Found Phys (2009) 39: 1280–1295
DOI 10.1007/s10701-009-9353-2

Four Causal Classes of Newtonian Frames

Bartolomé Coll · Joan Josep Ferrando · Juan Antonio Morales-Lladosa

Received: 3 March 2009 / Accepted: 17 September 2009 / Published online: 1 October 2009

Abstract The causal characters (spacelike, lightlike, timelike) of the coordinate lines, coordinate surfaces and coordinate hypersurfaces of a coordinate system in Relativity define what is called its *causal class*. It is known that, in any relativistic space-time, there exist one hundred and ninety nine such causal classes. But in Newtonian physics (where only spacelike and timelike characters exist) the corresponding causal classes have not been discussed until recently. Here it is shown that, in sharp contrast with the relativistic case, in Newtonian space-time the different causal classes of coordinate systems are drastically reduced to four. These causal classes admit simple geometric descriptions and physical interpretations. For example, it is shown that one can generate coordinate systems of the four causal classes by means of the sole *linear synchronization group*, i.e. by coordinate transformations that only change the origin of time linearly. The relativistic analogs of these examples are also considered.

Keywords Coordinate systems · Causal classifications · Newtonian frames · Synchronizations

1 Introduction

In Relativity, lines, surfaces and hypersurfaces of the space-time may be spacelike, lightlike or timelike. These characters, of clear physical meaning, are usually referred

B. Coll · J.J. Ferrando · J.A. Morales-Lladosa (✉)
Departament d'Astronomia i Astrofísica, Universitat de València, 46100 Burjassot, València, Spain
e-mail: antonio.morales@uv.es

B. Coll
e-mail: bartolome.coll@uv.es

J.J. Ferrando
e-mail: joan.ferrando@uv.es

to as causal characters. The causal character of lines, surfaces or hypersurfaces of the space-time is strongly related to the different physical ingredients with which one can physically construct them in the physical space-time (dust, clocks, rods, strings, light beams, light flashes, etc.).

By taking three, two or one coordinates as constants, coordinate systems locally define families of coordinate lines, coordinate surfaces or coordinates hypersurfaces, namely they define four three-parametric congruences of lines, six two-parametric families of surfaces and four one-parametric families of hypersurfaces. Of course, every one of these fourteen geometric elements has a specific causal character.

It seems to be an extended intuitive opinion that, in general, it is sufficient to know the causal characters either of the sole four congruences of coordinate lines, or of the sole four families of hypersurfaces to know the causal character of all the other geometric elements of the coordinate system. However, this opinion is mistaken: the fourteen causal characters are generically independent. Consequently, we must analyze all them in order to known the causal properties of a coordinate system generically.

The ordered set of these fourteen causal characters is called the causal class of the coordinate system. It was established some time ago that there are 199 different causal classes of coordinate systems in Relativity [1]. In Newtonian space-time, lines, surfaces and hypersurfaces may only be spacelike or timelike. It is thus evident that the number of causal classes of coordinate systems in classical physics is less than in the relativistic case. However, it seems that this subject has never been considered until recently [2].

Here we analyze this question and show that in Newtonian physics there exist only four different causal classes of coordinate systems. A precise geometric description of these four classes is given and some examples of each one are commented on. Finally, the four causal classes of Newtonian coordinates are contrasted with the Lorentzian ones and, specifically, with their four relativistic analogs.

The well known usual coordinate systems, essentially based on a three-space foliation plus a one-time congruence, are induced by, or induce, the standard *evolution conception* of Newtonian and relativistic physics. But other cuts or foliations, among the other three possible cuts or foliations in Newtonian theory or among the other one hundred and ninety eight possible cuts or foliations in Relativity, may help us to better describe and understand other aspects of the space-time, and even awaken our interest for variations of physical fields other than the timelike ones, intimately induced by the evolution conception.

This shows that the interest in the causal classification of coordinate systems is not only taxonomic. But, perhaps its most imminent application concerns the current methods for solving practical relativistic problems. The theory of Relativity is conceptually considered to be a physically autonomous theory. This means that, for its development, it needs no other physical concepts than the ones contained in its specific foundations, or those that can be coherently deduced from them. But in practice, despite efforts made in this sense [3–12], to develop physical applications must, for the moment, resort to Newtonian concepts and post-Newtonian methods. This situation reduces Relativity theory, with few exceptions, to a role of *corrective algorithm* for Newtonian theory, relegating its best specific concepts to a simple historically admirable, but otherwise ineffective, method of setting the main equations of the theory, the Einstein equations. In fact, irrespective of the revolutionary and paradigmatic concepts that General Relativity opposed to the Newtonian scope of the space-time, only quantitative first terms in Taylor's development of Einstein equations (with respect to a Newtonian background) essentially continue to be the unique element of General Relativity used to improve Newtonian results obtained under Newtonian concepts.

As long as this situation continues, it is highly convenient in post-Newtonian developments to choose coordinate systems with causal properties that are the same both for the starting Newtonian model and for the relativistically corrected metric structure. Otherwise, in going from Newtonian to relativistic results by adding higher corrective terms, one would add, to the quantitative corrective process involving the physical quantities of the problem, qualitative corrections due to an eventual change of causal characters of the geometric elements of the coordinate system. If such a change takes place, the physical interpretation of the vector or tensor *components* of the physical quantities of the problem, and therefore the proper tools for measure them, could change drastically.[1]

Fortunately this convenient choice of analogous causal classes has been made up to now, naturally but unconsciously. Simply because the starting Newtonian coordinate system has essentially been chosen as the Cartesian one, and the weak gravitational fields usually considered in astronomy cannot change their causal character with the lower order perturbed relativistic values of the metric. However, new problems, concerning black holes, binary systems, gravitational waves, positioning systems, formation flight satellites and space physics, could lead to starting from other Newtonian coordinate systems, better adapted to these problems or to include higher order terms. Then, changes in the causal character of some of the ingredients of the starting Newtonian coordinate system become possible when evaluated with the corrective algorithm generating the relativistic space-time metric.

In fact, in Numerical Relativity, a verification not only of the regularity but of the stability (constancy) of the whole causal class of the coordinate system would also be convenient in order to guarantee the physical interpretation, at least, of the components of the energetic quantities present in the second member of Einstein equations.

The paper is organized as follows. In Sect. 2 the notion of causal class of a frame is introduced and extended to coordinate systems. Section 3 characterizes the four causal classes of frames or coordinate systems in Newtonian space-time, and extends this result to an arbitrary dimension. In Sect. 4 the notions of coordinate parameter and gradient coordinate are emphasized in order to better understand the limits of assigning a causal character to the coordinates, and the first elements of the synchronization group are stressed for further applications. In Sect. 5 we show that the linear synchronization group is able to generate coordinate systems of *any* of the four causal classes, and the causal class of the ancestral local Solar time is obtained and

[1] Think, for example, that for the energy tensor, the usual interpretation of its components in terms of energy density, momentum density and stress quantities is only valid for *standard* frames. Standard frames privilege *one* observer (timelike congruence of lines) among all others, but constitute one sole class among the relativistic classes of possible frames; in all the others, and in particular in the real null frames of emission coordinates [3, 10–12], such an interpretation fails, because no observers are necessary at all.

commented. In Sect. 6 Newtonian and Lorentzian classes are contrasted among the relativistic analogs of the chosen Newtonian examples. Finally, in Sect. 7 we comment on the potential use of our results as training tools for a better understanding of physical space-time.

2 Notion of Causal Class

In Relativity, directions and planes or hyperplanes of directions at an event are said to be spacelike, null or timelike *oriented* if they are respectively exterior, tangent or secant to the light-cone of this event. These characters, of clear geometric meaning, are usually called *causal characters* because of their physical implications. They naturally extend to vectors and volume forms on these sets of directions.

Thus, every one of the vectors v_A of a frame $\{v_A\}$ $(A = 1, \dots, 4)$ has a particular causal character c_A. What about the causal characters C_{AB} $(A < B)$ of the six *associated* planes $\Pi(v_A, v_B)$ of the frame? Are they determined by the sole causal characters c_A of the vectors of the frame? Certainly not, because for example the plane associated to two spacelike vectors may have any causal character. So, in general, the specifications c_A and C_{AB} are independent.

Moreover, in order to give a complete description of the causal properties of the frames, one also needs to specify the causal characters c_A of the four covectors θ^A giving the dual frame $\{\theta^A\}$, $\theta^A(v_B) = \delta^A_B$. The c_A's are one-to-one related to the causal characters of the four associated 3-planes $\Pi(v_B, v_C, v_D)$ with $\theta^A(v_B) = \theta^A(v_C) = \theta^A(v_D) = 0$ which are not determined, in general, by the specification of both c_A and C_{AB}.

Thus, the causal characters of the fourteen directions, planes and hyperplanes associated to any frame are generically independent. They are described by the set of $(4 + 6 + 4 =)$ 14 symbols $\{\mathrm{c}_A, \mathrm{C}_{AB}, c_A\}$, which is called the *causal signature* of a frame $\{v_A\}$, which fully characterizes its *causal class*: the causal class of a frame is the set of all the frames that have the same causal signature. The causal signature is, generically, the minimum (necessary and sufficient) set of symbols that fully characterize the causal character of a frame, and it provides exhaustive information about the causal properties of its geometric elements (directions, planes and hyperplanes). Elsewhere [1], the following result was obtained.

Theorem 1 *In a four-dimensional Lorentzian space-time there exist* 199 *causal classes of frames*.

As a *natural frame* is nothing but the set of derivations along the parameterized lines of a coordinate system, the notion of causal class extends naturally to the set of coordinate lines of the coordinate system and so, to the coordinate system itself. But because this extension of the notion of causal class to a coordinate system is by construction a point by point extension, a coordinate system may present different causal classes at different points of its domain of definition. Indeed, some examples of this situation will be given below.

The assignment of one specific causal class to a coordinate system in a region of the space-time supposes that the causal characters of all the geometric elements of the coordinate system are the same at any point of the region or, in other words, that the region under consideration is a *causal homogeneous region* for the coordinate system in question.

By definition, the causal class of a coordinate system $\{x^\alpha\}_{\alpha=1}^4$ in a domain is the causal class $\{\mathrm{c}_\alpha, \mathrm{C}_{\alpha\beta}, c_\alpha\}$ of its associated natural frame at the events of the domain. The c_α's are the causal characters of the vectors $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$ of the natural frame $\{\partial_\alpha\}$ itself, and the c_α's are the causal characters of the 1-forms dx^α of the coframe $\{dx^\alpha\}$.

What is the situation in Newtonian physics concerning causal characters and causal classes? Of course, now the causal characters $\mathrm{c_A}$, $\mathrm{C_{AB}}$, c_A reduce to timelike or spacelike. Now, a causal class is also characterized by the fourteen quantities $\{\mathrm{c_A}, \mathrm{C_{AB}}, c_A\}$, but some of them determine the others systematically. Specifically, in Sect. 3 we shall show that for Newtonian frames one has the implications

$$\{\mathrm{c_A}\} \quad \Rightarrow \quad \{\mathrm{C_{AB}}, c_A\}, \qquad \{\mathrm{C_{AB}}\} \quad \Rightarrow \quad \{c_A\},$$

but

$$\{\mathrm{C_{AB}}\} \quad \nRightarrow \quad \{\mathrm{c_A}\}, \qquad \{c_A\} \quad \nRightarrow \quad \{\mathrm{c_A}, \mathrm{C_{AB}}\}.$$

These implications lead to a Newtonian situation that is remarkably simpler than the Lorentzian one. In fact, surprisingly enough at first glance, only four causally different classes of frames or coordinate systems are admissible in Newtonian space-time (see Sect. 3 below). It is startling that, in spite of this poverty of classes, only the *standard class* (i.e. the one adapted to the absolute space $\oplus$ time Newtonian decomposition) has been explicitly referred to in the literature. In the next section we will construct these four classes of Newtonian frames.

3 Causal Classes of Newtonian Frames

The differences in the geometric description of Lorentzian and Newtonian frames come from the causal structure induced by the metric description of the underlying physics.

In Relativity the space-time metric defines a one-to-one correspondence between vectors and covectors at every event. In contrast, in Newtonian physics no non-degenerate metric structure exists. The degenerate metric structure is given by a rank one covariant positive *time metric* T and an orthogonal rank three contravariant positive *space metric* γ^*, $T \times \gamma^* = 0$, where $\times$ stands for the cross product.[2]

The time metric T is necessarily of the form $T = \theta \otimes \theta$, where the 1-form θ, the *time current,* defines the unit of time. That this time is *uniform* for any observer, or *absolute*,[3] implies the exact character of the time current, $\theta = dt$, where t is any absolute time scale.[4] The hypersurfaces $t =$ *constant* constitute the *instantaneous spaces*, *simultaneity loci* or *spaces* at the instant t.

[2] The cross product $\times$, or matrix product, is the contraction of the adjacent vector spaces of the tensor product $\otimes$. In tensor components, $T \times \gamma^*$ is written as $T_{\alpha\rho}\gamma^{*\rho\beta}$.

[3] Absolute and uniform times are strongly related. See [13].

[4] A time scale is a rhythm generated by a unit interval together with a choice of origin.

It should be stressed that the above elements, T (or θ) and γ^*, already determine the Newtonian causal structure.[5] Here, we are only interested in the causal character, at each event, of directions, planes and hyperplanes induced by the sole Newtonian structure provided by θ and γ^*. In this structure, a vector v is *spacelike* if it is instantaneous with respect to the time current θ, i.e. if $\theta(v) = 0$. Otherwise, the vector is *timelike*. A timelike vector v is *future* (resp. *past*) *oriented* if $\theta(v) > 0$ (resp. $\theta(v) < 0$).

It is clear that a basis can have three spacelike vectors *at most* so that, denoting the causal characters (respectively spacelike, timelike) of vectors with Roman letters (e, t), it holds:

Lemma 1 *There exist four causal* types *of Newtonian frames*, *namely*:

$$\{\mathrm{teee}\}, \qquad \{\mathrm{ttee}\}, \qquad \{\mathrm{ttte}\}, \qquad \{\mathrm{tttt}\}.$$

In a Newtonian structure, correspondingly, a covector $\omega \neq 0$ is *timelike* if it has no instantaneous part with respect to the space metric γ^*, i.e. if $\gamma^*(\omega) = 0$. Otherwise, the covector ω is *spacelike*. The sole timelike codirection is that defined by the current θ at every event because γ^* has rank 3. Thus, if ω is timelike it is necessarily of the form $\omega = a\,\theta$ with $a \neq 0$. Then ω is *future* (resp. *past*) *oriented* if $a > 0$ (resp. $a < 0$).

It is then clear that a coframe has *at most* one timelike covector so that, denoting the causal characters (spacelike, timelike) of covectors with Italic letters (e, t), it holds:

Lemma 2 *There exist two causal* types *of Newtonian cobases*, *namely*:

$$\{teee\}, \qquad \{eeee\}.$$

Lemmas 1 and 2 show the lack of symmetry of causal types of Newtonian bases and cobases, in contrast to the rigorous symmetry of the relativistic case.

A r-plane Π is *spacelike* if every vector v in it is spacelike. Otherwise, Π is *timelike*, i.e. it contains timelike vectors. Two (resp. three) linearly independent spacelike vectors generate a spacelike 2-plane (resp. 3-plane).

A r-coplane Ω is *timelike* if it contains the time current θ. Otherwise Ω is *spacelike*.

The annihilator coplane Ω_Π of a r-plane Π is the $(4 - r)$-coplane

$$\Omega_\Pi \equiv \{\omega \,|\, \omega(v) = 0, \forall v \in \Pi\}.$$

Obviously, these definitions also apply to r-plane fields and r-coplane fields in causal homogeneous regions.

Accordingly, we have the following result.

[5] Nevertheless, for the formulation of the equations of motion, a flat and symmetric affine connection is also required in order to introduce inertia. In addition, in the four-dimensional formulation of Newtonian gravity, the requirement of another symmetric, non-flat connection is needed in order to introduce the gravitational field [14, 15] (for a concise presentation, see [16] and references therein), but we shall not need them in this work.

Table 1 The four causal classes of Newtonian frames. Roman letters (e, t), capital letters (E, T), calligraphic ($\mathcal{E}, \mathcal{T}$) and Italic (e, t) letters represent the causal characters (spacelike, timelike) respectively of the vectors of the frame, of their associated 2-planes, of their associated 3-planes and of the covectors of the coframe. This causal classification naturally extends to coordinate systems in causal homogeneous regions

	teee	ttee	ttte	tttt
$eeee$ ($\mathcal{TTTT}$)		TTTTTE	TTTTTT	TTTTTT
$teee$ ($\mathcal{TTTE}$)	TTTEEE			

Lemma 3 *A r-plane* Π *is spacelike* (resp. *timelike*) *iff* Ω_Π *is timelike* (resp. *spacelike*).

In particular, given a Newtonian frame $\{v_1, v_2, v_3, v_4\}$, a covector θ^α of its *dual* frame $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ is timelike (resp. spacelike) iff the 3-plane generated by $\{v_\beta\}_{\beta\neq\alpha}$ is spacelike (resp. timelike).

On account of the above considerations, the causal characters of the four vectors of a Newtonian frame unambiguously determine the causal characters of their six associated 2-planes and the causal characters of their four associated 3-planes. Consequently, we have the following result.

Theorem 2 *In the* 4*-dimensional Newtonian space-time there exist four, and only four, causal classes of frames.*

The four Newtonian causal classes are represented in Table 1 which is read as follows.

1. The first column shows the causal characters $c_A = \{e\,e\,e\,e\}$, $c_A = \{t\,e\,e\,e\}$ of the covectors of the coframe (or correspondingly, of the causal characters $\bar{c}_A = \{\mathcal{TTTT}\}$, $\bar{c}_A = \{\mathcal{TTTE}\}$ of the four 3-planes of the frame or of the four families of coordinate hypersurfaces of a coordinate system). As stated in Lemma 2, only these two sets are possible, up to permutations.
2. The first row shows the causal characters $\mathrm{c_A} = \{\mathrm{t\,e\,e\,e}\}$, $\mathrm{c_A} = \{\mathrm{t\,t\,e\,e}\}$, $\mathrm{c_A} = \{\mathrm{t\,t\,t\,e}\}$, $\mathrm{c_A} = \{\mathrm{t\,t\,t\,t}\}$ of the vectors of the frames or, correspondingly, the causal characters of the congruences of coordinate lines of a coordinate system. As stated in Lemma 1, only four such sets are possible, up to permutations.
3. Each non empty (p, q)-cell ($p=1, 2;\ q=1, 2, 3, 4$) shows the set of causal characters C_{AB} of the associated 2-planes of vectors of the q-th frame, which corresponds to the p-th coframe or, correspondingly, the set of causal characters of the six coordinate surfaces of a coordinate system.
4. Permutations of the vectors of the frame or of the covectors of the coframe induce permutations of the associated 2-planes and 3-planes, but do not alter their causal class. Thus, permutations of the lines or hypersurfaces of a coordinate system induce permutations of the coordinate surfaces of the system, but do not alter its causal class.

For instance, standard frames, i.e. those that are locally performed with three rods and one clock at rest with respect to the rods, belong to the causal class {teee, TTTEEE, *teee*}. The history of the clock is a timelike coordinate line. The other coordinate lines are spacelike straight lines tangent to the rods at every (clock's) instant.

As already mentioned, the simplicity of the Newtonian causal structure compared with the Lorentzian one lies in the fact that the causal type of a Newtonian frame fully determines its causal class. This is related to the fact that, in Newtonian space-time, any set of spacelike vectors always generates a spacelike subspace. Consequently, the number of causally different Newtonian classes of frames is equal to the dimension of the space. This is a general property, independent of the dimension n of the space-time. Denoting by $\{k\,\mathrm{t}, (n-k)\,\mathrm{e}\}$ the causal type of a basis with k timelike vectors and $n-k$ spacelike ones,[6] we therefore have:

Theorem 3 *In the n-dimensional Newtonian space-time there exist n causal classes of frames. A basis whose causal type is $\{k\,\mathrm{t}, (n-k)\,\mathrm{e}\}$, $k = 1, \ldots, n$, has $\binom{n-k}{r}$ spacelike associated r-planes and $\binom{n}{r} - \binom{n-k}{r}$ timelike associated r-planes $(r = 1, \ldots, n)$.*

Now we will go on to construct some examples of linear coordinate transformations that change the causal class of a starting coordinate system and also we will give examples of coordinate systems of the unusual causal classes. But first we need to specify some simple but important notions.

4 Coordinate Parameters, Gradient Coordinates and Synchronizations

When the coordinate system is already known, say $\{x^\alpha\}$, the coordinate geometric elements are easily discerned: the four one-parameter families of coordinate hypersurfaces are given by $\{x^\alpha = \text{constant}\}$, the six two-parameter families of coordinate surfaces are given by $\{x^\alpha = \text{constant}, x^\beta = \text{constant}\}$, and the four three-parameter families of coordinate lines are given by $\{x^\alpha = \text{constant}, x^\beta = \text{constant}, x^\gamma = \text{constant}\}$ for superscripts α, β, γ which are all different.

4.1 Coordinate Parameters and Gradient Coordinates

In fact, in any space-time, every coordinate x^α plays two extreme roles: that of a (coordinate) hypersurface for every constant value, of gradient dx^α, and that of a (coordinate) line when the other coordinates remain constant, of tangent vector ∂_α. This simple fact shows that, despite our deep-seated custom of associating a causal character to a coordinate, saying that *it is* timelike, lightlike or spacelike, *this denomination is not generically coherent.* Causal characters are generically associated with directions or sets of directions of geometric objects, but not with space-time variables or parameters associated to them. In the case of a coordinate x^α, this generic incoherence appears because its two natural variations in the coordinate system, dx^α and ∂_α,

[6] The comma between different causal characters is placed in this condensed expression for visual clarity only.

have *generically different* causal characters. *Only* when both causal characters coincide, it is conceptually clear to extend to x^α itself the denomination of the common causal character of its two mentioned variations.

Consequently, we shall say generically of a coordinate x^α that it is a c_α *gradient coordinate* and a c_α *coordinate parameter* when the causal characters of its variations dx^α and ∂_α are c_α and c_α, respectively.

In addition, concerning a coordinate t which is a timelike coordinate parameter and a timelike (resp. spacelike) gradient coordinate, we shall say also that it defines a *spacelike* (resp. *timelike*) *synchronization* (the coordinate hypersurfaces $t =$ constant being the synchronous event loci of the coordinate lines $t =$ variable. See below).

It is to be noted that the denomination "timelike coordinate parameter" instead of the commonly used "timelike coordinate" is the correct one when t defines a timelike synchronization. Indeed, in that case t may be a constant or even a decreasing parameter along future oriented timelike trajectories of the space-time coordinate region, an odd property for a "time coordinate".

A paradigmatic example of this situation is the best known timelike coordinate parameter: the *local Solar time*, which will be considered in Sect. 5. However, prior to its analysis, we should first present the group of (pure) synchronizations and its finite dimensional subgroup, the group of (pure) linear synchronizations.

4.2 The Synchronization Group

Consider a set of clocks in some region of a space-time. Their histories constitute a set of timelike lines on the region, naturally parameterized by the time t of the clocks. A *synchronization* is the stipulation of the locus of events where the clocks display the time $t = t_0$ for some chosen constant value t_0.

Here we are interested in 'smooth situations', in which the smallness of the clocks, their number and their histories are such that they can be efficiently described by a (sufficiently differentiable) congruence of timelike lines, $\gamma(t)$, and for which the locus of events $t = t_0$ defining the synchronization constitutes a (sufficiently differentiable, transverse) hypersurface, $\varphi(x) = t_0$. Once the trajectories are so synchronized, the loci of events $t =$*constant* for *any* constant defines a one-parameter family of hypersurfaces, to which the initial hypersurface $\varphi(x) = t_0$ belongs; let $\varphi(x) = t$ be its equation.

Any one of these hypersurfaces $\varphi(x) = t$ is said to define the *same* synchronization as the hypersurface $\varphi(x) = t_0$. Denoting by $\dot\gamma$ the tangent vector to the histories of the clocks, $\dot\gamma \equiv \frac{d}{dt}\gamma(t)$, such space-time function $\varphi(x)$ verifies $\mathcal{L}(\dot\gamma)\varphi = 1$, where $\mathcal{L}(\dot\gamma)$ is the Lie derivative[7] with respect to $\dot\gamma$.

Conversely, it is easy to see that the level hypersurfaces $\psi(x) = k$, $k =$ *constant*, of any function $\psi(x)$ that verifies $\mathcal{L}(\dot\gamma)\psi = 1$, define a synchronization for the (congruence of histories of the) clocks, i.e. there exists a canonical parameter t for the field $\dot\gamma$, $\frac{d}{dt}\gamma(t) = \dot\gamma$, such that $k = t$.

Consequently, for a congruence of (histories of) clocks of tangent vector field $\dot\gamma$, *the set of all its possible synchronizations is the set of all the scalar functions* $\psi(x)$

[7] On functions φ the Lie derivative reduces to a directional derivative, $\mathcal{L}(\dot\gamma)\varphi = \dot\gamma(d\varphi) = \dot\gamma^\rho\partial_\rho\varphi$.

such that $\mathcal{L}(\dot{\gamma})\psi = 1$. And it is obvious that, if φ is such a synchronization, *any* other synchronization ψ is of the form $\psi = \varphi + \omega$, where ω is an invariant function of the field $\dot{\gamma}$, $\mathcal{L}(\dot{\gamma})\omega = 0$. The group of transformations of (pure) synchronizations for the congruence of clocks, or *synchronization group*, is thus isomorphic to the additive group of functions $\{\omega\}$ which are invariant for the congruence $\dot{\gamma}$: if φ is an initial synchronization and ω any $\dot{\gamma}$-invariant function, any other synchronization ψ is obtained by $\psi = T_\omega \varphi \equiv \varphi + \omega$.

In order to express the synchronization group as a transformation group of the space-time, let us start from a coordinate system $\{x^\alpha\}$ ($\alpha = 0, 1, \ldots, n-1$) adapted both, to the field $\dot{\gamma}$, say $\dot{\gamma} = \partial_0$, and to the synchronization φ, thus $d\varphi = dx^0$. In this coordinate system, the $\dot{\gamma}$-invariant character of a function ω is expressed by its independence of the timelike coordinate parameter x^0, $\omega = \omega(x^i)$ ($i = 1, \ldots, n-1$). The new coordinate system $\{X^\alpha\}$, generated by ω and adapted both to $\dot{\gamma}$ and to $T_\omega \varphi = \psi$ then takes the form

$$X^0 = x^0 + \omega(x^i), \qquad X^i = x^i. \tag{1}$$

These are the space-time transformation equations of the synchronization group.

Nevertheless, for our purpose to generate the Newtonian causal classes with easy, it is sufficient to consider the simplest subgroup of the synchronization group (1), the *linear synchronization group*:

$$X^0 = x^0 + a_i x^i, \qquad X^i = x^i. \tag{2}$$

Its matrix form may be analyzed as follows. Let $\mathbf{1}$ be the $n \times 1$ column matrix of components $(1, 0, \overset{n-1}{\ldots}, 0)$, and consider the set of all the $1 \times n$ matrices $\mathbf{a}$ orthogonal to $\mathbf{1}$, $\mathbf{a} \cdot \mathbf{1} = 0$; they are obviously of the form $\mathbf{a} = (0, a)$ with $a \equiv (a_1, \ldots, a_{n-1})$. Then, the *linear synchronization algebra* is the (commutative) algebra of matrices of the form $\mathbf{1} \otimes \mathbf{a}$, so that the matrices L of the linear synchronization group are of the form $L = \exp\{\mathbf{1} \otimes \mathbf{a}\} = I + \mathbf{1} \otimes \mathbf{a}$, which clearly correspond to matrices of minimal polynomial $(L - I)^2 = 0$. In obvious matrix notation, (2) may be written $\mathbf{X} = L\mathbf{x}$.

From (2) we have the relations between the natural frames and coframes of two coordinate systems related by a linear synchronization:

$$\partial_{X^0} = \partial_{x^0}, \qquad \partial_{X^i} = -a_i \partial_{x^0} + \partial_{x^i}, \tag{3}$$

$$dX^0 = dx^0 + a_i dx^i, \qquad dX^i = dx^i. \tag{4}$$

Remark that, until now, all the considerations about the synchronization group remain valid for both Newtonian and relativistic space-times and are applicable to *any* starting coordinate system.

5 Examples of Newtonian Coordinate Systems of Different Causal Classes

5.1 Generating Newtonian Causal Classes by the Linear Synchronization Group

The linear synchronization group provides one of the simplest ways of generating *all* the Newtonian causal classes.

In the following, we will always start from a *standard inertial coordinate system* $\{x^\alpha\} \equiv x^0 = t$, so that its natural frame is of the causal type {te...e}.

Let us apply the transformation (2) to this coordinate system. By construction (definition of a change of synchronization) the new coordinate X^0 is a timelike coordinate parameter. However, X^0 is found to be a spacelike gradient coordinate whenever $a \neq 0$, because then, according to (4), one has $dX^0 \wedge dt \neq 0$. On the other hand, every new coordinate X^i is a timelike coordinate parameter whenever the corresponding component a_i of a does not vanish, because ∂_{X^i}, which is given by the second expression in (3), is timelike in this case, $\gamma^*(\partial_{X^i}) \neq 0$. Nevertheless X^i remains a spacelike gradient coordinate, because $\forall i, dX^i \wedge dt \neq 0$.

We see thus that, in the n-dimensional Newtonian space-time, starting from a standard coordinate system $\{t, x^i\}$ of causal type {t, $(n-1)$e}, *the linear synchronization transformations* (2) *for every one of the vectors* $a = (1, \overset{k-1}{\ldots}, 1, 0, \overset{n-k}{\ldots}, 0)$, $(k = 1, \ldots, n)$*, define a coordinate system* $\{X^\alpha\}$ *of causal type* {kt, $(n-k)$e}*, belonging to the k-th causal class of the n possible ones*, according to Theorem 3. Then, for every $r = 1, \ldots, n$, the $\binom{n}{r}$ associated r-planes are of causal type $\left\{\left[\binom{n}{r} - \binom{n-k}{r}\right]\mathrm{T}, \binom{n-k}{r}\mathrm{E}\right\}$. For $n = 4$, this obviously gives the four causal classes of Table 1.

It is worth noting that all the different causal classes have been obtained by simple, *pure*, changes of synchronization of the *same* system of clocks, excluding any other change of coordinates or of observers. Apparently, this is not an intuitive idea for most of us.

5.2 The Causal Class of the Ancestral Local Solar Time

The local Solar time, i.e. the time shown by a sundial, is the oldest timelike coordinate parameter known by humanity, and still remains indefinitely alive and currently in use (be it in the form of stepped time zones). As we mentioned, this local Solar time is a paradigmatic example of the situations where the current but particular notion of "timelike coordinate" becomes incoherent.

Specifically, we consider here the causal class of a coordinate system at rest with respect to a spherical Earth in uniform rotation when the (absolute) clocks are synchronized by the *local Solar time or sundial synchronization*: at any place they watch the same fixed time, 12 h, when the Sun is just on the local meridian (neither the inclination of the ecliptic nor the translational motion of the Earth are taken into account).

Let $\{t, r, \theta, \phi\}$ be a standard coordinate system where $\{r, \theta, \phi\}$ are the usual geocentric inertial spherical coordinates; it belongs to the standard causal class {teee, TTTEEE, *teee*}.

The *geocentric rotating spherical coordinate system* $\{t, r, \theta, \Phi\}$, is obviously given by the (pure) rotation

$$\Phi = \phi - \omega t, \tag{5}$$

where ω is the Earth's angular velocity. Here the coordinate lines where only t varies are no longer inertial, but the timelike helices that they describe remain synchronized by absolute simultaneity. This and the fact that the sole new coordinate Φ verifies $d\Phi \wedge dt \neq 0$, make the causal class of this rotating coordinate system to remain the standard one.

Now, starting from this coordinate system $\{t, r, \theta, \Phi\}$, a linear synchronization transformation of the form

$$T = t + \frac{\Phi}{\omega} = \frac{\phi}{\omega}, \tag{6}$$

generates the *Solar time geocentric rotating spherical coordinate system* $\{T, r, \theta, \Phi\}$. In this Solar time coordinate system, the time T is synchronized by a sundial for a fixed Sun placed at the meridian of longitude $\phi = 0$. Obviously, the coordinate lines where only T varies, which are described by the observers at rest with respect to the Earth, of tangent vector ∂_T, are timelike.

On the other hand, the coordinate lines where only Φ varies, of tangent vector ∂_Φ, are also timelike, because the inverse transformation is, from (6), $\{t = T - \Phi/\omega, r, \theta, \phi = \omega T\}$ and it follows that $\omega\partial_\Phi = -\partial_t$: they form, in fact, the congruence of inertial observers, as one might anticipate. This and Table 1, enables us to state that *the causal class of the ancestral local Solar time coordinate system is* {ttee, TTTTTE, *eeee*}.

It is worth noting that, in Newtonian physics as well as in Relativity, the more natural and ancestral synchronization is generated by *timelike* hypersurfaces, a fact that seems to be systematically forgotten in theoretical physics, where a synchronization is always defined by *spacelike* hypersurfaces.

6 Lorentzian Causal Classes with Newtonian Analogues

Among the 199 Lorentzian causal classes of space-time frames one can find the four Newtonian ones. This means that, in any relativistic space-time, one can always choose local coordinate systems of any of the four Newtonian causal classes.

Here we shall analyze in Minkowski space-time the same situations that we have already analyzed in the Newtonian case. The new lightlike character is denoted by a Roman letter l for vectors and coordinate lines, by a capital letter L for the associated 2-planes and coordinate 2-surfaces and by an Italic letter *l* for covectors.

6.1 The Linear Synchronization Group in Minkowski Space-Time

Consider Minkowski space-time in an inertial coordinate system $\{x^0, x^i\}$ such that the metric is $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, \ldots, 1)$, so that the associated natural frame is of causal type $\{\mathrm{t\,e} \ldots \mathrm{e}\}$. If now the linear synchronization group (2) acts on this coordinate system, the natural frame and coframe of the new system $\{X^\alpha\}$ are given by (3) and (4). Then, their respective direct scalar products give the covariant and contravariant components, $g_{\alpha\beta}$ and $g^{\alpha\beta}$ respectively, of the metric in this new system:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & a \\ a & I - a \otimes a \end{pmatrix}, \qquad g^{\alpha\beta} = \begin{pmatrix} -1 + a^2 & a \\ a & I \end{pmatrix} \tag{7}$$

where $a \equiv (a_1, \ldots, a_{n-1})$, $a^2 \equiv \sum_{i=1}^{n-1} a_i^2$ and I is the $n-1$ identity matrix.

We can see from (7) that, *as in the Newtonian case, the new coordinate X^0 is a timelike coordinate parameter*. However, X^0 *is a spacelike gradient coordinate only* *when* $|a| > 1$*, while in the Newtonian case the condition is simply* $a \neq 0$*. When* $|a| = 1$ *or* $|a| < 1$*,* X^0 *is a null or timelike gradient coordinate, respectively*. Obviously, the first of these last two situations is forbidden in the Newtonian case, and the second one cannot be attained by the linear synchronization group (up to, trivially, by the identity transformation, $a = 0$).

On the other hand, *every new coordinate* X^i *remains, like in the Newtonian case, a spacelike gradient coordinate*. However X^i *is a timelike coordinate parameter only when* $|a_i| > 1$*, while in the Newtonian case the condition is simply* $a_i \neq 0$*. When* $|a_i| = 1$ *or* $|a_i| < 1$*,* X^i *is a null or spacelike coordinate parameter, respectively.* Both situations are also absent in the Newtonian case (up to for $a = 0$).

Finally, the coordinate two-forms satisfy:

$$dX^i \wedge dX^j = dx^i \wedge dx^j, \tag{8}$$

$$dX^0 \wedge dX^i = dx^0 \wedge dx^i + a_j\, dx^i \wedge dx^j, \tag{9}$$

$$(dX^i \wedge dX^j)^2 = 1, \tag{10}$$

$$(dX^0 \wedge dX^i)^2 = -1 + a^2 - a_i^2. \tag{11}$$

Consequently, the $(n-2)$-coordinate surfaces $X^i = constant$, $X^j = constant$ $(i, j$ given) are timelike and the $(n-2)$-coordinate surfaces $X^0 = constant$, $X^i = constant$ $(i$ given) are timelike, null or spacelike if $a^2 - a_i^2$ is greater, equal or smaller than 1, respectively. This information, insufficient for $n > 4$, fully determines the causal class of the coordinate system $\{X^0, X^i\}$ for $n = 4$:

All the causal classes obtained by a linear synchronization transformation have a causal signature of the form:

$$\{\mathrm{t}\,\mathrm{c}_1\,\mathrm{c}_2\,\mathrm{c}_3,\ \mathrm{T}\,\mathrm{T}\,\mathrm{T}\,\mathrm{C}_{12}\,\mathrm{C}_{13}\,\mathrm{C}_{23},\ c_0\,e\,e\,e\}, \tag{12}$$

where the non-fixed causal characters, $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3, \mathrm{C}_{12}, \mathrm{C}_{13}, \mathrm{C}_{23}, c_0$ *depend on the* a_i *parameters as follows:*

$$\mathrm{c}_i = \begin{cases} \mathrm{t}, & |a_i| > 1, \\ \mathrm{l}, & |a_i| = 1, \\ \mathrm{e}, & |a_i| < 1, \end{cases} \qquad \mathrm{C}_{ij} = \begin{cases} \mathrm{T}, & a_i^2 + a_j^2 > 1, \\ \mathrm{L}, & a_i^2 + a_j^2 = 1, \\ \mathrm{E}, & a_i^2 + a_j^2 < 1, \end{cases}$$
$$c_0 = \begin{cases} t, & |a| < 1, \\ l, & |a| = 1, \\ e, & |a.| > 1. \end{cases} \tag{13}$$

Then, a detailed analysis of the compatible characters shows that *the number of different causal classes that may be generated by a linear synchronization transformation is* 29*, in contrast with the only* 4 *Newtonian ones*.

Evidently the four Newtonian analogues exist in Relativity. In fact, *the Lorentzian causal classes of same causal signature as the four Newtonian ones correspond to* *the following values of the parameters a_i:*

$$
\begin{aligned}
&\{\mathrm{tttt},\ \mathrm{TTTTTT},\ e\,e\,e\,e\} && \text{if} \quad \forall\, i, && |a_i| > 1,\\
&\{\mathrm{ttte},\ \mathrm{TTTTTT},\ e\,e\,e\,e\} && \text{if} \quad \begin{cases} \exists!\, i, & |a_i| < 1,\\ \forall\, j \neq i, & |a_j| > 1, \end{cases}\\
&\{\mathrm{ttee},\ \mathrm{TTTTTE},\ e\,e\,e\,e\} && \text{if} \quad \begin{cases} \exists!\, i, & |a_i| > 1,\\ j, k \neq i, & a_j^2 + a_k^2 < 1, \end{cases}\\
&\{\mathrm{teee},\ \mathrm{TTTEEE},\ t\,e\,e\,e\} && \text{if} \quad \forall\, i, && |a_i| < 1.
\end{aligned}
\tag{14}
$$

6.2 Local Solar Time Synchronization

In the Newtonian Example 5.2 of the rotating Earth, the latitude of the observer plays in fact no role for time synchronization. For this reason, we shall consider here, in the place of the Earth, a rigidly rotating disk and, instead of spherical coordinates, cylindrical ones.

So, let $\{t, \phi, \rho, z\}$ be an inertial laboratory referred to a standard cylindrical coordinate system in Minkowski space-time; it belongs to the standard causal class {teee, TTTEEE, *teee*}.

The *rotating cylindrical coordinate system* $\{t, \Phi, \rho, z\}$, adapted to the congruences of the observers in rigid rotation motion is defined by the transformation (5). In the Newtonian case this system remains in the standard class, as happens for the considered rotating spherical coordinate system.

As it is well known, in Minkowski space-time the presence of the light cylinder $\rho = 1/\omega$ allows the existence of other causal classes. Indeed, the covariant and contravariant components of the metric tensor in this rotating coordinate system are, respectively:

$$
g_{\mu\nu} = \begin{pmatrix} -1+\omega^2\rho^2 & \omega\rho^2 & 0 & 0\\ \omega\rho^2 & \rho^2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1 & \omega & 0 & 0\\ \omega & \frac{1}{\rho^2} - \omega^2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}
$$

The causal character c_α of the vectors of the coordinate frame is given by the sign of the diagonal elements $g_{\alpha\alpha}$ of the metric matrix; correspondingly, the causal character $\mathrm{C}_{\alpha\beta}$ of the coordinate 2-surfaces is given by the signs of the second order diagonal minors, $g_{\alpha\alpha} g_{\beta\beta} - (g_{\alpha\beta})^2$; and finally the causal character c_α of the coordinate co-frame is given by the signs of the diagonal elements $g^{\alpha\alpha}$ of the inverse metric matrix $g^{\mu\nu}$. Consequently, the rotating cylindrical coordinate system belongs to the following classes:

$$
\{\mathrm{teee},\ \mathrm{TTTEEE},\ t\,e\,e\,e\} \quad \text{if} \quad \rho < 1/\omega, \tag{16}
$$

$$
\{\mathrm{leee},\ \mathrm{TLLEEE},\ t\,l\,e\,e\} \quad \text{if} \quad \rho = 1/\omega, \tag{17}
$$

$$
\{\mathrm{eeee},\ \mathrm{TEEEEE},\ t\,t\,e\,e\} \quad \text{if} \quad \rho > 1/\omega. \tag{18}
$$

Note that, *in the rotating system, t remains a timelike gradient coordinate*, which determines the events that are simultaneous with respect to the inertial observer at rest on the rotation axis. Nevertheless, *t only remains a timelike coordinate parameter inside the light cylinder, $\rho < 1/\omega$.*

Now, starting from this rotating system $\{t, \Phi, \rho, z\}$, let us perform the Solar time linear synchronization change (6). In the new coordinate system $\{T, \Phi, \rho, z\}$, the covariant and contravariant components of the metric tensor are, respectively:

$$g_{\mu\nu} = \begin{pmatrix} -1+\omega^2\rho^2 & \frac{1}{\omega} & 0 & 0 \\ \frac{1}{\omega} & -\frac{1}{\omega^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} \frac{1}{\omega^2\rho^2} & \frac{1}{\omega\rho^2} & 0 & 0 \\ \frac{1}{\omega\rho^2} & \frac{1}{\rho^2}-\omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

It then follows that, *in the interior $\rho < 1/\omega$ of the light cylinder, the Solar time rotating coordinate system $\{T, \Phi, \rho, z\}$ belongs to the same causal class* {ttee, TTTTTE, *eeee*} *as the Solar time geocentric rotating system of Newtonian space-time*.

7 Comments Around our Results

We have shown that in Newtonian space-time there are only four causal classes of coordinate systems (Theorem 2).

Of these four classes, the standard one, of causal signature {teee, TTT EEE, *teee*}, seems to be the only class of which many people are aware or, at least, the only one having any physical interest.

We do not think so. On the contrary, notwithstanding its undeniable importance, we believe that their almost exclusive use in physics, excessively reinforcing the space-time cut into space plus time, exaggerates the physical interest of the evolution vision (i.e. of the leading role of time dependence of spatial configurations in the description of space-time changes of physical systems).

Other cuts of the space-time may be, and are of intrinsic interest. This is the case, for example, of the Solar system synchronization, foliating the space-time by *timelike* instants. And more importantly, the case of the positioning systems, cutting any (history of an) extended object by four (histories of) electromagnetic pulses or sound waves [8–12].

Once enlarged to any causal character, the very concept of synchronization, foliating space-time by instants largely unrelated to simultaneity, is found to be a gentle but powerful instrument which can help us to practise ‘seeing’ space-time from different, unconventional, viewpoints. In fact, the simple linear synchronization group is already able to generate coordinate systems of *any* Newtonian class.

Once we become used to handling arbitrary synchronizations, we can try to learn to describe nature without using *any* synchronization at all. This is possible by means of the emission coordinates, associated to the above mentioned positioning systems. Elsewhere [17] we have considered similar constructions in Newtonian and relativistic physics and analyze all the possible causal classes of emission coordinate systems according to the speed of the (sound or light) used signal and the different emitters configurations.

Acknowledgements This work has been supported by the Spanish Ministerio de Educación y Ciencia, MEC-FEDER project FIS2006-06062.

References

1. Coll, B., Morales, J.A.: 199 Causal classes of space-time frames. Int. J. Theor. Phys. **31**, 1045–1062 (1992)
2. Coll, B., Martín, J.J. (eds.): In: Proc. Int. School on Relativistic Coordinates, Reference and Positioning Systems, Salamanca, 2005. Universidad de Salamanca (in press)
3. Coll, B.: Elements for a theory of relativistic coordinate systems. Formal and physical aspects. In: Pascual-Sánchez, J.F., Floría, L., San Miguel, A., Vicente, F. (eds.) Proceedings of the XXIII Spanish Relativity Meeting ERE-2000 on Reference Frames and Gravitomagnetism, pp. 53–65. World Scientific, Singapore (2001). http//coll.cc
4. Bahder, T.B.: Navigation in curved space-time. Am. J. Phys. **69**, 315–321 (2001)
5. Rovelli, C.: GPS observables in general relativity. Phys. Rev. D **65**, 044017 (2002)
6. Blagojević, M., Garecki, J., Hehl, F.W., Obukhov, Y.N.: Real null coframes in general relativity and GPS type coordinates. Phys. Rev. D **65**, 044018 (2002)
7. Lachièze-Rey, M.: The covariance of GPS coordinates and frames. Class. Quantum Gravity **23**, 3531–3544 (2006)
8. Coll, B., Ferrando, J.J., Morales, J.A.: Two-dimensional approach to relativistic positioning systems. Phys. Rev. D **73**, 084017 (2006)
9. Coll, B., Ferrando, J.J., Morales, J.A.: Positioning with stationary emitters in a two-dimensional space-time. Phys. Rev. D **74**, 104003 (2006)
10. Coll, B., Pozo, J.M.: Relativistic positioning systems: the emission coordinates. Class. Quantum. Gravity **23**, 7395–7416 (2006)
11. Bini, D., Geralico, A., Ruggiero, M.L., Tartaglia, A.: Emission vs Fermi coordinates: applications to relativistic positioning systems. Class. Quantum. Gravity **25**, 205011 (2008)
12. Coll, B., Ferrando, J.J., Morales, J.A.: Emission coordinates in Minkowski space-time. In: Kunze, K.E., Mars, M., Vázquez-Mozo, M.A. (eds.) Physics and Mathematics of Gravitation, Proceedings of the XXXIth Spanish Relativity Meeting ERE-2008. AIP Conference Proceedings, vol. 1122, pp. 225–228. AIP, New York (2009)
13. Coll, B.: Temps uniforme et indice d'un fluid en relativité. Manuscript (1972). http://coll.cc
14. Cartan, E.: Sur les variétés à connexion affine et la théorie de la relativité generalisée. Ann. Sci. École Norm. Sup. **41**, 1–25 (1924)
15. Trautman, A.: Sur la théorie newtonienne de la gravitation. C.R. Acad. Sci. Paris **257**, 617–720 (1963). For completeness, see Trautman, A.: Comparison of Newtonian and relativistic theories of space-time. In: B. Hoffmann (ed.) Perspectives in Geometry and Relativity. Essays in Honor of Vlácav Hlavaty, pp. 413–424. Bloomington, Indiana Univ. Press, 1966
16. Rüede, C., Strautmann, N.: On Newton-Cartan Cosmology. Helv. Phys. Acta **70**, 318–335 (1997). See also arXiv:gr-qc/9604054
17. Coll, B., Ferrando, J.J., Morales, J.A.: Newtonian and Lorentzian emission coordinates. Phys. Rev. D **80**, 064038 (2009)