ISOLATED SINGULARITIES OF BINARY DIFFERENTIAL EQUATIONS OF DEGREE n

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ABSTRACT. We study isolated singularities of binary differential equations of degree n which are totally real. This means that at any regular point, the associated algebraic equation of degree n has exactly n different real roots (this generalizes the so called positive quadratic differential forms when $n = 2$). We introduce the concept of index for isolated singularities and generalize Poincaré-Hopf theorem and Bendixon formula. Moreover, we give a classification of phase portraits of the n-web around a generic singular point. We show that there are only three types, which generalize the Darbouxian umbilics D_1, D_2 and D_3.

1. INTRODUCTION

The study of the principal foliations near an isolated umbilic point of a surface M immersed in $\mathbb{R}^3$ leads us to the consideration of quadratic binary differential equations (BDE) of the form

$$a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2 = 0,$$

where $a(x,y), b(x,y), c(x,y)$ are smooth functions in some open subset $U \subset \mathbb{R}^2$ which are defined, after taking a parametrization of M, by means of the coefficients of the first and second fundamental form of M. Since the principal lines are orthogonal in the induced metric of M, we have that the discriminant $\Delta = b(x,y)^2 - a(x,y)c(x,y) \geq 0$, with equality if and only if (x,y) corresponds to an umbilic of M, so that $a(x,y) = b(x,y) = c(x,y) = 0$ and hence, (x,y) is a singularity of the BDE. It was Darboux [5] who classified the generic singularities and discovered there are only three topological types, known as the Darbouxian umbilics D_1, D_2 and D_3 (see [1] and [12] for a modern and precise study of this classification).

In fact, we can consider quadratic BDE of this type for general functions $a(x,y)$, $b(x,y)$ and $c(x,y)$, with the discriminant property: $\Delta \geq 0$ with equality if and only if $a(x,y) = b(x,y) = c(x,y) = 0$. The quadratic forms with this property are called *positive* and have been studied by many authors [2, 6, 9, 11, 13]. A positive quadratic differential form defines a pair of transverse foliations in the region of regular points. Moreover, Guíñez showed has that in this more general situation, the only generic singularities are again the Darbouxian umbilics D_1, D_2 and D_3.

The aim of this paper is to generalize this to degree n BDE of the form

$$a_0(x,y)dx^n + a_1(x,y)dx^{n-1}dy + \cdots + a_n(x,y)dy^n = 0,$$

where $a_i(x,y)$ are smooth functions defined on $U \subset \mathbb{R}^2$ such that for any $(x,y) \in U$, either it is a singular point (that is, $a_i(x,y) = 0$ for any $i = 1, \ldots, n$) or the associated algebraic

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equation has exactly n different real roots. If the functions $a_i(x, y)$ have this property, then we say that the symmetric differential n-form $\omega = \sum_{i=1}^{n} a_i(x, y)dx^{n-i}dy^i$ is *totally real.*

When $n = 1$, a differential n-form is always totally real and it induces an oriented foliation in the plane with singularities. For $n = 2$, totally real is equivalent to positive in the Guíñez sense and hence, the BDE defines a pair of transverse (non oriented) foliations. However, for $n \geq 3$, the corresponding BDE induces locally a n-web in the regular region (that is, a set of n foliations $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ which are pairwise transverse). It seems that isolated singularities of n-webs in the plane have not been considered previously in the literature. Moreover, we feel that the use of degree n BDE is a good approach to treat this subject.

The topological configuration of a n-web ($n \geq 3$) can be extremely complicated, even in the regular case. When $n = 3$, the curvature of the web is a function which is a topological invariant. Hence, even for regular webs we find that the topological classification has functional moduli. It is known that a regular 3-web is parallelizable or hexagonal (that is, equivalent to three families of parallel straight lines) if and only if the curvature is zero. We should also mention that because of the rigidity of webs (any homeomorphism between two regular webs is in fact a diffeomorphism [7]) the topological and differentiable classifications are the same.

We show here that for $n \geq 3$, the classification of generic singularities of totally real differential n-forms gives again only three types, which we call the generalized Darbouxian D_1, D_2 and D_3. Here, generic means a generic choice of coefficients in the linear part of the functions $a_i(x, y)$. Moreover, the classification has to be understood not as a topological classification, but just as a description of the phase portrait of the foliations around the singular point.

One of the main ingredients of the classification is the index of an isolated singular point. It is defined as a rational number of the form k/n, where $k \in \mathbb{Z}$ and it can be interpreted as the rotation number of a continuously chosen vector tangent to the leaves, when we make a trip around the singular point. We also show the generalization of the Poincaré-Hopf theorem: if M is a compact surface and ω is a totally real n-form with a finite number of singular points, then the sum of the indices is equal to the Euler characteristic $\chi(M)$.

Another important point in the paper is the use of complex coordinates. By setting $z = x+iy$ and $\overline{z} = x-iy$, we can express any n-form as $\omega = A_0dz^n + A_1dz^{n-1}d\overline{z} + \cdots + A_nd\overline{z}^n$, where $A_j = \overline{A}_{n-j}$ are differentiable functions. Then the index of an isolated singular point is equal to $-\deg(A_0)/n$, where $\deg(A_0)$ is the mapping degree of A_0. This implies that generically, the index is $\pm 1/n$.

The final ingredient for the classification is the use of the polar blow-up method to study singularities with a non degenerate principal part (see [3] and [11] for related results for vector fields or quadratic forms). We obtain a generalization of the Bendixon formula, which says that the index is equal to $1 + (e - h)/2n$ where e, h are the number of elliptic and hyperbolic sectors respectively. On the other hand, for a non degenerate singularity, the blow-up produces a n-form which has only singularities of saddle/node type. The configuration of these singularities gives a description of the phase portrait of the foliations around the singular point.

We finish the paper with a section dedicated to higher order principal lines and umbilics of surfaces M immersed in some Euclidean space $\mathbb{R}^N$. This was the original motivation of the authors to study singularities of differential n-forms. Other geometrical motivations of the same kind can be found also in [15] or [10].

2. Totally real differential forms

Definition 2.1. Let M be a C^∞ surface. A *(symmetric) differential n-form* on M is a differentiable section of the symmetric tensor fiber bundle $S^n(T^*M)$. If we take coordinates x, y on some open subset $U \subset M$, any differential n-form can be written in a unique way as

$$w = \sum_{i=0}^{n} f_i dx^i dy^{n-i},$$

where $f_i : U \to \mathbb{R}$ are smooth functions.

We will say that $p \in M$ is a *singular point* of ω if $\omega(p) = 0$. We will denote by $\mathrm{Sing}(\omega)$ the set of singular points of ω.

In general, if $p \in M$, $\omega(p) : T_pM \to \mathbb{R}$ is a form of degree n. Let $p \in M \setminus \mathrm{Sing}(\omega)$, we say that ω is *totally real at* p if there are n linear forms $\lambda_1, \dots, \lambda_n \in T_pM^*$ which are pairwise linearly independent and such that $\omega(p) = \lambda_1 \dots \lambda_n$. We say that ω is *totally real* if it is totally real at any point $p \in M \setminus \mathrm{Sing}(\omega)$.

A linear differential form ($n = 1$) is always totally real. In the case $n = 2$, a quadratic differential form is totally real if it is *positive* in the sense of [9]. Take local coordinates x, y defined on some open subset $U \subset M$ and assume that ω is given by

$$\omega = Adx^2 + 2Bdxdy + Cdy^2,$$

for some smooth functions $A, B, C : U \to \mathbb{R}$. Then ω is totally real in U if and only if for any $p \in U$, either $A(p) = B(p) = C(p) = 0$ or $B^2(p) - A(p)C(p) > 0$.

Definition 2.2. A (1-dimensional) *n-web* on a surface M is a set of n (1-dimensional) foliations $\mathcal{W} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ on M such that they are pairwise transverse at any point of M.

If ω is a totally real differential n-form on M, then we can locally associate a n-web on $M \setminus \mathrm{Sing}(\omega)$ in the following way. For each $p \in M \setminus \mathrm{Sing}(\omega)$, there are pairwise linearly independent linear forms $\lambda_1, \dots, \lambda_n \in T_pM^*$ such that $\omega(p) = \lambda_1 \dots \lambda_n$. Moreover, it is possible to choose these linear forms so that they depend smoothly on p (and hence define differential linear forms) on some open neighbourhood $U \subset M$. Then, the n-web is just defined by taking $\mathcal{F}_i$ as the foliation determined by λ_i on U (that is, the tangent vectors to $\mathcal{F}_i$ are the null vectors of λ_i).

Note that in general, it is not possible to extend this to a global n-web on $M \setminus \mathrm{Sing}(\omega)$ (unless it is simply connected). Moreover, two totally real differential n-forms ω_1 and ω_2 define the same n-web on U if and only if there is a non-zero smooth function $f : U \to \mathbb{R}$ such that $\omega_1 = f\omega_2$ on U.

Remember that if ω is a differential n-form on N and $f : M \to N$ is a differentiable map between surfaces, then $f^*\omega$ is the n-form on M given by $f^*\omega(p)(X) = \omega(f(p))(f_*X)$ for any $p \in M$ and $X \in T_pM$, and being $f_* : T_pM \to T_{f(p)}N$ the differential of f at the point p.

Definition 2.3. Let ω_1, ω_2 be two totally real differential n-forms defined on surfaces M, N respectively. We say that they are *C^∞-equivalent* (resp. *topologically equivalent*) if there is a C^∞ diffeomorphism (resp. homeomorphism) $\phi : M \to N$ such that

(1) $\phi(\mathrm{Sing}(\omega_1)) = \mathrm{Sing}(\omega_2)$,
(2) $\phi : M \setminus \mathrm{Sing}(\omega_1) \to N \setminus \mathrm{Sing}(\omega_2)$ preserves locally the leaves of the foliations of the n-webs defined by ω_1, ω_2.

It is obvious that if ϕ is a C^∞ diffeomorphism, then condition (2) is equivalent to the existence of a nonzero smooth function $f : M \setminus \mathrm{Sing}(\omega_1) \to \mathbb{R}$ such that $\phi^*(\omega_2) = f\omega_1$ on $M \setminus \mathrm{Sing}(\omega_1)$.

3. The index of an isolated singular point

We will define an index for isolated singular points of totally real differential forms, which generalize the index in the case of linear or quadratic forms.

Definition 3.1. Let ω be a totally real differential n-form on a surface M and $p \in M$ an isolated singular point. Assume that M is orientable and choose some orientation. Moreover, we choose a Riemannian metric g on M and orthogonal coordinates x, y on some open neighbourhood U of p in M, compatible with the orientation. Now, let $\alpha : [0, \ell] \to M$ be a simple, closed and piecewise regular curve, such that $\alpha([0, \ell]) \subset U$ is the boundary of a simple region R, which contains p as the only singular point in the interior. Moreover, we assume that α goes through the boundary of R in positive sense. Since α is a closed curve, it is obvious that we can extend it to $\alpha : \mathbb{R} \to M$, with $\alpha(t + \ell) = \alpha(t)$.

For each $t \in \mathbb{R}$ we choose a unit tangent vector $X(t)$ which is a solution of the equation $\omega(\alpha(t))(X) = 0$ at the point $\alpha(t)$. Since it is an algebraic equation of degree n, we can choose $X(t)$ so that it defines a differentiable unit vector field along α.

If we start with $t = 0$, after a complete turn, $X(\ell)$ must coincide with one of the $2n$ unit vectors which are solution of $\omega(\alpha(0))(X) = 0$. Because of transversality, after $2n$ turns in positive sense, we must return to the initial vector, that is, $X(2n\ell) = X(0)$. Now, let $\theta(t)$ be a differentiable determination of the angle from $\frac{\partial}{\partial x}|_{\alpha(t)}$ to$X(t)$. Then, $\theta(2n\ell)$ and $\theta(0)$ differ in an integer multiple of 2π. We define the *index* of ω in p by

$$\mathrm{ind}(\omega, p) = \frac{\theta(2n\ell) - \theta(0)}{4\pi n}.$$

It follows from the definition that the index is always a rational number of the form $s/2n$, with $s \in \mathbb{Z}$.

Lemma 3.2. *The index* $\mathrm{ind}(\omega, p)$ *does not depend on the choice of:*

(1) *the determination of the angle* θ*,*
(2) *the vector field* X*,*
(3) *the coordinates* x, y*,*
(4) *the curve* α*,*
(5) *the Riemannian metric* g*,*
(6) *the orientation of* M*.*

Proof. Note that two determinations of the angle must differ in an integer multiple of 2π. Thus, it is clear that the index does not depend on the determination.

We show now that the index does not depend on the vector field X. Suppose that we consider two vector fields $X_1(t), X_2(t)$. Note that they are solutions of an algebraic equation of degree n and they are differentiable with respect to the parameter t. Thus if $X_1(t) = \pm X_2(t)$ at some point of the curve, then this should be true for any point. In this case, the corresponding determinations of the angles should differ in an integer multiple of π, giving the same index.

Thus, we can assume that $X_1(t) \neq \pm X_2(t)$, for all $t \in \mathbb{R}$. Then, we can choose the determinations so that

$$0 < |\theta_1(t) - \theta_2(t)| < \pi,$$

for all $t \in \mathbb{R}$. Moreover, suppose that

$$\frac{\theta_i(2n\ell) - \theta_i(0)}{4\pi n} = \frac{s_i}{2n}$$

with $s_1, s_2 \in \mathbb{Z}$. Then,

$$|s_1 - s_2| = \frac{1}{2\pi} |\theta_1(2n\ell) - \theta_1(0) - \theta_2(2n\ell) + \theta_2(0)| < 1,$$

and necessarily $s_1 = s_2$.

To show that the index does not depend on the coordinates, let $Y_0 \in T_{\alpha(0)}M$ be any nonzero tangent vector. Let us denote by $Y(t)$ the parallel transport of Y_0 along $\alpha(t)$ and let $\psi(t)$ be a determination of the angle from $\frac{\partial}{\partial x}|_{\alpha(t)}$ to $Y(t)$. Following [4, Eq (2), page 271], we have that

$$\psi(\ell) - \psi(0) = \int_R K d\sigma,$$

where K denotes the gaussian curvature of M and $d\sigma$ is the area element. From this we deduce

$$2n \int_R K d\sigma - 2\pi s = (\psi - \theta)(2n\ell) - (\psi - \theta)(0), \tag{1}$$

being $s/2n$ the index. Since the angle $\psi - \theta$ does not depend on the coordinates x, y, we get that the index does not depend either.

Let now α and β be two curves satisfying the conditions of the definition of the index. We will show that the index given by both curves is the same. Suppose first that the curves are disjoint. Then it is obvious that we can construct a family of curves α_t, with $t \in [0,1]$, depending continuously on t, which verify the conditions of the definition of index and such that $\alpha_0 = \alpha$ and $\alpha_1 = \beta$. Taking into account that it is possible to express the index by means of an integral expression, we deduce that the index with respect to α_t depends continuously on t. Since the index can only take rational values, we deduce that it must be constant. In the case that the curves α and β are not disjoint, we can take a third curve small enough so that it is disjoint with α and β and then apply the above argument.

The independence with respect to the Riemannian metric g has an analogous argument. In fact, if g and h are two Riemannian metrics, we can consider the family of Riemannian metrics $g_t = (1-t)g + th$ so that $g_0 = g$ and $g_1 = h$. Again by means of an integral expression of the index, we see that the index with respect to the metric g_t depends continuously on t and hence, it must be constant.

Finally, it only remains to show that it does not depend on the orientation. In fact, if we change the orientation, we have to change α by $\tilde{\alpha}(t) = \alpha(\ell - t)$ and θ by $\tilde{\theta}(t) = -\theta(\ell - t)$. Then,

$$\tilde{\theta}(2n\ell) - \tilde{\theta}(0) = -\theta(\ell - 2n\ell) + \theta(\ell) = -\theta(0) + \theta(2n\ell).$$

□

As a consequence of this lemma, we deduce that the index is well defined and it only depends on the differential form ω. Moreover, the definition can be extended to the case that M is not orientable, just by taking a local orientation in a neighbourhood of the singular point.

On the other hand, the definition of index can be also extended to the case that p is a regular point, although in such case the index is always zero. In fact, we can take coordinates in such a way that $\partial/\partial x$ coincides with X along α and hence, $\theta(t) \equiv 0$.

Finally, another immediate consequence of the above lemma is that the index is invariant by equivalence. Let ω_1, ω_2 be two totally real differential n-forms defined on surfaces M, N respectively, which are equivalent through the diffeomorphism $\phi : M \to N$. Then, for each $p \in \mathrm{Sing}(\omega_1)$,

$$\mathrm{ind}(\omega_1, p) = \mathrm{ind}(\omega_2, \phi(p)).$$

Remark 3.3. We give here a formula which can be very useful to compute the index. Let us denote by $X_1(t), \dots, X_{2n}(t)$ the unit vector fields along α which are solution of $\omega(\alpha(t))(X) = 0$. We assume that they are ordered so that

$$\theta_1(t) < \theta_2(t) < \cdots < \theta_{2n}(t) < \theta_1(t) + 2\pi,$$

where $\theta_j(t)$ denotes the determination of the angle of each vector field $X_j(t)$. In particular, we have that

$$\theta_1(\ell) = \theta_i(0) + 2\pi m,$$

for some $m \in \mathbb{Z}$ and $i \in \{1, \dots, 2n\}$. Then, the index is given by

$$\mathrm{ind}(\omega, p) = m + \frac{i-1}{2n}.$$

In fact, we introduce the notation $\theta_{2n+1}(t) = \theta_1(t) + 2\pi$, $\theta_{2n+2}(t) = \theta_2(t) + 2\pi$, and in general, $\theta_{2qn+j}(t) = \theta_j(t) + 2q\pi$, for any $q \in \mathbb{Z}$ and $j \in \{1, \dots, 2n\}$. Then,

$$\begin{aligned}
\theta_1(\ell) &= \theta_i(0) + 2\pi m, \\
\theta_1(2\ell) &= \theta_i(\ell) + 2\pi m = \theta_{2i-1}(0) + 4\pi m, \\
&\dots \\
\theta_1(2n\ell) &= \theta_{2n(i-1)+1}(0) + 4\pi n m = \theta_1(0) + 2\pi(2nm + i - 1).
\end{aligned}$$

From this, we arrive to

$$\mathrm{ind}(\omega, p) = \frac{\theta_1(2n\ell) - \theta_1(0)}{4\pi n} = m + \frac{i-1}{2n}.$$

We finish this section by showing the generalization of the well known Poincaré-Hopf Theorem for vector fields or quadratic differential forms [14, 4].

Theorem 3.4. *Let M a compact surface and let ω be a totally real differential n-form with a finite number of singular points $p_1, \dots, p_m$. Then,*

$$\chi(M) = \sum_{i=1}^{m} \mathrm{ind}(\omega, p_i),$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of M.

Proof. The proof given here is just an adaptation of the proof given in [4, page 279] for the case of vector fields. We show first the theorem in the case that M is orientable.

We choose some orientation and a Riemannian metric on M. Let $\{\varphi_i : U_i \to \mathbb{R}^2\}_{i \in I}$ an atlas on M so that each chart is orthogonal and compatible with the orientation. Moreover, we take a triangulation $\mathcal{T}$ such that:

(1) Each triangle $T \in \mathcal{T}$ is contained in some coordinate neighbourhood.

(2) Each triangle $T \in \mathcal{T}$ contains at most one singular point p_T. (In the triangles with no singular points we choose any interior point p_T.)
(3) The boundary of each triangle $T \in \mathcal{T}$ has no singular points and is positively oriented.

Let X_T be a vector field along the boundary of each triangle $T \in \mathcal{T}$ which is a solution of equation $\omega(X) = 0$. Moreover, we choose it in such a way that if T_1, T_2 are adjacent triangles, then X_{T_1}, X_{T_2} coincide along the common edge. From Equation (1) we obtain

$$\int_T K d\sigma - 2\pi \operatorname{ind}(\omega, p_T) = \frac{\Delta_T}{2n},$$

for any $T \in \mathcal{T}$, where Δ_T denotes the variation of the angle from X_T to some parallel vector field after going through the boundary of T $2n$ times in positive sense.

Now, summing up for any $T \in \mathcal{T}$ and taking into account that each edge is common to two triangles with opposite orientations, we arrive to

$$\int_M K d\sigma - 2\pi \sum_{T \in \mathcal{T}} \operatorname{ind}(\omega, p_T) = \sum_{T \in \mathcal{T}} \frac{\Delta_T}{2n} = 0.$$

Finally, the result is a consequence of the Gauss-Bonnet Theorem:

$$\int_M K d\sigma = 2\pi \chi(M).$$

In the case that the surface M is not orientable, we consider $\pi : \tilde{M} \to M$ a double covering, where $\tilde{M}$ is an orientable and compact surface. Then $\chi(\tilde{M}) = 2\chi(M)$ and since π is a local diffeomorphism, each singular point p_i of ω gives exactly two singular points of the induced n-form $\pi^*\omega$ with the same index. Thus, this case is a consequence of the orientable case. $\square$

4. Differential forms in complex coordinates

We identify $\mathbb{R}^2 = \mathbb{C}$ and use the following notation

$$\begin{aligned} z &= x + iy, & \overline{z} &= x - iy, \\ dz &= dx + idy, & d\overline{z} &= dx - idy, \\ \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), & \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \end{aligned}$$

With this notation, it is obvious that any differential n-form on an open subset $U \subset \mathbb{C}$ can be written in a unique way in this coordinates as

$$\omega = A_0 dz^n + A_1 dz^{n-1} d\overline{z} + \cdots + A_n d\overline{z}^n,$$

for some differentiable functions $A_j : U \to \mathbb{C}$ such that $A_j = \overline{A}_{n-j}$ for all $j = 0, \ldots, n$.

The following theorem is a generalization of [14, VII.2.3] in the case $n = 2$.

Theorem 4.1. *Let ω be a totally real differential n-form on an open subset $U \subset \mathbb{C}$ and let $p \in U$ be an isolated singular point. Then, p is an isolated zero of A_0 and*

$$\operatorname{ind}(\omega, p) = -\frac{\deg(A_0, p)}{n},$$

where $\deg(A_0, p)$ *denotes the local degree of A_0 at p.*

Proof. Let $\delta > 0$ small enough and let $\alpha(t) = p + \delta e^{it}$, for $t \in \mathbb{R}$. We denote by $X_1(t), \dots, X_n(t)$ unit vector fields along α which are pairwise linearly independent and are solution of the equation $\omega(\alpha(t))(X) = 0$. We also denote by $\theta_j(t)$ a differentiable determination of the angle of $X_j(t)$, so that

$$X_j(t) = e^{i\theta_j(t)}\frac{\partial}{\partial z} + e^{-i\theta_j(t)}\frac{\partial}{\partial \overline{z}}.$$

It is obvious that $X_j(t)$ annihilates the linear form $\lambda_j(t)$ along α given by

$$\lambda_j(t) = e^{i\phi_j(t)}dz + e^{-i\phi_j(t)}d\overline{z},$$

being $\phi_j(t) = \pi/2 - \theta_j(t)$. Thus, by using elementary properties of the algebraic equations of degree n, we deduce that along α it is possible to factor ω as

$$\omega(\alpha(t)) = f(t)\lambda_1(t)\dots\lambda_n(t),$$

for some non vanishing function $f : \mathbb{R} \to \mathbb{R}$.

On the other hand, by comparing the coefficient of dz^n in the above expression, we have that

$$A_0(\alpha(t)) = f(t)e^{i(\phi_1(t)+\dots+\phi_n(t))}.$$

From this we see that $A_0(\alpha(t)) \neq 0$, for all $t \in \mathbb{R}$, which shows the first statement. Moreover, a differentiable determination of the angle of $A_0(\alpha(t))$ is given by

$$\beta(t) = \phi_1(t) + \dots + \phi_n(t) + \pi q,$$

for some $q \in \mathbb{Z}$.

Finally,

$$\begin{aligned}\deg(A_0, p) &= \frac{\beta(4\pi n) - \beta(0)}{4\pi n} = \sum_{j=1}^{n} \frac{\phi_j(4\pi n) - \phi_j(0)}{4\pi n} \\ &= -\sum_{j=1}^{n} \frac{\theta_j(4\pi n) - \theta_j(0)}{4\pi n} = -n \operatorname{ind}(\omega, p).\end{aligned}$$

□

Corollary 4.2. *The index of any isolated singular point of a totally real differential n-form on a surface M has the form s/n, with $s \in \mathbb{Z}$. Moreover, for each $s \in \mathbb{Z}$ there is a totally real differential n-form with an isolated singular point of index s/n.*

Proof. The first part is an immediate consequence of the above theorem. To see the second part, just consider $M = \mathbb{C}$, $p = 0$ and

$$\omega = \begin{cases} z^s dz^n + \overline{z}^s d\overline{z}^n, & \text{if } s \geq 0, \\ \overline{z}^{|s|} dz^n + z^{|s|} d\overline{z}^n, & \text{if } s < 0. \end{cases}$$

□

According to Definition 3.1, an isolated singular point of an n-web will have an index of the form $s/2n$, with $s \in \mathbb{Z}$. The above corollary says that in the case that the n-web is induced from a totally real differential n-form, the index will be of the form s/n, with $s \in \mathbb{Z}$. This can be interpreted as some kind of orientability condition for the n-web defined by a differential n-form.

For instance, when $n = 1$, a linear differential form in M induces an orientable foliation in a neighbourhood of each point of M. In this case, the index of an isolated singular point is an integer. For $n = 2$, a positive quadratic differential form induces a pair of (non necessarily orientable) transverse foliations in a neighbourhood of each point of M. The index associated to each one of the foliations is the same (because of transversality) and it is a half-integer (see [14, VII.2.2]).

Corollary 4.3. *Let ω be a totally real differential n-form on a surface M and $p \in M$ an isolated singular point. Let $\alpha : [0, \ell] \to M$ be a curve satisfying the conditions for the definition of the index and let $X(t)$ be a unit vector field along α, solution of $\omega(\alpha(t))(X) = 0$. Then $X(n\ell) = X(0)$ and*

$$\operatorname{ind}(\omega, p) = \frac{\theta(n\ell) - \theta(0)}{2\pi n},$$

where $\theta(t)$ denotes a determination of the angle of $X(t)$.

Proof. This is consequence of the above corollary and Remark 3.3. Let us denote by $X_1(t), \dots, X_{2n}(t)$ the unit vector fields along α which are solution of $\omega(\alpha(t))(X) = 0$, being $X(t) = X_1(t)$. We suppose that they are ordered so that

$$\theta_1(t) < \theta_2(t) < \cdots < \theta_{2n}(t) < \theta_1(t) + 2\pi,$$

where $\theta_j(t)$ is the determination of the angle of each vector field $X_j(t)$. Then,

$$\operatorname{ind}(\omega, p) = m + \frac{i-1}{2n},$$

where $\theta_1(\ell) = \theta_i(0) + 2\pi m$, with $m \in \mathbb{Z}$ and $i \in \{1, \dots, 2n\}$. Moreover, we introduce the notation $\theta_{2qn+j}(t) = \theta_j(t) + 2q\pi$, for any $q \in \mathbb{Z}$ and $j \in \{1, \dots, 2n\}$.

From the above corollary we see that $i - 1$ must be even and hence, we can write $i - 1 = 2q$, with $q \in \mathbb{Z}$. Thus,

$$\theta_1(n\ell) = \theta_{n(i-1)+1}(0) + 2\pi mn = \theta_1(0) + 2\pi(mn + q),$$

giving $X_1(n\ell) = X_1(0)$. □

Definition 4.4. We say that a singular point p of a totally real differential n-form ω is *simple* if the linear part of ω at p is itself a totally real differential n-form having p as an isolated singular point. Suppose that in complex coordinates

$$\omega = A_0 dz^n + A_1 dz^{n-1} d\overline{z} + \cdots + A_n d\overline{z}^n,$$

for some differentiable functions $A_i : U \to \mathbb{C}$. We also assume, for simplicity, that $p = 0$. Then, each one of these functions A_i has a Taylor expansion at the origin

$$A_i = a_i z + b_i \overline{z} + \dots$$

with $a_i, b_i \in \mathbb{C}$. The linear part of ω at p is the differential n-form

$$\omega_1 = (a_0 z + b_0 \overline{z}) dz^n + (a_1 z + b_1 \overline{z}) dz^{n-1} d\overline{z} + \cdots + (a_n z + b_n \overline{z}) d\overline{z}^n.$$

Corollary 4.5. *Any simple singular point of a totally real differential n-form on a surface M has index $\pm 1/n$.*

Proof. We take complex coordinates, suppose that $p = 0$ and the linear part of ω at p is

$$\omega_1 = (a_0 z + b_0 \overline{z}) dz^n + (a_1 z + b_1 \overline{z}) dz^{n-1} d\overline{z} + \cdots + (a_n z + b_n \overline{z}) d\overline{z}^n.$$

If ω_1 is totally real and p is an isolated singular point, by Theorem 4.1, p is an isolated zero of the linear function $a_0 z + b_0 \overline{z}$ and hence, such linear function is regular. Since it is the linear part of the function A_0, p is a regular point of A_0. Thus, $\deg(A_0, p) = \pm 1$ and $\operatorname{ind}(\omega, p) = \pm 1/n$. □

5. Non degenerate differential forms

Let ω be a totally real differential n-form on some open subset $U \subset \mathbb{C}$ and let $p \in U$ be an isolated singular point. We can extend the notation introduced in the above section and denote by ω_k the homogeneous part of degree k of ω. That is, each one of the coefficients A_j admits a Taylor expansion at p and ω_k is the n-form whose coefficients are the homogeneous parts of degree k in the expansion of the A_j.

Definition 5.1. We say that ω is *semi-homogeneous* at p if there is $k \geq 1$ such that $\omega_i = 0$ for $i = 1, \ldots, k-1$ and ω_k is a totally real differential n-form having p as an isolated singular point. Note that when $k = 1$, this is equal to the definition of simple singular point.

Assume for simplicity that $p = 0$ and let

$$\omega_k = A_0^k dz^n + A_1^k dz^{n-1} d\overline{z} + \cdots + A_n^k d\overline{z}^n,$$

where A_i^k are homogeneous polynomials of degree k. We define the *characteristic polynomial of* ω as the (real) homogeneous polynomial of degree $k+n$

$$P_\omega = A_0^k z^n + A_1^k z^{n-1}\overline{z} + \cdots + A_n^k \overline{z}^n.$$

Let us denote by $\pi : \mathbb{R}^2 \to \mathbb{C}$ the *polar blow-up*, that is, $\pi(r,t) = re^{it}$. We fix $\delta > 0$ small enough such that $\pi((-\delta,\delta) \times \mathbb{R}) \subset U$ and $p = 0$ is the only singular point of ω in such set.

Lemma 5.2. *If ω is semi-homogeneous with principal part ω_k, then*

$$\tilde{\omega}(r,t) = \begin{cases} \dfrac{1}{r^k}\omega(re^{it}), & \text{if } r \neq 0, \\ \omega_k(e^{it}), & \text{if } r = 0, \end{cases}$$

defines a totally real differential n-form along π on $(-\delta,\delta) \times \mathbb{R}$ with no singular points.

Proof. Suppose that ω is given by

$$\omega = A_0 dz^n + A_1 dz^{n-1} d\overline{z} + \cdots + A_n d\overline{z}^n$$

and let us denote by A_j^k the homogeneous part of degree k of A_j. By the Hadamard Lemma it follows that

$$A_j(re^{it}) = r^k B_j(r,t),$$

for some differentiable functions $B_j : (-\delta,\delta) \times \mathbb{R} \to \mathbb{R}$ such that $B_j(0,t) = A_j^k(e^{it})$. In particular

$$\tilde{\omega}(r,t) = B_0(r,t) dz^n + B_1(r,t) dz^{n-1} d\overline{z} + \cdots + B_n(r,t) d\overline{z}^n.$$

□

As a consequence of the above lemma, if ω is semi-homogeneous, we can choose n unit vector fields $X_1(r,t), \ldots, X_n(r,t)$ along π on $(-\delta,\delta) \times \mathbb{R}$ which are pairwise linearly independent and solution of $\tilde{\omega}(r,t)(X) = 0$. Moreover, we denote by $\theta_j(r,t)$ a differentiable determination of the angle of each vector field $X_j(r,t)$. Then we showed in the proof of Theorem 4.1, that it is possible to factor $\tilde{\omega}$ as

$$\tilde{\omega} = f \lambda_1 \ldots \lambda_n,$$

being λ_j the linear forms given by

$$\lambda_j = e^{i\phi_j} dz + e^{-i\phi_j} d\overline{z},$$

with $\phi_j = \pi/2 - \theta_j$ and $f : (-\delta,\delta) \times \mathbb{R} \to \mathbb{R}$ a non vanishing function.

Definition 5.3. The pull-back through π of the n-form $\tilde{\omega}$ defines an n-form $\pi^*\tilde{\omega}$ on $(-\delta,\delta)\times\mathbb{R}$, which is called *the polar n-form of ω*. Analogously, we call *linear polar forms of ω* the linear forms $\pi^*\lambda_1,\dots,\pi^*\lambda_n$, in such a way that

$$\pi^*\tilde{\omega} = f\pi^*\lambda_1\dots\pi^*\lambda_n,$$

An easy computation gives that for each $j=1,\dots,n$

$$\pi^*\lambda_j = 2(\cos\varphi_j dr - r\sin\varphi_j dt),$$

where $\varphi_j=\phi_j+t$. Thus, each one of these polar linear forms has singular points $(0,t)$ with $\varphi_j(0,t)=\pi/2+q\pi$, $q\in\mathbb{Z}$.

Note that a point $(0,t)$ can be a singular point of only one of the polar linear forms. In fact, suppose that

$$\varphi_{j_1}(0,t)=\pi/2+q_1\pi, \quad \varphi_{j_2}(0,t)=\pi/2+q_2\pi,$$

for some $q_1,q_2\in\mathbb{Z}$. Then

$$\theta_{j_1}(0,t)-\theta_{j_2}(0,t)=(q_2-q_1)\pi,$$

which implies that the corresponding vector fields are linearly dependent and hence, $j_1=j_2$.

Moreover, under some conditions it is possible to determine the topological type of these singular points. Let Λ_j the vector field given by

$$\Lambda_j = r\sin\varphi_j\frac{\partial}{\partial r}+\cos\varphi_j\frac{\partial}{\partial t}.$$

Then, the jacobian matrix at a singular point is

$$D\Lambda_j(0,t)=\pm\begin{pmatrix}1 & -\frac{\partial\varphi_j}{\partial r}\\ 0 & -\frac{\partial\varphi_j}{\partial t}\end{pmatrix}.$$

As a consequence, we have that $(0,t)$ is a hyperbolic singular point of $\pi^*\lambda_j$ if and only if $\frac{\partial\varphi_j}{\partial t}\neq 0$. Moreover, $(0,t)$ is of *saddle type* when $\frac{\partial\varphi_j}{\partial t}>0$ and of *node type* when $\frac{\partial\varphi_j}{\partial t}<0$.

Lemma 5.4. *Let ω be a semi-homogeneous totally real differential n-form and $p=0$ an isolated singular point. Then $z=e^{it}$ is a root of the characteristic polynomial P_ω if and only if $(0,t)$ is a singular point of one of its polar linear forms. Moreover, it is a simple root if and only if $(0,t)$ is a hyperbolic singular point of such polar linear form.*

Proof. In general, we have that $\pi^*dz=e^{it}(dr+irdt)$ and $\pi^*d\overline{z}=e^{-it}(dr-irdt)$. In particular, restricted to $r=0$, we get

$$\pi^*\tilde{\omega}(0,t)=\left(\sum_{j=0}^{n}A_j^k(e^{it})(e^{it})^j(e^{-it})^{n-j}\right)dr^n = P_\omega(e^{it})dr^n.$$

On the other hand, by using the factor of $\pi^*\tilde{\omega}$ in the polar linear forms, we see that

$$\pi^*\tilde{\omega}(0,t)=f(0,t)\cos\varphi_1(0,t)\dots\cos\varphi_n(0,t)dr^n,$$

which implies that

$$P_\omega(e^{it})=2^nf(0,t)\cos\varphi_1(0,t)\dots\cos\varphi_n(0,t).$$

Thus, it is obvious that $z=e^{it}$ is a root of P_ω if and only if $(0,t)$ is a singular point of one of the polar linear forms.

Moreover, since P_ω is a homogeneous polynomial it is easy to check that z is a simple root if and only if $\frac{d}{dt}\left(P_\omega(e^{it})\right) \neq 0$. But if we differentiate in the above expression, we arrive to

$$\frac{d}{dt}\left(P_\omega(e^{it})\right) = \pm 2^n f(0,t)\frac{\partial\varphi_j}{\partial t}(0,t).$$

Therefore, it is a simple root if and only if $(0,t)$ is a hyperbolic singular point, by the above remark. $\square$

Remark 5.5. Suppose that $z = e^{it}$ is a root of the characteristic polynomial P_ω. By the above lemma, $(0,t)$ is a singular point of one the polar linear forms, that is, $\varphi_j(0,t) = \pi/2 + q\pi$, for some $j \in \{1,\dots,n\}$, and $q \in \mathbb{Z}$. For each $p \in \mathbb{Z}$, $e^{i(t+p\pi)} = \pm z$ is also a root of P_ω and hence, there are $j_p \in \{1,\dots,n\}$, and $q_p \in \mathbb{Z}$ such that $\varphi_{j_p}(0,t+p\pi) = \pi/2+q_p\pi$. This implies that

$$\varphi_j(0,t) - \varphi_{j_p}(0,t+p\pi) = (p-q_p)\pi,$$

for any $p \in \mathbb{Z}$. But looking at the way that the functions φ_j are constructed, if this is true for some point $t \in \mathbb{R}$, then it must be true for any $t \in \mathbb{R}$. Then, by taking derivatives with respect to t,

$$\frac{\partial\varphi_j}{\partial t}(0,t) = \frac{\partial\varphi_{j_p}}{\partial t}(0,t+p\pi).$$

Thus, $(0,t)$ is a singular point of $\pi^*\lambda_j$ of saddle or node type if and only if $(0,t+p\pi)$ is a singular point of $\pi^*\lambda_{j_p}$ of saddle or node type respectively. In conclusion, the singularity type only depends on the direction determined by $z = e^{it}$.

Definition 5.6. Let ω be a totally real differential n-form with an isolated singular point p. We say ω is *non degenerate* at p if it is semi-homogeneous and the characteristic polynomial has only simple roots.

Theorem 5.7. *Let ω be a totally real differential n-form with a non degenerate singular point p. Then,*

$$\operatorname{ind}(\omega,p) = 1 - \frac{S^+ - S^-}{n},$$

where S^+ and S^- denote the numbers of characteristic directions of saddle and node type respectively.

Proof. Denote by S_j^+ and S_j^- the numbers of singular points of saddle and node type respectively of the polar linear form $\pi^*\lambda_j$ in the interval $[0,2\pi n)$. Then,

$$\sum_{j=1}^n S_j^+ = 2nS^+, \quad \sum_{j=1}^n S_j^- = 2nS^-.$$

Remember that such points are given by the points $(0,t)$ such that $\varphi_j(0,t) = \pi/2 + q\pi$, with $q \in \mathbb{Z}$. Moreover, it is of saddle type when φ_j is increasing at such point and of node type when it is decreasing. This implies that

$$\varphi_j(0,2\pi n) - \varphi_j(0,0) = \pi(S_j^+ - S_j^-),$$

for all $j = 1,\dots,n$.

Now, by Corollary 4.3,

$$\begin{aligned}\operatorname{ind}(\omega,p) &= \frac{1}{n}\sum_{j=1}^{n}\frac{\theta_j(0,2\pi n)-\theta_j(0,0)}{2\pi n} = -\frac{1}{n}\sum_{j=1}^{n}\frac{\phi_j(0,2\pi n)-\phi_j(0,0)}{2\pi n}\\ &= -\frac{1}{n}\sum_{j=1}^{n}\frac{\varphi_j(0,2\pi n)-2\pi n-\varphi_j(0,0)}{2\pi n} = 1-\frac{1}{n}\sum_{j=1}^{n}\frac{\varphi_j(0,2\pi n)-\varphi_j(0,0)}{2\pi n}\\ &= 1-\frac{1}{n}\sum_{j=1}^{n}\frac{S_j^+-S_j^-}{2n} = 1-\frac{S^+-S^-}{n},\end{aligned}$$

since $\phi_j(0,t)=\frac{\pi}{2}-\theta_j(0,t)$ and $\varphi_j(0,t)=\phi_j(0,t)+t$. □

Definition 5.8. Let ω be a totally real differential n-form with a non degenerate singular point p. By a *sector* we mean each one of the regions bounded by two consecutive characteristic directions S_1 and S_2. We say a sector is

(1) *hyperbolic:* if both S_1 and S_2 are of saddle type;
(2) *parabolic:* if one of S_1 and S_2 is of saddle type and the other one is of node type;
(3) *elliptic:* if both S_1 and S_2 are of node type.

Let S^+ and S^- denote the number of characteristic directions of saddle and node type respectively and let h and e denote the numbers of hyperbolic and elliptic sectors respectively. It is obvious that $e-h=2(S^- - S^+)$. Thus, we get the following immediate consequence of Theorem 5.7, which generalizes the well known Bendixson formula for the index when $n=1$.

Corollary 5.9. *Let ω be a totally real differential n-form with a non degenerate singular point p. Then,*

$$\operatorname{ind}(\omega,p)=1+\frac{e-h}{2n},$$

where e and h are the numbers of elliptic and hyperbolic sectors respectively.

Remark 5.10. When ω has a non degenerate principal part, it is possible to improve the formula for the index given in Remark 3.3. Let $X_1(r,t),\dots,X_n(r,t)$ be unit vector fields along π on $(-\delta,\delta)\times\mathbb{R}$ which are pairwise linearly independent and solution of $\tilde{\omega}(r,t)(X)=0$. Moreover, we suppose that they are chosen so that

$$\theta_1(0,t)<\theta_2(0,t)<\cdots<\theta_n(0,t)<\theta_1(0,t)+\pi,$$

where $\theta_j(r,t)$ denotes the determination of the angle of each vector field $X_j(r,t)$. Note that for $r=0$, these vector fields are solution of an equation with homogeneous coefficients, which implies that

$$\theta_1(0,\pi)=\theta_i(0,0)+\pi m,$$

for some $m\in\mathbb{Z}$ and $i\in\{1,\dots,n\}$. Then, it follows that

$$\operatorname{ind}(\omega,p)=m+\frac{i-1}{n}.$$

6. Phase portrait of non degenerate singular points

In general, the foliations of an n-form can present very complicated configurations around a singular point. In the case that ω has a non degenerate singular point, the n foliations are obtained as the image of the integral curves of the polar linear forms through the polar blow-up. Moreover, since the characteristic polynomial has only simple roots, then the problem is simpler, because the polar linear forms only have singularities of saddle or node type.

Definition 6.1. Let ω be a totally real differential n-form with a non degenerate singular point p. Let x be a point near p and let L one of the n leaves of the web passing through x. We say that L is

(1) *hyperbolic:* if p is not an accumulation point of L;
(2) *parabolic:* if p is an accumulation point on just one side of L;
(3) *elliptic:* if p is an accumulation point on both sides of L.

If the leaf L is hyperbolic (respectively parabolic, elliptic), then it corresponds to an integral curve of one of the polar linear forms with a saddle-saddle (respectively saddle-node, node-node) connection. In order to have a complete description, we need to know how many sectors the leaf is going to pass through when connecting the two singular directions (Figure 6.1).

Lemma 6.2. *Let x be a point near p and let L be a hyperbolic leaf through x connecting two saddles. Assume that L passes through k sectors containing n_1 saddles and n_2 nodes (so that $n_1 + n_2 = k - 1$). Then,*

$$k = n + 2n_2.$$

Proof. Let R be the union of the closed sectors that L passes trhough, which is bounded by the two saddles S_1 and S_2. Since R is simply connected, we can separate the web in R into n foliations $\mathcal{F}_1, \dots, \mathcal{F}_n$. We will assume that L is a leaf of $\mathcal{F}_1$. Then $\mathcal{F}_1$ also contains the saddles S_1, S_2 and all its other leaves of $\mathcal{F}_1$ are also hyperbolic.

Let $\mathcal{F}_i$ be one of the other foliations, with $i = 2, \dots, n$. We can use the leaves of $\mathcal{F}_i$ to define a continuous map $\phi_i : L \to S_1 \cup S_2$. Given $y \in L$, we take the leaf L_i of $\mathcal{F}_i$ passing through y. Because of transversality, either L_i intersects $S_1 \cup S_2$ in a single point which we define as $\phi_i(y)$ or p is an accumulation point of L_i, in which case we define $\phi_i(y) = p$ (see Figure 2).

Since the leaves of $\mathcal{F}_i$ are disjoint, we have two possibilities: either $\phi_i^{-1}(p)$ is just one point and $\mathcal{F}_i$ contains just one saddle, or $\phi_i^{-1}(p)$ is an interval, so that $\mathcal{F}_i$ contains one node and two saddles (see Figure 3).

Finally, assume there are a foliations of the first type and b of the second type, with $a + b = n - 1$. Then, $n_1 = a + 2b$ and $n_2 = b$, which gives the desired result. □

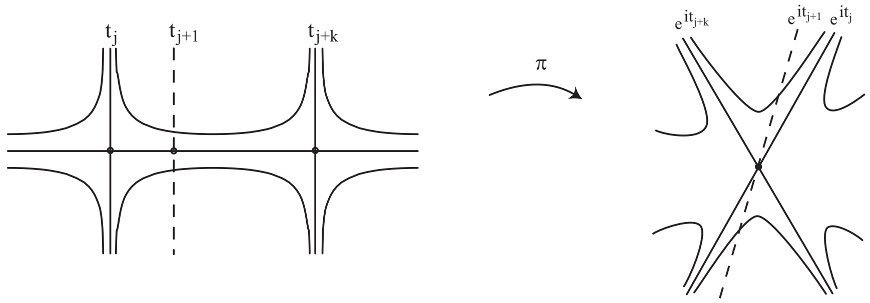
Lemma 6.3. *Let x be a point near p and let L be an elliptic leaf through x connecting two nodes. Assume that L passes through k sectors containing n_1 saddles and n_2 nodes (so that $n_1 + n_2 = k - 1$). Then,*

$$k = n + 2n_1.$$

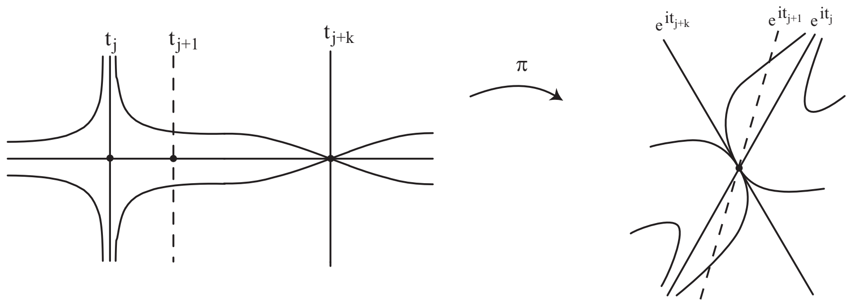
Proof. We assume that $p = 0$ and that ω has the following principal part

$$\omega_k = A_0^k dz^n + \cdots + A_n^k d\overline{z}^n,$$

(1) saddle-saddle;



(2) saddle-node;



(3) node-node.

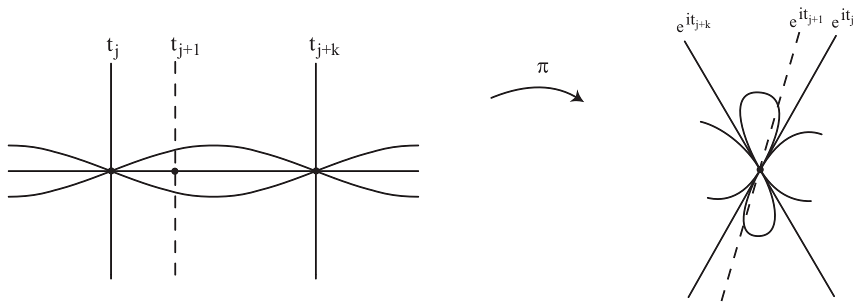


FIGURE 1

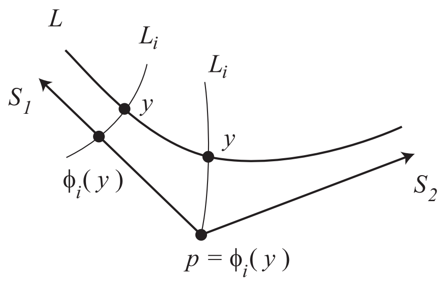


FIGURE 2

where A_i^k are homogeneous polynomials of degree k. We take now the inversion $z = 1/w$, which gives:

$$dz^n = -\frac{dw}{w^{2n}} - \frac{\overline{w}^{2n}dw}{(w\overline{w})^{2n}},$$

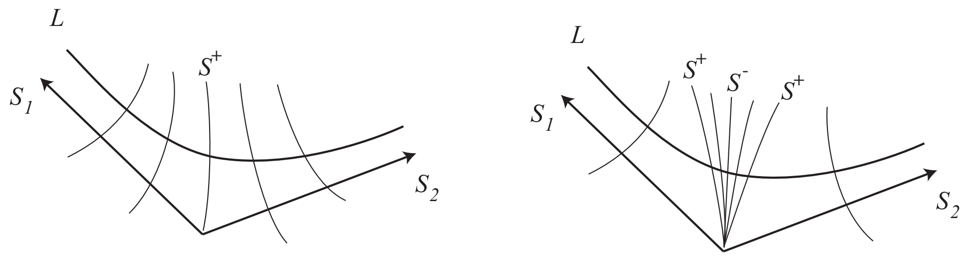


FIGURE 3

and

$$A_i^k(z) = A_i^k(\frac{1}{w}) = \frac{A_i^k(\overline{w})}{(w\overline{w})^k}.$$

Then we obtain that in $\mathbb{C}\setminus\{0\}$, ω_k is equivalent to the differential form

$$\sigma_k = A_0^k(\overline{w})\overline{w}^{2n}dw^n + \cdots + A_n^k(\overline{w})w^{2n}d\overline{w}^n).$$

Note that σ_k is also totally real with non degenerate principal part and the characteristic polynomial has the same roots as ω_k, although the inversion transforms saddles into nodes and nodes into saddles. Moreover, elliptic leaves of the foliations of ω_k are transformed into hyperbolic leaves of σ_k and vice versa. Thus, the result is a consequence of the above lemma. In Figure 4 we present the result of taking the inversion of Figure 3.

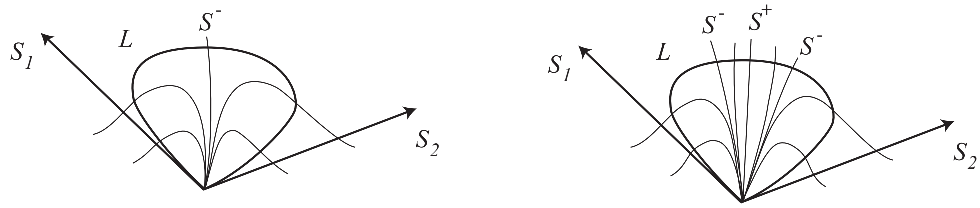


FIGURE 4

□

Lemma 6.4. *Let x be a point near p and let L be a parabolic leaf through x connecting a saddle and a node. Assume that L passes through k sectors containing n_1 saddles and n_2 nodes (so that $n_1 + n_2 = k - 1$). Then,*

$$k = 1 + 2n_2.$$

Proof. We follow a similar argument to that of the proof of Lemma 6.2. We denote by R the union of sectors containing the leaf L, which is bounded by the saddle S_1 and the node S_2. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be the n foliations determined by ω in R so that L is a leaf of $\mathcal{F}_1$.

For each one of the foliations $\mathcal{F}_i$, with $i = 2, \dots, n$ we have again two possibilities as listed in Figure 5. In one case $\mathcal{F}_i$ does not contain any characteristic direction, while in the other cased it contains one saddle and one node. If we denote by a, b the number of foliations of each type respectively, we have that $a+b = n-1$ and $n_1 = n_2 = b$. Therefore, we get $k = 1 + 2n_2$.

□

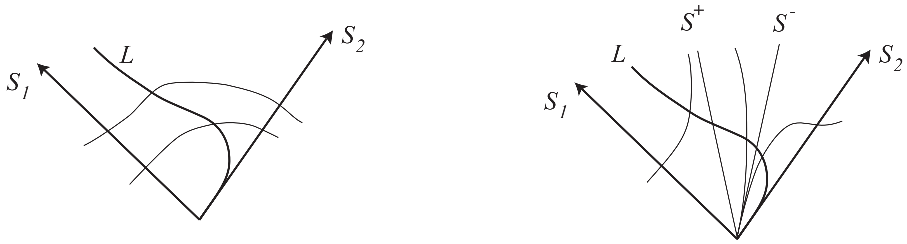


FIGURE 5

Remark 6.5. Once we know how many directions of saddle or of node type we have, as well as their relative position around the singular point p, the three above lemmas allow us to complete the phase portrait of all the leaves of the n web determined by ω. We call this the *phase portrait* of ω at p. When $n \leq 2$, it is well known that this is enough for topological classification, that is, if two differential n-forms have the same phase portrait at a point, then they are locally topologically equivalent. For $n \geq 3$, this is not true anymore because the curvature of the web is a topological invariant.

7. PHASE PORTRAITS NEAR HYPERBOLIC SINGULAR POINTS

In this section we give the possible phase portraits of "generic" singular points of totally real differential n-forms.

Definition 7.1. We say that p is a *hyperbolic* singular point of a totally real differential n-form ω if it is simple and the characteristic polynomial P_ω has only simple roots.

Theorem 7.2. *Let p be a hyperbolic singular point of a totally real differential n-form ω ($n \geq 2$). Then, there are only three possible phase portraits of the foliations of ω around p:*

1. *Type D_1 or lemon: there are $n-1$ directions of saddle type with hyperbolic leaves passing through n sectors.*
2. *Type D_2 or monstar: there are n directions of saddle type and one of node type; the hyperbolic leaves pass through $n+2$ sectors, while the parabolic leaves pass through one sector.*
3. *Type D_3 or star: there are $n+1$ directions of saddle type with hyperbolic leaves passing through n sectors.*

Proof. Let S^+ and S^- be the numbers of directions of saddle and node type respectively. The sum $S^+ + S^-$ is the total number of roots of the characteristic polynomial P_ω, which has degree $n+1$. Since the roots are simple,

$$0 \leq S^+ + S^- \leq n+1, \quad S^+ + S^- \equiv n+1 \mod 2.$$

Assume that $S^+ + S^- = n+1$. If $S^- \geq 2$, then $S^+ \leq n-1$ and by Theorem 5.7,

$$\mathrm{ind}(\omega, p) = 1 - \frac{S^+ - S^-}{n} \geq 1 - \frac{n-1-2}{n} = \frac{3}{n}.$$

This is not possible, by Corollary 4.5, since the index can only be $\pm 1/n$. Thus, the only possibilities are $S^+ = n+1$, $S^- = 0$ or $S^+ = n$, $S^- = 1$ which correspond to the types D_3 and D_2 respectively. Note that the index in each case is $-1/n$ or $1/n$ respectively.

Next case is $S^+ + S^- = n-1$. As above, if we suppose that $S^- \geq 1$, then $S^+ \leq n-2$ and hence,

$$\operatorname{ind}(\omega, p) = 1 - \frac{S^+ - S^-}{n} \geq 1 - \frac{n-2-1}{n} = \frac{3}{n}.$$

The only possibility is $S^+ = n-1$, $S^- = 0$ which correspond to the type D_1 and has index $1/n$.

Finally, assume that $S^+ + S^- \leq n-3$. Then necessarily $S^- \geq 0$, $S^+ \leq n-3$ and hence,

$$\operatorname{ind}(\omega, p) = 1 - \frac{S^+ - S^-}{n} \geq 1 - \frac{n-3-0}{n} = \frac{3}{n}.$$

Therefore, it is clear that there are no more possibilities.

The discussion about the number of sectors of hyperbolic or parabolic leaves is a consequence of above lemmas. □

The above classification in the case $n = 2$ gives the classification obtained by Darboux for the curvature lines around generic umbilic points of an immersed surface in $\mathbb{R}^3$ (see [1] and [12]). A proof for the general case of hyperbolic singular points of quadratic forms can found in [9].

Example 7.3. Consider $\omega_1 = \overline{z}dz^n + zd\overline{z}^n$. By Theorem 4.3,

$$\operatorname{ind}(\omega_1, 0) = -\deg(\overline{z}, 0)/n = 1/n.$$

Moreover, the characteristic polynomial is

$$P_{\omega_1} = \overline{z}z^n + z\overline{z}^n = \overline{z}z(z^{n-1} + \overline{z}^{n-1}),$$

which has $n-1$ real simple roots. Thus, for any n, ω_1 has a hyperbolic singular point of type lemon or D_1.

Now, let $\omega_{2,\epsilon} = (iz - (1+\epsilon)i\overline{z})dz^n + (-i\overline{z} + (1+\epsilon)iz)d\overline{z}^n$, with $\epsilon > 0$. In this case, the index is again $1/n$ and the characteristic polynomial is

$$\begin{aligned} P_{\omega_{2,\epsilon}} &= (iz - (1+\epsilon)i\overline{z})z^n + (-i\overline{z} + (1+\epsilon)iz)\overline{z}^n \\ &= (iz - i\overline{z})(z^n + \overline{z}^n) + \epsilon iz\overline{z}(z^{n-1} - \overline{z}^{n-1}). \end{aligned}$$

Given n, it follows that for ϵ small enough, $P_{\omega_{2,\epsilon}}$ has exactly $n+1$ real simple roots. Then, $\omega_{2,\epsilon}$ has a hyperbolic singular point of type monstar or D_2.

Finally, we consider $\omega_3 = zdz^n + \overline{z}d\overline{z}^n$. The index is now $-1/n$ and $P_{\omega_3} = z^{n+1} + \overline{z}^{n+1}$. For any n, it has $n+1$ simple real roots, so that ω_3 is of type star or D_3.

In Figure 6, we can find pictures of the foliations for the three examples D_1, D_2, D_3 in the cases $n = 2$ (top) and $n = 3$ (bottom) obtained with *Mathematica* (D_1 and D_3) and with the program *Homogeneous equations lines* by A. Montesinos [16] (D_2 with $\epsilon = 1/2$).

8. Higher order principal lines and umbilics

Let $g : M \to \mathbb{R}^N$ be a C^∞ immersion of a surface M in Euclidean space $\mathbb{R}^N$. We consider the distance squared unfolding $D : \mathbb{R}^N \times M \to \mathbb{R}^N \times \mathbb{R}$ given by

$$D(x, p) = (x, d_x(p)) = (x, \frac{1}{2}\|x - g(p)\|^2).$$

We use Thom-Boardman notation for singularities. Then, it follows that $\Sigma^2(D)$ is the subset of $\mathbb{R}^N \times M$ of pairs (x, p) such that the jacobian matrix of d_x has kernel rank 2 at p, which is nothing but the *normal bundle* of M in $\mathbb{R}^N$.

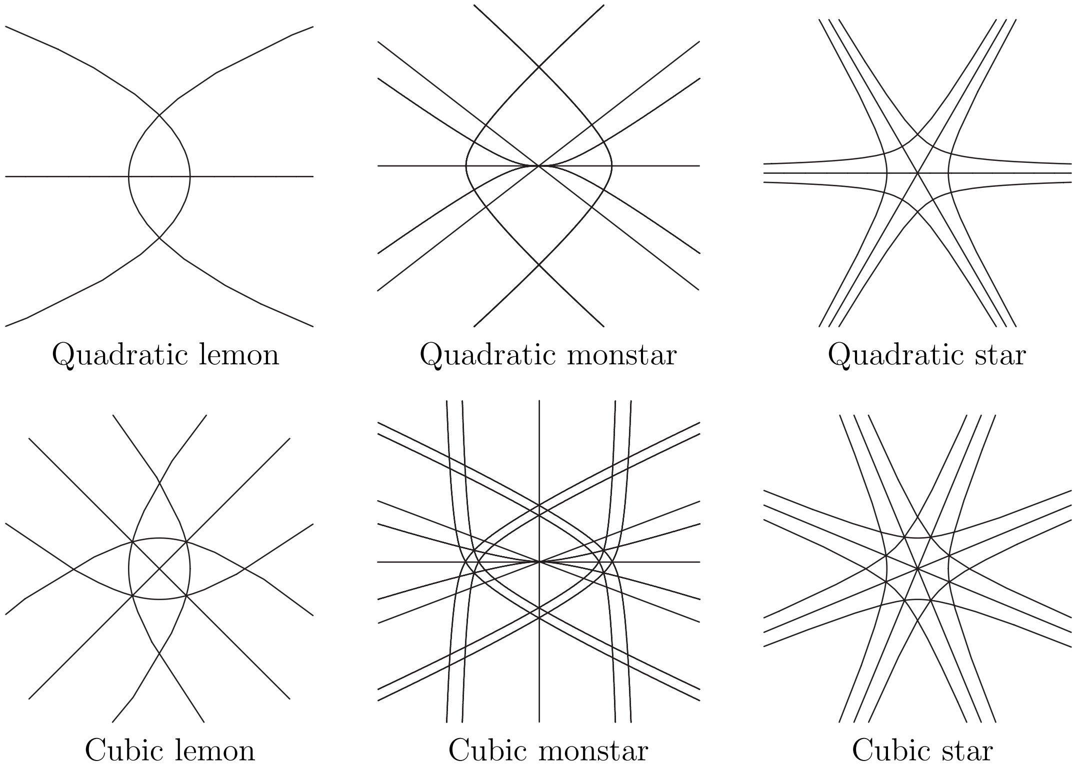


Figure 6

Assume $N = 3$. Then $\Sigma^{2,1}(D)$ is the subset of $\Sigma^2(D)$ given by pairs (x, p) such that the hessian matrix of d_x has kernel rank 1 at p. This is known as the *focal set* of M in $\mathbb{R}^3$ and corresponds to the subset of pairs (x, p) such that x is a centre of principal curvature at a non umbilic point $p \in M$. Moreover, we can also consider the *contact directions*, which are defined as the tangent directions $X \in T_pM$ such that $X \in \ker \operatorname{Hess}(d_x)_p$. When p is not parabolic, then these contact direction correspond to the principal lines of M (when p is parabolic, principal lines are in fact contact directions of the height function, in which case the sphere becomes a plane and x goes to infinity).

By taking local coordinates u, v in an open subset $U \subset M$, it is possible to find the differential equation of principal lines:

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0,$$

where E, F, G and L, M, N are respectively the coefficients of the first and second fundamental forms of M in $\mathbb{R}^3$. The singular points of this equation are the umbilic points of M where the surface has a contact of type $\Sigma^{2,2}$ with some sphere of $\mathbb{R}^3$ (if the umbilic is non flat). However, for our purposes, it is better to consider the following equivalent differential equation:

$$\begin{vmatrix} g_{1u} & g_{2u} & g_{3u} & 0 & 0 \\ g_{1v} & g_{2v} & g_{3v} & 0 & 0 \\ g_{1uu} & g_{2uu} & g_{3uu} & E & dv^2 \\ g_{1uv} & g_{2uv} & g_{3uv} & F & -dudv \\ g_{1vv} & g_{2vv} & g_{3vv} & G & du^2 \end{vmatrix} = 0.$$

This matrix (excluding the last column) was introduced in [8] to define the notion of k-rounding of an immersion $g: M \to \mathbb{R}^N$. It is a higher order generalization of umbilic point and for an appropriate choice of the ambient dimension N these points are generically isolated.

For instance, assume now that $k = 3$ and consider an immersion of a surface M in $\mathbb{R}^7$. We define the *third order contact directions* as the tangent directions $X \in T_pM$ such that $X \in \ker J^3(d_x)_p$ and $(x,p) \in \Sigma^{2,2,1}(D)$. Here J^3 is the operator defined in local coordinates

$$J^3(f) = \begin{pmatrix} f_{uuu} & f_{uuv} \\ f_{uuv} & f_{uvv} \\ f_{uvv} & f_{vvv} \end{pmatrix}.$$

Note that $f_u = f_v = f_{uu} = f_{uv} = f_{vv} = 0$, this definition does not depend on the coordinates.

Note that we can do the same construction by taking the height function unfolding $H: S^6 \times M \to S^6 \times \mathbb{R}$ given by

$$H(\mathbf{v}, p) = (\mathbf{v}, h_v(p)) = (\mathbf{v}, \langle \mathbf{v}, g(p) \rangle).$$

We also include in the above definition of third order contact directions those $X \in T_pM$ such that $X \in \ker J^3(h_{\mathbf{v}})_p$ and $(\mathbf{v}, p) \in \Sigma^{2,2,1}(H)$.

Assume that M is locally parameterized locally by a map $g: U \subset \mathbb{R}^2 \to \mathbb{R}^7$. We use the following notation: $\varphi_{\alpha\beta} = \langle g_\alpha, g_\beta \rangle$ are the coefficients of the first fundamental form and

$$\varphi_{\alpha\beta\gamma} = \langle g_{\alpha\beta}, g_\gamma \rangle + (\varphi_{\alpha\beta})_\gamma.$$

Theorem 8.1. *With the above notation, the differential equation for the third order contact directions is*

$$\begin{vmatrix} g_{1u} & \cdots & g_{7u} & 0 & 0 \\ g_{1v} & \cdots & g_{7v} & 0 & 0 \\ g_{1uu} & \cdots & g_{7uu} & \varphi_{uu} & 0 \\ g_{1uv} & \cdots & g_{7uv} & \varphi_{uv} & 0 \\ g_{1vv} & \cdots & g_{7vv} & \varphi_{vv} & 0 \\ g_{1uuu} & \cdots & g_{7uuu} & \varphi_{uuu} & dv^3 \\ g_{1uuv} & \cdots & g_{7uuv} & \varphi_{uuv} & -dudv^2 \\ g_{1uvv} & \cdots & g_{7uvv} & \varphi_{uvv} & du^2dv \\ g_{1vvv} & \cdots & g_{7vvv} & \varphi_{vvv} & -du^3 \end{vmatrix} = 0.$$

Proof. Let $X = ag_u + bg_v$ be a non-zero tangent vector at p with $a, b \in \mathbb{R}$. It follows from the definition that X is a third order contact direction if and only if $f_u = f_v = f_{uu} = f_{uv} = f_{vv} = 0$, $(f_{uuu}, f_{uuu}, f_{uuu}, f_{uuu}) \neq 0$ and

$$\begin{pmatrix} f_{uuu} & f_{uuv} \\ f_{uuv} & f_{uvv} \\ f_{uvv} & f_{vvv} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for either $f = d_x$ or $f = h_{\mathbf{v}}$. Note that this last equation is equivalent to

$$\begin{pmatrix} f_{uuu} \\ f_{uuv} \\ f_{uvv} \\ f_{vvv} \end{pmatrix} = \lambda \begin{pmatrix} b^3 \\ -ab^2 \\ a^2b \\ -a^3 \end{pmatrix},$$

for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$. In order to simplify the expressions we introduce the notation $\sigma_{\alpha\beta\gamma}$, which are defined by

$$\begin{pmatrix} \sigma_{uuu} \\ \sigma_{uuv} \\ \sigma_{uvv} \\ \sigma_{vvv} \end{pmatrix} = \begin{pmatrix} b^3 \\ -ab^2 \\ a^2b \\ -a^3 \end{pmatrix}.$$

Then we can express shortly our conditions by $f_\alpha = f_{\alpha\beta} = 0$ and $f_{\alpha\beta\gamma} = \lambda\sigma_{\alpha\beta\gamma}$.

On the other hand, we recall that for $f = d_x$ we have

$$\begin{aligned} f_\alpha &= -\langle g_\alpha, x - g\rangle, \\ f_{\alpha\beta} &= -\langle g_{\alpha\beta}, x - g\rangle + \varphi_{\alpha\beta}, \\ f_{\alpha\beta\gamma} &= -\langle g_{\alpha\beta\gamma}, x - g\rangle + \varphi_{\alpha\beta\gamma}, \end{aligned}$$

while for $f = h_{\mathbf{v}}$,

$$\begin{aligned} f_\alpha &= \langle g_\alpha, v\rangle, \\ f_{\alpha\beta} &= \langle g_{\alpha\beta}, v\rangle, \\ f_{\alpha\beta\gamma} &= \langle g_{\alpha\beta\gamma}, v\rangle. \end{aligned}$$

Assume we have a third order contact line with $f = d_x$. Then,

$$\begin{pmatrix} g_\alpha & 0 & 0 \\ g_{\alpha\beta} & \varphi_{\alpha\beta} & 0 \\ g_{\alpha\beta\gamma} & \varphi_{\alpha\beta\gamma} & \sigma_{\alpha\beta\gamma} \end{pmatrix} \begin{pmatrix} x - g \\ -1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which implies that the matrix has determinant equal to zero. For $f = h_{\mathbf{v}}$ we take $(\mathbf{v}, 0, \lambda)$ instead of $(x - g, -1, \lambda)$.

Conversely, if the determinant of the matrix is zero, then there is $(X, Y, Z) \neq 0$ such that

$$\begin{pmatrix} g_\alpha & 0 & 0 \\ g_{\alpha\beta} & \varphi_{\alpha\beta} & 0 \\ g_{\alpha\beta\gamma} & \varphi_{\alpha\beta\gamma} & \sigma_{\alpha\beta\gamma} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $Y \neq 0$, we take $x = -X/Y + g$ and $\lambda = -Z/Y$, which gives a third order contact line for $f = d_x$. Otherwise, $Y = 0$ implies, necessarily $X \neq 0$ so that we can define $\mathbf{v} = X/\|X\|$ and $\lambda = Z$. This gives a third order contact line for $f = h_{\mathbf{v}}$. □

We see that third order contact lines are defined by means of a cubic differential form and can be interpreted as some kind of "third order principal directions". The singular points corresponds to the "third order umbilics" (that is, points $p \in M$ where $g(M)$ has a third order contact $\Sigma^{2,2,2}$ with some hypersphere or hyperplane of $\mathbb{R}^7$). In general, this cubic differential form is not always totally real (as it happens with principal lines of a surface in $\mathbb{R}^3$). However, in the case that it is, we find that for a generic immersion the singularities are hyperbolic and the phase portrait of the 3-web is described in Theorem 7.2, in analogy with the classical Darbouxian classification of principal foliations near generic umbilics.

Corollary 8.2. *Let $g : M \to \mathbb{R}^7$ be a generic immersion. Let p be a third order umbilic such that the third order contact lines are defined by a totally real cubic differential form near p. Then, p is hyperbolic in the sense of Definition 7.1.*

Proof. Given a map $g : M \to \mathbb{R}^7$, we denote by $j^4 g : M \to J^4(M, \mathbb{R}^7)$ its 4-jet extension. We also denote by $\mathcal{U} \subset J^4(M, \mathbb{R}^7)$ with the following property: $p \in M$ is a third order umbilic of g if and only if $j^4 g(p) \in \mathcal{U}$. It follows that $\subset U$ is an algebraic subset of codimension 2 in $J^4(M, \mathbb{R}^7)$.

We also denote by $\mathcal{U}_1$ the subset of $\mathcal{U}$ such that $j^4 g(p) \in \mathcal{U}_1$ if and only if p is not a simple singularity of the cubic differential form which defines third order contact lines. Analogously, we define $\mathcal{U}_2$ as the subset of $\mathcal{U}$ where the characteristic polynomial of the cubic differential form has not simple roots.

In both cases, $\mathcal{U}_i$ is an algebraic subset of $J^4(M, \mathbb{R}^7)$ of codimension 3. In fact, the equations of $\mathcal{U}$ are functions which only depend on the derivatives of g up to order 3, whilst the equations of $\mathcal{U}_i$ involve in a non-trivial way the 4th order derivatives. This implies that $\operatorname{codim} \mathcal{U}_i > \operatorname{codim} \mathcal{U}$. The result follows now from Thom transversality theorem by requesting transversality to both $\mathcal{U}_1$ and $\mathcal{U}_2$. $\square$

This construction can be generalized easily to any k. We just need to consider an immersion $g : M \to \mathbb{R}^N$, with $N = \frac{(k+2)(k+1)}{2} - 3$. Then, the k-th order contact lines are defined by means of a symmetric differential form of degree k, whose singularities correspond to the k-roundings of M in $\mathbb{R}^N$ (see [8] for more details).

References

[1] J.W. Bruce, D.L. Fidal, *On binary differential equations and umbilics*, Proc. Roy. Soc. Edingburgh Sect. A **111** (1989), no.1–2, 147–168.

[2] J.W. Bruce, F. Tari, *On binary equations*, Nonlinearity **8** (1995), 255–271.

[3] M. Brunella, M. Miari, *Topological equivalence of a vector field with its principal part defined by Newton polyhedra*, J. Differential Equations **85** (1990), 338–366.

[4] M.P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall Inc. 1976.

[5] G. Darboux, *Sur la forme des lignes de courbure dans la voisinage d'un ombilic,* Leçons sur la Theorie des Surfaces, IV, Note 7, Gauthier Villars, Paris, 1896.

[6] A.A. Davydov, *Qualitative theory of control systems*, Translations of Mathematical Monographs, vol. 141, American Mathematical Society, 1994.

[7] J.P. Dufour, *Rigidity of Webs*, Web Theory and Related Topics, 106–113, World Scientific, Singapore, 2001.

[8] T. Fukui and J.J. Nuño-Ballesteros, *Isolated roundings and flattenings of submanifolds in euclidean spaces*, Tohoku Math. J. **57** (2005), 469–503.

[9] V. Guíñez, *Positive quadratic differential forms and foliations with singularities on surfaces*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 19–44.

[10] C. Gutierrez, I. Guadalupe, R. Tribuzy, V. Guíñez, *Lines of curvature on surfaces immersed in R^4*, Bol. Soc. Brasil. Mat. (N.S.) **28** (1997), no. 2, 233–251.

[11] C. Gutierrez, R.D.S. Oliveira, M. Teixeira, *Positive quadratic differential forms: topological equivalence through Newton Polyhedra*, preprint.

[12] C. Gutierrez and J. Sotomayor, *Structural stable configurations of lines of principal curvature*, Bifurcation, ergodic theory and applications (Dijon, 1981), 195–215, Asterisque, 98-99, Soc. Math. France, Paris, 1982.

[13] P. Hartman and A. Wintner, *On the singularities in nets of curves defined by differential equations*, Amer. J. Math. **75** (1953), 277–297.

[14] H. Hopf, Differential Geometry in the Large, Lect. Notes Math. 1000, Springer-Verlag 1983.

[15] J.A. Little, *On singularities of submanifolds of higher dimensional euclidean space*, Anna. Mat. Pura et Appl. (ser. 4A) **83** (1969), 261–336.

[16] A. Montesinos-Amilibia, *Homogeneous equations lines*, computer program available by anonymous ftp at `ftp://topologia.geomet.uv.es/pub/montesin`.

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