Superficies MANUAL

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1. Introduction

Superficies (that in Spanish means "Surfaces") is a program for visualizing the intrinsic and extrinsic geometry of surfaces in $\mathbb{R}^3$ and also the geometry of a rectangular domain in $\mathbb{R}^2$ endowed with a pseudo-Riemannian metric. It aims to be a teaching tool, but you are free to use it as you like, from designing a pot to analyzing the form of a singularity.

For drawing surfaces, you write down the equations that define a surface and the domain or range where you need to compute it; then, **Superficies** draws the surface in a realistic manner. After drawing it, you can interact with **Superficies** for obtaining geometrical data. That is, as you move the mouse, **Superficies** shows in real time the coordinates and curvatures of the surface point the mouse is walking on. Changing the menu option, you can obtain asymptotic lines and directions, curvature lines and directions, mean curvature lines and directions[1], geodesics, geodesic balls, exponential mapping, lines of constant geodesic curvature, and (with some luck) a geodesic between two points fixed by you. For pseudo-Riemannian planes, parametric surfaces or tubes, you may also draw the parallel transport of a vector along the surface. You can get a wireframe view of the surface, that sometimes allows a better understanding of the surface form. The surfaces admit two free parameters, and you may change them interactively and see in real time the effect in the surface. You may move and zoom the surfaces.

You may change the ambient geometry by choosing "Minkowski space", so that from then on, **Superficies** assumes that the ambient $\mathbb{R}^3$ has the metric:

$$dx^2 + dy^2 - dz^2.$$

Accordingly, you may also get then the "light rays" on the surface, that is the lines on the surface that have null tangents.

You can define the surface in three manners, implicit and parametric and tube. In the implicit way, you must provide **Superficies** with one or several functions in $\mathbb{R}^3$, say $f(x, y, z)$ and $g(x, y, z)$, and fix the range as a sphere or a cube in $\mathbb{R}^3$ with edges parallel to the axes where the surfaces $f = 0$ and $g = 0$ are to be drawn.

In parametric form, you write down three functions $x(u, v), y(u, v), z(u, v)$ that parameterize the surface, and fix the domain in the u, v plane, which must be a rectangle with edges parallel to the axes.

If you choose "Tube..." in the menu, then you must write down the equations $x(t), y(t), z(t)$ of a curve, and then another function $r(t)$. The program will draw

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[1] Mean curvature direction: a direction on a point whose normal curvature equals the arithmetic mean curvature of the surface at that point. Harmonic mean curvature direction: the same with the harmonic mean.

a surface generated by circles $\mathbf{c}(t)$. The circle $\mathbf{c}(t)$ is centered at the point $\mathbf{p}(t) = (x(t), y(t), z(t))$, has radius $r(t)$ and lies in the plane orthogonal to the tangent to the curve at that point $\mathbf{p}(t)$.

When you want to see the geometry of a pseudo-Riemannian rectangle U in $\mathbb{R}^2$, you give the functions $E(x, y), F(x, y), G(x, y)$, and fix the domain U as a rectangle as before. Then, **Superficies** assumes that you have on U a pseudo-Riemannian metric

$$g = Edx^2 + 2Fdxdy + Gdy^2,$$

and draws the rectangle colored according to the scalar curvature of g. As in the case of surfaces, you may get in real time the coordinates and curvature of the point where the mouse is walking on, and the light rays, geodesics, etc. Of course, you haven't curvature, nor mean curvature nor asymptotic lines because they belong in extrinsic geometry, but, as in parametric surfaces or tubes, you have a real time parallel transport.

You may save the surface as a JPEG picture, print it, etc.

2. Defining the surfaces

Double-click or "Open" **Superficies** in the usual way. You will see a window with two white panes for a parameterized surface. The left panel will be called the parameters panel; and the right one, the drawing panel. In the "File" menu you will find the four main options "Parametric Surface...", "Implicit Surface...", "Pseudo-Riemannian Plane..." and "Tube...". When you select one of them, the window adjusts so that you may enter the appropriate data, that is the equations defining the surface and the dominion or range you are interested in. You may have only a window open at a time, so that you will be asked to "Save" the previous one in order not to lose its picture or data before working in a window of different type. The document so saved will have extension ".gtv".

Superficies is rather forgiving about the input of equations. That is, it knows about different spellings of some functions, say "arctan" = "atan". It does not need the times "*" symbol between factors nor even a blank, except in very rare cases; however, if you feel more secure, put a blank space between factors. It recognizes the string "pi" as 3.1415... but not the Greek letter π itself. But it does need that the arguments of functions be put between parentheses. Also, it requires that numbers be written as in the following examples: 7.23, -5890.3, 28, 0.2345, etc. So, number formats as .42, 0.876E-3, etc. are not allowed. Besides the functions:

plus (+), minus (−), times (*), divide (/), and power (aˆb or a**b),

you can use the following functions (after an equal sign between parentheses I have put different spellings) :

sqr (square), sqrt (square root), sin (= sen), cos, tan (= tg), cot (= cotan = cotg), arcsin (= asin = arcsen = asen), arccos (= acos = arcos), arctan (= atan = arctg), exp, ln (= log), sinh (= sh), cosh (= ch), tanh (= th), arcsinh (= asinh = argsh), arccosh (= acosh = argch), arctanh (= atanh = argtanh = argth), abs (absolute value).

You can also use integrals for defining the functions. Suppose that in a formula on the independent variables x, y, z you want to use some integral. The first condition is that it must contain at least one of these variables and no more than two. For instance, the variables y, z. Then you will use this pattern:

int (f(y,z,r); g(y,z); h(y,z)).

That is, it begins with the name *int* or *integral* (both are ok), followed by parentheses and three functions between them, separated by semicolons. The first is the integrand, and the variable upon which the integral is computed must be named r. The second means the lower limit and the third, the upper limit. So the example above means the function of y and z given by

$$\int_{g(y,z)}^{h(y,z)} f(y,z,r)dr.$$

The functions f, g, h that define the integral should be written according with the instructions given above, but with some conditions: they must be free from *parameters* (see below) and they must be free from integrals, that is it is not permitted to put integrals inside integrals. However, in a formula you can put several integrals. Thus, the following would be a valid formula on the variables x, y, z :

cos(Int(x cos(y r); sin(x-y); cos(x+y)))+ z Integral(r xˆ2; x+z; zˆ3).

Suppose that you did select the default, "Parametric Surface..." At the left of the window bottom you see three edit spaces labeled "x(u,v) = ", etc. By default, they show formulas that you can change. For instance, write

$$x(u,v) = \cos(\mathrm{u})\sin(\mathrm{pi\ v}),$$
$$y(u,v) = \sin(\mathrm{u})\sin(\mathrm{piv}),$$
$$z(u,v) = 2\cos(\mathrm{piv}).$$

Then, fix the domain writing

$$0 < u < 2\mathrm{pi}, \quad -1 < v < 1$$

Click in the button "Draw". After some instants, **Superficies** will show the surface in the right panel of the window. You will see another blank panel to the left that represents the domain of the coordinates "u", and "v". Both panels are "synchronized", that is, with the exception of moving the surface, which occurs in the right panel, everything that happens in one of them is mapped in real time on the other panel by means of the map $\mathbf{p}(u,v) = (x(u,v), y(u,v), z(u,v))$ or its inverse.

When the mouse passes near the right bottom corner of the parameter or the surface panel it changes for signaling that you may resize that panel. However, I have put some limits to their sizes that seem reasonable to me (if you think otherwise, send me an e-mail).

Now, select "Implicit Surface..." in the "File" menu, enter a formula for a function on the variables "x", "y", "z", fix the cube or the sphere in $\mathbb{R}^3$ where the surface is to be drawn, and click the "Draw" button. If you are tired of waiting, press the "Stop" button that terminates the task, or click in another application window and leave **Superficies** computing in the background. If you want to draw simultaneously several such surfaces (no more than 10), you may separate their functions by the ampersand sign &. For instance, if you input the following text

$$x^2 + y^2 + z^2 - 1 \quad \& \quad x - 2y + z$$

or, equivalently

$$(x^2 + y^2 + z^2 - 1)(x - 2y + z),$$

the program will draw the unit sphere centered at the origin together with the plane with equation $x - 2y + z = 0$.

Select "Tube...", enter formulas for the functions "x(t)", "y(t)", "z(t)", and "r(t)" and click in "Draw". Once the tube is drawn, it will behave for all purposes as a parametric surface. This shall be understood in the following without further comment.

From version 5.0 on, but only for parametric surfaces or tubes (and from version 6, implicit surfaces also), you may include two parameters in the functions defining the surface: they must be named p or q. There will be shown, for each of them, three edit fields: for its minimum, actual and maximum value. However, parameters are not permitted under integrals.

Now, select "Pseudo-Riemannian Plane..."; if you want so, edit the functions on the variables "x", "y" that define the metric, and the rectangle in $\mathbb{R}^2$ that defines the domain, and click "Draw". The panel will be colored according to the curvature.

Superficies will display in the status panel at the window bottom some messages that may be of use to you, for instance when it does not agree with the input of a function or of the domain.

3. About the rendering

For pseudo-Riemannian planes, **Superficies** draws the rectangle in a color that represents the curvature and whether the metric is definite or not at each point. That is, when the metric is definite (Riemannian) the color is reddish for the points with positive curvature, yellow for negative and white for near zero curvature; when the metric is not definite (pseudo-Riemannian proper) the color is blue for positive, purple for negative and white for near zero curvature. Points where the metric is degenerate are skipped so that they appear white.

For implicit and parametric surfaces, **Superficies** draws each side of the surface in a different color, golden or turquoise by default, but you can pick other colors. It is assumed that the surface is lighted by an ambient light and by a point-like light source that casts shadows in a realistic manner. The computation of shadows is costly; this is the reason that the program suppress it when you are moving the surface.

For parametric surfaces, there is sometimes a problem when the program cannot decide what part of the surface is nearest to the viewer. Then, it might show in an apparently random manner adjacent little triangles golden or turquoise. Usually this is due to "repetitions", that is the surface passes several times by the same points. The only thing that can be done then is to limit the domain for avoiding those repetitions or to increase the "Grid" density in the edit panel of that name at the bottom right corner of the window (the maximum density is 127).

If after a time of drawing lines on the surface you want to wipe out all them, select "Refresh" in the "File" menu or use the shortcut ⟨Control⟩ + ⟨R⟩. You may prefer to wipe only the last one. This is the result of the shortcut ⟨Control⟩ + ⟨Z⟩ : if, on typing it, the mouse was on the panel for 2D drawings, only the last drawn curve in that panel is erased; if it was in the 3D panel, only the last one in that panel; if it was in other place, then both last curves are erased. In order to redraw the window or print them with good resolution, those lines are stored in memory and may occupy a large part of it. Thus, I have put a limit to this memory allocation whose trespassing may pop up on some occasion as a warning. When you "Refresh" the surface, the program releases all the allocated space and can store lines anew.

If the surface formula contains some of the parameters p or q, you will see in the upper (for parameter p) or in the left (for parameter q) margin of the left panel two little brown ticks. By clicking and dragging them, you will be able to change in real time those parameters and see the change in the surface: the extreme left (bottom) tick position points to the minimum value of the p (q) parameter, and the extreme right (top) points to the maximum value, according to the values stored in the respective edit fields. Also, the edit field for the value of the moved parameter will be updated accordingly.

When a surface is already drawn you can use the menu "Save 2D Drawing as JPEG..." or "Save 3D Drawing as JPEG..." to save the drawing on the parameter panel of the surface panel in that format. You may send to the printer the drawing on the parameter panel with the "Print 2D drawing..." menu, or the surface panel drawing with"Print 3D drawing..."

4. Preferences

In the menu "Preferences" you have first the decision whether consider the ambient $\mathbb{R}^3$ as Euclidean or Minkowskian. In the first case, the inner product is the usual, that is $(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = a_1a_2 + b_1b_2 + c_1c_2$, and in the second it is $(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = a_1a_2 + b_1b_2 - c_1c_2$, so that the coordinate "z" act as "time" in the relativistic sense.

When drawing lines you can choose among three options. The lines can be drawn in both sides of the surface, or you can choose one of them: "Draw on Normal Side" draws lines in the side toward which the gradient of $f(x, y, z)$ (implicit surfaces) or the normal associated to the parametrization (parametric) is pointing. "Draw on the Other Side" speaks by itself.

In parametric or implicit surfaces, you may choose to see a cube that contains tightly the surface. In pseudo-Riemanian planes and parametric surfaces, you have the item "Show Unit Circle", so that after selecting it you may see in the parameter panel, as you move the mouse, the unit circle of the tangent plane (for pseudo-Riemannian points, the "circle" includes also the vectors with square length equal to -1, so that it consists of two hyperbolas). You may reduce the circle radius by pressing ⟨Control⟩ + ⟨Down Arrow⟩, increase it by means of ⟨Control⟩ + ⟨Up Arrow⟩ or going again to the unit pressing ⟨Control⟩ + ⟨Left Arrow⟩.

When the window is a surface, you have a hierarchical submenu headed "View Mode". On it you have the options "Standard" and"Wireframe". The first one is the default realistic view. The second draws a simple wireframe. You may rotate the surface, in both view modes, by pressing the right button of the mouse while dragging it; if in addition you maintain pressed ⟨Control⟩ then you will translate it; and if you keep pressed ⟨Shift⟩ instead, then you will zoom it by dragging the mouse up or downwards.

You may change the color gamut used in the rendering of the surfaces. First, you may choose between “Color by Lighting”, “Color by Gauss Curvature”, “Color by Mean Curvature” and “Color by Gauss Curvature Sign”. The basic idea is as follows. In the standard rendering ”by lighting”, the 3D appearance of the surface is mostly due to its illumination or lighting. The program computes that illumination for each pixel, and gives to it a value from 1 (no illumination) to 511 (full illumination). Then it looks up on a list of 511 colors, and draws the pixel in the corresponding color of the list. That list is the “gamut” on which the surface is drawn. To tell the truth, there are two such lists, one for each side of the surface, that we distinguish by the name “obverse” and “reverse”.

In the same manner, we have a gamut for the other three color preference settings, one gamut for each because then we do not distinguish sides. In the case of “Color by Gauss Curv Sign”, the color is blue in the elliptic regions (Gauss curvature positive), an red in hyperbolic ones (Gauss curvature negative).

Below those three menu options you will see the item “Palettes”. On clicking it, a window appears with five tabs: the two first, for each side of the “lighting” rendering; the third and fourth for Gauss and mean curvature color, and the fifth for the curves that may be drawn on the surface. The colors for the sign of Gauss curvature are not modifiable. With the exception of the fifth tab, on each of them you will see, at left, an hexagon with a dot in it; at center, a vertical gauge with a moving cursor; at right, a colored polygon on which a curve connects several dots. The essential thing is this curve: it represents the gamut. That is, the end which is a little circle represents the first color (number 1) of the gamut; the other end, a little square, the last color (number 511); the intermediate vertices represent the colors of numbers 72, 144, 216, 288, 360, 436; the remaining color numbers are more or less evenly spaced in the corresponding intervals. So, the color number N will be the color of the colored polygon where the point N of the curve is on.

You may modify the curve by dragging with the mouse any of the dots (control points) of the curve. If the chosen gamut corresponds with the chosen color system, you will see in real time the effect of the change in the surface. Also, if the chosen gamut corresponds to the color system chosen for the surface, you may change the background color by simply left-clicking with the mouse on any colored point of this color-picker window while pressing the ⟨Control⟩ key.

The hexagon and the gauge are means to change the colored polygon on which the customizable gamut curve is drawn. This is the rationale: the control point in the RGB cube (the hexagon), together with the center of the RGB cube, define a line, the “axis”. The position of the cursor in the vertical gauge defines an angle between 0 and 2π, and this angle defines the orientation of a plane that contains the axis. The intersection of this plane with the RGB cube boundary is drawn on the hexagon; the intersection of the plane with the interior of the cube is drawn in the right part of the window and on it it is superposed the customizable gamut curve.

The fifth tab in ‘Palettes” is meant to change the color of the curves. The control dot in the hexagon select the color hue, and then the control dot in the meter controls its brightness. With the exception of the arrows of the parallel transport and the intersections with planes that appear when using “Clipping”, “Normal Curvatures” and “Meusnier Theorem”, all the other curves will be drawn in the fixed color until the user changes it again.

The "Geometry" and "Draw Mesh" menus work only when the surface is in "Standard" mode.

5. Geometry

As you move the mouse, you will see some geometric information about the point signaled currently by the mouse. In all cases we have the parameters, the coordinates of its image (surfaces) and the curvature K, that means scalar curvature (pseudo-Riemannian) or Gauss curvature (surfaces). When the point is pseudo-Riemannian proper, then the value of K is preceded by a bullet. For surfaces we have in addition the mean curvature H, and the two principal curvatures k_1 and k_2.

In the menu "Geometry" only one item can be active at a given moment. By default, there is nothing selected. If your selection is "Coordinate Lines", when you click on the surface in the drawing panel or in the parameter panel, **Superficies** draws the coordinate lines that pass by that point (pseudo-Riemannian or parametric), or the intersection of the surface with the three coordinate planes that pass by that point (implicit).

Below it, you will find the menu item "Light Rays". For surfaces it only works when the "Minkowski option" is set. Select "Light Rays". As you move the mouse, if there are light rays passing by the point, then **Superficies** draws two little segments that represent the directions of null length of the tangent plane. When you click the mouse in one of those points, **Superficies** draws the light rays that pass by that point.

There is also the option of leave the program drawing the curve for ever, or until it encounters an obstacle or the user stops it. This occurs when the user maintains pressed the key ⟨Shift⟩ when clicking the mouse for initiating the curve.

It works in like manner when one of the items "Curvature Lines", "Mean Curvature Lines", "Harmonic Mean Curv Lines" or "Asymptotic Lines" is selected (only for surfaces). As the mouse moves, you will see principal directions, arithmetic mean and harmonic mean curvature directions or asymptotic directions at the point signaled by the mouse, if there are any. Preferred principal and mean curvature directions exist only on points that are not umbilical and in the case of harmonic mean curvature if in addition the Gauss curvature is non-negative. Asymptotic directions only on points with non-positive Gauss curvature. Clicking on the surface will result in the drawing of the currently chosen lines passing by the point. As in the case of the unit circle, you may increase or reduce the size of the segments making the principal or asymptotic directions in the parameter space by pressing ⟨Control⟩ and one of the up, down or left arrow keys.

In some cases, those lines are closed, so that they would go on for ever; thus, the program strives to detect that and stops them when they reach the initial point. If it fails, "Stop" them and this will stop the branch that is being drawn currently (there are four branches to be drawn from each point).

Next you have several options related to geodesics. All work the same in the four types of surfaces. Suppose that you have selected one of them, say "Geodesics". Click and drag the mouse; you will see a segment starting at the point that you clicked first and that sticks to the mouse; it is meant to fix the initial tangent of the geodesic that you will get when you release the mouse. On some occasions, for instance if the surface is compact, the geodesic might go on for ever, and you will need to click the "Stop" button.

By default, the geodesic is drawn only in the sense of the initial vector. If you need that Superficies draw a geodesic in both senses, that is forward and backward, keep pressed the key ⟨Alt⟩ while you set the initial direction.

In like manner you can get the "Exponential Map", that is the result of applying the exponential map to the segment that sticks to the mouse: it will be the trace of a geodesic with that segment as initial condition and whose length is the length of the segment.

"Geodesic between Points" will attempt to find and draw the minimizing geodesic that joins the points of the surface signaled by the ends of the sticking segment and remains in the domain or range of the surface. Note that that geodesic does not always exist, or even if it does, **Superficies** may be unable to find it; anyway, if **Superficies** finds one it is not sure whether in fact it minimizes distance.

Suppose that you draw all geodesics that start from a given point stopping them when they have run for the same fixed length, say r. The ends of those geodesics form the "Geodesic Ball" centered at the initial point and with radius r. In **Superficies** you will fix the initial point clicking the mouse and the radius by dragging it; the radius will be the length of the sticking segment. When the point is Minkowskian, **Superficies** draws the ball with squared radius equal to r^2, and also the ball with squared radius equal to $-r^2$.

When you select "Lines of Const Geod Curv", by clicking and dragging you get the line of constant geodesic curvature with initial point and direction given by the sticking segment, and with a radius of curvature equal to the length of that segment. To change the "hand" of that curvature, that is from curving right to curving left, press the ⟨Alt⟩ key when clicking the mouse for dragging it.

If the window is a parametric surface, a tube or a pseudo-Riemannian plane, then you have the option "Parallel Transport". Click the mouse and drag; a tiny segment pointing to right is left at the initial point, and you will see in real time the result of the parallel displacement of that segment along the curve that follows the mouse. Note however that in parametric surfaces or tubes, you must click and drag the mouse in the parameter panel.

When (parametric surface or tube) the functions include parameters and you change them and there are some lines already drawn in the surface, the surface will change, and the color of all those lines will change to gray, thus indicating that they will not longer be curvature lines, that were formerly drawn in red for meaning, say, curvature lines or whatever. New lines will be drawn, however, in the current color selected in the palette.

For surfaces you have three more items, "Clipping", "Normal Curvatures" and "Meusnier Theorem". "Clipping" allows you to clip the surface with one or more planes; after selecting this item, you will see the surface enclosed in a sphere with a circle drawn on it (note that this circle appears as an ellipse); the plane of that circle is the clipping plane; if it cuts the surface, the intersection is drawn in white. When the mouse passes near the circle that determines the clipping plane, the circle changes color (from gray to orange); then you can translate it with the pressed mouse. When the mouse is not on the clipping circle, if you press the mouse and drag, the whole sphere is rotated, the clipping plane with it; in this manner, the clipping plane may be put in any desired position. The draw button shows the caption “Clip it”; on pressing it, you obtain the surface clipped by the selected plane. There is a hint about the part that will remain and the part that will be clipped: the sphere that surround the surface is drawn in two colors, gray and yellow, according to the side with respect to the clipping plane: the region in the surface in the yellow side will remain; that in the gray side will disappear.

You may repeat the clipping. While there are clipping planes, the stop button shows the text “No Clips”: it is to redraw the surface without any clips at all. In the case of parametric surfaces, the program shows in gray the part of the domain whose image has disappeared by the clipping process.

With the “Normal Curvatures” option, you may clip the surface by a plane defined by a vector tangent to the surface and another vector orthogonal to the former. To do this, click and drag the mouse on the surface. You will see a little red circle in the first point clicked, a blue line that determines the chosen direction tangent to the surface at that point, a red line orthogonal to the surface and a gray half-transparent disk that contains both lines and cuts the surface along a curve drawn in white. Below the curvatures data you will see the normal curvature k_n for that tangent (the blue line). On releasing the mouse, the draw button will show the caption “Clip it”, as before. On pressing it, the surface is clipped by the current section, so that the part to the right of that tangent disappears. This choice of the normal section can also be made in the parameter panel (parametric or tube). When the mouse moves over the tangent (blue line) or over the red line they change color to orange, meaning that you may then drag them to change the cut plane accordingly; the normal curvature is then updated.

The option “Meusnier Theorem” is meant to illustrate that theorem. You click on the surface (this may also be made in the parameter space) and drag. After releasing the mouse button, a lot of things happen. First, the direction tangent to the surface determined by the dragging appear as a blue line; the surface is cut and clipped by a plane that is orthogonal to that line and passes by the first point you clicked, which we will name as p ; a gray line tangent to the surface and orthogonal to the blue line is also displayed; a gray transparent half-disk that contains the blue line and (initially) the normal to the surface at p cuts the surface along a curve c drawn in white. The straight edge of the half-disk is initially orthogonal to the surface at p. Along it you will see two red dots and a red arrow; one of the dots is fixed at p; the other one stands for the curvature center of c at p and its distance to p is accordingly equal to the inverse of the curvature k of c at p; the arrow stands for the curvature vector of c at p so that its length is equal to k. By its tip passes a teal colored line parallel to the gray line (which is tangent to the surface at p). Now, if you move the mouse over the straight edge of the gray half-disk, you will see that the red arrow changes to orange, meaning that then you may move it by pressing the mouse button and dragging. In doing so, you obtain a family of curves on the surface that share the same tangent line at p , that is, the blue line. Simultaneously, you will see that the curvature centers describe a circle and that the tip of the curvature vector marches along the teal line. Since the orthogonal projection of all the points of that line upon the normal to the surface at p go to the same point, we see that all plane sections of the surface that share the same tangent line at p have the same normal curvature (Meusnier Theorem). You will also notice another red dot that appears as the orthogonal projection of the tip of the curvature vector of c upon the tangent plane at p; all these dots lie upon the same line, the gray line tangent to the surface and orthogonal to the tangent of c at p. The vector joining p with this last dot is the geodesic curvature vector of c at p.

The last two items under the menu "Geometry" are used for drawing two, only one, or none of the two foliations (a foliation here is a family of curves that do not intersect) that appear as light rays, curvature lines, asymptotic lines or mean curvature lines.

If you select any item in the menu "Draw Mesh" (only for surfaces) you will get a coordinate mesh for parametric surfaces, or a mesh built with the intersections of the surface with equally spaced coordinate planes. The number of lines in each direction is the number appearing in the chosen item and you may choose in the upper part of the menu list the classes of lines that you want.

6. Final notes

This program was written in Borland Delphi language, with the Delphi XE2 edition. It may be used freely. The program, its source code, this manual and some samples can be freely retrieved at the following site:

<http://www.uv.es/montesin/>

References

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, Inc, Englewood Cliffs, New Jersey, 1976.

[2] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

7. Appendix: list of shortcuts

Purpose	**Left mouse**	**Right mouse**	**Drag**	**Keys**
Erase all curves				<Ctrl> + <R>
Erase last curve				<Ctrl> + <Z>
Curves 1st order	•			
Curves 2nd order	•		•	
Parallel transport	•		•	
Don't halt curves 1st order	•			<Shift>
Geodesic both senses	•		•	<Alt>
Change const geod curv	•		•	<Alt>
Increase circle radius				<Up>
Decrease circle radius				<Down>
Default circle radius				<Left>
Rotate surface		•	•	
Translate surface		•	•	<Ctrl>
Increase-Decrease surface		•	•	<Shift>
Background surface	Palettes			<Ctrl>
Increase cross size				<Up>
Decrease cross size				<Down>
Default cross size				<Left>

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