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Newtonian and relativistic emission coordinates

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Emission coordinates are those generated by positioning systems. Positioning systems are physical systems constituted by four emitters broadcasting their respective times by means of sound or light signals. We analyze the incidence of the space-time causal structure on the construction of emission coordinates. The Newtonian case of four emitters at rest is analyzed and contrasted with the corresponding situation in special relativity.

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I. INTRODUCTION

The study of space-time coordinate systems and the different protocols associated with their physical construction is a broad and open field in current physics [1–10]. Here, we consider those coordinate systems constructed from *positioning systems* that are basically defined by four clocks (emitters) broadcasting their respective times by means of some type of signal (electromagnetic, sonic). At each space-time event reached by the signals, the received 4 times define the *emission coordinates* of this event (with respect to the given positioning system).

A complete description of any coordinate system must mention the protocols for the physical construction of its geometric elements (coordinate lines, coordinate surfaces, and coordinate hypersurfaces) which may have associated different causal characters (spacelike, timelike, null). Thus, for example, these coordinate elements may be performed, among other ways, by means of clocks for timelike lines, laser pulses for null lines, rods or inextensible threads for spacelike lines, laser beams for timelike surfaces, light-front signals for null hypersurfaces, and so on.

Positioning systems, and their emission coordinates, may be constructed both in Newtonian and relativistic physics and their definition does not involve the use of any synchronization convention. The physical protocols allowing the realization of emission coordinates involve the velocity of the used signal and the configuration (kinematics) of the emitters. Thus, the coordinate hypersurfaces of an emission coordinate system are the four families of space-time cones with vertices on the events of the world lines of the emitters, the *emission cones* generated by the signals. For homogeneous and nondispersive media, the emission cones are wholly determined by the speed of the broadcast signals.

A coordinate system has associated 4 one-parametric families of hypersurfaces, 6 two-parametric families of surfaces, and 4 three-parametric families of lines. The set of causal characters of these 14 geometric elements is said the *causal signature* of the coordinate system. A *causal class* is the set of all coordinate systems having same causal signature.

The space-time coordinate systems have been classified from the causal point of view, and the result is [11,12]: *the number of causal classes of Newtonian and relativistic coordinate systems is* 4 *and* 199, *respectively.*

The assignment of one specific causal class to a coordinate system in a region of the space-time supposes that the causal characters of all the geometric elements of the coordinate system (lines, surfaces and hypersurfaces) are the same at any point of the region or, in other words, that the region under consideration is a *causal homogeneous region* for the coordinate system in question. Therefore, there are 4 or 199 causally different ways to parametrize the events in a causal homogeneous region of the space-time, according to the classical or relativistic description that we want to make.

In dealing with evolution formalisms, one usually considers standard coordinate systems which are adapted to a $3 + 1$ splitting of the space-time in space plus time. The space-time is thus represented as (absolutely or relatively) foliated by a one-parametric family of spacelike hypersurfaces (instants of a synchronization). This provides the standard space-time decomposition that, of course, is not the only admissible one.[1] As the causal classification of space-time coordinate systems shows, other three causally different decompositions exist in Newtonian physics, and other 198 exist in relativity.[2]

Among these unusual space-time decompositions, those associated with positioning systems deserve special attention. The corresponding four-dimensional region is sliced by the histories (emission cones) of the broadcast signals.

[1]See Ref. [11] in connection with the role that the synchronization group plays in the physical realization of unusual Newtonian and relativistic space-time parametrizations.

[2]Remember that other formulations already exist in relativity (different from the $3 + 1$ one) offering complementary (or alternative) advantages to solve Einstein equations (see also footnote 5 below).

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The relativistic theory of positioning systems using electromagnetic signals has been analyzed elsewhere [4–6,13–15], remarking their incidence in the current global navigation systems. Of course, the corresponding theory with subluminous signals deserves relevant interest in connection with sonic positioning systems, both in the Newtonian and in the relativistic regimes.

Here, we analyze the causal properties of the emission coordinates and, for the sake of simplicity, we consider the case of four emitters at rest. Elsewhere, we have obtained (in the Minkowski space-time) the transformation between inertial and light emission coordinates for arbitrary motions of the emitters [16,17].

The results reported in this paper lay the foundations for a Newtonian theory of positioning systems, an issue with potential applications. Indeed, nowadays there is an increasing interest in the study and development of indoor and ultrasonic positioning systems (see, for instance, [18]).

The paper is organized as follows. In Sec. II we summarize the main results on the causal classification of Newtonian and relativistic space-time frames and coordinate systems. For a more detailed discussion, see [11,12]. In Sec. III we analyze the causal properties of Newtonian emission coordinates. Any emission coordinate domain always presents three regions corresponding to the three nonstandard Newtonian causal classes. The relativistic situation is studied in Sec. IV, where one may distinguish 103 causal classes of emission coordinates. One causal class corresponds to luminous signals and is always causally homogeneous. The remaining 102 causal classes may be physically constructed using sound signals. Then, depending on the velocity of the emitted signal and on the configuration of the emitters, the domain of the emission coordinates presents different causal homogeneous regions. Finally, in Sec. V we present a discussion of our results and some future perspectives.

II. THE CAUSAL CLASSIFICATION OF SPACE-TIME COORDINATE SYSTEMS

In this section, we summarize the main results about the causal classification of Newtonian and relativistic frames and coordinate systems. The proof of these results has been presented in Refs. [11,12].

A. Notion of causal class

In the four-dimensional space-time, the four vectors of a frame $\{v_A\}_{A=1}^4$ also define six planes Π_{AB} generated by the pairs $\{v_A, v_B\}$, $A \neq B$, and form hyperplanes Π_{ABC} generated by the triplets $\{v_A, v_B, v_C\}$, $A \neq B \neq C \neq A$; these last ones are biunivocally determined by the four orthogonal one forms, θ^A, whose set constitutes the dual frame $\{\theta^A\}_{A=1}^4$ of $\{v_A\}_{A=1}^4$, $\theta^A(v_B) = \delta_B^A$. The set of the causal characters of these 14 geometric elements $\{v_A, \Pi_{AB}, \theta^A\}$ is called the *causal signature* of the frame and, in an abridged form, it is denoted as $\{\mathrm{c}_A, \mathrm{C}_{AB}, c_A\}$. Here, the symbol c_A is the causal character of the vectors v_A; C_{AB}, with $A < B$, is the causal character of the plane generated by v_A and v_B; and c_A is the causal character of the covector θ^A. Evidently, c_A also provides the causal character of the hyperplane generated by the vectors v_B, v_C, and v_D which are different from v_A.

By definition, the *causal class* of a frame is the set of all the frames that have the same causal signature; and the causal class of a coordinate system $\{x^\alpha\}_{\alpha=1}^4$ in a causal homogeneous domain of the space-time is the causal class $\{\mathrm{c}_\alpha, \mathrm{C}_{\alpha\beta}, c_\alpha\}$ of its associated natural (or coordinate) frame at the events of the domain. The c_α's are the causal characters of the vectors $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$ of the natural frame $\{\partial_\alpha\}$ itself, and the c_α's are the causal characters of the 1-forms dx^α of the coframe $\{dx^\alpha\}$. Four families of coordinate hypersurfaces are associated with this coframe, and their mutual intersections give six families of coordinate surfaces whose causal characters are precisely given by $\mathrm{C}_{\alpha\beta}$ (of course, the mutual intersections of these surfaces give the four congruences of coordinate lines of causal character c_α). We have chosen the following order for the causal characters of a causal class: $\{\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3\mathrm{c}_4, \mathrm{C}_{12}\mathrm{C}_{13}\mathrm{C}_{14}\mathrm{C}_{23}\mathrm{C}_{24}\mathrm{C}_{34}, c_1c_2c_3c_4\}$.

We use the following notation. Roman letters (e, t, l) denote the causal characters (spacelike, timelike, null) of vectors and coordinate lines. Capital letters (E, T, L) denote the causal characters (spacelike, timelike, null) of the associated planes and coordinate surfaces. And Italic letters (e, t, l) denote the causal characters (spacelike, timelike, null) of covectors which also allows us to determine the causal characters (timelike, spacelike, null, respectively) of the coordinate hypersurfaces.

B. 4 causal classes of Newtonian coordinate systems

The Newtonian causal structure admits only two causal characters. Spacelike directions are those defined by any pair of simultaneous events. Timelike directions are those defined by pairs of nonsimultaneous events (there is no absolute cone associated with light propagation or with any other physical signal). Consequently, the study of the causal classification of coordinate systems in the Newtonian space-time is simpler than the relativistic one.[3]

The Newtonian causal structure is determined by an exact 1-form $\theta = dt$, called the *time current* and a rank three contravariant positive *space metric* γ^* satisfying the orthogonality condition $\gamma^*(\theta) = 0$. Each hypersurface $t =$ const is the loci or space of simultaneous events at the instant t.

At every event, the causal character of directions, planes and hyperplanes is given in terms of θ and γ^* according to the following definitions.

[3]To our knowledge, and in spite of its simplicity, the causal classification of Newtonian coordinates has been first given in [11].

A vector v is *spacelike* if it is instantaneous with respect to the time current θ, i.e. if $\theta(v) = 0$. Otherwise, the vector is *timelike*.

Correspondingly, a covector $\omega \neq 0$ is *timelike* if it has no instantaneous part with respect to the space metric γ^*, i.e. if $\gamma^*(\omega) = 0$. Otherwise, the covector ω is *spacelike*. The sole timelike codirection is that defined by the current θ at every event, because γ^* has rank 3.

A r-plane Π is *spacelike* if every vector v in it is spacelike. Otherwise, Π is *timelike*, i.e. it contains timelike vectors.

A r-coplane Ω is *timelike* if it contains the time current θ. Otherwise Ω is *spacelike*.

These notions are also naturally valid for vectors fields and 1-forms in causal homogeneous regions, and so, they are obviously extended for curves and surfaces.

The analysis presented in [11] shows that only the following causal signatures

$$\{\text{teee}, \text{TTTEEE}, \textit{teee}\} \qquad \{\text{ttee}, \text{TTTTTE}, \textit{eeee}\}$$

$$\{\text{ttte}, \text{TTTTTT}, \textit{eeee}\} \qquad \{\text{tttt}, \text{TTTTTT}, \textit{eeee}\}$$

are admissible by the Newtonian causal structure.

Standard frames, i.e. those constructed with three rods and one clock at rest with respect to the rods, belong to the causal class {teee, TTTEEE, *teee*}. The history of the clock is a timelike coordinate line. The other coordinate lines are spacelike straight lines tangent to the rods at every (clock's) instant.

Geometrically, this causal class may be visualized as follows. The natural coframe is of causal type {*teee*}. This means that the family of (instantaneous) hyperplanes generated by the directions of the three rods is spacelike, and the three families of hyperplanes (each one being the history of the plane generated by two rods) are timelike. Then, the mutual cuts of these coordinate hyperplanes give the six families of coordinate planes, (three of them being timelike and the other three being spacelike, {TTTEEE}). The coordinate planes cut in four congruences of coordinate lines (one being timelike and the others being spacelike, {teee}).

In [11], we have considered the other three nonstandard Newtonian frames, which may be physically constructed from a lineal change of the standard inertial synchronization. The Newtonian positioning systems and their associated emission coordinates provide another physical example of frames with these, up to now, unusual causal signatures (see Sec. III below).

It is to be remarked that the four Newtonian causal signatures also exist in relativity, according with the causal classification of the Lorentzian frames (see Fig. 1 in the next subsection).

C. 199 causal classes of relativistic coordinate systems

The relativistic causal structure admits three causal characters. Directions, planes or hyperplanes at a given event are spacelike, null or timelike if they are, respectively, exterior, tangent or secant to the light cone of this event.

In Lorentzian geometry, and with the signature $(-+++)$, a vector $a \neq 0$ is spacelike, null or timelike if $a^2 \equiv a \cdot a$ is, respectively, positive, zero or negative, $a \cdot b$ standing for the Minkowski scalar product of a and b. Also, the plane generated by the vectors a and b is spacelike, null or timelike if $(a \wedge b)^2 = a^2 b^2 - (a \cdot b)^2$ is positive, zero or negative. And finally, the hyperplane generated by a, b and c is spacelike, null or timelike if $(a \wedge b \wedge c)^2$ is positive, zero or negative, with

$$(a \wedge b \wedge c)^2 = \begin{vmatrix} a^2 & a \cdot b & a \cdot c \\ a \cdot b & b^2 & b \cdot c \\ a \cdot c & b \cdot c & c^2 \end{vmatrix}.$$

Now, the space-time metric defines a one-to-one correspondence between vectors and covectors at every event that obviously allows us to define the causal character of codirections and coplanes. Furthermore, all these pointlike notions are naturally extended to tensor fields and r-forms on homogeneous causal domains.

In [12] we have obtained the 199 causal signatures compatible with a Lorentzian 4-dimensional space-time structure. They are shown in Fig. 1 which provides the causal classification of the relativistic coordinate systems. The reading of Fig. 1 is as follows.[4]

(1) The first column shows the sets of causal characters c_A of the covectors of the coframe (that also gives the sets of causal characters of the four hyperplanes of the frame or of the four families of coordinate hypersurfaces of a coordinate system). Only 15 sets are possible, up to permutations [12].

(2) The first row shows the sets of causal characters c_A of the vectors of the frame (that also gives the sets of causal characters of the congruences of coordinate lines of a coordinate system). Only 16 sets are possible, up to permutations [12].

(3) Each nonempty (p, q)-cell $(1 \leq p \leq 15,\ 1 \leq q \leq 16)$ shows the set of causal characters C_{AB} of the planes of vectors of the qth frame, which corresponds to the pth coframe or, correspondingly, the set of causal characters of the six coordinate surfaces of a coordinate system.

(4) Permutations of the vectors of the frame or of the covectors of the coframe induce permutations of the

[4]The notation used in [12] is lightly different, but completely equivalent, to the present notation. In [12], the signs $(+, -, 0)$ stand for the causal characters (spacelike, timelike, null) of the associated planes and coordinate surfaces, while here they are denoted by capital letters (E, T, L), respectively.

	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tlll	ttll	tttl	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLEE TTTTEE LLLLLE TLLLLE TTLLLE TTTLLE TTTTLE TTTTTE LLLLLL TLLLLL TTLLLL TTTLLL TTTTLL TTTTTL TTTTTT	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE TTLTEE TTLETL TTTTEE TTLLLE TTLELL TTTLLE TTLTLE TTLLTE TTLETL TTTTLE TTLTTE TTLETT TTTTTE TTLLLL TTTLLL TTLTLL TTLLTL TTTTLL TTLTTL TTLLTT TTTTTL TTLTTT TTTTTT		TTTEEE TTTLEE TTTTEE TTTLLE TTTTLE TTTTTE TTTLLL TTTTLL TTTTTL TTTTTT	TTLTLE TTLLTE TTTTLE TTTTTE TTLTLL TTLLTL TTTTLL TTLLTT TTLTLT TTTTTL TTTTLT TTTTTT	TTTTLE TTTTTE TTTTLL TTTTTL TTTTLT TTTTTT	TTTTTE TTTTTL TTTTTT	TTLTLL TTTTLL TTTTTL TTTTTT	TTTTLL TTTTTL TTTTTT	TTTTTL TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT
leee	EEEEEE LEEEEE EEELEE TEEEEE LELEEE LEELEE EELLEE TLEEEE TEELEE EETLEE TTEEEE LLLEEE LLELEE LELLEE TLLEEE TLELEE TELLEE LETLEE TTLEEE TTELEE TETLEE TTTEEE LLLLEE TLLLEE LLTLEE TTLLEE TLTLEE TTTLEE	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE	TEELLE TLELLE TTELLE TLLLLE TTLLLE TTTLLE	TTTEEE TTTLEE	TTLLLE TTTLLE	TTTLLE										
teee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE	TTLEEE TTTEEE		TTTEEE												
llee	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE EELLEE ELELEE TLEEEE LLELEE LELLEE TLELEE TELLEE	TLLEEE TLLLEE			TLLLLE											
tlee	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE TLEEEE	TLLEEE														
ttee	EEEEEE LEEEEE TEEEEE															
llle	EEEEEE LEEEEE LLEEEE LLELEE															
tlle	EEEEEE LEEEEE LLEEEE															
ttle	EEEEEE LEEEEE															
ttte	EEEEEE															
llll	EEEEEE															
tlll	EEEEEE															
ttll	EEEEEE															
tttl	EEEEEE															
tttt	EEEEEE															

FIG. 1 (color online). The 199 causal classes of relativistic frames and coordinates (cf. Ref. [12]).

planes and hyperplanes, but do not alter their causal class. Correspondingly, permutations of the lines or hypersurfaces of a coordinate system induce permutations of the coordinate surfaces of the system, but do not alter its causal class.

Note that in Fig. 1 one must distinguish those coframes of type {*leee*} having dual frame of the type {leee} from those having dual frame of the type {elee}. These two cases are not causally equivalent. When the frame is {leee}, the hyperplane generated by the three spacelike vectors is null, and the null vector of the frame and the null covector of the coframe have associated, by the metric, different null directions. When the frame is {elee}, such a hyperplane is timelike, and the null directions are metrically equivalent.

For example, a coordinate system whose causal class is {elee, TEELLE, *leee*} has associated a family of null coordinate hypersurfaces and three families of timelike hypersurfaces. Their mutual cuts give one family of timelike surfaces, two families of null surfaces, and three families of spacelike surfaces. The intersections of these surfaces give a congruence of null lines and three congruences of spacelike lines. In this case, the null vector and the null covector are metrically identified. This class includes some particular types of generalized Bondi-Sachs coordinates.[5] According to this example, the above distinction is very relevant to obtain the complete classification of the generalized Bondi-Sachs coordinates in 13 causal classes [22].

Relativistic standard frames have the causal signature {teee, TTTEEE, *teee*}. Other familiar relativistic frames are those constituted by two null vectors and two spacelike ones. Among them, the more popular are those that belong to the causal class {llee, TLLLLE, *llee*} (that allows us to define the complex Newman-Penrose tetrads). Usually, such frames are constructed by taking advanced and retarded null coordinates (u, v) on a family of timelike surfaces and two angular coordinates (θ, ϕ) on the orthogonal (spacelike) surfaces.[6]

On the other hand, the causal signature of the real null frames (those constituted by four independent real null vectors) is {llll, TTTTTT, *eeee*}. Some time ago, Zeeman [23] considered real null frames as a useful tool to determine the causality group of the Minkowski space-time.[7] The causal signature associated with real null frames may be seen as the algebraic dual of the causal signature of the light emission coordinate systems, {eeee, EEEEEE, *llll*}, that will be considered in detail in subsection IV B below. Of course, we can also consider Lorentzian frames constituted by four timelike or four spacelike vectors.[8]

These examples show the role that the causal classification of relativistic frames plays in the physical construction of admissible space-time coordinates. However, it becomes apparent that only a few of the causal classes have been commonly employed until now. In fact, the overwhelming majority of the relativistic causal classes, explicitly given in Fig. 1, remains vastly unexplored. This paper may be seen as a piece to induce the study and physical construction of these other space-time coordinate systems.

III. NEWTONIAN EMISSION COORDINATES

In this section we deal with the three nonstandard parametrizations of Newtonian space-time domains and we show that they may be physically constructed by means of emission coordinates. The corresponding relativistic situation is addressed in the next section.

A. Emission-inertial coordinate transformation

Suppose an inertial medium in which a class of signals (sound, light) propagates at constant velocity v. Let $\kappa(t)$ be the space-time pointlike trajectory of an emitter clock that uses such signals to continuously broadcast its time t. In the space-time, the front waves describe thus sound-cones or light-cones carrying the value $t = \text{const}$. Four such emitters $\kappa^A(t)$ $(A = 1, 2, 3, 4)$ fill the space-time with four (one-parameter) families of cones $t^A = \text{const}$ which generically define a space-time system of *emission coordinates*.

Let us take every event as the vertex of the past cone of velocity v corresponding to the signals in question. This cone cuts the four histories $\kappa^A(t)$ of the clocks at the clock times t^A. Then, the set $\{t^A\}$ constitutes the four emission coordinates of the event.

Here we will consider the simple case of four emitters at rest with respect to the inertial medium referred to a standard coordinate system $\{t, x^i\} = \{t, \vec{r}\}$, of world lines

$$\kappa^A(t) = (t, \vec{c}^A). \tag{1}$$

Then, the signal emitted by the clock κ^A at the instant t^A at velocity v describes in the space-time a cone of equation

[5]The Bondi-Sachs coordinates [19,20] were introduced to split Einstein equations in characteristic form using null hypersurfaces, in connection with astrophysical scenarios described by bounded radiating systems and asymptotic flatness. These coordinate systems and their generalizations [21] are very appropriate for investigating certain global properties at null infinity.

[6]Notice that, starting from two null families of coordinate hypersurfaces and two timelike ones, that is, from a coframe of the type {*llee*}, there exist 14 causal classes of coordinate systems, different from {llee, TLLLLE, *llee*}, requiring different physical constructions (see Fig. 1).

[7]*Light coordinates* [3], those that are built from the intersection of four beams of light, provide a physical realization of the causal class {llll, TTTTTT, *eeee*}.

[8]The interest of the wholly symmetric realizations of these frames was stressed by Derrick [7] and also by Finkelstein and Gibbs [24]. The causal classification of Lorentzian symmetric frames in seven causal classes was studied in [8]. More recently, Rovelli [9] and Blagojević *et al.* [10] have considered physical constructions of null symmetric coframes in the context of positioning systems.

$$v(t - t^A) = |\vec{r} - \vec{c}^A|, \quad (2)$$

so that the emission coordinates $\{t^A\}$ are related to the inertial ones $\{t, \vec{r}\}$ by

$$t^A = t - \frac{1}{v}|\vec{r} - \vec{c}^A|. \quad (3)$$

To know the causal class of the emission coordinates $\{t^A\}$ it is convenient to consider the coordinate r-forms.

B. Emission coordinate hypersurfaces

From (3), the coframe of 1-forms $\{dt^A\}$ may be written

$$dt^A = dt + \omega^A, \qquad \omega^A \equiv -\frac{1}{v}u^A, \quad (4)$$

where u^A is the 1-form associated to the generically unit spacelike vector $\vec{u}^A$, given by

$$\vec{u}^{\,A} \equiv \frac{\vec{r} - \vec{c}^A}{|\vec{r} - \vec{c}^A|}, \quad (5)$$

$u^A = \gamma(\vec{u}^A)$, γ being the 3-dimensional inverse of the structure metric γ^* associated to the inertial observers ∂_t, $\gamma.\gamma^* = I - \theta \otimes \partial_t$, and θ being the time current.[9] The Jacobian matrix of the transformation (3) is not defined at the events $(t, \vec{r})$ where $\vec{r} = \vec{c}^A$, that is to say, along the clock world lines κ^A. Below we shall see other events where the Jacobian matrix is not defined. Out of these world lines one has $\omega^A \neq 0$ and thus dt^A is spacelike (it is not collinear to the time current). Consequently, it follows.

Proposition 1. The coframe of the Newtonian emission coordinate system is of causal type $\{eeee\}$.

C. Emission coordinate surfaces

The coplanes of the coordinate system are determined by the 2-forms

$$dt^A \wedge dt^B = dt \wedge (\omega^B - \omega^A) + \omega^A \wedge \omega^B, \quad (6)$$

so that the coplane AB is generically spacelike, and can be timelike only when $\omega^A \wedge \omega^B = 0$, that is to say on the timelike plane of events Π_{AB} that contains the world lines of κ^A and κ^B. Because the clocks are at rest with respect to the starting inertial system, at any $t =$ const their positions $\kappa^A(t) \equiv A$ will generically define the four vertices (A, B, C, D) (all $\neq$) of a 3-dimensional tetrahedron (see Fig. 2). Denote by ℓ_{AB} the straight line passing through A and B

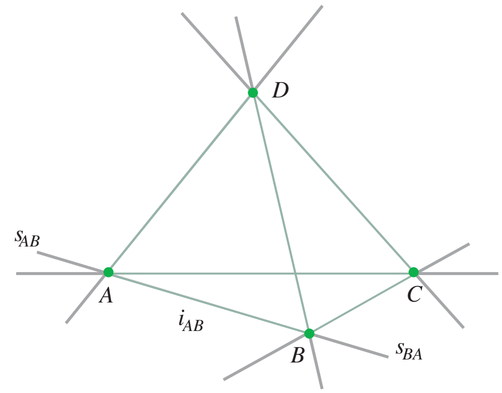

FIG. 2 (color online). At any instant $t =$ const, the positions $\kappa^A(t) \equiv A$ ($A = 1, 2, 3, 4$) of the four clocks generically define the four vertices A, B, C, D (all $\neq$) of a 3-dimensional tetrahedron. If the clocks are at rest in an inertial system, the outer open segments s_{AB} and s_{BA} of the straight line ℓ_{AB} containing the edge i_{AB} between the vertices A and B represent the shadows of the signals B and A produced by A and B, respectively.

and, in it, by i_{AB} the corresponding open edge of the tetrahedron and by s_{AB} (*resp.* s_{BA}) the other open segment contiguous to A (*respect.* contiguous to B). It is then clear that the timelike plane Π_{AB} is the history of the straight line ℓ_{AB}, and we will denote by I_{AB} the history of i_{AB}, the (timelike) open strip of Π_{AB} whose boundaries are κ^A and κ^B. Similarly, S_{AB} (*resp.* S_{BA}) will denote the (timelike) open strip of Π_{AB} contiguous to κ^A (*resp.* contiguous to κ^B). Now we see that the condition $\omega^A \wedge \omega^B = 0$ takes place along ℓ_{AB}, thus on the events of Π_{AB}. In addition, because from (4) all the ω^A have same length, one has $\omega^A = -\omega^B$ on i_{AB}, thus on the events of I_{AB}, and $\omega^A = \omega^B$ on the two other open segments s_{AB} and s_{BA}, thus on the events of S_{AB} and S_{BA}, where one has

$$dt^A \wedge dt^B = 0, \quad (7)$$

and the coordinate system degenerates. These open strips of Π_{AB}, S_{AB}, and S_{BA}, are also the half-planes describing the history of the *shadows* that the clocks A and B make, respectively, to the signals of the clocks B and A. These considerations on expressions (6) and (7) show that either all the coordinate coplanes are spacelike, or one of them is timelike, so that, on account of a general algebraic property[10] stated in [11], the following result occurs.

Proposition 2. Generically the type of the coordinate planes of the Newtonian emission coordinates is {TTTTTT} but on the events of the six timelike strips I_{AB}, and only on them, the type is {TTTTTE}, the coordinate system being degenerate on the shadows S_{AB} and S_{BA} and undetermined on the world lines κ^A.

[9]Note that, while γ^* is an intrinsic element of the geometry of Newtonian space-time, its 'three-dimensional inverse' γ is an *observer-dependent* quantity, given by $\gamma.\gamma^* = I - \theta \otimes u$, where u is the unit velocity of the chosen observer. Two different observers have associated two different degenerate four-dimensional covariant metrics γ of rank three, although their induced spatial components on the instantaneous space take the same value, as it is well experienced in the usual Newtonian three-dimensional formalism.

[10]This property says that a r-plane Π is spacelike (*resp.* timelike) iff the annihilator coplane $\Omega_\Pi \equiv \{\omega | \omega(v) = 0 \ \forall \ v \in \Pi\}$ is timelike (*resp.* spacelike).

D. Emission coordinate lines

To analyze the coordinate lines of a Newtonian emission coordinate system, let us consider the dual 3-forms:

$$dt^A \wedge dt^B \wedge dt^C = \omega^A \wedge \omega^B \wedge \omega^C + dt \wedge (\omega^A \wedge \omega^B + \omega^B \wedge \omega^C + \omega^C \wedge \omega^A). \quad (8)$$

The hyperplane of covectors ABC is generically spacelike, and can be timelike only when $\omega^A \wedge \omega^B \wedge \omega^C = 0$, which happens on the events of the timelike hyperplane Π_{ABC} that contains the world lines κ^A, κ^B, κ^C. In the stationary 3-dimensional sections $t =$ constant, these events correspond to the planes ℓ_{ABC} that contain the three clocks A, B, C, and thus the three lines ℓ_{AB}, ℓ_{BC}, ℓ_{CA}, including the tetrahedral faces i_{ABC} that their edges i_{AB}, i_{BC}, and i_{CA} delimit, and the six strips s_{AB}, s_{BA}, s_{BC}, s_{CB}, s_{CA}, s_{AC}. We already know that, apart from on the clocks A, B, C themselves, on these last six strips the coordinate coplanes degenerate; are there any other events in which the coordinate hyperplanes of covectors are degenerate? In other words, there where $\omega^A \wedge \omega^B \wedge \omega^C = 0$ out of the edges, can the other term in (8) also vanish? We have

$$\omega^C = \alpha\omega^A + \beta\omega^B, \quad (9)$$

so that (8) becomes

$$dt^A \wedge dt^B \wedge dt^C = (1 - \alpha - \beta) dt \wedge \omega^A \wedge \omega^B, \quad (10)$$

which cannot degenerate, being $\omega^A \wedge \omega^B \neq 0$, unless $\alpha + \beta = 1$. But

$$1 = (\omega^C)^2 = \alpha^2 + \beta^2 + 2\alpha\beta(\omega^A \cdot \omega^B) = 1 + 2\alpha\beta(\omega^A \cdot \omega^B - 1),$$

admits no solution, because $\alpha \neq 0 \neq \beta$ and necessarily $\omega^A \cdot \omega^B < 1$. The tangent vectors to the coordinate lines being at every event causally related to the hyperplanes by the aforementioned property[11] gives the following result.

Proposition 3. The coordinate lines of the emission coordinates in Newtonian space-times are generically of type {tttt}, but on the events of the timelike hyperplanes Π_{ABC} containing three emitters they are generically of type {ttte}, and are of type {ttee} on the events of the timelike strips I_{AB} generated by every pair of clocks.

It is pertinent here to note that, *in Newtonian space-time, the emission coordinate system generated by a positioning system is never everywhere causally homogeneous*, but always has three regions corresponding to the nonstandard three causal classes. Only the emission coordinate systems generated by relativistic positioning systems based on light signals are always causally homogeneous, as we will see in the next section.

The geometry of the coordinate surfaces and coordinate lines of the emission coordinates is simple. Because generated by the two by two or three by three intersections of the coordinate hypersurfaces, which are isotropic cones of parallel axes, *the coordinate surfaces and coordinate lines of the emission coordinates are hyperboloids and hyperbolas, respectively.* As already seen, these hyperbolas are generically timelike lines, up to at their base point, where they become spacelike.

E. Coordinate volume element and Jacobian

As we have seen, the transformation (3) from a standard inertial coordinate system $\{t, x^i\} = \{t, \vec{r}\}$ to an emission coordinate system $\{t^A\}$ is degenerate on the clock shadows S_{AB}, timelike space-time surfaces generated by every clock for the signals coming to it from the others. Thus the question: is transformation (3) degenerate at other events than those of the shadows S_{AB}? To see it, let us consider the emission coordinate volume element, η_e:

$$\begin{aligned}\eta_e &\equiv dt^A \wedge dt^B \wedge dt^C \wedge dt^D \\ &= dt \wedge [-\omega^A \wedge \omega^B \wedge \omega^C + \omega^B \wedge \omega^C \wedge \omega^D - \omega^C \wedge \omega^D \wedge \omega^A + \omega^D \wedge \omega^A \wedge \omega^B] \\ &= -dt \wedge [(\omega^A - \omega^D) \wedge (\omega^B - \omega^D) \wedge (\omega^C - \omega^D)].\end{aligned}$$

It is then evident that the Jacobian is degenerate, as we already know, there where $\omega^A = \omega^B$, that is to say, on the clock shadows S_{AB}, for any pair $A \neq B$. But, as the above expression for η_e shows, the Jacobian can be also degenerate there where the three vectors $\omega^A - \omega^D$ are linearly dependent. It can be seen (for example in [25]), that this happens on the events where the signals coming from the four clocks are seen or heard as coming from four points located on a circle of the celestial sphere of the event (quotient of the instantaneous space of the event by the radial distance to the event).

IV. RELATIVISTIC EMISSION COORDINATES

Let us consider now the relativistic analog of the emission coordinates of the above section. Now, every emitter κ^A is supposed to continuously broadcast, in an inertial nondispersive medium, their proper time τ^A by means of sound or light signals that propagate in the medium at constant velocity $v \leq 1$.

It is to be stressed that, in a flat metric, the coordinate transformation between inertial and light emission coordinates may be obtained for any arbitrarily prescribed kinematics of the emitters [16,17]. However, here we are interested in the study of causal properties and, as in Sec. III, the four emitters will be considered at rest with respect to the medium referred to a standard coordinate system $\{t, x^i\} = \{t, \vec{r}\}$. The inertial time t is also the proper time of the four emitters and their world lines take the expression (1): $\kappa^A(t) = (t, \vec{c}^A)$. Then, the equation of the cones that describe the signals is (2), and the emission coordinates $\{t^A\}$ are related to the inertial ones $\{t, \vec{r}\}$ by (3).

[11] See footnote 10.

A. Coordinate hypersurfaces, surfaces and lines

To know the causal class of the emission coordinate system $\{t^A\}$ we can start from the coframe of 1-forms $\{dt^A\}$ given in (4) and (5), that provide the causal character of the coordinate hypersurfaces $t^A = \text{const}$. Out of the clock world lines κ^A, where the transformation (3) is not defined, dt^A is spacelike or null because:

$$(dt^A)^2 = -1 + \frac{1}{v^2} \geq 0 \qquad (11)$$

Consequently, we have this statement.

Proposition 4. The coframe of the relativistic emission coordinate system with $v < 1$ is of causal type $\{eeee\}$. When $v = 1$, the coframe has causal type $\{llll\}$.

In other words, all the coordinate hypersurfaces of the relativistic emission coordinates are timelike when $v < 1$, and null when $v = 1$.

The coplanes of the coordinate system are determined by the 2-forms (6) that satisfy

$$(dt^A \wedge dt^B)^2 = -\frac{1}{v^4}(\mu_{AB}^2 - 2v^2\mu_{AB} + 2v^2 - 1), \qquad (12)$$

where

$$\mu_{AB} \equiv u_A \cdot u_B. \qquad (13)$$

Note that μ_{AB} is the cosine of the angle between the signals coming from the emitters A and B. The study of the polynomial (12) in μ_{AB} leads to the following:

Proposition 5. The coplane AB of a relativistic emission coordinate system is spacelike, null, or timelike according as μ_{AB} is greater, equal or smaller than $2v^2 - 1$.

The causal character of the planes may be directly obtained from the dual version of the above statement. Consequently, the emission coordinate surfaces defined by constant t^A and t^B are spacelike, null, or timelike according to whether μ_{CD} is smaller, equal to, or greater than $2v^2 - 1$. Now, it is understood that the pairs of indexes AB and CD are constrained to take complementary pairs of values (for instance, $A = 2$, $B = 3$ and $C = 1$, $D = 4$).

To analyze the coordinate lines, let us consider the dual 3-forms (8). We have

$$(dt^A \wedge dt^B \wedge dt^C)^2 = \frac{1}{v^4}\left(\frac{1 - v^2}{v^2}\Delta_D - \Lambda_D\right),$$
$$D \neq A, B, C,$$

where Δ_D and Λ_D depend on μ_{AB} as

$$\Delta_D \equiv (u_A \wedge u_B \wedge u_C)^2$$
$$= 1 + 2\mu_{AB}\mu_{BC}\mu_{CA} - (\mu_{AB}^2 + \mu_{BC}^2 + \mu_{CA}^2)$$
$$\Lambda_D \equiv 2(1 - \mu_{AB})(1 - \mu_{BC})(1 - \mu_{CA}).$$

From them, we arrive to the following result.

Proposition 6. The hyperplane of covectors ABC of a relativistic emission coordinate system is spacelike, null, or timelike according as Λ_D/Δ_D is smaller, equal to, or greater than $(1 - v^2)/v^2$.

As a consequence, the dual version of this result says that the emission congruence of coordinate lines defined by variable t^A and constant t^B (with $B \neq A$) is timelike, null, or spacelike according to whether Λ_A/Δ_A is smaller, equal to, or greater than $(1 - v^2)/v^2$.

B. The 103 causal classes of relativistic emission coordinates

First, let us consider the light case $v = 1$. It is clear that, from (11), we have $(dt^A)^2 = 0$ so that *the coframe of the relativistic emission coordinate systems with* $v = 1$ *is of causal type* $\{llll\}$. Because the μ_{AB} are all smaller than 1, it follows from Proposition 5 that all the coplanes are timelike, and consequently all the planes are spacelike, that is $\mathrm{C}_{AB} = \mathrm{E}$. On the other hand, Λ_A and Δ_A are both positive and then, from proposition 6, all the hyperplanes of covectors are timelike, and the c_A cannot but be spacelike, $\mathrm{c}_A = \mathrm{e}$. This result, obtained for an inertial homogeneous medium and four static clocks, may be shown true also for arbitrary clocks in general space-times [5]. We have thus:

Proposition 7. All the relativistic positioning systems with light signals define in their whole domains a sole causal class, of causal signature

$$\{\mathrm{eeee}, \mathrm{EEEEEE}, llll\}.$$

These relativistic positioning systems, of great interest for future space research and navigation, have been considered elsewhere [4–6,13–16]. On the other hand, in order to construct the theory of the current sonic positioning systems, it is reasonable to think that a Newtonian approach suffices (see, for instance, [18]). However, as we are going to show, the relativistic approach offers considerable conceptual advantages over the Newtonian one providing, in addition, new examples of relativistic causal classes that might be physically constructed.

Now, let us consider the sonic case $v < 1$. Taking into consideration the results of subsection IV A (propositions 4, 5, and 6) we arrive to the following statement.

Proposition 8. The causal classes of the relativistic emission coordinate systems with $v < 1$ are of the form:

$$\{\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3\mathrm{c}_4, \mathrm{C}_{12}\mathrm{C}_{13}\mathrm{C}_{14}\mathrm{C}_{23}\mathrm{C}_{24}\mathrm{C}_{34}, eeee\}$$

where the causal characters, c_A, C_{AB} depend on the cosines μ_{AB} of the angles between the signals coming from the emitters A and B as

$$\mathrm{c}_A = \begin{cases} \mathrm{t} & \frac{\Lambda_A}{\Delta_A} < \frac{1-v^2}{v^2} \\ \mathrm{l} & \frac{\Lambda_A}{\Delta_A} = \frac{1-v^2}{v^2} \\ \mathrm{e} & \frac{\Lambda_A}{\Delta_A} > \frac{1-v^2}{v^2} \end{cases} \qquad (14)$$

$$\mathrm{C}_{AB} = \begin{cases} \mathrm{T} & \mu_{CD} > 2v^2 - 1 \\ \mathrm{L} & \mu_{CD} = 2v^2 - 1 \\ \mathrm{E} & \mu_{CD} < 2v^2 - 1 \end{cases} \qquad (15)$$

with $C, D \neq A, B$.

A detailed analysis of the compatible characters of the geometric elements leads us to the following result:

Proposition 9. Depending on the different configurations of the stationary emitters and/or of the different values of the velocity $v < 1$, the relativistic emission coordinate systems may present space-time regions of 102 different causal classes.

In fact, the result follows by taking into account the causal classification of Fig. 1. The compatible causal characters C_{AB} are explicitly given in the $(1, q)$-cells with $q \neq 3$. In fact, the (1, 3)-cell is empty. The potential sets C_{AB} in this cell may be (and has been) counted in the (1, 2)-cell, by virtue of the permutation freedom of the coframe spacelike characters $\{eeee\}$. For this reason, both cells in Fig. 1 are separated by a dashed vertical segment.

For same reasons as in the Newtonian case, the coordinate lines of emission coordinates are also hyperbolas here. Nevertheless, their causal types differ: in the Newtonian case every hyperbola is everywhere timelike except at its base point, where it is spacelike; in the relativistic case with $v < 1$ the corresponding point becomes enlarged to a whole spacelike interval, bounded by two points where it is null, the rest of the branches being timelike. In the relativistic case $v = 1$ the hyperbolas are spacelike everywhere. Obviously, this fact is on the basis of the richness (103 causal classes) of the relativistic positioning systems.

C. Causal signatures with Newtonian analogous

Finally, it is worth mentioning that some emitters configurations and sound velocities $v < 1$ generate space-time regions of the same causal signatures that the three Newtonian ones analyzed in Sec. III. More specifically, we have this result.

Proposition 10. There are three relativistic causal classes of emission coordinates with $v < 1$ which have Newtonian causal signatures. They are related to how the events receive the sound signals, according to the following three sets of conditions:

$$\{\text{tttt}, \text{TTTTTT}, eeee\} \quad \text{if } \forall\ A, \qquad \frac{\Lambda_A}{\Delta_A} < \frac{1 - v^2}{v^2}$$

$$\{\text{ttte}, \text{TTTTTT}, eeee\} \quad \text{if} \begin{cases} \exists ! A, & \frac{\Lambda_A}{\Delta_A} > \frac{1-v^2}{v^2} \\ \forall B \neq A, & \frac{\Lambda_B}{\Delta_B} < \frac{1-v^2}{v^2} \end{cases}$$

$$\{\text{ttee}, \text{TTTTTE}, eeee\} \quad \text{if} \begin{cases} \text{for } I = A, B, & \frac{\Lambda_I}{\Delta_I} < \frac{1-v^2}{v^2} \\ \forall\ C \neq A, B, & \frac{\Lambda_C}{\Delta_C} > \frac{1-v^2}{v^2} \\ \mu_{AB} < 2v^2 - 1 \end{cases}$$

It is easy to find (in Fig. 1) the above causal classes and also the standard one {teee, TTTEEE, *teee*}. Of course, the signatures of these relativistic causal classes are those of the Newtonian case.

V. COMMENTS AND WORK IN PROGRESS

We have analyzed positioning systems and their associated emission coordinates from the causal point of view. Positioning systems in relativity may be of 103 causal classes. Three of them correspond to the three Newtonian causal classes, and only one of them, the {eeee, EEEEEE, *llll*}, corresponds to relativistic positioning systems based on light signals.

Thus, the causal classes associated with emission coordinates constitute a broad, but strict, subset of the set of the 199 causal classes of relativistic coordinates systems whose main aspects have been summarized.

A point of interest is that the use of emission coordinates shows that one can locate events in the space-time *without* any use of the concept of synchronization. Furthermore, positioning systems allow their users to know their own coordinates (emission coordinates) without delay. From laboratory domains, Earth surface physics or global navigation systems to space physics, solar system or celestial astronomy, positioning systems allow the explicit construction of the correspondence between the events of the observable physical world and the points of its mathematical space-time model in the physical theory in use.

The ability to take hold of Newtonian space-time without the use of the simultaneity foliation, or any other synchronization, may seem rather academic. But such ability in the relativistic space-time seems urgent. Simply because, in relativity, relative simultaneity synchronizations, be they introduced as an approximate concept or as an exact one, have neither more nor less physical reality than the celestial crystal spheres of the Ptolemaic epicyclic theory.

Such synchronizations are conventional constructions whose realization in fact demands the *a priori* knowledge of (a good number of) the physical quantities that one usually wants to know. As such constructions, they can play a role in the ''*a posteriori*'' physical interpretation of some physical quantities, but are unusable as a *starting* basis for referring physical observations of an unknown environment.

The direct confrontation of physicists with their environment in order to understand it gravitationally is a basic problem yet unsolved in relativity. Such a confrontation needs a locating structure that, in order to not to chase its own tail, should be constructed *before* the gravitational properties are measured. As analyzed elsewhere (see, for example, [4–6,13–17]) this locating structure is constituted by the relativistic positioning systems broadcasting light signals in vacuum.

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