Proposition 2.2. (1) *Let $u : M \to NM$ be a local section in a neighborhood of $p \in M$. Then*

$$\det(g_p - u_p \cdot \alpha_p) = \det(d(\zeta \circ u)_p^\top).$$

(2) *Let $u \in \mathcal{F}_p(M)$. Then there is $t \in T_pM \backslash \{0\}$ such that $g_p(t) = u \cdot \alpha_p(t) \in T_p^*M$; in the remaining of the statement of this proposition, t will have this property.*
(3) $u \cdot \eta_p(t) = 1$. *In particular, $u \neq 0,\ \eta_p(t) \neq 0$.*
(4) $\eta_p(t) \notin (d\eta_p)(T_tT_pM) \subset T_{\eta_p(t)}N_pM$, *under the usual identification of N_pM with $T_{\eta_p(t)}N_pM$.*

Proof. 1) If, as before, we take a local frame w_i of NM in a neighborhood U of p, we can write u as $u = \sum_i u^i w_i,\ u^i \in C^\infty(U)$. Thus, $\zeta \circ u = \mathrm{id} + \sum_i u^i w_i$, whence if $X \in T_pM$, we will have

$$d(\zeta \circ u)_p^\top(X) = \Big(X + \sum_i \big((D_X u^i)w_{ip} + u^i{}_p D_X w_i\big)\Big)^\top = X + A_{u_p}(X).$$

Therefore, the determinant of $d(\zeta \circ u)_p^\top$ is that of the matrix $d(\zeta \circ u)(t_a) \cdot t_b = \delta_{ab} - u_p \cdot \alpha_p(t_a, t_b)$.

2) We can look at $g_p - u \cdot \alpha_p$ as a linear map from T_pM to T_p^*M. Since it maps linearly a vector space into other of same dimension and has zero determinant, we conclude that there is some non-zero $t \in T_pM$ such that $(g_p - u \cdot \alpha_p)(t) = 0$.

3) If the 1-form $(g_p - u \cdot \alpha_p)(t)$ acts upon the vector t itself, we get $t \cdot t - u \cdot \alpha_p(t,t) = 0$, whence, by dividing by $t \cdot t \neq 0$, we obtain the claim.

4) For making the calculations easier we can assume that $t \cdot t = 1$. Then, if $X \in T_t T_p M$, we have

$$d\eta_p(X) = \frac{2}{(t \cdot t)^2}((t \cdot t)\alpha_p(t, X) - (t \cdot X)\alpha_p(t,t)) = 2(\alpha_p(t, X) - (t \cdot X)\eta_p(t)).$$

Suppose that this is equal to $\eta_p(t)$. By inner multiplication of this with u we get $u \cdot \eta_p(t) = t \cdot t = 1$, while the same multiplication with $d\eta_p(X)$ yields $2(u \cdot \alpha_p(t) - g_p(t))(X) = 0$, which is absurd. □

In general, $\mathcal{F}(M)$ will be a hypersurface, possibly with singularities, of NM, whose intersection with each fiber N_pM will be an algebraic hypersurface of degree k. Thus, in the case of a surface $M \in \mathbb{R}^{2+n}$, the intersection $\mathcal{F}_p(M)$ of $\mathcal{F}(M)$ with N_pM will be a quadric.

3. Inverted pedal

In this section we are interested solely in the study of $\mathcal{F}_p(M)$. This justifies the use of the following simplified notation:

$$T = T_pM,\ N = N_pM = T^\perp,\ \alpha = \alpha_p,\ \eta(t) = \frac{\alpha(t,t)}{t \cdot t},\ \mathcal{F} = \mathcal{F}_p(M),\ g = g_p.$$

Definition 3.1. Let P be a smooth manifold, $\mu : P \to \mathbb{R}^n$ a smooth map, and $F \in \mathbb{R}^n$. For each $p \in P$, let $\mathrm{ped}_\mu(p)$ be the nearest point to F from among those of the affine subspace tangent to $\mu(P)$ at $\mu(p)$,

$$\{\mu(p) + d\mu(X)\ :\ X \in T_pP\}.$$

The map thus obtained $\mathrm{ped}_\mu : P \to \mathbb{R}^n$ is called the *pedal map of μ with pedal point F*. Let $\tilde{P}_\mu = \{p \in P\ :\ \mathrm{ped}_\mu(p) \neq F\}$. If $R : \mathbb{R}^n \backslash \{F\} \to \mathbb{R}^n \backslash \{F\}$ is the inversion with respect to the hypersphere with center F and unit radius, the composition $R \circ \mathrm{ped}_\mu : \tilde{P} \to \mathbb{R}^n \backslash \{F\}$ (and sometimes, also its image) will be called the *inverted pedal of μ with pedal point F. If F is the origin, we will drop the specification of the pedal point.*

Let us show the relation between the focal set of M and the inverted pedal of η, that is the pedal point will be the origin of N. Let $0 \neq t \in T$ and let $0 \neq z = \eta(t) + d\eta(X)$, with $X \in T_t(PT)$, a point in the pedal of η. We exclude $z = 0$ because we will need to apply to it the inversion R. Then z is the point nearest to the origin in the affine tangent space to $\eta(PT)$ at $\eta(t)$. Therefore we must have $z \cdot d\eta(T_tPT) = d(z \cdot \eta)(T_tPT) = 0$. In particular, $z \cdot d\eta(t) = 0$, whence

$z \cdot z = z \cdot \eta(t) \neq 0$. Also, $d(z \cdot \eta)_t = 0$. Hence t is a critical point of the map $t \mapsto z \cdot \eta(t)$. But one sees easily that this entails

$$g(t,t) z \cdot \alpha(t,t') - (t \cdot t') z \cdot \alpha(t,t) = 0$$

for any $t' \in T_t T$, i.e. $(z \cdot \alpha - z \cdot \eta(t) g)(t) = 0$, and this requires the vanishing of $\det(z \cdot \alpha - z \cdot \eta(t) g)$. By dividing that determinant by $(-z \cdot \eta(t))^k$, we conclude that

$$\det(g - \frac{z}{z \cdot \eta(t)} \cdot \alpha) = 0,$$

that is

$$\frac{z}{z \cdot \eta(t)} = \frac{z}{z \cdot z} = R(z) \in \mathcal{F}.$$

We have thus proved:

Proposition 3.2. *Let $z \in N_p M$ a point in the inverted pedal of η_p. Then $z \in \mathcal{F}_p(M)$.*

Let us see whether there is some form of converse of this.

Let $x \in N$ be a point in $\mathcal{F}$. Then $\det(g - x \cdot \alpha) = 0$. Let $t \in T,\ t \cdot t = 1$, be such that $g(t) = x \cdot \alpha(t)$. We know that then $x \cdot \eta(t) = 1$ and $\eta_p(t) \notin (d\eta_p)(T_t T_p M)$. Let $z = \mathrm{ped}_\eta(t)$; if $z = 0$ we would have $\eta(t) \in (d\eta_p)(T_t T_p M)$, which is absurd. As we have see before we will have $(z \cdot z) g(t) = z \cdot \alpha(t)$, from which we obtain $g(t) = \frac{z}{z \cdot z} \cdot \alpha(t)$. Therefore $(\frac{z}{z \cdot z} - x) \cdot \alpha(t) = 0$, that is $\frac{z}{z \cdot z} - x$ must be orthogonal to the subspace of N generated by $\alpha(t)$, that is to the subspace $\alpha(t,T)^\perp$.

If $\dim \alpha(t,T) = r$, the condition on $\frac{z}{z \cdot z} - x$ means that it should belong to an $(n-r)-$dimensional affine subspace of N. We can write this in the form

$$x = \frac{z}{z \cdot z} + u = R(z) + u,\ u \in \alpha(t,T)^\perp.$$

As before, let $\widetilde{PT} = \{[t] \in PT\ :\ \mathrm{ped}_\eta(t) \neq 0\}$, so that $R(\mathrm{ped}_\eta([t])) \in \mathcal{F}$. Let us put $B_t = R(\mathrm{ped}_\eta([t])) + \alpha(t,T)^\perp$. We have proved thus

Theorem 3.3. *$\mathcal{F}$ is the union of the inverted pedal of η with $\cup_{t \in \widetilde{PT}} B_t$.*

This describes completely $\mathcal{F}(M)$. Note that $\dim \alpha(t,T) \leq k$. Hence, if for example $k = 2$ (M is thus a surface) and $n = 2$ then generically the dimension of $\mathcal{F}_p$ will be that of $\eta(PT_p M)$, that is one; thus, then $\mathcal{F}_p$ will be a conic. If $n = 3$, it will be generically a ruled quadric surface, because, if $R(z) \in \mathcal{F}_p$, then $R(z) + \alpha_p(t, T_p M)^\perp \in \mathcal{F}_p$, and $\alpha_p(t, T_p M)^\perp$ will have generically dimension equal to one.

4. Focal set at a semiumbilic point of a surface in $\mathbb{R}^4$.

From now on, M will be a surface immersed in $\mathbb{R}^4$, that is $k, n = 2$. If $p \in M$ and (t_1, t_2) is an orthonormal basis of $T_p M$, le us put $b_1 = \eta_p(t_1) = \alpha_p(t_1, t_1)$, $b_2 = \eta_p(t_2) = \alpha_p(t_2, t_2)$, $b_3 = \alpha_p(t_1, t_2)$. If $t = t_1 \cos\theta + t_2 \sin\theta$, we will have $t \cdot t = 1$ and $\eta_p(t) = b_1 \cos^2\theta + b_2 \sin^2\theta + b_3 \sin 2\theta = H + B\cos 2\theta + C \sin 2\theta$, where

$$H = \frac{1}{2}(b_1 + b_2), \quad B = \frac{1}{2}(b_1 - b_2), \quad C = b_3.$$

The image of the map η_p is thus an ellipse in $N_p M$ centered in H called *curvature ellipse.* That ellipse can degenerate to a segment or to a point. The vector H, which does not depend on the choice of the orthonormal basis (t_1, t_2), is called *mean curvature vector.* The other two, B and C, depend on that choice.The point p is said to be *semiumbilic* if the curvature ellipse at p degenerates to a segment, that is if B and C are linearly dependent. If in addition the straight line containing that segment passes by the origin, p is said to be an *inflection point.* If the curvature ellipse degenerates to a point the p is called *umbilic.* Let us see how is the focal set $\mathcal{F}_p(M)$ at a semiumbilic point p.

Assume that $B \neq 0$. Then, we can put $C = qB$ and the ellipse is written as $\eta_p(t) = H + (\cos 2\theta + q \sin 2\theta)B$. The end points of this segment are obtained when the derivative of this function with respect to θ vanishes, that is when $-\sin 2\theta + q \cos 2\theta = 0$, that is when $\tan 2\theta = q$. Therefore when

$$\cos 2\theta = \frac{1}{\sqrt{1+q^2}}, \quad \sin 2\theta = \frac{q}{\sqrt{1+q^2}},$$

or their opposites. The extreme points of the ellipse are thus

$$H \pm \sqrt{1+q^2}\, B = H \pm \sqrt{B \cdot B + C \cdot C}\frac{B}{\sqrt{B \cdot B}} = H \pm \sqrt{B \cdot B + C \cdot C}\, n,$$

where n is any unit vector parallel to the segment that is the ellipse. This formula does not depend on the choice between B and C. We put $v_1 = H + \sqrt{1+q^2}\, n$ and $v_2 = H - \sqrt{1+q^2}\, n$. We have found unit vectors $s_1, s_2 \in T_pM$ such that $\eta_p(s_1) = v_1$ and $\eta_p(s_2) = v_2$. Then $\mathrm{ped}_{\eta_p}(s_1)$ is the point nearest to the origin among those of the set

$$\{\eta_p(s_1) + (d\eta_p)_{s_1}(t') : t' \in \mathbb{R}\} = \{v_1\}.$$

There is only one point in that set. Moreover, we have $\alpha_p(s_1, t') = 0$ when t' is tangent at s_1 to the unit circle in T_pM, that is $\alpha_p(s_1, Js_1) = 0$. Since we know that $\alpha_p(s_1, s_1) = v_1$, we conclude that $\alpha_p(s_1, T_pM) = \mathbb{R}v_1$ and from this that $\alpha_p(s_1, T_pM)^\perp = \mathbb{R}Jv_1$. In the same manner, we get $\alpha_p(s_2, T_pM)^\perp = \mathbb{R}Jv_2$. Therefore, we have two straight lines in $\mathcal{F}_p$ given by

$$\lambda \mapsto \frac{v_1}{v_1 \cdot v_1} + \lambda J v_1, \quad \lambda \mapsto \frac{v_2}{v_2 \cdot v_2} + \lambda J v_2.$$

If $\tan 2\theta \neq q$ then the affine straight line tangent to the curve $\eta_p(t)$ is constant. Therefore, the point nearest to the origin in that line is always the same, and by continuity this implies that that point is the intersection of the two straight lines that we have just computed. Therefore we have:

Proposition 4.1. *Let p be a non-of-inflection umbilic point in the surface M immersed in $\mathbb{R}^4$. Then*

$$\mathcal{F}_p(M) = \left\{ \frac{v_1}{v_1 \cdot v_1} + \lambda J v_1 : \lambda \in \mathbb{R} \right\} \cup \left\{ \frac{v_2}{v_2 \cdot v_2} + \lambda J v_2 : \lambda \in \mathbb{R} \right\}.$$

If p is an inflection point then the vectors v_1 and v_2 are linearly dependent. Therefore, $\mathcal{F}_p(M)$ consists of two parallel straigh lines. Finally, if p is umbilic then $v_1 = v_2$, so that both lines coincide.

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