Asymptotic curves on surfaces in $I\!\!R^5$ and strong principal directions on surfaces in $I\!\!R^4$

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1 The Taylor expansion of the exponential map

Let M be a surface in $\mathbb{R}^n$ and let $m \in M$. We know that there is an open neighborhood U_m of $0 \in T_mM$ such that the exponential map $\exp_m : U_m \to \mathbb{R}^n$ is an one-to-one immersion. We recall also that $\exp_m(x) = \gamma_x(1)$, where $\gamma_x : [0,1] \to \mathbb{R}^n$ is the geodesic in M with initial condition $\gamma(0) = m$, $\gamma_x'(0) = x$. We shall consider the Taylor expansion of $\exp_m$ around the origin of T_mM. It will be written as

$$\exp_m(x) = m + I_m(x) + \frac{1}{2}Q_m(x) + \frac{1}{6}K_m(x) + \ldots,$$

where I_m, Q_m, K_m are respectively linear, quadratic and cubic forms in T_mM with values in $\mathbb{R}^n$.

Our purpose is to write these forms in terms more familiar with the usual techniques of differential geometry. Let $x \in U_m$ and put $x = tv$, where $t \in \mathbb{R}$ and $v \in T_mM$ is a unit vector. Then, as it is well known, $\exp_m(x) = \exp_m(tv) = \gamma_v(t)$. Therefore

$$\gamma_v(t) = m + I_m(v)t + \frac{1}{2}Q_m(v)t^2 + \frac{1}{6}K_m(v)t^3 + O(t^4).$$

Hence, $\gamma_v'(0) = v = I_m(v)$, so that $I_m : T_mM \to \mathbb{R}^5$ is the inclusion. We also have $\gamma_v''(0) = Q_m(v)$ and $\gamma_v'''(0) = K_m(v)$.

Now, γ_v is a geodesic in M and this implies that at every t we have $\gamma_v''(t) \in N_{\gamma_v(t)}M$, where N_pM denotes the subspace of $\mathbb{R}^n$ orthogonal to T_pM.

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In fact, we have then $\gamma_v''(t) = \alpha_{\gamma_v(t)}(\gamma'(t), \gamma'(t))$, where α denotes the second fundamental form of M. Hence,

$$Q_m(v) = \gamma_v''(0) = \alpha_m(v, v).$$

Thus, it is clear that the second order geometry of M around m is determined by the value at m of the second fundamental form of M. Let us study the third order geometry.

2 Third order geometry

Let $\xi \in T_m M$. We may make the parallel transport of ξ along the geodesic γ_v in order to have a parallel vector field $X(t)$ along that geodesic. This means that $X(0) = \xi,\ X(t) \in T_{\gamma_v(t)} M$ and that $X'(t) \in N_{\gamma_v(t)} M$. Then, we will have $X \cdot \gamma_v'' = 0$. Differentiating, we get

$$\begin{aligned} X \cdot \gamma_v''' &= -X' \cdot \gamma_v'' = -X' \cdot \alpha(\gamma_v', \gamma_v') = -(D_{\gamma_v'} X) \cdot \alpha(\gamma_v', \gamma_v') \\ &= -\alpha(X, \gamma_v') \cdot \alpha(\gamma_v', \gamma_v'). \end{aligned}$$

Hence, by evaluation at $t = 0$ we have

$$\xi \cdot K_m(v) = \xi \cdot \gamma_v'''(0) = -\alpha_m(\xi, v) \cdot \alpha_m(v, v).$$

We observe thus that the tangential part of the third order geometry at m depends only on the second order geometry at m.

Now, let $\zeta \in N_m M$. As before, we define the vector field $Z(t)$ along the curve γ_v as the parallel transport of ζ. Thus, for any t we will have $Z(t) \in N_{\gamma_v(t)} M$ and $Z'(t) \in T_{\gamma_v(t)} M$. Hence $Z' \cdot \gamma_v'' = 0$. Thus

$$Z \cdot \gamma_v''' = (Z \cdot \gamma_v'')' = (Z \cdot \alpha(\gamma_v', \gamma_v'))' = Z \cdot \big(\nabla_{\gamma_v'} \alpha\big)(\gamma_v', \gamma_v'),$$

where ∇ is the connection induced in the vector bundle $NM \otimes T^*M \otimes T^*M \to M$ by the connection $\nabla^\top$ in the vector bundle TM and the connection $\nabla^\perp$ in the vector bundle NM. Let us explain this. Let β be a section of the vector bundle $NM \otimes T^*M \otimes T^*M \to M$. Then, β is completely determined by its action upon any two sections of TM, namely X, Y, and any section of NM, namely W, given by $(W, X, Y) \mapsto W \cdot \beta(X, Y)$. Then, if $A \in \mathfrak{X}(M)$, the section $\nabla_A \beta$ of $NM \otimes T^*M \otimes T^*M \to M$ is determined by the knowledge of $W \cdot \big(\nabla_A \beta\big)(X, Y)$ for any X, Y, W as before. Well then, we define

$$\begin{aligned} &W \cdot \big(\nabla_A \beta\big)(X, Y) \\ &\quad = D_A\big(W \cdot \beta(X, Y)\big) \\ &\qquad - (\nabla_A^\perp W) \cdot \beta(X, Y) - W \cdot \beta(\nabla_A^\top X, Y) - W \cdot \beta(X, \nabla_A^\top Y). \end{aligned}$$

It is now easy to prove that this defines a linear connection for sections of that fibre bundle.

Our result follows once one takes account that Z and γ_v' are parallel along γ_v and that $(Z \cdot \alpha(\gamma_v', \gamma_v'))' = D_{\gamma_v'}(Z \cdot \alpha(\gamma_v', \gamma_v'))$.

We have thus that $\zeta \cdot K_m(v) = \zeta \cdot \big(\nabla_v \alpha\big)(v, v)$. We conclude that, for any $u \in \mathbb{R}^n$ and for any $x \in U_m$, we have

$$\begin{aligned} u \cdot \exp_m(x) =& u \cdot m + u \cdot x + \frac{1}{2} u \cdot \alpha_m(x, x) \\ & - \frac{1}{6} \alpha_m(u^\top, x) \cdot \alpha_m(x, x) + \frac{1}{6} u^\perp \cdot (\nabla_x \alpha)(x, x) + O(|x|^4) \end{aligned}$$

Note that we have not used anywhere the dimension of M. Thus all these results are valid for any submanifold in $\mathbb{R}^n$.

Below, we will denote by $f_{3,u}$ the third order approximation of the function $x \in T_m M \mapsto u \cdot (\exp_m(x) - m)$, that is

$$f_{3,u}(x) = u \cdot x + \frac{1}{2} u \cdot \alpha_m(x, x) - \frac{1}{6} \alpha_m(u^\top, x) \cdot \alpha_m(x, x) + \frac{1}{6} u^\perp \cdot (\nabla_x \alpha)(x, x).$$

3 Application to the asymptotic directions for a surface in $I\!R^5$

Let M be a surface immersed in $\mathbb{R}^5$. Let us reword the characterization of asymptotic directions at a point $m \in M$ given in [1], p. 1006.

Definition 3.1 *Let* $0 \neq u \in \mathbb{R}^5$. *Then,* u *determines a binormal direction* at m iff the following conditions are true: (i) 0 is a singular point of $f_{3,u}$; (ii) there is a non-vanishing vector $x \in T_m M$ such that $u \cdot \alpha_m(x, y) = 0$ for any $y \in T_m M$ and such that $f_{3,u}(x) = 0$. We say that such a vector x defines an *asymptotic direction* at m.

Assume that $0 \neq x \in T_m M$ defines an asymptotic direction. Then there is $u \in \mathbb{R}^5$ with the two properties of the above definition. These are equivalent clearly to the requirements that $u \in N_m M$, that $u \cdot \alpha_m(x, .) = 0$ and that $u \cdot (\nabla_x \alpha)(x, x) = 0$. Now, let t_1, t_2 be any basis of $T_m M$. Then the three vectors $\alpha_m(x, t_1)$, $\alpha_m(x, t_2)$, $(\nabla_x \alpha)(x, x) \in N_m M$ must have a non-vanishing vector $u \in N_m M$ orthogonal to them all. Since $\dim N_m M = 3$, we conclude that the necessary and sufficient condition for x being an asymptotic direction is that those three vectors be linearly dependent, that is

$$\det \big(\alpha_m(x, t_1), \alpha_m(x, t_2), (\nabla_x \alpha)(x, x)\big) = 0. \tag{1}$$

This is a homogeneous polynomial of degree five in the (two) components of x, that is, the asymptotic lines on the surface M immersed in $\mathbb{R}^5$ must satisfy a homogeneous binary equation of degree 5.

Now, we are going to prepare this formula for the effective computation of those asymptotic lines. Assume that (t_1, t_2) is an orthonormal reference of TM in a neighborhood of m, and that we put $b_1 = \alpha(t_1, t_1)$, $b_2 = \alpha(t_2, t_2)$, $b_3 = \alpha(t_1, t_2)$. Then, $\alpha = b_1 \otimes t_1^\sharp \otimes t_1^\sharp + b_2 \otimes t_2^\sharp \otimes t_2^\sharp + b_3 \otimes (t_1^\sharp \otimes t_2^\sharp + t_2^\sharp \otimes t_1^\sharp)$, where $t_i^\sharp$ is the 1-form defined by $t_i^\sharp(z) = t_1 \cdot z$.

We have $\nabla_x^\perp b_i = (D_x b_i)^\perp = D_x b_i - (D_x b_i \cdot t_1)t_1 - (D_x b_i \cdot t_2)t_2$. Now we note that the equation 1 may be equivalently written as

$$\det\big(t_1, t_2, \alpha_m(x, t_1), \alpha_m(x, t_2), (\nabla_x \alpha)(x, x)\big) = 0, \tag{2}$$

and this means that in equation 2 we may replace $D_x b_i$ by $\nabla_x^\perp b_i$. Also, $(\nabla_x^\top t_i^\sharp)(x) = D_x(t_i^\sharp(x)) - t_i^\sharp\big((D_x t_i)^\top\big) = x(t_i^\sharp(x)) - t_i^\sharp(D_x t_i) = (D_x t_i^\sharp)(x) = (D_x t_i) \cdot x = (t_j \cdot D_x t_i)(t_j \cdot x)$, where $i = 1, 2$ and $i \neq j \in \{1, 2\}$. Therefore, if we put $x = ct_1 + dt_2$, we have

$$\big(D_x(t_1^\sharp \otimes t_1^\sharp)\big)(x, x)b_1 = 2c(D_x t_1^\sharp)(x)b_1 = 2kcd\, b_1,$$

where we have put

$$k = (t_2 \cdot D_x t_1) = -(t_1 \cdot D_x t_2) = c(t_2 \cdot D_{t_1} t_1) + d(t_2 \cdot D_{t_2} t_1).$$

In the same manner we have

$$\big(D_x(t_2^\sharp \otimes t_2^\sharp)\big)(x, x)b_2 = -2kcd\, b_2,$$

and

$$\big(D_x(t_1^\sharp \otimes t_2^\sharp)\big)(x, x)b_3 = \big(D_x(t_2^\sharp \otimes t_1^\sharp)\big)(x, x)b_3 = k(d^2 - c^2)b_3.$$

Hence, for the computation of 2, instead of $(\nabla_x \alpha)(x, x)$ we may write

$$c^2 D_x b_1 + d^2 D_x b_2 + 2cd D_x b_3 + 2kcd(b_1 - b_2) + 2k(d^2 - c^2)b_3.$$

We compute first the part

$$\det(\alpha_m(x, t_1), \alpha_m(x, t_2), 2kcd(b_1 - b_2) + 2k(d^2 - c^2)b_3).$$

It is equal to

$$\begin{aligned}
&\det\big(cb_1 + db_3, cb_3 + db_2, 2kcd\, b_1 - 2kcd\, b_2 + 2k(d^2 - c^2)b_3\big)\\
=&2kcd^3 \det(b_3, b_2, b_1) - 2kc^3 d \det(b_1, b_3, b_2) + 2kcd(d^2 - c^2) \det(b_1, b_2, b_3)\\
=&(-2kcd^3 + 2kc^3 d + 2kcd^3 - 2kc^3 d) \det(b_1, b_2, b_3)\\
=&0.
\end{aligned}$$

Therefore we may write the equation of asymptotic directions 2 as

$$\det(t_1, t_2, cb_1 + db_3, cb_3 + db_2, c^2 D_x b_1 + d^2 D_x b_2 + 2cd D_x b_3) = 0,$$

where every expression may be computed without special difficulty.

4 The equations in terms of a chart

Assume that M is parameterized by a chart $X : \mathbb{R}^2 \to \mathbb{R}^5$ with coordinates (u, v). Then, let $x = aX_u + bX_v$. The equation of asymtpotic lines may be written as

$$\det(X_u, X_v, \alpha(x, X_u), \alpha(x, X_v), (\nabla_x \alpha)(x.x)) = 0$$

For brevity, we shall use the following notation

$$\det{}^{\perp}(P, ..., Q) = \det(X_u, X_v, P, ..., Q).$$

Now $\alpha(x, X_u) = (aX_{uu} + bX_{vu})^{\perp}$ and an analogous formula for $\alpha(x, X_u)$. Hence, the equation is

$$\det{}^{\perp}(aX_{uu} + bX_{vu}, aX_{uv} + bX_{vv}, (\nabla_x \alpha)(x.x)) = 0.$$

The last vector field in the determinant can be seen as a sum of terms like $a^3(\nabla_{X_u}\alpha)(X_u, X_u)$, $a^2b(\nabla_{X_u}\alpha)(X_u, X_v), \ldots$. Let us express them in terms of the derivatives of X. We have for the first one

$$\begin{aligned}(\nabla_{X_u}\alpha)(X_u, X_u) &= \nabla^{\perp}_{X_u}\big(\alpha(X_u, X_u)\big) - 2\alpha(X^{\top}_{uu}, X_u)\\ &= \Big(D_{X_u}X^{\perp}_{uu} - 2D_{X_u}X^{\top}_{uu}\Big)^{\perp} = \Big(X_{uuu} - 3D_{X_u}X^{\top}_{uu}\Big)^{\perp}.\end{aligned}$$

This contributes to the determinant by the term

$$a^3 \ \det{}^{\perp}(aX_{uu} + bX_{vu}, aX_{uv} + bX_{vv}, X_{uuu} - 3D_{X_u}X^{\top}_{uu}).$$

Thus, we need to compute $\big(D_{X_u}X^{\top}_{uu}\big)^{\perp}$. Suppose that $X^{\top}_{uu} = p_{uu}X_u + q_{uu}X_v$ for some functions p_{uu}, q_{uu}. Then, $\big(D_{X_u}X^{\top}_{uu}\big)^{\perp} = (p_{uu}X_{uu} + q_{uu}X_{uv})^{\perp}$. Hence, the term of the determinant will be

$$\begin{aligned}&a^3 \ \det{}^{\perp}(aX_{uu} + bX_{vu}, aX_{uv} + bX_{vv}, X_{uuu} - 3(p_{uu}X_{uu} + q_{uu}X_{uv})\\ =&a^5 \det{}^{\perp}(X_{uu}, X_{uv}, X_{uuu}) + a^4 b \det{}^{\perp}(X_{uu}, X_{vv}, X_{uuu} - 3q_{uu}X_{uv})\\ &+ a^3b^2 \det{}^{\perp}(X_{uv}, X_{vv}, X_{uuu} - 3p_{uu}X_{uu})\end{aligned}$$

We need to compute p_{uu} and q_{uu}. We will have the following linear system, where $E = X_u \cdot X_u,\ F = X_u \cdot X_v,\ G = X_v \cdot X_v$:

$$\begin{aligned} Ep_{uu} + Fq_{uu} &= X_{uu} \cdot X_u \\ Fp_{uu} + Gq_{uu} &= X_{uu} \cdot X_v. \end{aligned}$$

whose solution is $p_{uu} = \frac{1}{S}\big((X_{uu} \cdot X_u)G - (X_{uu} \cdot X_v)F\big)$ and $q_{uu} = \frac{1}{S}\big((X_{uu} \cdot X_v)E - (X_{uu} \cdot X_u)F\big)$, where $S = EG - F^2$.

For $\left(D_{X_v} X_{vv}^{\top}\right)^{\perp}$ the computation is almost identical. That is, the contribution to the determinant is

$$\begin{aligned} &b^3\ \det{}^{\perp}(aX_{uu} + bX_{vu}, aX_{uv} + bX_{vv}, X_{vvv} - 3(p_{vv}X_{uv} + q_{vv}X_{vv}) \\ =&a^2b^3 \det{}^{\perp}(X_{uu}, X_{uv}, X_{vvv} - 3q_{vv}X_{vv}) + ab^4 \det{}^{\perp}(X_{uu}, X_{vv}, X_{vvv} - 3p_{vv}X_{uv}) \\ &+ b^5 \det{}^{\perp}(X_{uv}, X_{vv}, X_{vvv}), \end{aligned}$$

where

$$p_{vv} = \frac{1}{S}\big((X_{vv} \cdot X_u)G - (X_{vv} \cdot X_v)F\big), \quad q_{vv} = \frac{1}{S}\big((X_{vv} \cdot X_v)E - (X_{vv} \cdot X_u)F\big)$$

Let us study the part that depends on $a^2b(\nabla_{X_u}\alpha)(X_u, X_v)$. We have

$$\begin{aligned} (\nabla_{X_u}\alpha)(X_u, X_v) &= \nabla^{\perp}_{X_u}\big(\alpha(X_u, X_v)\big) - \alpha(X_u, X^{\top}_{uv}) - \alpha(X^{\top}_{uu}, X_v) \\ &= \Big(D_{X_u}X^{\perp}_{uv} - D_{X_u}X^{\top}_{uv} - D_{X_v}X^{\top}_{uu}\Big)^{\perp} \\ &= \Big(X_{uuv} - 2D_{X_u}X^{\top}_{uv} - D_{X_v}X^{\top}_{uu}\Big)^{\perp}. \end{aligned}$$

We need to calculate the functions p_{uv}, q_{uv} such that $X^{\top}_{uv} = p_{uv}X_u + q_{uv}X_v$. They will be

$$p_{uv} = \frac{1}{S}\big((X_{uv} \cdot X_u)G - (X_{uv} \cdot X_v)F\big), \quad q_{uv} = \frac{1}{S}\big((X_{uv} \cdot X_v)E - (X_{uv} \cdot X_u)F\big).$$

The contribution to the determinant will be

$$\begin{aligned} &a^2b\ \det{}^{\perp}(aX_{uu} + bX_{vu}, aX_{uv} + bX_{vv}, X_{uuv} - 2(p_{uv}X_{uu} + q_{uv}X_{uv}) - p_{uu}X_{uv} - q_{uu}X_{vv}) \\ &=a^4b \det{}^{\perp}(X_{uu}, X_{uv}, X_{uuv} - q_{uu}X_{vv}) + a^3b^2 \det{}^{\perp}(X_{uu}, X_{vv}, X_{uuv} - (2q_{uv} + p_{uu})X_{uv}) \\ &\quad + a^2b^3 \det{}^{\perp}(X_{uv}, X_{vv}, X_{uuv} - 2p_{uv}X_{uu}) \end{aligned}$$

In the same manner we compute the term with $ab^2(\nabla_{X_v}\alpha)(X_u,X_v)$. We will have

$$(\nabla_{X_v}\alpha)(X_u,X_v)=\Big(X_{uvv}-2D_{X_v}X_{uv}^{\top}-D_{X_u}X_{vv}^{\top}\Big)^{\perp}.$$

The contribution to the determinant will be

$$\begin{aligned}
&ab^2\ \det{}^{\perp}(aX_{uu}+bX_{vu},aX_{uv}+bX_{vv},X_{uvv}-2(p_{uv}X_{uv}+q_{uv}X_{vv})-p_{vv}X_{uu}-q_{vv}X_{uv})\\
&=a^3b^2\det{}^{\perp}(X_{uu},X_{uv},X_{uvv}-2q_{uv}X_{vv})+a^2b^3\det(X_{uu},X_{vv},X_{uvv}-(2p_{uv}+q_{vv})X_{uv})\\
&\quad+ab^4\det{}^{\perp}(X_{uv},X_{vv},X_{uvv}-p_{vv}X_{uu})
\end{aligned}$$

We compute now the term with $ab^2(\nabla_{X_u}\alpha)(X_v,X_v)$. We will have

$$\begin{aligned}
(\nabla_{X_u}\alpha)(X_v,X_v)&=\nabla^{\perp}_{X_u}\big(\alpha(X_v,X_v)\big)-2\alpha(X_v,X_{uv}^{\top}))\\
&=\Big(D_{X_u}X_{vv}^{\perp}-2D_{X_v}X_{uv}^{\top}\Big)^{\perp}\\
&=\Big(X_{uvv}-D_{X_u}X_{vv}^{\top}-2D_{X_v}X_{uv}^{\top}\Big)^{\perp}.
\end{aligned}$$

Its contribution to the determinant is the same as in the preceding case. In the same manner we will get that the contribution corresponding to

$$a^2b(\nabla_{X_v}\alpha)(X_u,X_u)$$

will be the same as that corresponding to

$$a^2b(\nabla_{X_u}\alpha)(X_u,X_v),$$

which we already have computed.

Our task from now on will be to compute the coefficients on $a^5,a^4b,\ldots,b^5$ in the determinant. First, the coefficient of a^5. It will be

$$\det{}^{\perp}(X_{uu},X_{uv},X_{uuu}).$$

That of b^5 will be

$$\det{}^{\perp}(X_{uv},X_{vv},X_{vvv}).$$

The coefficient of a^4b will be

$$\begin{aligned}
&\det{}^{\perp}(X_{uu},X_{vv},X_{uuu}-3q_{uu}X_{uv})\\
&+3\det{}^{\perp}(X_{uu},X_{uv},X_{uuv}-q_{uu}X_{vv})\\
=&\det{}^{\perp}(X_{uu},X_{vv},X_{uuu})+3\det{}^{\perp}(X_{uu},X_{uv},X_{uuv}).
\end{aligned}$$

In the same manner, the coefficient of ab^4 will be

$$\det{}^{\perp}(X_{uu}, X_{vv}, X_{vvv}) + 3\det{}^{\perp}(X_{uv}, X_{vv}, X_{uvv}).$$

That of a^3b^2 will be

$$\begin{aligned} & \det{}^{\perp}(X_{uv}, X_{vv}, X_{uuu} - 3p_{uu}X_{uu}) \\ & + 3\det{}^{\perp}(X_{uu}, X_{uv}, X_{uvv} - 2q_{uv}X_{vv}) \\ & + 3\det{}^{\perp}(X_{uu}, X_{vv}, X_{uuv} - (2q_{uv} + p_{uu})X_{uv}) \\ = \; & \det{}^{\perp}(X_{uv}, X_{vv}, X_{uuu}) \\ & + 3\det{}^{\perp}(X_{uu}, X_{uv}, X_{uvv}) \\ & + 3\det{}^{\perp}(X_{uu}, X_{vv}, X_{uuv}). \end{aligned}$$

Finally, that of a^2b^3 will be

$$\det{}^{\perp}(X_{uu}, X_{uv}, X_{vvv}) + 3\det{}^{\perp}(X_{uv}, X_{vv}, X_{uuv}) + 3\det{}^{\perp}(X_{uu}, X_{vv}, X_{uvv}).$$

5 Strong principal directions of surfaces in $\mathbb{R}^4$

A study of these directions is given in [2] The equation to solve is now

$$\det\big(\alpha(x, Jx), (\nabla_x\alpha)(x, x)\big) = 0.$$

Let us use the same notation as in the preceding section. Then $x = aX_u + bX_v$. Also

$$JX_u = \frac{-FX_u + EX_v}{d} \qquad JX_v = \frac{FX_v - GX_u}{d},$$

where $d = \sqrt{EG - F^2}$. Therefore

$$\begin{aligned} \alpha(x, Jx) &= \frac{\alpha\big(aX_u + bX_v, a(-FX_u + EX_v) + b(FX_v - GX_u)\big)}{d} \\ &= \frac{1}{d}\big(- (a^2F + abG)X_{uu} + (a^2E - b^2G)X_{uv} + (abE + b^2F)X_{vv}\big)^{\perp}. \end{aligned}$$

Thus an equivalent equation is given by

$$\det{}^{\perp}\Big(r_{aa}a^2 + r_{ab}ab + r_{bb}b^2, (\nabla_x\alpha)(x, x)\Big) = 0,$$

where

$$r_{aa} = -FX_{uu} + EX_{uv}, \quad r_{ab} = -GX_{uu} + EX_{vv}, \quad r_{bb} = FX_{vv} - GX_{uv}.$$

Now, according with the results of the above section, we can write the left hand side of the equation as

$$
\begin{aligned}
&a^3 \det{}^{\perp}(r_{aa}a^2 + r_{ab}ab + r_{bb}b^2, Y_{uuu})\\
&+3a^2b \det{}^{\perp}(r_{aa}a^2 + r_{ab}ab + r_{bb}b^2, Y_{uuv})\\
&+3ab^2 \det{}^{\perp}(r_{aa}a^2 + r_{ab}ab + r_{bb}b^2, Y_{uvv})\\
&+b^3 \det{}^{\perp}(r_{aa}a^2 + r_{ab}ab + r_{bb}b^2, Y_{vvv}),
\end{aligned}
$$

where we have put

$$
\begin{aligned}
Y_{uuu} &= X_{uuu} - 3(p_{uu}X_{uu} + q_{uu}X_{uv}),\\
Y_{uuv} &= X_{uuv} - 2p_{uv}X_{uu} - (2q_{uv} + p_{uu})X_{uv} - q_{uu}X_{vv},\\
Y_{uvv} &= X_{uvv} - 2q_{uv}X_{vv} - (2p_{uv} + q_{vv})X_{uv} - p_{vv}X_{uu},\\
Y_{vvv} &= X_{vvv} - 3(q_{vv}X_{vv} + p_{vv}X_{uv}).
\end{aligned}
$$

Then the coefficient in a^5 in the equation will be

$$
\begin{aligned}
&\det{}^{\perp}\big(r_{aa}, X_{uuu} - 3(p_{uu}X_{uu} + q_{uu}X_{uv})\big)\\
&\quad = \det{}^{\perp}(r_{aa}, X_{uuu}) + 3(Fq_{uu} + Ep_{uu}) \det{}^{\perp}(X_{uu}, X_{uv}).
\end{aligned}
$$

The coefficient in a^4b will be

$$
\begin{aligned}
&\det{}^{\perp}(r_{ab}, Y_{uuu}) + 3\det{}^{\perp}(r_{aa}, Y_{uuv})\\
&\quad = \det{}^{\perp}(r_{ab}, X_{uuu}) + 3\det{}^{\perp}(r_{aa}, X_{uuv})\\
&\qquad + \det{}^{\perp}(-GX_{uu} + EX_{vv}, -3(p_{uu}X_{uu} + q_{uu}X_{uv}))\\
&\qquad + 3\det{}^{\perp}(-FX_{uu} + EX_{uv}, -2p_{uv}X_{uu} - (2q_{uv} + p_{uu})X_{uv} - q_{uu}X_{vv})\\
&\quad = \det{}^{\perp}(r_{ab}, X_{uuu}) + 3\det{}^{\perp}(r_{aa}, X_{uuv})\\
&\qquad + 3\big(F(2q_{uv} + p_{uu}) + 2Ep_{uv} + Gq_{uu}\big) \det{}^{\perp}(X_{uu}, X_{uv})\\
&\qquad + 3(Ep_{uu} + Fq_{uu}) \det{}^{\perp}(X_{uu}, X_{vv}).
\end{aligned}
$$

That of a^3b^2 will be:

$$\begin{aligned}
&\det{}^{\perp}(r_{bb}, Y_{uuu}) + 3\det{}^{\perp}(r_{ab}, Y_{uuv}) + 3\det{}^{\perp}(r_{aa}, Y_{uvv}) \\
&\quad = \det{}^{\perp}(r_{bb}, X_{uuu}) + 3\det{}^{\perp}(r_{ab}, X_{uuv}) + 3\det{}^{\perp}(r_{aa}, X_{uvv}) \\
&\qquad - 3\det{}^{\perp}(FX_{vv} - GX_{uv}, p_{uu}X_{uu} + q_{uu}X_{uv}) \\
&\qquad + 3\det{}^{\perp}(-GX_{uu} + EX_{vv}, -2p_{uv}X_{uu} - (2q_{uv} + p_{uu})X_{uv} - q_{uu}X_{vv}) \\
&\qquad + 3\det{}^{\perp}(-FX_{uu} + EX_{uv}, -2q_{uv}X_{vv} - (2p_{uv} + q_{vv})X_{uv} - p_{vv}X_{uu}) \\
&\quad = \det{}^{\perp}(r_{bb}, X_{uuu}) + 3\det{}^{\perp}(r_{ab}, X_{uuv}) + 3\det{}^{\perp}(r_{aa}, X_{uvv}) \\
&\qquad + 3\big(2Gq_{uv} + F(2p_{uv} + q_{vv}) + Ep_{vv}\big)\det{}^{\perp}(X_{uu}, X_{uv}) \\
&\qquad + 3(Fp_{uu} + Gq_{uu} + 2Ep_{uv} + 2Fq_{uv})\det{}^{\perp}(X_{uu}, X_{vv}) \\
&\qquad + 3(Fq_{uu} + Ep_{uu})\det{}^{\perp}(X_{uv}, X_{vv})
\end{aligned}$$

That of a^2b^3 :

$$\begin{aligned}
&\det{}^{\perp}(r_{aa}, Y_{vvv}) + 3\det{}^{\perp}(r_{ab}, Y_{uvv}) + 3\det{}^{\perp}(r_{bb}, Y_{uuv}) \\
&\quad = \det{}^{\perp}(r_{aa}, X_{vvv}) + 3\det{}^{\perp}(r_{ab}, X_{uvv}) + 3\det{}^{\perp}(r_{bb}, X_{uuv}) \\
&\qquad - 3\det{}^{\perp}(-FX_{uu} + EX_{uv}, q_{vv}X_{vv} + p_{vv}X_{uv}) \\
&\qquad + 3\det{}^{\perp}(-GX_{uu} + EX_{vv}, -2q_{uv}X_{vv} - (2p_{uv} + q_{vv})X_{uv} - p_{vv}X_{uu}) \\
&\qquad + 3\det{}^{\perp}(FX_{vv} - GX_{uv}, -2p_{uv}X_{uu} - (2q_{uv} + p_{uu})X_{uv} - q_{uu}X_{vv}) \\
&\quad = \det{}^{\perp}(r_{aa}, X_{vvv}) + 3\det{}^{\perp}(r_{ab}, X_{uvv}) + 3\det{}^{\perp}(r_{bb}, X_{uuv}) \\
&\qquad + 3(Fp_{vv} + Gq_{vv})\det{}^{\perp}(X_{uu}, X_{uv}) \\
&\qquad + 3(Fq_{vv} + 2Gq_{uv} + Ep_{vv} + 2Fp_{uv})\det{}^{\perp}(X_{uu}, X_{vv}) \\
&\qquad + 3(2Ep_{uv} + F(2q_{uv} + p_{uu}) + Gq_{uu})\det{}^{\perp}(X_{uv}, X_{vv})
\end{aligned}$$

That of ab^4 can be obtained by symmetry from that of a^4b, so that it is

$$\begin{aligned}
&\det{}^{\perp}(r_{ab}, X_{vvv}) + 3\det{}^{\perp}(r_{bb}, X_{uvv}) \\
&+ 3\big(F(2p_{uv} + q_{vv}) + 2Gq_{uv} + Ep_{vv}\big)\det{}^{\perp}(X_{uv}, X_{vv}) \\
&+ 3(Gq_{vv} + Fp_{vv})\det{}^{\perp}(X_{uu}, X_{vv}).
\end{aligned}$$

And that of b^5, from that of a^5. It is

$$\det{}^{\perp}(r_{bb}, X_{vvv}) + 3(Fp_{vv} + Gq_{vv})\det{}^{\perp}(X_{uv}, X_{vv}).$$

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