UNFOLDING PLANE CURVES WITH CUSPS AND NODES

J.J. NUÑO-BALLESTEROS

Abstract. Given an irreducible surface germ $(X,0)\subset(\mathbb{C}^3,0)$ with 1-dimensional singular set Σ, we denote by $\delta_1(X,0)$ the delta invariant of a transverse slice. We show that $\delta_1(X,0)\geq m_0(\Sigma,0)$, with equality if and only if $(X,0)$ admits a corank 1 parameterisation $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ whose only singularities outside the origin are transverse double points and semicubic cuspidal edges. Then, we use the local Euler obstruction $\mathrm{Eu}(X,0)$ in order to characterize those surfaces which have finite codimension with respect to $\mathscr{A}$-equivalence or as a frontal type singularity.

1. Introduction

Any irreducible complex plane curve singularity $(Y,0)$ can be parameterised, that is, it can be seen as the image of a finite and generically 1-1 map germ $\gamma:(\mathbb{C},0)\to(\mathbb{C}^2,0)$. Then, we can look at it either as a finitely determined map germ with respect to the $\mathscr{A}$-equivalence or also as a frontal type singularity (using Zakalyukin's terminology [16]) of finite codimension in some sense. This phenomenon becomes explicit when we consider a suitable deformation Y_t, parameterised by a stable map γ_t. In the first case, Y_t is a Morsification of Y, since the degenerated singularity splits into a finite number of nodes, that is, transverse double points A_1. In the second case, besides the nodes, we also allow the birth of simple cusps A_2, which are stable singularities in this context. As an example, we see in fig. 1 the two deformations of the E_6 singularity, parameterised by $\gamma(v)=(v^3,v^4)$.

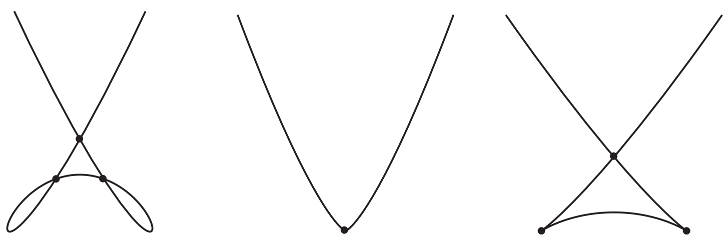

Figure 1

The total space of the deformation $(X,0)$ is an irreducible surface in $(\mathbb{C}^3,0)$ with 1-dimensional singular locus Σ which has special properties. It can be parameterised as the image of a map germ $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ given by $f(u,v)=(u,\gamma_u(v))$. If γ_u is a Morsification, then f is $\mathscr{A}$-finite, that is, it has finite codimension with respect to the $\mathscr{A}$-equivalence. Otherwise, if γ_u

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is a deformation as a frontal, then f is itself a frontal type surface of finite codimension as a frontal (see Section 3). We show in fig. 2 the two surfaces constructed with the two deformations of E_6. On the left hand side, we have the $P_3(c)$ singularity of D. Mond [12] and on the right hand side we have the swallowtail.

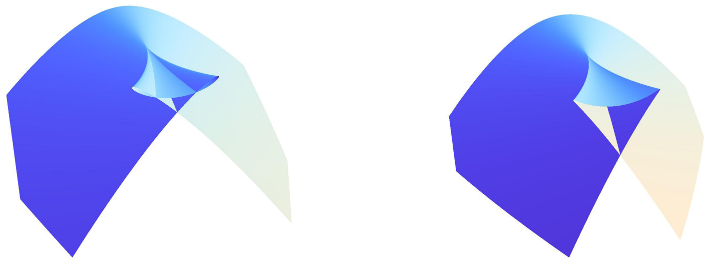

FIGURE 2

Another interesting property of $(X, 0)$ is the equality $\delta_1(X, 0) = m_0(\Sigma, 0)$, where $\delta_1(X, 0)$ is the transverse delta invariant (i.e., the delta invariant of a generic plane section) and $m_0(\Sigma, 0)$ is the multiplicity of its singular locus. Since this is the minimal possible value for $\delta_1(X, 0)$, we say that $(X, 0)$ is a δ_1-minimal surface. In fact, we show in theorem 2.1 that for any irreducible surface $(X, 0)$ with non isolated singularity, we have $\delta_1(X, 0) \geq m_0(\Sigma, 0)$, with equality if and only if $(X, 0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and such that the only singularities outside the origin are transverse double points or semicubic cuspidal edges.

In the last part of the paper, we use the local Euler obstruction $\mathrm{Eu}(X, 0)$ in order to characterize those surfaces among the δ_1-minimal ones which are stable unfoldings of plane curves or frontals. We show that if $(X, 0)$ is δ_1-minimal, then

$$1 \leq \mathrm{Eu}(X, 0) \leq m_0(X, 0).$$

Moreover, we deduce (see corollary 4.3):

(1) $(X, 0)$ is the image of a corank 1 $\mathscr{A}$-finite map germ if and only if it is δ_1-minimal and $\mathrm{Eu}(X, 0) = 1$.
(2) $(X, 0)$ is the image of a corank 1 frontal of finite codimension if and only if it is δ_1-minimal and $\mathrm{Eu}(X, 0) = m_0(X, 0)$.

Note that Jorge-Pérez and Saia proved in [8] that if $(X, 0)$ is the image of a corank 1 $\mathscr{A}$-finite map germ, then $\mathrm{Eu}(X, 0) = 1$. The results presented here are also related to those of [11], where we consider the classification and the invariants of corank 1 $\mathscr{A}$-finite map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ by looking at the transverse slice.

2. δ_1-MINIMAL SURFACES

Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a singular surface. Given $0 \in H \subset \mathbb{C}^3$ a generic plane we consider the plane curve $Y = X \cap H$ and we call it a *transverse slice* of X. The delta invariant of Y at 0 is an invariant of $(X, 0)$ which is independent of the choice of H. We denote $\delta_1(X, 0) := \delta(Y, 0)$ and call it the *transverse delta invariant.*

Given an analytic set germ $(V,0) \subset (\mathbb{C}^n,0)$ we denote by $m_0(V,0)$ its multiplicity. We recall that this can be computed by means of a generic linear projection $\ell : \mathbb{C}^n \to \mathbb{C}^d$, where $d = \dim(V,0)$. Then $m_0(V,0) = \#V \cap H_t$ where $H_t = \ell^{-1}(t)$ and $t \in \mathbb{C}^d$ is a generic value.

Theorem 2.1. *Let $(X,0) \subset (\mathbb{C}^3,0)$ be an irreducible surface with singular locus $(\Sigma,0)$ of dimension 1, then*

$$\delta_1(X,0) \geq m_0(\Sigma,0).$$

Moreover, the equality holds if and only if $(X,0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ such that the only singularities outside the origin are transverse double points and semicubic cuspidal edges.

Proof. We consider a linear projection $\ell : \mathbb{C}^3 \to \mathbb{C}$ such that $H = \ell^{-1}(0)$ is a generic plane and $Y = X \cap H$ is a transverse slice of X. Moreover, for each $t \in \mathbb{C}$ we can take $H_t = \ell^{-1}(t)$ in such a way that $Y_t = X \cap H_t$ defines a flat deformation of $(Y,0)$.

Since $(X,0)$ is irreducible, it has a normalization $n : (\tilde{X},0) \to (X,0)$, where $(\tilde{X},0)$ is a normal surface and n is finite and generically 1-1. By taking the composition $\tilde{p} = p \circ n : (\tilde{X},0) \to (\mathbb{C},0)$ we have also a flat deformation of $\tilde{Y} = n^{-1}(Y)$.

We use now a result of Lejeune-Lê-Teissier [6] (see also [3, 4.1.14]): for any $t \neq 0$ small enough,

$$\delta(Y,0) = \delta(\tilde{Y},0) + \sum_{p \in S(Y_t)} \delta(Y_t,p), \tag{1}$$

where $S(Y_t)$ denotes the singular set of Y_t. Obviously, $S(Y_t) = Y_t \cap \Sigma = H_t \cap \Sigma$ and for each $p \in S(Y_t)$, $\delta(Y_t,p) \geq 1$. Therefore,

$$\delta(Y,0) \geq \#H_t \cap \Sigma = m_0(\Sigma,0).$$

We have the equality in the case that $(X,0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ and the only singularities of $(X,0)$ outside the origin are transverse double points and semicubic cuspidal edges. In fact, after taking a linear coordinate change in $\mathbb{C}^3$ and after reparameterisation, we can assume that f is given in the form

$$f(u,v) = (u, p(u,v), q(u,v)),$$

for some function germs p, q and such that generic plane is $x = 0$ (here we denote by (x,y,z) the coordinates in $\mathbb{C}^3$). Then $\tilde{Y}$ is the curve $u = 0$ which is smooth and thus $\delta(\tilde{Y},0) = 0$.

On the other hand, for each $t \neq 0$, the deformation Y_t is given by $x = t$. The only singularities of Y_t are cusps and nodes, both having delta invariant equal to 1. By (1), $\delta(Y,0) = m_0(\Sigma,0)$.

We see now the converse. If $\delta(Y,0) = m_0(\Sigma,0)$, we deduce from (1) that $\delta(\tilde{Y},0) = 0$ and $\delta(Y_t,p) = 1$ for each $t \neq 0$ and $p \in S(Y_t)$. In other words, $\tilde{Y}$ is smooth at 0 and the only singularities of Y_t are cusps and nodes when $t \neq 0$.

Since $\delta(\tilde{Y},0) = 0$, we have from (1) that Y_t is a delta constant family of curves in the sense of Teissier. By [3, 7.1.3], Y_t admits a normalization in family. But the unicity of the normalization implies that $\tilde{X}$ is smooth at 0 and we can assume $\tilde{X} = \mathbb{C}^2$. Thus, $(X, 0)$ is the image of $f = i \circ n : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, where i denotes the inclusion map.

Because of $\tilde{Y}$ is smooth at 0, f must have corank 1. Moreover, the only singularities of f outside the origin will be semicubic cuspidal edges and transverse double points (having as transverse slice cusps and nodes, respectively). □

Definition 2.2. We say that a surface $(X, 0) \subset (\mathbb{C}^3, 0)$ is δ_1-*minimal* if it is irreducible with 1-dimensional singular locus Σ and $\delta_1(X, 0) = m_0(\Sigma, 0)$.

It follows from the proof of theorem 2.1 that the following statements are equivalent:

(1) $(X, 0)$ is δ_1-minimal.
(2) $(X, 0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that the only singularities outside the origin are semicubic cuspidal edges and transverse double points.
(3) $(X, 0)$ is the image of an unfolding of a plane curve with only cusps and nodes.

Example 2.3. Let $(X, 0)$ be the surface parameterised by the double fold map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by $f(u, v) = (u^2, v^2, u^5 + v^5 + 2u^3v^3)$ (see [10]). Then $(X, 0)$ is irreducible, its singular set Σ has dimension 1 and all the singularities outside the origin are semicubic cuspidal edges and transverse double points (see fig. 3). But since f has corank 2, we expect to get $\delta_1(X, 0) > m_0(\Sigma, 0)$.

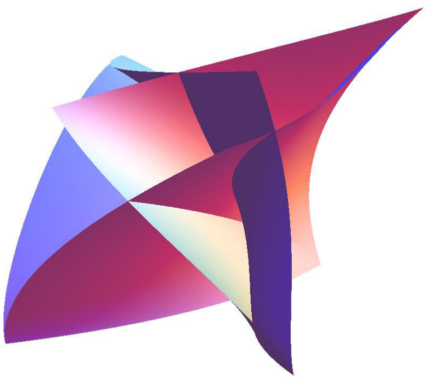

FIGURE 3

In fact, according to [10], Σ is the curve in $(\mathbb{C}^3, 0)$ given by the zeros of the 3×3 minors of the following matrix:

$$\begin{pmatrix} -z & x^2 & y^2 & 2xy \\ x^3 & -z & 2x^2y & y^2 \\ y^3 & 2xy^2 & -z & x^2 \\ 2x^2y^2 & y^3 & x^3 & -z \end{pmatrix}$$

With the aid of the computer algebra system SINGULAR [4], we compute $m_0(\Sigma, 0) = 13$. On the other hand, $(X, 0)$ is given by the determinant of the above matrix:

$$x^{10} - 8x^8y^3 + 16x^6y^6 - 2x^5y^5 - 2x^5z^2 - 16x^4y^4z$$
$$- 8x^3y^8 - 8x^3y^3z^2 + y^{10} - 2y^5z^2 + z^4 = 0.$$

In order to compute the transverse slice, we just substitute $z = ax + by$ for some generic coefficients $a, b \in \mathbb{C}$. Again with the aid of Singular we get $\delta_1(X, 0) = 14$.

We can associate two invariants to each δ_1-minimal surface $(X, 0)$. Let $\ell : \mathbb{C}^3 \to \mathbb{C}$ be a generic linear projection and put $H_t = \ell^{-1}(t)$ and $Y_t = X \cap H_t$. Since $(X, 0)$ is δ_1-minimal, the only singularities of Y_t for $t \neq 0$ small enough are cusps and nodes.

Definition 2.4. We define the *numbers* of *transverse cusps* and *transverse nodes* of $(X, 0)$, respectively as:

- $\kappa =$ number of cusps (A_2) of Y_t,
- $\nu =$ number of nodes (A_1) of Y_t.

It is obvious that the numbers κ, ν are well defined and do not depend on the choice of the linear projection ℓ nor the parameter t. Moreover, we also deduce from the proof of theorem 2.1 that

$$\kappa + \nu = \delta_1(X, 0).$$

If $(X, 0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, then after taking a linear coordinate change in $\mathbb{C}^3$ and after reparameterisation, we can assume that f is given in the form

$$f(u, v) = (u, \gamma_u(v)),$$

where $\gamma_u(v)$ is the parameterisation of the plane curve $Y_u = X \cap \{x = u\}$.

Proposition 2.5. *Let $(X, 0)$ be a δ_1-minimal surface, parameterised by $f(u, v) = (u, \gamma_u(v))$, where $x = 0$ is a generic plane. The following statements are equivalent:*

1. $\kappa = 0$,
2. *f is $\mathscr{A}$-finite,*
3. *for each $t \neq 0$, γ_t is $\mathscr{A}$-stable.*

Proof. The equivalence between (1) and (3) follows from the fact that the only $\mathscr{A}$-stable singularities of plane curves are nodes. The equivalence between (1) and (2) is a consequence of the Mather-Gaffney determinacy criterion: the map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is $\mathscr{A}$-finite if and only if there is a proper representative $f : U \to V$ such that $f^{-1}(0) = \{0\}$ and the restriction to $U \setminus \{0\}$ is $\mathscr{A}$-stable. But since the cross-caps and the transverse triple points are isolated, by shrinking U if necessary, this is equivalent to that f has only transverse double points on $U \setminus \{0\}$. □

Example 2.6. Let $(X, 0)$ be an irreducible surface with 1-dimensional singular set whose transverse slice has type E_6. We parameterise the curve by $\gamma(v) = (v^3, v^4)$ and take the mini-versal deformation:

$$\Gamma(v; a, b, c) = (v^3 + av, v^4 + bv^2 + cv).$$

Then, after a linear coordinate change, $(X,0)$ admits a parameterisation of the form

$$f(u,v)=(u,v^3+a(u)v,v^4+b(u)v^2+c(u)v),$$

for some $a,b,c\in\mathbb{C}\{u\}$, with $a(0)=b(0)=c(0)=0$.

The discriminant of the deformation Δ is the set of points $(a,b,c)\in\mathbb{C}^3$ such that the curve $\gamma_{a,b,c}(v)=(v^3+av,v^4+bv^2+cv)$ is not $\mathscr{A}$-stable. According to [11], Δ has equation $P_1P_2P_3=0$, where:

$$P_1=16a^3-48a^2b+36ab^2+27c^2,$$
$$P_2=32a^3-48a^2b+24ab^2-4b^3+27c^2,$$
$$P_3=a-b.$$

The three factors P_1,P_2,P_3 correspond to the strata of singular points, self-tangencies and triple points, respectively.

If we also denote $P_i=P_i(a(u),b(u),c(u))$, we have three types of δ_1-minimal surfaces:

(1) $(X,0)$ is δ_1-minimal with $\kappa=0$ and $\nu=3$ if and only if $P_1P_2P_3\neq 0$.
(2) $(X,0)$ is δ_1-minimal with $\kappa=1$ and $\nu=2$ if and only if $P_1=0$, but $(c,2a-3b)\neq(0,0)$ and $P_2P_3\neq 0$.
(3) $(X,0)$ is δ_1-minimal with $\kappa=2$ and $\nu=1$ if and only if $(c,2a-3b)=(0,0)$, but $P_2P_3\neq 0$.

3. Frontals

In this section, we consider frontal type singularities. This concept was introduced by Zakalyukin and Kurbatskiĭ in [16] and it is the generalization of a front. Roughly speaking, a frontal is the projection of a Legendrian submanifold with singularities. We refer also to Ishikawa's paper [5] for basic definitions and notations about Legendre singularities.

Let $PT^*\mathbb{C}^{n+1}$ be the projectivized cotangent bundle of $\mathbb{C}^{n+1}$ with the canonical contact structure defined by the contact form α and denote the projection by $\pi: PT^*\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$. By definition, a holomorphic map germ $\mathcal{L}:(\mathbb{C}^n,0)\to PT^*\mathbb{C}^{n+1}$ is said to be *integral* if $\mathcal{L}^*\alpha\equiv 0$. This is means that $\mathcal{L}=(f,[\nu])$ where $f:(\mathbb{C}^n,0)\to\mathbb{C}^{n+1}$ is a holomorphic map germ and $\nu:(\mathbb{C}^n,0)\to T^*\mathbb{C}^{n+1}$ is a holomorphic non-zero 1-form along f such that $\nu(df\circ\xi)=0$, for any $\xi\in V_n$, the space of all germs of vector fields in $(\mathbb{C}^n,0)$. If ν is given in coordinates by $\nu=\sum_{j=1}^{n+1}\nu_j dx_j$, this is also equivalent to

$$\sum_{j=1}^{n+1}\nu_j\frac{\partial f_j}{\partial u_i}=0,\quad \forall i=1,\dots,n.$$

Definition 3.1. We say that a map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ is a *frontal* map germ if there is an integral map germ $\mathcal{L}:(\mathbb{C}^n,0)\to PT^*\mathbb{C}^{n+1}$ such that $\pi\circ\mathcal{L}=f$. If in addition $\mathcal{L}$ is an embedding, then we say that f is a *front*.

When $\mathcal{L}$ is an integral embedding, then its image in $PT^*\mathbb{C}^{n+1}$ is called a Legendrian submanifold. If it is not an embedding, then it is usual to call the image a Legendrian submanifold with singularities. A hypersurface singularity $(X,0)$ in $(\mathbb{C}^{n+1},0)$ is called a *frontal* (resp. *front*) if there is a frontal (resp. front) map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ whose image is $(X,0)$.

Remark 3.2. If the map germ f is itself an embedding, then it is always a frontal and the class $[\nu]$ is determined univocally by the components of the cross product:

$$\frac{\partial f}{\partial u_1}\wedge\cdots\wedge\frac{\partial f}{\partial u_n}.$$

If f is not an embedding, but it is generically immersive (for instance, when it is finite and generically 1-1), then the class $[\nu]$ is also univocally determined, if it exists.

Example 3.3. Let us see some examples:

(1) Any irreducible plane curve singularity is always a frontal. Assume $(Y,0)$ is parameterised in $(\mathbb{C}^2,0)$ by $\gamma(v)=(p(v),q(v))$, where

$$\begin{aligned}p(v)&=a_nv^n+a_{n+1}v^{n+1}+\dots,\\ q(v)&=b_mv^m+b_{m+1}v^{m+1}+\dots\end{aligned}$$

with $a_n,b_m\neq 0$ and $n\leq m$. Then we take the 1-form:

$$\nu=\frac{1}{v^{n-1}}(-q'(v)dx+p'(v)dy).$$

Note that $(Y,0)$ is a front if and only if $m=n+1$.

(2) The double fold surface $(X,0)$ of Example 2.3 is a corank 2 frontal surface in $(\mathbb{C}^3,0)$. In fact, since

$$\frac{\partial f}{\partial u}\wedge\frac{\partial f}{\partial v}=uv(-2u(5u^2+6v^3),-2v(6u^3+5v^2),4),$$

we may take

$$\nu=-2u(5u^2+6v^3)dx-2v(6u^3+5v^2)dy+4dz.$$

(3) Not every parameterised surface $(X,0)\subset(\mathbb{C}^3,0)$ is a frontal. For instance, given the cross-cap $f(u,v)=(u,v^2,uv)$ we have

$$\frac{\partial f}{\partial u}\wedge\frac{\partial f}{\partial v}=(-2v^2,-u,2v).$$

There is no a non-zero holomorphic 1-form ν such that

$$\nu\left(\frac{\partial f}{\partial u}\right)=\nu\left(\frac{\partial f}{\partial v}\right)=0.$$

In general, we have the following criterion for corank 1 hypersurfaces.

Proposition 3.4. *Consider a hypersurface $(X,0)\subset(\mathbb{C}^{n+1},0)$ parameterised by a corank 1 map germ $f(u,v)=(u,p(u,v),q(u,v))$, with $u\in\mathbb{C}^{n-1}$, $v\in\mathbb{C}$. Then $(X,0)$ is a frontal if and only if either $\frac{\partial p}{\partial v}$ divides $\frac{\partial q}{\partial v}$ or $\frac{\partial q}{\partial v}$ divides $\frac{\partial p}{\partial v}$.*

Proof. We have that

$$\frac{\partial f}{\partial u_1}\wedge\cdots\wedge\frac{\partial f}{\partial u_{n-1}}\wedge\frac{\partial f}{\partial v}=\left(\Delta_1,\dots,\Delta_{n-1},-\frac{\partial q}{\partial v},\frac{\partial p}{\partial v}\right)$$

where $\Delta_i=\frac{\partial q}{\partial v}\frac{\partial p}{\partial u_i}-\frac{\partial q}{\partial u_i}\frac{\partial p}{\partial v}$.

Assume, for instance, that $\frac{\partial q}{\partial v} = \lambda \frac{\partial p}{\partial v}$ for some function λ. Then, $\Delta_i = \mu_i \frac{\partial p}{\partial v}$, with $\mu_i = \lambda \frac{\partial p}{\partial u_i} - \frac{\partial q}{\partial u_i}$ and thus, we can take

$$\nu = \mu_1 dx_1 + \cdots + \mu_{n-1} dx_{n-1} - \lambda dx_n + dx_{n+1}.$$

Conversely, suppose that there is non-zero 1-form ν such that $\mathcal{L} = (f, [\nu])$ is integral. Then, there is a function α, such that

$$\Delta_i = \alpha \nu_i, \ i = 1, \dots, n-1, \quad -\frac{\partial q}{\partial v} = \alpha \nu_n, \quad \frac{\partial p}{\partial v} = \alpha \nu_{n+1},$$

and hence,

$$\alpha \nu_i = -\alpha \left(\nu_n \frac{\partial p}{\partial u_i} + \nu_{n+1} \frac{\partial q}{\partial u_i} \right), \ i = 1, \dots, n-1.$$

If $\alpha = 0$, we have $\frac{\partial p}{\partial v} = \frac{\partial p}{\partial v} = 0$ and the result is obvious. Otherwise, if $\alpha \neq 0$, we have that

$$\nu_i = -\nu_n \frac{\partial p}{\partial u_i} - \nu_{n+1} \frac{\partial q}{\partial u_i}, \ i = 1, \dots, n-1.$$

Since $\nu(0) \neq 0$, then necessarily either $\nu_n(0) = 0$ or $\nu_{n+1}(0) \neq 0$ so that either $\frac{\partial p}{\partial v} | \frac{\partial q}{\partial v}$ or $\frac{\partial q}{\partial v} | \frac{\partial p}{\partial v}$. □

Example 3.5. We apply this criterion to see some examples of frontal surfaces:

(1) The swallowtail is $(X, 0)$ is a frontal surface (see the right hand side of fig. 2). In fact, it is parameterised by $f(u, v) = (u, v^3 + uv, v^4 + \frac{2}{3} uv^2)$ and we have $\frac{\partial p}{\partial v} = 3v^2 + u$ and $\frac{\partial q}{\partial v} = \frac{4}{3} v(3v^2 + u)$.

(2) The folded Whitney umbrella is the surface $(X, 0)$ in $(\mathbb{C}^3, 0)$ parameterised by $f(u, v) = (u, v^2, uv^3 + v^5)$ (see fig. 4). This is also a frontal since $\frac{\partial p}{\partial v} = 2v$ and $\frac{\partial q}{\partial v} = v(3uv + 5v^3)$.

Now we define the codimension of a frontal as the codimension of the Legendrian singularity whose projection is the frontal, with respect to Legendre equivalence. Let us denote $W = PT^*\mathbb{C}^{n+1}$ for simplicity and let $\mathcal{L} : (\mathbb{C}^n, 0) \to (W, w_0)$ be the integral map germ given by $\mathcal{L} = (f, [\nu])$. We recall the following notations from [5]:

(1) $VI_{\mathcal{L}}$ is the space of all integral infinitesimal deformations of $\mathcal{L}$, that is, germs of vector fields along $\mathcal{L}$ which preserve the contact structure.

(2) VL_{W,w_0} is the space of all germs of Legendre vector fields in (W, w_0).

Definition 3.6. We define the *$\mathscr{F}_e$-codimension* of f as

$$\mathscr{F}_e - \operatorname{codim}(f) = \dim_{\mathbb{C}} \frac{VI_{\mathcal{L}}}{\{d\mathcal{L} \circ \xi + \tilde{\eta} \circ \mathcal{L} : \ \xi \in V_n, \ \tilde{\eta} \in VL_{W,w_0}\}}.$$

If the $\mathscr{F}_e$-codimension is finite, we say that f is *$\mathscr{F}$-finite* and if the $\mathscr{F}_e$-codimension is zero, then we say that f is *$\mathscr{F}$-stable.*

According to [5], the space $VI_{\mathcal{L}}$ can be interpreted as the space of all infinitesimal integral deformations of $\mathcal{L}$ and the subspace

$$\{d\mathcal{L} \circ \xi + \tilde{\eta} \circ \mathcal{L} : \ \xi \in V_n, \ \tilde{\eta} \in VL_{W,w_0}\}$$

should be understood as the extended tangent space to the orbit of $\mathcal{L}$ under the action of Legendre equivalences. It follows from the definition that f is $\mathscr{F}$-stable if and only if $\mathcal{L}$ is infinitesimally Legendre stable in the sense of [5]. By [5, 4.1], any corank 1 $\mathscr{F}$-stable frontal is the projection of an open Whitney umbrella.

All the above definitions are also valid if instead of germs we consider multigerms $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, y)$, where $S \subset \mathbb{C}^n$ is any finite set and $y \in \mathbb{C}^{n+1}$. We use the above remark to classify the $\mathscr{F}$-stable singularities of curves and surfaces. Note that all the $\mathscr{F}$-stable singularities of frontal surfaces except folded Whitney umbrellas are generic fronts and their classification is well known (see for instance [1]).

Proposition 3.7. (1) *The $\mathscr{F}$-stable singularities of a frontal curve are cusps and nodes.*
(2) *The $\mathscr{F}$-stable singularities of a frontal surface are either: semicubic cuspidal edges, swallowtails, folded Whitney umbrellas or their transverse self-intersections (see fig. 4).*

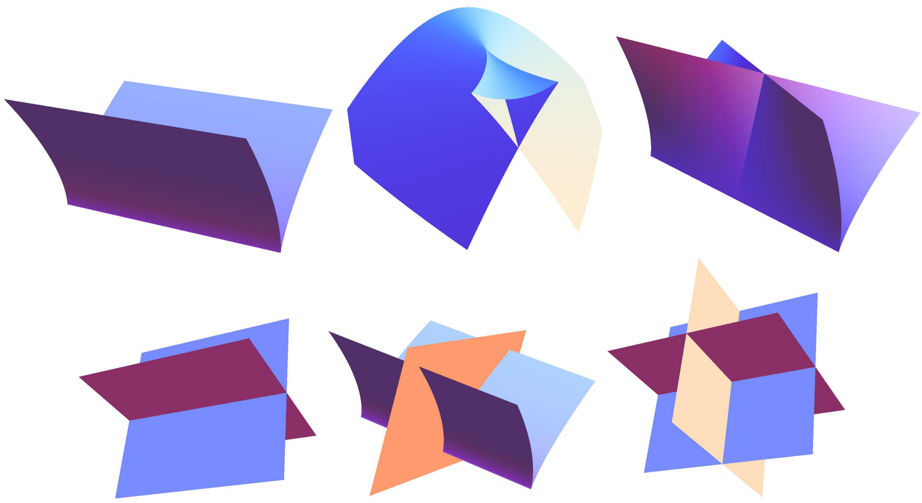

FIGURE 4

The following property is an adapted version of the Mather-Gaffney finite determinacy criterion for frontals (see [15]).

Proposition 3.8. *A frontal $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{F}$-finite if and only if there is a proper and finite-to-one representative $f : U \to V$ such that $f^{-1}(0) = \{0\}$ and the multigerm at any point $y \in V \setminus \{0\}$ is $\mathscr{F}$-stable.*

By shrinking the neighbourhoods U, V if necessary, all the isolated $\mathscr{F}$-stable singularities can be avoided. Then, we have the following direct consequence of propositions 3.7 and 3.8.

Corollary 3.9. (1) *A frontal curve is $\mathscr{F}$-finite if and only if it has isolated singularity.*

(2) *A frontal surface of corank 1 is $\mathscr{F}$-finite if and only if the only singularities outside the origin are transverse double points and semicubic cuspidal edges.*

Recall that if $(X,0)$ is δ_1-minimal then $0\le \kappa \le m_0(X,0)-1$, where κ is the number of cusps. Then, we have the following property, which is, in some sense, dual to Proposition 2.5.

Proposition 3.10. *Let $(X,0)$ be a δ_1-minimal surface parameterised by $f(u,v)=(u,\gamma_u(v))$, where $x=0$ is a generic plane. The following statements are equivalent:*

(1) $\kappa = m_0(X,0)-1$,
(2) *f is a $\mathscr{F}$-finite frontal,*
(3) *f is a frontal unfolding of γ_0 and for each $t\neq 0$, γ_t is $\mathscr{F}$-stable.*

Proof. Since $(X,0)$ is δ_1-minimal, the only singularities outside the origin are transverse double points and semicubic cuspidal edges. Moreover, for each t, the transverse slice Y_t is parameterised by $\gamma_t(v)=(p(t,v),q(t,v))$ and it has only cusps and nodes if $t\neq 0$. By 3.7 and 3.9, in order to show the equivalence between the three statements, we only need to show that $\kappa = m_0(X,0)-1$ if and only if f is a frontal.

Given $h\in\mathcal{O}_2$, we denote by $o_v(h)$ the order of h in v, that is, the order of $h(0,v)\in\mathcal{O}_1$. Assume that $o_v(p)=m$ and $o_v(q)=k$ with $m\le k$. Then, because of the genericity assumption, we have that $m_0(X,0)=m$.

For a fixed small enough $t\neq 0$, κ is equal to the number of solutions of $p_v(t,v)=q_v(t,v)=0$ in v. If $h=\gcd(p_v,q_v)$, then κ is less than or equal to the number of solutions of $h(t,v)=0$ in v. In particular,

$$\kappa\le o_v(h)\le o_v(p_v)=m-1=m_0(X,0)-1.$$

Thus, we have the following equivalences:

$$\kappa=m_0(X,0)-1 \iff o_v(h)=o_v(p_v)\iff p_v|q_v \iff \ f \text{ is a frontal.}$$

□

4. Local Euler obstruction

The local Euler obstruction was first introduced by McPherson [9] as an ingredient in the construction of characteristic classes of singular algebraic varieties. Here we prefer to use the approach of Lê-Teissier [7] in terms of polar multiplicities. Given an analytic set germ $(V,0)\subset(\mathbb{C}^n,0)$ of dimension d, its local Euler obstruction is computed as an alternate sum

$$\mathrm{Eu}(V,0)=\sum_{i=0}^{d-1}(-1)^i m_i(V,0),$$

where $m_i(V,0)$ denotes the ith-polar multiplicity (see [7] for definitions and details). In particular, for a surface $(X,0)$,

$$\mathrm{Eu}(X,0)=m_0(X,0)-m_1(X,0),$$

and hence, $\mathrm{Eu}(X,0)\le m_0(X,0)$.

In the next theorem, we compute the local Euler obstruction of a δ_1-minimal surface in terms of the number of transverse cusps κ. To do this, we first characterize the number ν of transverse nodes in terms of the number of vanishing cycles of the transverse slice Y_t.

Lemma 4.1. *Let $(X,0)$ be a δ_1-minimal surface. Then, for each $t \neq 0$ small enough, the Euler characteristic of Y_t is*

$$\chi(Y_t) = 1 - \nu.$$

Proof. Let us denote $\delta = \delta_1(X,0) = \delta(Y,0)$. Since $(X,0)$ is δ_1-minimal, we have seen in the proof of theorem 2.1 that $(Y,0)$ is irreducible and hence its Milnor number is $\mu(Y,0) = 2\delta$ (by Milnor's formula).

On the other hand, $\chi(Y_t)$ is related to the Milnor number by the following formula [3]:

$$\mu(Y,0) - \sum_{p\in S(Y_t)} \mu(Y_t,p) = \dim_{\mathbb{C}} H^1(Y_t;\mathbb{C}) = 1 - \chi(Y_t).$$

For each $t \neq 0$ small enough, the only singularities of Y_t are simple cusps, with Milnor number 2, and nodes, with Milnor number 1. Hence, we obtain

$$\mu(Y,0) - \sum_{p\in S(Y_t)} \mu(Y_t,p) = 2\delta - (2\kappa + \nu) = \nu.$$

□

Theorem 4.2. *Let $(X,0)$ be a δ_1-minimal surface. Then,*

$$\mathrm{Eu}(X,0) = 1 + \kappa.$$

In particular, $1 \leq \mathrm{Eu}(X,0) \leq m_0(X,0)$.

Proof. We use a formula of Brasselet-Lê-Seade [2] which is valid whenever $(X,0)$ is equidimensional and has 1-dimensional singular locus Σ. We take $t \neq 0$ small enough and assume that $Y_t \cap \Sigma = \{x_1,\dots,x_m\}$. Then,

$$\mathrm{Eu}(X,0) = \chi(Y_t) - m + \sum_{i=1}^{m} \mathrm{Eu}(X,x_i).$$

Note that $Y_t \cap \Sigma$ is the singular locus of Y_t and since each singular point has delta invariant 1, we have $m = \delta_1(X,0) = \kappa+\nu$. By lemma 4.1, $\chi(Y_t) = 1-\nu$. For each $i = 1,\dots,m$, $\mathrm{Eu}(X,x_i) = 2$ either if X is a semicubic cuspidal edge or a transverse double point at x_i. Thus,

$$\mathrm{Eu}(X,0) = 1 - \nu - (\kappa+\nu) + 2\kappa + 2\nu = 1 + \kappa.$$

□

As a consequence, we arrive to the following result which characterizes those surfaces that are stable unfoldings of plane curves or frontals.

Corollary 4.3. *Let $(X,0) \subset (\mathbb{C}^3,0)$ be an irreducible surface with singular locus of dimension 1. Then:*

(1) *$(X,0)$ is the image of a corank 1 $\mathscr{A}$-finite germ if and only if it is δ_1-minimal and $\mathrm{Eu}(X,0) = 1$.*
(2) *$(X,0)$ is the image of a corank 1 $\mathscr{F}$-finite front if and only if it is δ_1-minimal and $\mathrm{Eu}(X,0) = m_0(X,0)$.*

Proof. It follows directly from 2.1, 2.5, 3.10 and 4.2. □

We finish with a last result, where we consider irreducible surfaces with 1-dimensional locus in any ambient space and without any finiteness assumption. Given a space curve $(Y,0)\subset(\mathbb{C}^N,0)$, the *first polar multiplicity* was introduced by the author and Tomazella in [14]:

$$m_1(Y,0):=\mu(\ell|_{(Y,0)}),$$

where $\ell:\mathbb{C}^N\to\mathbb{C}$ is a generic linear form and $\mu(\ell|_{(Y,0)})$ is the Milnor number in the sense of Mond and van Straten [13]. Then, it is showed that

$$m_1(Y,0)=\mu(Y,0)+m_0(Y,0)-1, \tag{2}$$

where $\mu(Y,0)$ is now the Milnor number of a space curve as defined by Buchweitz and Greuel [3].

Proposition 4.4. *Let $(X,0)\subset(\mathbb{C}^{N+1},0)$ be a equidimensional surface with 1-dimensional singular set Σ. Then for $t\neq 0$,*

$$m_1(X,0)=m_1(Y,0)-\sum_{x\in S(Y_t)} m_1(Y_t,x),$$

where Y_t is the transverse slice of $(X,0)$.

Proof. This is a consequence again of the Brasselet-Lê-Seade formula together with (2):

$$\begin{aligned}
m_1(X,0)&=m_0(X,0)-\mathrm{Eu}(X,0)\\
&=m_0(X,0)-\chi(Y_t)+\sum_{x\in S(Y_t)}(\mathrm{Eu}(X,x)-1)\\
&=m_0(Y_0,0)-1+(1-\chi(Y_t))+\sum_{x\in S(Y_t)}(m_0(Y_t,x)-1)\\
&=m_0(Y_0,0)-1+\mu(Y_0,0)-\sum_{x\in S(Y_t)}(\mu(Y_t,x)-m_0(Y_t,x)+1)\\
&=m_1(Y_0,0)+\sum_{x\in S(Y_t)} m_1(Y_t,x).
\end{aligned}$$

□

Corollary 4.5. *With the above hypothesis, the following statements are equivalent:*

(1) $m_1(X,0)=0$.
(2) $(X,0)$ *defines a* m_1*-constant deformation of* $(Y,0)$.

Moreover, if $N=2$ and $(X,0)$ admits a parameterisation, then any of the two above statements is also equivalent to the following one:

(3) $(X,0)$ *is a frontal.*

Proof. The equivalence between he two first statements follows directly from 4.4. According to Lê-Teissier [7], the condition $m_1(X,0)=0$ is also equivalent to the fact that $(X,0)$ has a finite number of limiting tangent planes at the origin. But in the particular case that $(X,0)$ admits a parameterisation $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$, then this condition is equivalent to that $(X,0)$ is a frontal. □

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