Comments on space–time signature

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In terms of three signs associated to two vectors and to a 2-plane, a formula for the signature of any four-dimensional metric is given. In the process, a simple expression for the sign of the Lorentzian metric signature is obtained. The relationship between these results and those already known are commented upon.

I. INTRODUCTION

To evaluate the signature of a metric tensor, two general methods may be used. One of them (Lagrange method[1,2]) implies the explicit calculation of tensor's *diagonal form*, the signature being then given by computation of the positive and negative elements. Through this method one obtains an *orthonormal frame* and the causal character of its vectors. The second one (Jacobi method[1]) consists of *signature sequence* construction, to which an appropiate rule is applied depending on the "degeneracy" of the sequence.

The Lagrange method has, in some cases, two disadvantages, namely, (i) its *indirect* character: the signature is obtained as a by-product of the explicit construction of an orthonormal frame, a construction that may be hard to obtain and sometimes devoid of interest, (ii) its *nonuniform* character: at some levels of the computation, the presence of vanishing diagonal terms (equivalent to that of null vectors) requires a change of algorithm.

The Jacobi method is *direct*, but remains *nonuniform*: depending on the character of the zeros in the signature sequence (absence of them, isolated zeros, two consecutive zeros,...) one has to apply different particular rules (Jacobi's rule, Gundelfinger's rule, Frobenius rule,...).[1,3,4]

The purpose of this paper is to give a *direct, general,* and *simple* expression for the signature of any four-dimensional metric. Here *direct* means without calculating either an orthonormal frame or any nonstrictly necessary quantity; *general* means that the sole expression is valid for *all* the possible forms of the metric (involving or not vanishing diagonal terms), and *simple* means which is easy to express and evaluate.

There are some situations of interest in which one has to evaluate the signature of a metric. This is the case, for example, in the equivalence problem for metrics where one has to answer whether two given metrics may be related by a local diffeomorphism. Irrespective of the method that one wishes to use (Cartan's orthonormal frame in general, or intrinsic scalar invariant coordinates in nonisometric cases), the direct evaluation of the signature of both metrics is a wise *preliminary* step: Their nonequivalence becomes readily evident if their signatures differ in absolute value.

The evaluation of the signature is also necessary in the *integration* of Einstein equations for (nondiagonal) prescribed forms of metric tensors, where one has to verify the *Lorentzian* character of the solutions and to obtain the *sign* of its signature to complete and discuss its physical interpretation (e.g., causal character of eigenvectors of its energy tensor).[5]

Also, in *finite perturbation* algorithms, where one starts from a given metric g_0 and its variation h, one has to verify that the new metric $g=g_0+h$ conserves the signature of g_0. This happens, in particular, in numerical relativity when one works in gauges with *nonzero* shift (general harmonic gauges, comoving coordinates for rotating fluids, minimum shear condi-

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tion). There, the finite interval of integration δt could transform, in principle, the signature of the given metric.[6]

The signature is also the only constraint that a metric imposes on the causal types of frames, so that the study of the *existence* of a frame corresponding to a given causal type for a metric is reduced to that of the frame's compatibility with the metric signature. This study is far from being trivial.[7]

For space–time, the Lorentzian character of the metric is insured by the negative sign of its determinant, so that only the *sign* of the signature remains to be evaluated. This sign may be deduced of course, from known signature theorems but, as indicated above, they do not admit in general, an easy formulation. The first result we shall obtain in this paper is a unique simple expression equivalent to the three different rules (Jacobi, Gundelfinger, Frobenius) needed in *general* to determine the *sign* of metric signature directly. This expression is given in Sec. II (Theorem 1).

It is in Sec. III that we give the general expression for the signature of any four-dimensional metric (Theorem 2). This expression is given as a function of three indices associated, respectively, to an arbitrary vector, a 2-plane containing it, and any vector of its orthogonal. This expression is not only useful for the space–time metric, but also to determine the elliptic or null signature of some symmetric tensors of interest (e.g., instantons, particular neutrino fields).

The relative tediousness of the standard, known methods may be checked on the following metric form:

$$\begin{pmatrix} \alpha & \beta & \beta & \nu \\ \beta & \alpha & \beta & \nu \\ \beta & \beta & \alpha & \nu \\ \nu & \nu & \nu & \mu \end{pmatrix}.$$

In contrast with them we shall show, in Sec. IV the efficiency with which our formula allows us to give a complete study of its signature.

Finally, in Sec. V, we comment on the relationship between our results and those already known, and indicate some applications in the study of the causal structure of Lorentzian frames.

II. SIGN OF A LORENTZIAN SIGNATURE

By definition, the *signature* σ of an n-dimensional metric that diagonalizes in p positive and q negative squares is given by $\sigma \equiv p-q$. A metric is denominated *Lorentzian* if $\sigma = \pm(n-2)$ and, in the case of an even dimension, it must have a negative determinant; for the space–time (four-dimensional Lorentzian metric) this last condition is also sufficient, so that we have in this case $\sigma = 2\epsilon$ with $\epsilon = \pm 1$. In this section our aim is to give a simple expression for ϵ.

Let Π be an arbitrary 2-plane, and u and v two vectors, respectively, tangent and orthogonal to Π. To every one of these elements, say x_a, let us associate a *causal index* i_a such that $i_a = +1, -1$, or 0 according to the positive, negative, or vanishing value of the scalar product (x_a, x_a);[8] denote, respectively, i_1, i_2, i_3 the causal indices corresponding to u, Π, v.[9] The triplet (i_1, i_2, i_3) will be called a *causal sequence*.

It is clear that the Lorentzian structure of the space–time forbids some causal sequences. Thus, if Π is spacelike ($i_2 = 1$), all its vectors u are space-like ($i_1 = \epsilon$) and since the 2-plane $\Pi_\perp$, orthogonal to Π, is timelike, v may be chosen with any orientation ($i_3 = 1, 0, -1$); if Π is timelike ($i_2 = -1$), it is u which may be chosen arbitrarily ($i_1 = 1, 0, -1$), v being necessarily spacelike ($i_3 = \epsilon$); if Π is null ($i_2 = 0$), $\Pi_\perp$ being also null, all the vectors of $\Pi \cup \Pi_\perp$ except the null direction $\Pi \cap \Pi_\perp$ ($i_1 = 0$, $i_3 = 0$), are spacelike ($i_1 = \epsilon$; $i_3 = \epsilon$). We have thus the following lemma.

Lemma 1: The causal sequences (i_1,i_2,i_3) corresponding to a Lorentzian metric of signature $\sigma=2\epsilon$ are of the form:

$$(\epsilon,1,i),(i,-1,\epsilon),(\epsilon,0,\epsilon),(\epsilon,0,0),(0,0,\epsilon),(0,0,0),$$

where ϵ takes the values 1, -1 and i the values 1, 0, -1.

A causal sequence will be called regular if its causal indices do not vanish simultaneously, or, in other words, if the module function

$$\Phi\equiv i_1^2+i_2^2+i_3^2 \tag{1}$$

is strictly positive. Obviously we then have:

Corollary: A causal sequence (i_1,i_2,i_3) corresponding to a Lorentzian metric determines the signature sign ϵ if, and only if, it is a regular causal sequence: $\Phi>0$.

As a direct comparision shows, the complement of the causal sequences of Lorentzian metrics of Lemma 1 in the set of 3^3 arbitrary triplets of indices (i_1,i_2,i_3) is the set of the eight triplets $(0,1,i)$, $(i,-1,0)$, $(\epsilon,0,-\epsilon)$, where ϵ takes the values $+1$, -1, and i the values $+1$, 0, -1. Now, looking for a function that vanishes on them, the structure of the two first classes leads to cancel i_2i_3 or $-i_2i_1$ with $-i_3$ or $-i_1$, respectively, which suggest an expression of the form $i_2(i_3-i_1)-(i_3+i_1)$; this expression vanishes also for the third class $(\epsilon,0,-\epsilon)$, and considering its values for the causal sequences of Lemma 1, one obtains the two following results.

Lemma 2: The triplets of indices (i_1,i_2,i_3) that cannot be regular causal sequences corresponding to a Lorentzian metric are those for which the function

$$\Sigma\equiv i_1+i_3+i_2(i_1-i_3) \tag{2}$$

vanishes.

Theorem 1: The sign of the signature σ of a Lorentzian metric is the sign of the function Σ on its regular causal sequences (i_1,i_2,i_3):

$$\text{sgn}\ \sigma=\text{sgn}\ \Sigma.$$

III. SIGNATURE OF A FOUR-DIMENSIONAL METRIC

The causal sequences corresponding to elliptic metrics of signature $\sigma=4\epsilon$ are obviously of the form $(\epsilon,1,\epsilon)$, and may also correspond to Lorentzian metrics, as was shown in Lemma 1. On the other hand, as a 2-plane Π containing a null vector cannot have a positive causal index,[8] the triplets $(0,1,i)$ cannot be causal sequences of four dimensional metrics. It follows that, from the eight triplets that do not correspond to Lorentzian metrics, the five triplets $(i,-1,0)$, $(\epsilon,0,-\epsilon)$ correspond to metrics of vanishing signature.

The other causal sequences of vanishing signature metrics cannot have the form $(\epsilon,1,\epsilon)$, $(\epsilon,1,0)$, or $(\epsilon,0,\epsilon)$; these forms are forbidden because, in this signature, the orthogonal to a totally spacelike or a null-spacelike 2-plane is, respectively, a totally timelike or a null-timelike 2-plane and vice versa. We have thus the following result.

Lemma 3: The causal sequences (i_1,i_2,i_3) corresponding to a metric of vanishing signature are of the form:

$$(\epsilon,1,-\epsilon),(i,-1,i'),(\epsilon,0,-\epsilon),\quad(\epsilon,0,0),(0,0,\epsilon),(0,0,0),$$

where the i's take the values 1, 0, -1, and ϵ the values 1, -1.

As Lorentzian and vanishing signature metrics have common causal sequences, they must be distinguished necessarily by an additional parameter. Let δ be the *determinant index* of the metric g, taking the values $+1$ or -1 according to the positive or negative character of $\det g$. As $\delta=-1$ for Lorentzian metrics, we are looking for a function Σ_δ such that $\Sigma_{-1}=\Sigma$; a simple choice is

$$\Sigma_\delta \equiv i_1+i_3+i_2(i_1+\delta i_3). \tag{3}$$

For elliptic causal sequences, $(\epsilon,1,\epsilon)$, one has $\Sigma_1=4\epsilon$. For the causal sequences of Lemma 3, Σ_1 vanishes except for $(\epsilon,0,0)$ and $(0,0,\epsilon)$, for which $\Sigma_1=\epsilon$; for these last cases, the module function Φ takes the value $\Phi=1$, while for the elliptic sequences it is $\Phi=3$: the expression $(1/2)(\Phi-1)\Sigma_\delta$ then gives the value of elliptic and vanishing signatures.

For the regular Lorentzian sequences of Lemma 1, Σ_{-1} takes the value 2ϵ except for $(\epsilon,0,0)$ and $(0,0,\epsilon)$, for which $\Sigma_{-1}=\Sigma_1$; there is a unique quadratic function in Φ that takes the value 1 for $\Phi=2,3$ and the value 2 for $\Phi=1$: the expression $(1/2)(\Phi^2-5\Phi+8)\Sigma_\delta$ then gives the value of Lorentzian signatures. With the aid of the weights $(1\pm\delta)/2$, both expressions may be linked in the following theorem.

Theorem 2: In terms of any of its regular causal sequences (i_1,i_2,i_3), the signature σ of a four-dimensional metric of determinant index δ is given by

$$\sigma=\tfrac{1}{4}\{(1-\delta)\Phi^2+2(3\delta-2)\Phi-9\delta+7\}\Sigma_\delta, \tag{4}$$

where Φ is the modular function, $\Phi\equiv i_1^2+i_2^2+i_3^2$, and $\Sigma_\delta\equiv i_1+i_3+i_2(i_1+\delta i_3)$.

IV. AN EXAMPLE

(a) In this section we shall apply our results (theorems 1 and 2) to a metric admitting an invariant action of the permutation group S_3 over the tangent space. The metrics admitting, in general, invariant actions of S_r are better (i.e., more homogeneously) distributed on the space of solutions to the Einstein equations than those admitting (continuous group of) isometries; furthermore, permutation symmetries of these types are mathematically easier to detect than usual isometries. This is why these symmetries are interesting as a new label in the classification of metrics. A first set of results concerning the invariant action of S_n on an n-dimensional Lorentzian manifold has been considered elsewhere.[10]

In a local chart adapted to a frame invariant by S_3, the metric takes the form

$$g=\begin{pmatrix}\alpha & \beta & \beta & \nu\\ \beta & \alpha & \beta & \nu\\ \beta & \beta & \alpha & \nu\\ \nu & \nu & \nu & \mu\end{pmatrix} \tag{5}$$

for which, the determinant is given by

$$\det g=(\alpha-\beta)^2[(\alpha+2\beta)\mu-3\nu^2] \tag{6}$$

so that g is regular iff

$$\alpha\neq\beta, \quad (\alpha+2\beta)\mu\neq3\nu^2, \tag{7}$$

and its determinant index $\delta\equiv\operatorname{sgn}(\det g)$ is

$$\delta=\operatorname{sgn}[(\alpha+2\beta)\mu-3\nu^2]. \tag{8}$$

The causal sequence (i_1,i_2,i_3) corresponding to the first adjoint 2-plane of the frame is related (see Ref. 9) to the first three principal minors Δ_k of g by $i_1=\text{sgn}\,\Delta_1$, $i_2=\text{sgn}\,\Delta_2$, $i_3=\delta\,\text{sgn}\,\Delta_3$; we have thus

$$i_1=\text{sgn}\,\alpha,\quad i_2=\text{sgn}(\alpha^2-\beta^2),\quad i_3=\delta\,\text{sgn}(\alpha+2\beta), \tag{9}$$

where it is understood that $\text{sgn}\,x=1$, 0, or -1 if, respectively, $x>0$, $x=0$, or $x<0$.

In order to discuss the signature of g we have to consider, on account of Eq. (9), the relative position of α with respect to β, 0, $-\beta$, and -2β. Thus, if we write $\epsilon(\alpha-\beta)>0$ with $\epsilon=1$ or -1, we have to inspect the three values $\alpha=0$, $\alpha=-\beta$, $\alpha=-2\beta$ and the four regions $\epsilon\alpha<0$, $0<\epsilon\alpha<-\epsilon\beta$, $-\epsilon\beta<\epsilon\alpha<-2\epsilon\beta$, $-2\epsilon\beta<\epsilon\alpha$ they delimit.

(b) Let us consider the Lorentzian case. From Eq. (8) $\delta=-1$ and, with the above notation $\epsilon(\alpha-\beta)>0$, the causal indices (9) may be conveniently written

$$i_1=\epsilon\,\text{sgn}(\epsilon\alpha),\quad i_2=\text{sgn}[\epsilon(\alpha+\beta)],\quad i_3=-\epsilon\,\text{sgn}[\epsilon(\alpha+2\beta)]. \tag{10}$$

Now, by *direct inspection* of their signs for every one of the three values and four regions mentioned above, we obtain the following causal sequences (i_1,i_2,i_3):

$$\begin{aligned}
&(-\epsilon,-1,\epsilon) \quad \text{for } \epsilon\alpha<0,\\
&(0,-1,\epsilon) \quad \text{for } \alpha=0,\\
&(\epsilon,-1,\epsilon) \quad \text{for } 0<\epsilon\alpha<-\epsilon\beta,\\
&(\epsilon,0,\epsilon) \quad \text{for } \alpha=-\beta,\\
&(\epsilon,1,\epsilon) \quad \text{for } -\epsilon\beta<\epsilon\alpha<-2\epsilon\beta,\\
&(\epsilon,1,0) \quad \text{for } \alpha=-2\beta,\\
&(\epsilon,1,-\epsilon) \quad \text{for } \epsilon\alpha>-2\epsilon\beta.
\end{aligned}$$

According to our Theorem 1, the sign of the signature σ of g is that of the function $\Sigma\equiv i_1+i_3+i_2(i_1-i_3)$. Its direct evaluation for any of the above seven cases gives $\Sigma=2\epsilon$; consequently, the *signature* σ of the S_3-invariant Lorentzian metric g given by (5) is

$$\sigma=2\,\text{sgn}(\alpha-\beta). \tag{11}$$

(c) In the *non-Lorentzian cases* $\delta=1$, the causal indices i_1 and i_2 are also given by their expressions (10) but, by (9), i_3 changes it sign:

$$i_3=\epsilon\,\text{sgn}[\epsilon(\alpha+2\beta)].$$

Thus, the corresponding causal sequences only differ from the seven evaluated in the Lorentzian case in their third index. Consequently, they are, respectively, given by

$$(-\epsilon,-1,-\epsilon),(0,-1,-\epsilon),(\epsilon,-1,-\epsilon),(\epsilon,0,-\epsilon),(\epsilon,1,-\epsilon),(\epsilon,1,0),(\epsilon,1,\epsilon).$$

Nevertheless, from the expression (8) for δ, it is clear that now the value $\alpha+2\beta=0$ is not admissible for $\delta=1$, so that the corresponding sequence $(\epsilon,1,0)$ cannot occur. Any way, according to Lemma 3 and the generic form $(\epsilon,1,\epsilon)$ of the causal sequences corresponding to the elliptic case, this sequence $(\epsilon,1,0)$ is always (i.e., whatever the form of the matrix metric) forbidden in the cases $\delta=1$.

For the non-Lorentzian cases, the expression of the signature σ given by Theorem 2 reduces to $\sigma=(1/2)(\Phi-1)\Sigma_1$, where $\Phi=i_1^2+i_2^2+i_3^2$ and $\Sigma_1=(1+i_2)(i_1+i_3)$. For the above first five causal sequences, one directly finds $\Sigma_1=0$ (and thus $\sigma=0$), meanwhile for the last one, one finds $\Sigma_1=4\epsilon$ and $\Phi=3$ (and thus $\sigma=4\epsilon$). Consequently, the signature σ of the S_3-invariant non-Lorentzian metrics g given by (5) is

$$\sigma=\begin{cases}4\,\mathrm{sgn}(\alpha-\beta) & \text{for} \quad (\alpha-\beta)(\alpha+2\beta)>0\\ 0 \quad \text{for} & (\alpha-\beta)(\alpha+2\beta)<0.\end{cases} \tag{12}$$

(d) When one wishes a general study of form (5) metrics, both results, (11) and (12), may be compacted in the sole expression

$$\sigma=2\,\mathrm{sgn}(\alpha-\beta)+(1+\delta)\mathrm{sgn}(\alpha+2\beta). \tag{13}$$

It is to be noted that the calculations made in this section to obtain results (11) and (12), or (13), are manifestly simpler than the calculations needed to construct the elements of a orthonormal frame which are necessary to conclude the same results.

V. COMMENTS

Several methods were developed over a long period of time to find the signature of a quadratic form $g=g_{ij}\theta^i\theta^j$. Perhaps the best known is the Lagrange's one,[1,2] which begins with a procedure to diagonalize g, whether some of the g_{ij}'s differ from zero or not. The signature is then obtained by direct computation of the p positive and q negative squares of the diagonalized form. Obviously, in spite of its interest and simplicity, this method is excessively hard when one only wishes to known the signature of g, without caring about the construction of its orthogonal frames. This has been implemented in the preceding section.

The starting point of all the other general methods is the sequence of signature $\{1,\Delta_1,...,\Delta_n\}$, Δ_k being the kth principal minor of the matrix g_{ij}. If all the Δ_k's in that sequence are different from zero, Jacobi's rule[1] gives the signature of g in terms of the permanences and variations of the signs of the Δ_k's when running the sequence. If isolated (nonconsecutive) Δ_k's are zero, a result by Gundelfinger[3] states that the signature may still be obtained by Jacobi's rule, but now applying it to the *reduced* sequence (that for which all the null minors are suppressed). This is due, in part, to the relation (see Ref. 3)

$$\Delta_{k-1}\Delta_{k+1}=\Delta_k\frac{\partial\Delta_{k+1}}{\partial g_{kk}}-\left(\frac{\partial\Delta_{k+1}}{\partial g_{k,k+1}}\right)^2 \tag{14}$$

which shows that, when $\Delta_k=0$, one has $\Delta_{k-1}\Delta_{k+1}<0$: every isolated zero produces a variation of sign in the reduced sequence. Finally, if two but not three consecutive Δ_k's are zero, Jacobi's and Gundelfinger's rules fail, but a result by Frobenius[4] again allows one to find the signature. His results states that when $\Delta_{k-2}=\Delta_{k-1}=0$ but $\Delta_{k-3}\Delta_k\neq 0$, the signature is given by Jacobi's rule provided that the variations of signs in the subsequence $\{\Delta_{k-3},\Delta_{k-2},\Delta_{k-1},\Delta_k\}$ be taken 2 for $\Delta_{k-3}\Delta_k>0$ or 1 for $\Delta_{k-3}\Delta_k<0$.

In the four-dimensional case, the indices of a signature sequence are related to ours by $\{1,i_1,i_2,\delta i_3,\delta\}$. It is then clear, from Lemmas 1 and 3, that for Lorentzian and vanishing signatures, one is confronted with signature sequences having isolated as well as consecutive zeros. Thus, our theorems 1 and 2 constitute simple and compact versions of the Jacobi's, Gundelfinger's, and Frobenius's rules simultaneously.

The restrictions to signature sequences induced by relation (14) forbid our causal sequences $(0,1,i)$, $(\epsilon,0,\delta\epsilon)$, $(i,\delta,0)$, a result that we have obtained directly by causality arguments. The easy way we have followed to show our theorems is an improvement on the usual proofs of the above known results. Our compact expressions would certainly simplify computer programs for signature.

The set $\{u,\Pi,v\}$, where u belongs to the 2-plane Π and v is orthogonal to it, may always be considered as a subset of the set $\{u,u_i;\ \Pi_i,\Pi_{ij};\ v_i,v\}$, $i=1,2,3$, associated to a frame $\{u,u_i\}$, where $\{v_i,v\}$, is its algebraic dual and Π_i,Π_{ij} are the 2-planes generated, respectively, by the pairs (u,u_i) and (u_i,u_j). This extended set contains all the elements needed to classify completely, from the causal point of view, the Lorentzian frames. This classification amounts the obtention of all the causally different coordinate systems that may be used in the space–time or, in a more geometric language, to obtain all the different relative positions of a frame with respect to the light cone. Such a classification has been made elsewhere[7] and, as was shown there, the corresponding table may be used as a signature table. Our present Theorem 1 plays the rôle of a leading constraint condition.

Twenty four causal sequences (i_1,i_2,i_3) may be obtained by permutations of a frame. When *all of them* are identical, the frame is called *causally symmetric.* A subclass of them, those which are also *metrically symmetric*, have already been considered.[10] In the analysis of causal symmetry groups of a frame, Theorem 1 simplifies many arguments; the corresponding results will be presented elsewhere.

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[1] F. R. Gantmacher, *Matrix Theory*, Vol. I (Chelsea, New York, 1959).
[2] É. Cartan, *The Theory of Spinors* (Dover, New York, 1966).
[3] S. Gundelfinger, J. für Reine Angew. Math. **91**, 221 (1881).
[4] G. Frobenius, J. für Reine Angew. Math. **114**, 187 (1895).
[5] For a large set of equations including some important ones (such as the general expression of Einstein equations, $S=\kappa T$), the overall sign of a Lorentzian metric is not important. But there are also many questions whose answer requires to know the sign of the Lorentzian signature; otherwise, how can one recognize if a vector is spacelike or timelike? or, how can one choose between the tensor $(\rho+p)u\otimes u-pg$ or $(\rho+p)u\otimes u+pg$ in order to represent a perfect fluid? The questions for which the overall sign of a Lorentzian metric is not important are exactly those questions related to even (differential) *concomitants* of the metric (i.e., concomitants of the metric and their derivatives which are even functions of their arguments). All questions related to odd concomitants of a Lorentzian metric *depend drastically* on the overall sign of the metric.
[6] Let us note that, this change may occur *only* in sign. As a consequence, the control of the negative sign of the determinant in relativity is not sufficient to insure the permanence of the whole signature.
[7] B. Coll and J. A. Morales, Int. J. Theor. Phys. 1045 (1992).
[8] Remember that a 2-plane Π may be defined by a *simple* (vanishing determinant) 2-form, so that we may write $\Pi=u\wedge w$, w being another vector tangent to Π; then $(\Pi,\Pi)=(u,u)(w,w)-(u,w)^2$.
[9] In practice, Π being arbitrary, one may take favorably Π as being *any* of the six adjoints 2-planes of the reference frame; in this case, the signs of three successive principal minors of order 1, 2, and 3 of the metric matrix may be taken as being i_1, i_2, and $-i_3$.
[10] B. Coll and J. A. Morales, J. Math. Phys. **32**, 2450 (1991).