Symmetric frames on Lorentzian spaces

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Symmetric frames (those whose vectors are metrically indistinguishable) are studied both, from the algebraic and differential points of view. Symmetric frames which, in addition, remain indistinguishable for a given set of concomitants of the metric are analyzed, and the necessary and sufficient conditions for a space-time to admit them are given. A new version of the cosmological principle then follows. Natural symmetric frames (induced by local charts) are also considered, and the space-times admitting them are obtained.

I. INTRODUCTION

The so-called *physically admissible* frames of the space-time consist of one timelike and three spacelike vectors. The last ones span the local space, while the first one indicates the direction of the space-time along which they are dragged. Physically admissible frames are thus related to (and, at the same time, induce) an *evolution* point of view.

Similarly, the *null frames* consist of two null and two spacelike vectors, the null ones being usually oriented along the principal directions of the gravitational or electromagnetic field. Null frames are thus well adapted to *radiative propagation* situations.

We have pointed out elsewhere[1] that the feeling of comfort associated to the use of such frames has a large historical base, but not a serious scientific one: As a *physical object*, the space-time may be equally framed by *any one* of the frames of the existing 199 Lorentzian causal classes.[2,3]

Among these 199 classes, there are 7 very particular ones, which contain a striking type of frames: Those whose vectors are metrically indistinguishable (i.e., they have the same length and the same mixed scalar products). Such frames are called *symmetric frames*.

The cosmological principle suggests in part that some properties of space-time would be best described in such frames that no direction be privileged. As, by definition, this is the case for symmetric frames, one may expect them to play an interesting role in cosmology. Symmetric frames appear well adapted to *isotropy* considerations.

The first problem concerning symmetric frames on non-elliptic metrics is their existence. We shall see that only Lorentzian metrics admit them: The Lorentzian character of a metric appears thus equivalently related to the notion of symmetric frames. In any non-Lorentzian case, the number of metrically indistinguishable vectors, and the dimension of the subspace they span, appear related to the null index of the corresponding metric.

Every symmetric frame has an axis. Its use makes easier the obtention of some properties and, in particular the obtention of the form of the elements of the transformation group that relates symmetric frames.

A complete classification of symmetric frames from a causal point of view shows that, in dimension m, there exist $2m-1$ types of them. For $m=4$, this result completes the one given by Derrick in his pionnering work on symmetric frames.[4]

It may occur that the vectors of a symmetric frame be indistinguishable even for a (differential) concomitant of the metric. When this happens for a concomitant $\mathscr{C}$, the frame is said to be $\mathscr{C}$-invariant, and it is said to be p-$\mathscr{C}$-invariant if, in addition, it is invariant for the covariant derivatives of $\mathscr{C}$ up to the order p. Ricci-invariant symmetric frames are perfect fluids, and a symmetric frame is p-Ricci invariant if it is one-Ricci invariant. This last result remains valid for Riemman-invariant symmetric frames, which are, in general, conformally flat perfect fluids. The following remarkable results follows: *the space-times admitting frames whose vectors are indistinguishable for any concomitant of the metric are the Friedmann–Robertson–Walker universes.* The Cosmological Principle is thus intimately related to the *complete* indistinguishability of the vectors of symmetric frames.

A symmetric frame is said to be natural if it is the natural frame of a coordinate system or, equivalently, if the Lie brackets of its vectors vanish. The axis of a natural symmetric frame is restricted to be shear-free and vorticity-free, and the Lorentzian spaces admitting natural symmetric frames may be located: They are those that admit an umbilical and conformally flat synchronization (foliation by spacelike hypersurfaces).

In the four-dimensional case, the Weyl tensor with respect to the axis is of electric type and thus the Petrov–Bel type of these space-times is **I**, **D**, or **O**. In particular, all the spherically symmetric space-times admit natural symmetric frames.

The paper is organized as follows: In Sec. II we study the relations between signature and null index of the metric and the number of indistinguishable vectors or the dimension of the subspace they span. In Sec. III we analyze some properties of the axis and its orthogonal space, obtain the general form of the transformations between symmetric frames, and find their different types from a causal point of view. Finally, Secs. IV and V are devoted to p-$\mathscr{C}$-invariant and natural symmetric frames, respectively.

A part of the results of this paper was presented, without proof, in the Spanish E. R. E.[5] and the French J. R.[6] annual relativistic meetings, and appeared in Ref. 7.

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II. METRICALLY INDISTINGUISHABLE VECTORS

The metric properties of a set of vectors are completely characterized by their scalar products. We shall consider here the following.

Definition: A set of vectors is said *metrically indistinguishable* if they have the same length and the same reciprocal scalar products. A frame whose vectors are metrically indistinguishable is called a *symmetric frame.*

In dimension $m \equiv n+1$, let $\{\xi_A\}_{A=1}^m$ be a symmetric frame for the metric g and let us denote

$$\alpha \equiv g(\xi_A,\xi_A), \quad \beta \equiv g(\xi_A,\xi_B), \tag{1}$$

for $A \neq B$. Let $\{\theta^A\}_{A=1}^m$ be the algebraic dual coframe of $\{\xi_A\}_{A=1}^m$, $\theta^A(\xi_B) = \delta_B^A \equiv \delta_{AB}$, then the metric $g = g_{AB}\theta^A \otimes \theta^B$ is such that

$$g_{AB} = (\alpha - \beta)\delta_{AB} + \beta 1_A 1_B, \tag{2}$$

where $1_A \equiv 1$ for any value of A.

Denote by Δ_{k+1} the principal minor of order $k+1$ of g_{AB},

$$\Delta_{k+1} \equiv \det(g_{ij})_{i,j=1}^{k+1} = \det(g_{ir} - g_{i,r+1}, g_{i,k+1})_{i=1,r\leqslant k}^{k+1}.$$

Taking into account that $g_{ir} - g_{i,r+1} = (\alpha - \beta)(\delta_{ir} - \delta_{i,r+1})$, and developing the above determinant by its first row there results

$$\Delta_{k+1} = (\alpha - \beta)\Delta_k + \beta(\alpha - \beta)^k$$

and a recurrent argument leads to the following.

Lemma 1: The $(k+1)$th principal minor of g_{AB} is given by

$$\Delta_{k+1} = (\alpha + k\beta)(\alpha - \beta)^k. \tag{3}$$

From this lemma Corollary 1 follows directly.

Corollary 1: g_{AB} is nonsingular if, and only if, one has

$$D \equiv (\alpha + n\beta)(\alpha - \beta) \neq 0. \tag{4}$$

Thus the sequence of signature $\{1, \Delta_1, \Delta_2, ..., \Delta_m\}$ of the metric g_{AB} contains at most one null minor, and its signature may be obtained from the Jacobi–Gundelfinger theorem,[8] which states that *when there are no consecutive zeros in the signature sequence of a metric in dimension m, its signature is given by* $\sigma = m - 2v$, *v being the number of variations of sign in the reduced sequence* (that whose null minors have been suppressed).

Suppose first $\beta > 0$. If $\alpha > \beta$, from Lemma 1 all the principal minors are positive and then we have $\sigma = m$; if $\alpha < -n\beta$, then $\Delta_i\Delta_{i+1} < 0$ $(i = 0,...,n)$ and it results $\sigma = m - 2m = -m$, meanwhile, if $-k\beta \leqslant \alpha < (1-k)\beta$ for a given $k < n$, then $\Delta_k\Delta_{k+1} \geqslant 0$ and $\Delta_i\Delta_{i+1} < 0$ $\forall i \neq k$, and it results that g_{AB} is a Lorentzian metric of signature $\sigma = m - 2n = 2 - m$. Thus

$$\sigma = \begin{cases} 1+n, & \text{if} \quad \alpha > \beta > 0, \\ 1-n, & \text{if} \quad -n\beta < \alpha < \beta, \\ -1-n, & \text{if} \quad \alpha < -n\beta < 0. \end{cases}$$

When $\beta < 0$, a similar discussion applied to $(-\alpha, -\beta)$ gives the signature of $-g_{AB}$, and then

$$\sigma = \begin{cases} 1+n, & \text{if} \quad \alpha > -n\beta > 0, \\ -1+n, & \text{if} \quad -n\beta > \alpha > \beta, \\ -1-n, & \text{if} \quad \alpha < \beta < 0. \end{cases}$$

The case $\beta = 0$ corresponds to a elliptic metric of signature $m\epsilon_\alpha$, ϵ_α being the sign of α, $\epsilon_\alpha \equiv \text{sgn}(\alpha)$. These considerations are collected in the following.

Theorem 1: In dimension $m = n+1$, the signature σ of a metric that admits a symmetric frame is given by

$$\sigma = \text{sgn}(\alpha + n\beta) + n\,\text{sgn}(\alpha - \beta). \tag{5}$$

From this follows the important result of the next Corollary.

Corollary 2: The only nonelliptic metrics admitting a symmetric frame are the Lorentzian ones.

It is interesting to note that this corollary involves an alternative, more geometric, idea of what is a Lorentz metric: Instead of associating it to a diagonalization and a particular computation of signs, our corollary associates it to the existence of symmetric frames.

In fact, from (4) the metric g_{AB} is elliptic or Lorentzian according to the inequalities $D > 0$ or $D < 0$, respectively. Furthermore, in the Lorentzian case (with $m > 2$) we have

$$\epsilon_\sigma \equiv \text{sgn}(\sigma) = \text{sgn}(\alpha - \beta)$$
$$= -\text{sgn}(\beta) = -\text{sgn}(\alpha + n\beta). \tag{6}$$

Let us consider a symmetric frame $\{\xi_A\}_{A=1}^m$ and let $v = \lambda^A\xi_A$ be such that $\{\xi_1,...,\xi_m,v\}$ be a set of $m+1$ metrically indistinguishable vectors. From (1) it follows

$$\alpha = g(v,v) = \lambda^A g(v,\xi_A) = \beta 1_A \lambda^A,$$

$$\beta = g(v,\xi_A) = \alpha\lambda^A + \beta \sum_{B \neq A} \lambda^B = (\alpha - \beta)\lambda^A + \alpha, \quad \forall A,$$

and, because of $\alpha \neq \beta$, one obtains

$$\lambda^A = -1 \;\; \forall A \;\; \text{and} \;\; \alpha = -m\beta. \tag{7}$$

From (3) and (7), it follows $D = (m+1)\beta^2 > 0$ and the metric is necessarily elliptic. We have thus the following lemma.

Lemma 2: The maximal number of metrically indistinguishable vectors[9] spanning a m-dimensional elliptic (resp. Lorentzian) space is $m+1$ (resp. m).

Now, let us consider a general metric of type (p,q), $p+q = m$, $p - q = \sigma$, and denote by i its *null index*, that is, the dimension of the maximal totally null subspaces: $i = \min\{p,q\}$. The dimension of the maximal elliptic (resp. Lorentzian) subspaces is $m - i$ (resp. $m - i + 1$); applying Lemma 2 to these subspaces we obtain the next theorem.

Theorem 2: In dimension m and metric of null index i, the maximal number N of metrically indistinguishable vectors generating a non-null subspace is given by $N = m - i + 1$.

Let d_M be the dimension of a maximal null subspace M generated by metrically indistinguishable vectors; when $p \neq q$, a vector v will be said spacelike (resp. timelike) if $\sigma g(v,v)$ is positive (resp. negative). Let $\{\xi_i\}_{i=1}^k$ be a basis of metrically indistinguishable vectors of a null subspace N_k. From Lemma 1, $\Delta_k = 0$ implies $\alpha = \beta$ or $\alpha = (1-k)\beta$. If $\alpha = 0$, N_k is a totally null subspace and then $d_M = i$. If $\alpha = \beta \neq 0$, the vectors $l_a \equiv \xi_a - \xi_k$ $(a = 1,..., k-1)$ are a basis of a totally null $(k-1)$-plane, N_{k-1}, and then $k \leqslant i+1$; when $\sigma \neq 0$, the greatest k is attained if N_{k-1} is maximal, thus $d_M = i+1$ and then the ξ_i's are spacelike since $g(\xi_i, l_a) = 0$; when $\sigma = 0$, N_{k-1} cannot be maximal, and so $d_M = i$. If $\alpha = (1-k)\beta$, $2 \leqslant k < m$, the vectors $r_i \equiv \xi_i - k^{-1} l$ generate an elliptic $(k-1)$-plane orthogonal to $l \equiv 1^i \xi_i$, but there exist at most $i-1$ timelike (resp. $m-i-1$ spacelike) vectors orthogonal to l and linearly independent, so that either $d_M = i \geqslant 2$ or $d_M = m-i$, depending on whether the r_i's are timelike or spacelike. Taking into account these results, we have the following.

Theorem 3: In dimension m and metric of null index i, the dimension d_M of the maximal null subspaces generated by metrically indistinguishable vectors is given by: (i) $d_M = i$ if the vectors are null or timelike; (ii) $d_M = i+1$ (resp. $d_M = m - i$) if, for nonzero signature, the vectors are spacelike and their lengths are equal to (resp. different from) their mutual scalar products.

III. ELEMENTS OF A SYMMETRIC FRAME

Suppose the following form of the inverse matrix of g:

$$g^{AB} = (\mu - \nu)\delta^{AB} + \nu 1^A 1^B, \qquad (8)$$

where $1^A \equiv \delta^{AB} 1_B$; then the only compatibility conditions of (2) and (8) are

$$\mu = (1/D)[\alpha + (n-1)\beta], \quad \nu = -\beta/D, \qquad (9)$$

and the next proposition follows.

Proposition 1: The algebraic dual coframe of a symmetric frame is a symmetric frame.

Note that

$$\mu - \nu = (\alpha - \beta)^{-1}, \quad \mu + n\nu = (\alpha + n\beta)^{-1}. \qquad (10)$$

Let ξ be an arbitrary vector. Two vectors ξ_1 and ξ_2 will be said *isometric* with respect to ξ if $g(\xi,\xi_1) = g(\xi,\xi_2)$. If $\xi = \lambda^A \xi_A$, where $\{\xi_A\}_{A=1}^m$ is a symmetric frame, then $g(\xi,\xi_A) = g(\xi,\xi_B)$ implies $\lambda^A = \lambda^B$. Thus we have the following.

Proposition 2: For any symmetric frame $\{\xi_A\}_{A=1}^m$ there exists a unique direction for which its vectors are isometric. This is the direction given by $\xi = 1^A \xi_A$ and it will be called the *axis* of the symmetric frame.

In the same way, the codirection defined by $\Theta \equiv 1_A \theta^A$ is the axis of the algebraic dual coframe $\{\theta^A\}$ of $\{\xi_A\}$. Thus

$$g(\xi) = g_{AB} 1^A \theta^B = (\alpha + n\beta) 1_A \theta^A = (\alpha + n\beta)\Theta.$$

Proposition 3: The axis of the algebraic dual coframe of a symmetric frame is the metric dual codirection of its axis.

Clearly, one has

$$g(\xi,\xi) = m g(\xi,\xi_A) = m(\alpha + n\beta), \qquad (11)$$

so that, from (6) and (11), it follows that $\mathrm{sgn}(g(\xi,\xi)) = -\epsilon_\sigma$, and then we have the following.

Proposition 4: In dimension $m > 2$, the axis of a Lorentzian symmetric frame is timelike.

Denoting $\lambda^2 \equiv m|\alpha + n\beta|$, the covector associated to the unit timelike vector $u \equiv \lambda^{-1}\xi$ is given by

$$g(u) = u_A \theta^A = -\epsilon_\sigma (\lambda/m)\Theta. \qquad (12)$$

Let $\mathscr{H}_\xi$ be the orthogonal hyperplane to the axis ξ of a symmetric frame. In the Lorentzian case, the induced metric on $\mathscr{H}_\xi$, $\gamma \equiv g + \epsilon_\sigma u \otimes u$, is given by

$$\gamma_{AB} = (\alpha - \beta)(\delta_{AB} - (1/m) 1_A 1_B). \qquad (13)$$

On the other hand $v \equiv v^A \xi_A \in \mathscr{H}_\xi$ iff $1_A v^A = 0$, so that the frame $\{x_A\}_{A=1}^m$ defined by

$$x_a = \frac{1}{\sqrt{a(a+1)}}\left[\sum_{i=1}^{a} \xi_i - a\xi_{a+1}\right], \quad x_m = \frac{1}{\sqrt{m}}\xi \qquad (14)$$

$(a = 1,\ldots,n)$ is orthogonal and satisfies

$$g(x_a,x_a) = \alpha - \beta, \quad g(x_m,x_m) = \alpha + n\beta. \qquad (15)$$

Let us consider the elliptic metric $\delta \equiv (\alpha-\beta)^{-1}(g - \beta\Theta \otimes \Theta)$. From (2) and (14) it follows that $\delta(\xi_A,\xi_B) = \delta(x_A,x_B) = \delta_{AB}$, which says to us that (14) is a δ-orthogonal transformation; the inverse of this transformation is thus its own transpose, and we have

$$\xi_1 = \sum_{a=1}^{n} \frac{1}{\sqrt{a(a+1)}} x_a + \frac{1}{\sqrt{m}} x_m,$$

$$\xi_m = \frac{1}{\sqrt{m}}(-\sqrt{n}\, x_n + x_m),$$

$$\xi_s = -\sqrt{\frac{s-1}{s}}\, x_{s-1} + \sum_{a=s}^{n} \frac{1}{\sqrt{a(a+1)}} x_a + \frac{1}{\sqrt{m}} x_m \quad (s = 2,\ldots,n). \qquad (16)$$

The inverse relations of (15) give us the components α and β of the metric in the symmetric frame

$$g(\xi_A,\xi_A) = (1/m)[g(x_m,x_m) + n g(x_a,x_a)],$$
$$g(\xi_A,\xi_B) = (1/m)[g(x_m,x_m) - g(x_a,x_a)], \quad A \neq B. \qquad (17)$$

Let $\{e_A\}_{A=1}^m$ be the projection of a symmetric frame on $\mathscr{H}_\xi$. From (11), it follows that $e_A = \xi_A - m^{-1}\xi$ and

$$\gamma(e_A,e_B) = g(e_A,e_B) = g_{AB} - (1/m)(\alpha + n\beta), \quad \forall A,B.$$

So we have the following.

Lemma 3: The projection of a symmetric frame on the hyperplane orthogonal to its axis is a set of metrically indistinguishable vectors for the induced elliptic metric on the hyperplane.

This lemma allows us to obtain the subgroup of $GL(m,\mathbb{R})$ relating symmetric frames. The vectors $\{e_A\}$ are the radii of a regular m-hedron since $1^A e_A = 0$ and $\gamma(e_A,e_A) = -n\gamma(e_A,e_B)$ for $A \neq B$, according to (7). On $\mathscr{H}_\xi$, apart from permutations, the rotations are the only transformations preserving this m-hedron, and the homotheties $\bar{e}_A = a e_A$ are associated to linear transformations of the type $\bar{\xi}_A = a\xi_A + b\xi$. Consequently, Theorem 4 follows.

Theorem 4: Modulo permutations, orthogonal transformations, and homotheties, any transformation relating symmetric frames is of the form

$$\bar{\xi}_A = \xi_A + [(\lambda - 1)/m] 1^R \xi_R, \quad \lambda \neq 0. \qquad (18)$$

In general, an orthogonal transformation modifies the axis of the frame but preserves the causal character of its vectors. On the contrary, transformation (18) preserves the axis, but may modify the causal character of the vectors of the frame.

Let $M_A{}^B$ be the matrix associated to (18):

$$M_A{}^B = (1/m)(\lambda + n)\delta_A{}^B + (\lambda - 1) 1_A 1^B.$$

From (3), it follows that det $M_A{}^B = \lambda$, and $M_A{}^B$ is nonsingular. The transformed metric components $\bar{\alpha} = g(\bar{\xi}_A, \bar{\xi}_A)$ and $\bar{\beta} = g(\bar{\xi}_A, \bar{\xi}_B)$ are related to (1) by

$$\bar{\alpha} - \alpha = \bar{\beta} - \beta = (1/m)(\alpha + n\beta)(\lambda^2 - 1). \qquad (19)$$

How many types of symmetric frames admits an m-dimensional Lorentzian space? Let us introduce the *causal index* $\kappa \in (0,m)$ of a Lorentzian symmetric frame by $\kappa \equiv 1 - \alpha/\beta$; from (3) and (6) one has $\mathrm{sgn}(\Delta_s) = (\epsilon_\sigma)^s \, \mathrm{sgn}(\kappa - s)$, so that the *adjoint* s-planes (those spanned by s vectors of the frame) are spacelike, null, or timelike iff $\kappa > s$, $\kappa = s$, or $\kappa < s$, respectively, and one has the following.

Theorem 5: In dimension m, there exist $2m - 1$ causal types of Lorentzian symmetric frames.

(i) Every open interval $(p, p+1)$ of values of κ $(p = 0,1,...,m-1)$ defines a causal type of frames whose adjoint s-planes are spacelike or timelike according as $s \leqslant p$ or $s \geqslant p+1$, respectively.

(ii) Every integer value of κ, $\kappa = p = 1,...,m-1$, defines a causal type of frames with null adjoint p-planes and whose adjoint s-planes are spacelike or timelike according as $s < p$ or $s > p$, respectively.

In the four-dimensional case this gives seven different types of symmetric frames. In them, the Minkowski metric diag.$(1, -1, -1, -1)$ adopts the form

$$\beta \begin{pmatrix} 1-\kappa & 1 & 1 & 1 \\ 1 & 1-\kappa & 1 & 1 \\ 1 & 1 & 1-\kappa & 1 \\ 1 & 1 & 1 & 1-\kappa \end{pmatrix},$$

where $\beta > 0$ and $0 < \kappa < 4$. The frames corresponding to $\kappa = 1$, $\kappa = 2$, and $\kappa = 3$ have null vectors, null planes, or null hyperplanes, respectively. Note that this classification is finer than the Derrick's[4] one, five of his ten types being obtained by adding the time orientation.

IV. CONCOMITANT-INVARIANT SYMMETRIC FRAMES

Symmetric frames are an algebraic concept; one can say that they "live" in the tangent space of Lorentzian manifolds. In this section, we shall "glue" them, by means of the differential concomitants of the metric, to the base manifold itself. We shall see how interesting restrictions appear.

The vectors of a symmetric frame are not necessarily indistinguishable for the differential concomitants of the metric. A symmetric frame whose vectors are indistinguishable for the Ricci tensor of the metric will be called a *Ricci-invariant symmetric frame*. In such a frame the metric is given by (2) and the Ricci tensor is written as

$$R_{AB} = (a-b)\delta_{AB} + b\, 1_A 1_B. \qquad (20)$$

From Lemma 1, (2), and (20) we compute directly

$$\det(R_{AB} - \lambda g_{AB}) = [a + nb - \lambda(\alpha + n\beta)]\,[a - b - \lambda(\alpha - \beta)]^n,$$

and taking into account (10), the eigenvalues of the Ricci tensor are given by

$$\lambda_1 = (a+nb)(\mu + n\nu), \quad \lambda_2 = (a-b)(\mu - \nu). \qquad (21)$$

Their respective eigenspaces are the axis ξ of the symmetric frame and its orthogonal hyperplane $\mathscr{H}_\xi$:

$$R^A{}_B 1^B = g^{AC} R_{CB} 1^B = (a+nb) g^{AC} 1_C = \lambda_1 1^A,$$
$$R^A{}_B v^B = g^{AC} R_{CB} v^B = (a-b) g^{AC} \delta_{CB} v^B = \lambda_2 v^A,$$

where v is an arbitrary vector of $\mathscr{H}_\xi$, $1_A v^A = 0$. Consequently, the Ricci tensor is spatially isotropic with respect to ξ, and thus

$$\mathrm{Ric} = rg + su \otimes u, \qquad (22)$$

where $r \equiv \lambda_2$, $s \equiv \lambda_1 - \lambda_2$, and u is the unit vector along the axis. Reciprocally, if Ric has the form (22) then any symmetric frame of axis $\{u\}$ is Ricci invariant. If one considers Einstein $(s=0)$ and Ricci flat $(s = r = 0)$ spaces as *degenerate* perfect fluids, one has, remembering Proposition 4, the following.

Theorem 6: A Lorentzian space admits Ricci-invariant symmetric frames if, and only if, it is a perfect fluid. These frames are all those whose axis is collinear to the fluid velocity.

Let us note that the above arguments, and in particular Eq. (22), are valid for *any* second-order tensor for which the vectors of the symmetric frame remain indistinguishable.

A Ricci-invariant symmetric frame whose vectors are indistinguishable for the covariant derivatives of the Ricci tensor up to the order p will be said a *p-Ricci invariant symmetric frame*. Let us consider one-Ricci invariant symmetric frames; denoting by ∇Ric the covariant derivative of the Ricci tensor, one must have

$$\nabla_P R_{QS} = \rho_1, \quad \nabla_P R_{QQ} = \rho_2,$$
$$\nabla_P R_{PQ} = \rho_3, \quad \nabla_P R_{PP} = \rho_4,$$

for any different values of the indices P,Q,S. Thus

$$\nabla_A R_{BC} = \tau 1_A 1_B 1_C + \eta 1_A \delta_{BC} + \phi(1_B \delta_{AC} + 1_C \delta_{AB}) + \psi \delta_{ABC}, \qquad (23)$$

where

$$\tau = \rho_1, \quad \eta = \rho_2 - \rho_1, \quad \phi = \rho_3 - \rho_1,$$
$$\psi = \rho_4 - 2\rho_3 - \rho_2 + \rho_1,$$

and $\delta_{ABC} \equiv \delta_{AB}\delta_{BC}$ (repeated indices are not summed).

From (22) it follows directly

$$\nabla_A R_{BC} = r_A g_{BC} + s_A u_B u_C + s(\nabla_A u_B u_C + u_B \nabla_A u_C), \qquad (24)$$

where $f_A \equiv \xi_A(f)$ is the derivative of the function f along ξ_A. Contracting indices in (23) and (24), and using (8) and (12) with $\epsilon_\sigma = -1$, we have

$$g^{AB}\nabla_A R_{BC} = r_C + (\dot{s} + s\theta)u_C = [m\lambda^{-2}(m\tau + \eta + \phi) + \mu(m\phi + \psi)]1_C, \qquad (25)$$

$$R_A = mr_A + s_A = [m\lambda^{-2}(m\tau + 2\phi) + \mu(m\eta + \psi)]1_A, \qquad (26)$$

where the dot denotes derivation along u $(\dot{s} \equiv u^A s_A)$, $\theta \equiv \nabla_A u^A$ is the expansion of u and $R \equiv g^{AB}R_{AB}$ is the scalar curvature. These relations imply that the exterior differential of r and s are given by

$$dr = \dot{r}u, \quad ds = \dot{s}u. \qquad (27)$$

Due to (25)–(27) the double contraction of the Bianchi identities, $R_C = 2g^{AB}\nabla_A R_{BC}$, is equivalent to

$$2s\theta = (m-2)\dot{r} - \dot{s}. \quad (28)$$

On the other hand, from (23) and (24), we can compute $u^A\nabla_A R_{BC}$ separately. Equating both expressions and taking into account (10), (12), (13), and (27) it follows

$$s\nabla_A u_B = [(\mu - \nu)/\lambda](m\phi + \psi)\gamma_{AB}. \quad (29)$$

If $s \neq 0$, u has only expansion and we obtain the following.

Theorem 7: In a nondegenerate perfect fluid, the axis of the one-Ricci invariant symmetric frames is geodesic, shear-free, and vorticity-free, and its expansion and the Ricci eigenvalues are functions of the potential of u.

Now by substitution of (22), (27), and (28) into (24) one has

$$\nabla_A R_{BC} = (\dot{r} + \dot{s})u_A u_B u_C + \dot{r} u_A \gamma_{BC} + (s/n)\theta(u_B\gamma_{AC} + u_C\gamma_{AB}), \quad (30)$$

and it is clear that higher covariant derivatives will provide no elements able to distinguish the vectors of the frame. We may thus state the next theorem.

Theorem 8: A symmetric frame is p-Ricci invariant if, and only if, it is one-Ricci invariant.

Let us consider now the frames whose vectors are indistinguishable for the metric and its Riemann tensor $\mathscr{R}$. We call them *Riemman invariant symmetric frames.*

Let Π_{AB} define the oriented two-plane generated by ξ_A and ξ_B, $\Pi_{AB} = \xi_A \wedge \xi_B$, and let us consider the Riemann tensor $\mathscr{R}$ as a bilinear symmetric form defined over the Π's. Denote $\mathscr{R}_{ABCD} \equiv \text{Riem}(\Pi_{AB},\Pi_{CD})$; a pair of Π's may have in common 0, 1, or 2 indices, and this allows us to consider the three following types of scalars: $\mathscr{R}_{PQPQ}$, $\mathscr{R}_{PQPR}$ and $\mathscr{R}_{PQRS}$, where P, Q, R, S stand for different indices. Due to the symmetries of the Riemann tensor we have the following.

Lemma 4: The components of the Riemann tensor with respect to a Riemann invariant symmetric frame satisfy

$$\mathscr{R}_{PQPQ} = \mathbb{A}, \quad \mathscr{R}_{PQPR} = \mathbb{B}, \quad \mathscr{R}_{PQRS} = 0,$$

for any distinct values of the indices P,Q,R,S.

From the following relations,

$$\delta^{AB}\mathscr{R}_{APBP} = n\mathbb{A}, \quad 1^A 1^B \mathscr{R}_{APBP} = n\mathbb{A} + (n-1)\mathbb{B},$$

$$\delta^{AB}\mathscr{R}_{APBQ} = (n-1)\mathbb{B}, \quad 1^A 1^B \mathscr{R}_{APBQ} = -\mathbb{A} + (n-1)\mathbb{B},$$

it results that in a Riemann invariant symmetric frame the Ricci tensor, $R_{AB} \equiv g^{CD}\mathscr{R}_{CADB}$ has the form (20) with

$$a \equiv R_{PP} = n\mu\mathbb{A} + (n-1)\nu\mathbb{B},$$
$$b \equiv R_{PQ} = -\nu\mathbb{A} + (n-1)(\mu - 2\nu)\mathbb{B} \quad (P \neq Q), \quad (31)$$

where μ and ν are given by (9). For $n > 1$, the linear system (31) for $\mathbb{A}$ and $\mathbb{B}$ is of rank two; so that when $a = b = 0$, its solution is $\mathbb{A} = \mathbb{B} = 0$. This says that the Riemann tensor is a linear and homogeneous function of the Ricci tensor, and so, the Weyl tensor vanishes. Then

$$\mathscr{R} = [1/(m-2)][\text{Ric} - [R/2(m-1)]g] \wedge g, \quad (32)$$

where R is the scalar curvature and $\wedge$ denotes the exterior product of double one-forms.[10] Remembering Theorem 6, one has the following result.

Theorem 9: For $m > 3$, the Lorentzian spaces admitting Riemann invariant symmetric frames are the conformally flat perfect fluids.

It is known[11] that, for $m = 4$, such spaces belong to the generalized Schwarzschild interiors or to the Stephani universes.[12]

From (32), it follows that the vectors of a Riemann invariant symmetric frame are indistinguishable for $\nabla\mathscr{R}$ if, and only if, they are indistinguishable for ∇Ric. A symmetric frame whose vectors are indistinguishable also for the Riemann tensor and its covariant derivatives up to the order p will be said a *p-Riemann invariant symmetric frame.* Theorem 7 and Eq. (30) imply that such a frame for $p = 1$ remains indistinguishable for the successive covariant derivatives of the Ricci tensor. And then, from (32) we have the next theorem.

Theorem 10: A symmetric frame is p-Riemann invariant if, and only if, it is one-Riemann invariant.

The Bianchi identities for a conformally flat space-time can be written

$$6(\nabla_A R_{BC} - \nabla_B R_{AC}) = R_A g_{BC} - R_B g_{AC}. \quad (33)$$

For $n = 3$, when Eq. (30) takes place, these identities are equivalent to Eq. (28), $s\theta = \dot{r} - \dot{s}/2$, which in terms of the density ρ and the pressure p of the perfect fluid may be written in the more familiar form $\dot{\rho} = -(\rho + p)\theta$. Thus the Bianchi identities do not impose additional kinematical restrictions on the expansion of u. Therefore, the space-times admitting a one-Riemann invariant symmetric frame are (conformally flat) perfect fluids with a geodesic, shear-free, and vorticity-free velocity whose expansion is an arbitrary function of the time coordinate only. As it is well known, these kinematical properties characterize the spatially homogeneous and isotropic cosmological models. Conversely, it is clear that the Friedmann–Robertson–Walker universes are conformally flat perfect fluids satisfying (30). Thus we have obtained the following interesting characterization.

Theorem 11: The space-times admitting one-Riemann invariant symmetric frames are the Friedmann–Robertson–Walker universes.

As a corollary of Theorems 10 and 11, the following remarkable result follows: *the cosmological principle states that the space-time admits frames whose vectors are indistinguishable for any differential concomitant of the metric.*

It should be noted that a geodesic, shear-free, and nonrotating perfect fluid velocity may exist only in a conformally flat space-time.[13] Then $\mathscr{R}$ is given by (32), and from Theorems 7 and 11, Theorem 12 follows.

Theorem 12: The space-times admitting one-Ricci invariant symmetric frames are the Einstein spaces and the Friedmann–Robertson–Walker universes.

V. NATURAL SYMMETRIC FRAMES

A symmetric frame that is the natural frame of a coordinate system is called a *natural symmetric frame.* First, let us consider the kinematic properties of the axis of such a frame. If $\{\xi_\mu\}_{\mu=1}^{m}$ is a natural symmetric frame, $\xi_\mu = \partial_\mu \equiv \partial/\partial x^\mu$, from (2) it results

$$g = [(\alpha - \beta)\delta_{\mu\nu} + \beta 1_\mu 1_\nu] dx^\mu \otimes dx^\nu, \quad (34)$$

where α and β are functions of the coordinates $\{x^\mu\}$ and $\{dx^\mu\}_{\mu=0}^{n}$ is the algebraic dual coframe of $\{\partial_\mu\}$. One has

$1^\mu \equiv 1_\mu = 1$ for any value of the index μ, so that $1_\mu 1^\mu = m \equiv n+1$ and $g_{\mu\nu} 1^\nu = (\alpha + n\beta) 1_\mu$. From Proposition 3 it follows directly that the axis $\xi = 1^\mu \partial_\mu$ of a natural symmetric frame is vorticity free. From (13), the induced metric γ on the hypersurfaces orthogonal to the axis is written as

$$\gamma = (\alpha - \beta)(\delta_{\mu\nu} - (1/m) 1_\mu 1_\nu) \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

and its Lie derivative with respect to the unit vector u along the axis is given by

$$\mathscr{L}(u)\gamma_{\mu\nu} = u^\rho \partial_\rho \ln|\alpha - \beta| \gamma_{\mu\nu}. \tag{35}$$

We have thus the next theorem.

Theorem 13: The axis of a natural symmetric frame is shear-free and vorticity-free.

In a domain of a Loretzian space, a *synchronization* is a foliation by spacelike hypersurfaces, and every one of these hypersurfaces are called *instants*. Equation (35) shows the umbilical character of (every instant of) the synchronization defined by (the hypersurfaces orthogonal to) the axis of the frame. Thus in a more geometric language Theorem 13 may be equivalently stated.

Theorem 14: In a Lorentzian space admitting natural symmetric frames there exists (locally) an umbilical synchronization.

A general result[14] states that an $(n+1)$-dimensional space admits an umbilical foliation iff there exist local coordinates $\{y^i, y^{n+1}\}$ such that the metric may be written as

$$g = A a_{ij} \, \mathrm{d}y^i \otimes \mathrm{d}y^j + B \, \mathrm{d}y^{n+1} \otimes \mathrm{d}y^{n+1} \quad (i,j = 1,...,n),$$

where A and B are arbitrary functions, and the a_{ij}'s are independent of the $(n+1)$th coordinate. If in addition the hypersurfaces of the umbilical foliation are conformally flat, the matrix a_{ij} can be taken diagonal and equal to the signature symbol of g. In particular, we have the following.

Lemma 5: An $(n+1)$-dimensional Lorentzian space admits an umbilical and conformally flat synchronization if, and only if, there exists coordinates $\{y^i, y^{n+1}\}$ such that the metric may be written as

$$g = \pm (a^2 \delta_{ij} \, \mathrm{d}y^i \otimes \mathrm{d}y^j - b^2 \, \mathrm{d}y^{n+1} \otimes \mathrm{d}y^{n+1}) \quad (i,j = 1,...,n),$$

where a and b are functions of $y^1,...,y^{n+1}$.

On the other hand, as the coefficients of the transformations (14) and (16) are constants, it follows that a symmetric frame is natural iff its associated orthogonal frame is natural. Thus we have the following geometric characterization.

Theorem 15: The Lorentzian space admitting natural symmetric frames are those in which there exists (locally) and umbilical and conformally flat synchronization.

In the four-dimensional case it is well known that, for the space-times admitting a vorticity-free and shear-free timelike direction, the magnetic part of the Weyl tensor with respect to it vanishes.[15] Then, we have the following.

Proposition 5: In the space-times admitting a natural symmetric frame, the Weyl tensor is of electric type with respect to the axis of the frame.

In particular, the *Petrov–Bel* type of these space-times is necessarily **I**, **D**, or O.

In relativity, spaces-times with the characteristic asked by Theorem 15 are already known: For example, the Stephani universes,[11] or the isotropic class considered by Derrick,[4] which contain the Friedmann–Robertson–Walker and Schwarzschid space-times. Also, as a direct consequence of a result by Takeno[16] concerning isotropic coordinates, one has the following corollary.

Corollary 3: All the spherically symmetric space-times admit natural symmetric frames.

It is interesting to observe that only in some cases, the condition for a symmetric frame of being natural not only implies the metric coefficients to verify at *every point* the equalities (1) but, in addition, obliges these coefficients to be *symmetric functions* of the corresponding local coordinates: the "tangent" symmetry goes down to the underlying manifold. This was shown by Derrick to happen for the isotropic class he considered in Ref. 4, but also occurs for other interesting spaces-times; a study of them will be published elsewhere.

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[1] B. Coll and J. A. Morales, "The 199 causal classes of frames of the space-time," unpublished paper; copies available upon request.
[2] See the Table of causal classes in Ref. 1.
[3] This statement is nothing but the algebraic or "tangent" version of the principle of general covariance for the underlying space-time.
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[6] B. Coll and J. A. Morales, "Repères Symmétriques Lorentziens," in Proceeding J.R.F. '88 Meeting, (Univ. Genève, CH-1211 Genève 4, Switzerland, 1989), pp. 143–148.
[7] B. Coll and J. A. Morales, C. R. Acad. Sci. Paris, Sér. I 306, 791 (1988).
[8] See, for instance, F. R. Gantmacher, *Théorie of Matrices* (Dunod, Paris, 1966), Vol. I. Jacobi's theorem concerns the case where all the principal minors are not null. When one or more minors are null but not consecutive, the result was established by S. Gundelfinger [see J. Reine Angew. Math. **91**, 221 (1881)]. For the case of two consecutive zeros, the theorem is due to G. Frobenius.
[9] In the elliptic case, they constitute, for $\alpha = 1$, the unit radii r_i $(1 \leqslant i \leqslant m+1)$ of a regular $(m+1)$-hedron. According to Eq. (7), they verify $1^i r_i = 0$ and $r_i \cdot r_j = -1/m$.
[10] If P and Q are double one-forms, P ∧ Q is defined as the double two-form having components
$$(\mathrm{P} \wedge \mathrm{Q})_{ijkl} = \mathrm{P}_{ik}\mathrm{Q}_{jl} + \mathrm{P}_{jl}\mathrm{Q}_{ik} - \mathrm{P}_{il}\mathrm{Q}_{jk} - \mathrm{P}_{jk}\mathrm{Q}_{il}.$$
[11] H. Stephani, Commun. Math. Phys. 4, 137 (1967).
[12] C. Bona and B. Coll, Gen. Relat. Gravit. 20, 293 (1988).
[13] See, for instance, D. Kramer, H. Stephani, M. A. H. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge U. P., Cambridge, 1980), p. 80.
[14] See L. P. Eisenhart, *Riemannian Geometry* (Princeton U. P., Princeton, NJ, 1949), p. 182.
[15] M. Trümper, J. Math. Phys. 6, 584 (1965).
[16] H. Takeno, "The theory of spherically symmetric space-times," Sci. Rep. Res. Inst. Theor. Phys. (Hiroshima U., Hiroshima, 1966), No. 5.