TOPOLOGICAL $\mathcal{K}$-CLASSIFICATION OF FINITELY DETERMINED MAP GERMS

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Abstract. We consider smooth map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ which are finitely C^0-$\mathcal{K}$-determined and we look at the classifiaction under C^0-$\mathcal{K}$-equivalence. The main tool is homotopy type of the link, which is obtained by intersecting the image of f with a small enough sphere centered at the origin. When $f^{-1}(0) = \{0\}$, the link is a smooth map between spheres and f is C^0-$\mathcal{K}$-equivalent to the cone of its link. When $f^{-1}(0) \neq \{0\}$, we consider a link diagram, which contains also some extra information, but again f is C^0-$\mathcal{K}$-equivalent to the generalized cone. As a consequence, we deduce some known results due to Nishimura (for $n = p$) or the first named author (for $n < p$). We also deduce some other new results of the same nature.

1. Introduction

A central question in Singularity theory is the local classification of mappings up to diffeomorphisms. However, this is a difficult problem and it presents a lot of rigidity. Then it seems natural to investigate the classification of mappings given by equivalence relations in which the change of coordinates are weaker than diffeomorphisms. In this work we study the topological $\mathcal{K}$-equivalence (or C^0-$\mathcal{K}$-equivalence). This equivalence relation is the topological version of classical $\mathcal{K}$-equivalence (or contact equivalence) introduced by Mather [9].

Here we study the C^0-$\mathcal{K}$-equivalence for the class of smooth map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ which are finitely determined with respect to this equivalence. This subject was studied by Nishimura [12] when $n = p$ and by the first named author of this paper in [4] when $n < p$. In both cases, these previous works consider a fundamental assumption: that f has a conic structure over its link. This fact is based in a result due to Fukuda [5].

In this paper we adapt the Fukuda's construction in such way that now the link is not anymore required stable as appear in [5, 6]. But, it is well defined up to homotopy and its homotopy class is used to determine the C^0-$\mathcal{K}$-class of f. In order to show this fact we divided our work in two parts: the case $f^{-1}(0) = \{0\}$ and the case $f^{-1}(0) \neq \{0\}$. When $f^{-1}(0) = \{0\}$, the Theorem 3.8 shows that the C^0-$\mathcal{K}$-class of f is determined by homotopy class of its link. As a consequence, the results of Nishimura and Costa previously cited are deduced and we also deduce some new results of the same nature. When $f^{-1}(0) \neq \{0\}$, we introduce the concept of link diagram of f (Definition 4.7) whose homotopic type determines the C^0-$\mathcal{K}$-class of f (Theorem 4.9). We use this construction to deduce that if $p = n - 1$, then two map germs are C^0-$\mathcal{K}$-equivalent if and only if they curves $f^{-1}(0)$ and $g^{-1}(0)$ have the same number of half-branches.

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Some of the ideas used in this work are inspired in previous papers of the authors, also with other collaborators (cf. [1, 3, 8, 11]). The main strategy used here is based on the following: the topological structure of a germ f can be determined by the topological type of its associated link. If f has an isolated singularity, Milnor [10] showed that the link is a fibration. If f is finitely C^0-$\mathcal{A}$-determined Fukuda (see [5, 6]) showed that the link is well defined up to stable isotopy and its isotopy class determines the topological $\mathcal{A}$-class of f. Here we assume a more general situation, where f is finitely C^0-$\mathcal{K}$-determined. In this case we show that the link is now well defined up to homotopy and essentially its homotopic class determines the C^0-$\mathcal{K}$-class of f.

2. Preliminaries

Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two smooth (or C^∞) map germs. We say that:

- f and g are *$\mathcal{A}$-equivalent* if there exist diffeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that the following diagram commutes

$$\begin{tikzcd}
(\mathbb{R}^n,0) \arrow[r, "f"] \arrow[d, "h"'] & (\mathbb{R}^p,0) \arrow[d, "k"'] \\
(\mathbb{R}^n,0) \arrow[r, "g"] & (\mathbb{R}^p,0)
\end{tikzcd}$$

- f and g are *$\mathcal{K}$-equivalent* if there exist diffeomorphisms $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{tikzcd}
(\mathbb{R}^n,0) \arrow[r, "(\mathrm{id}, f)"] \arrow[d, "h"'] & (\mathbb{R}^n \times \mathbb{R}^p,0) \arrow[r, "\pi_n"] \arrow[d, "H"'] & (\mathbb{R}^n,0) \arrow[d, "h"'] \\
(\mathbb{R}^n,0) \arrow[r, "(\mathrm{id}, g)"] & (\mathbb{R}^n \times \mathbb{R}^p,0) \arrow[r, "\pi_n"] & (\mathbb{R}^n,0)
\end{tikzcd}$$

  where $\mathrm{id} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the identity mapping of $\mathbb{R}^n$ and $\pi_n : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n, 0)$ is the canonical projection germ.
- f and g are *$\mathcal{V}$-equivalent* if there exist a diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$.

We have the following relations between these three concepts:

$$\mathcal{A}\text{-equivalence} \Rightarrow \mathcal{K}\text{-equivalence} \Rightarrow \mathcal{V}\text{-equivalence}.$$

In the three previous definitions, if we have homeomorphisms instead of diffeomorphisms, we say that f and g are *C^0-$\mathcal{A}$-equivalent*, *C^0-$\mathcal{K}$-equivalent* or *C^0-$\mathcal{V}$-equivalent*, respectively. We also have the analogous relations for these topological equivalences:

$$C^0\text{-}\mathcal{A}\text{-equivalence} \Rightarrow C^0\text{-}\mathcal{K}\text{-equivalence} \Rightarrow C^0\text{-}\mathcal{V}\text{-equivalence}.$$

When we investigate classification problems, a key notion in Singularity theory is finite determinacy. If f is finitely determined then we can assume that f is polynomial.

Definition 2.1. Let $\mathcal{G}= C^0$-$\mathcal{A}$ or C^0-$\mathcal{K}$. We say that a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is *k-$\mathcal{G}$-determined* if for any map germ g with $j^k g(0) = j^k f(0)$, g is $\mathcal{G}$-equivalent to f. We say that f is *finitely $\mathcal{G}$-determined* if it is k-$\mathcal{G}$-determined for some k. Here $j^k f(0)$ denotes the k-jet of f at 0.

Notice that if f is finitely k-$\mathcal{G}$-determined then f is $\mathcal{G}$-equivalent to $j^k f(0)$, which is polynomial.

The main goal of this paper is to study the C^0-$\mathcal{K}$-equivalence for smooth map germs which are finitely C^0-$\mathcal{K}$-determined. With respect to this subject, the paper of Nishimura [12] was a pioneer work. Recently, new works also treat of this theme (cf. [1, 3, 4, 11, 15]). Among these recent results, we present below a theorem due to Ruas and Vallete [15] which appears throughout this paper:

Theorem 2.2. *Let $F : U \times [0,1] \to \mathbb{R}^p$ be a continuous family, where $U \subset \mathbb{R}^n$ is an open neighbourhood of 0. Assume that $F^{-1}(0)$ is locally topologically trivial. Then F is C^0-$\mathcal{K}$-trivial.*

We denote by $J^r(n,p)$ the r-jet space from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^p, 0)$. For positive integers r and s with $s \geq r$, let $\pi_r^s : J^s(n,p) \to J^r(n,p)$ be the canonical projection defined by $\pi_r^s(j^s f(0)) = j^r f(0)$.

For a positive number $\epsilon > 0$ we set

$$D_n^\epsilon = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \epsilon\} \text{ and } S_\epsilon^{p-1} = \{x \in \mathbb{R}^p \mid \|x\|^2 = \epsilon\}.$$

T. Fukuda has proved the following cone structure theorem in his papers [5, 6]:

Theorem 2.3. *For any semialgebraic subset W of $J^r(n,p)$, there exist an integer s $(s \geq r)$ depending only n, p and r, and there exists a closed semialgebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s f(0)$ belonging to $(\pi_r^s)^{-1}(W) \setminus \Sigma_W$ we have the following properties:*

(A) *($n \leq p$) there is $\epsilon_0 > 0$ such that for any number ϵ with $0 < \epsilon \leq \epsilon_0$ we have*

- (A-i) *the set $\tilde{S}_\epsilon^{n-1} = f^{-1}(S_\epsilon^{p-1})$ is a smooth submanifold without boundary, which is diffeomorphic to the standard unit sphere S^{n-1}.*
- (A-ii) *The restricted mapping $f|\tilde{S}_\epsilon^{n-1} : \tilde{S}_\epsilon^{n-1} \to S_\epsilon^{p-1}$ is topologically stable (C^∞ stable if (n,p) is a nice pair in Mather's sense).*
- (A-iii) *If $\tilde{D}_\epsilon^{n-1} = f^{-1}(D_\epsilon^{p-1})$, then the restricted mapping $f|\tilde{D}_\epsilon^{n-1} : \tilde{D}_\epsilon^{n-1} \to D_\epsilon^p$ is C^0-$\mathcal{A}$-equivalent to the cone of $f|\tilde{S}_\epsilon^{n-1}$.*

(B) *($n > p$) for any sufficiently small positive numbers ϵ and δ, the upper bound of ϵ depending of f and the upper bound of δ depending of ϵ and f, we have:*

- (B-i) *$f^{-1}(0) \cap S_\epsilon^{n-1}$ is an $(n-p-1)$-dimensional manifold and it is diffeomorphic to $f^{-1}(0) \cap S_{\epsilon_0}$.*
- (B-ii) *$D_\epsilon^n \cap f^{-1}(S_\delta^{p-1})$ is a smooth manifold, in general with boundary and it is diffeomorphic to $D_{\epsilon_0}^n \cap f^{-1}(S_{\delta_0}^{p-1})$.*
- (B-iii) *the restriction $f|D_\epsilon^n \cap f^{-1}(S_\delta^{p-1}) : D_\epsilon^n \cap f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ is a topologically stable map (stable if n and p are in the nice dimensions) and its C^0-$\mathcal{A}$-class is independent of ϵ and δ.*

As Fukuda pointed out in [6], if $f : \mathbb{R}^n \to \mathbb{R}^p$ with $n > p$ but $f^{-1}(0) = \{0\}$, then the conditions (A-i), (A-ii) and (A-iii) remaind valid to f.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a finitely C^0-$\mathcal{K}$-determined map germ. Our paper is divided in two main parts: the case when $f^{-1}(0) = \{0\}$ and the case $f^{-1}(0) \neq \{0\}$. In both cases, our motivation is to show as the homotopy class of the link associated to f will be used to determine the C^0-$\mathcal{K}$-orbit of f. In order to do this, we compare our strategy with the works of Milnor [10] and Fukuda [5, 6].

3. THE CASE $f^{-1}(0) = \{0\}$

If $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a polynomial map germ, the finite C^0-$\mathcal{K}$-determinacy condition means that $\Sigma(f) \cap f^{-1}(0) = \{0\}$ when $n > p$ or $f^{-1}(0) = \{0\}$ when $n \leq p$, where $\Sigma(f)$ denotes the singular set of f (see [16]). In this section we will restrict ourselves to map germs satisfying $f^{-1}(0) = \{0\}$ (which includes also map germs with $n > p$).

If f is finitely C^0-$\mathcal{K}$-determined, then we can approximate f by another map g satisfying properties (A-i), (A-ii) and (A-iii) of Theorem 2.3 and which is C^0-$\mathcal{K}$-equivalent to f. The problem is that the link is not well defined up to C^0-$\mathcal{A}$-equivalence.

Instead of C^0-$\mathcal{A}$-equivalence of the link, we try to consider C^0-$\mathcal{K}$-equivalence for the links, but this does not make sense. The alternative is to consider homotopy between the links.

Definition 3.1. Given $f : X \to Y$ and $g : X' \to Y'$ continuous maps, they are *homotopically $\mathcal{A}$-equivalent* if there are $\phi : X \to X'$ and $\psi : Y \to Y'$ homeomorphisms such that g is homotopic to $\psi \circ f \circ \phi^{-1}$.

Theorem 3.2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ such that $f^{-1}(0) = \{0\}$. Then there exists $\epsilon_0 > 0$ such that for any ϵ with $0 < \epsilon \leq \epsilon_0$ we have:*

1. *$\tilde{S}^{n-1}_\epsilon$ is diffeomorphic to S^{n-1},*
2. *$f|\tilde{S}^n_\epsilon : \tilde{S}^n_\epsilon \to S^p_\epsilon$ is a smooth map whose homotopy $\mathcal{A}$-equivalence type is independent of ϵ.*
3. *$f|\tilde{D}^n_\epsilon : \tilde{D}^n_\epsilon \to D^p_\epsilon$ is C^0-$\mathcal{K}$-equivalent to the cone of $f|\tilde{S}^{n-1}_\epsilon$.*

Proof. We choose an open neighbourhood $0 \in U \subset \mathbb{R}^n$ such that $f^{-1}(0) = \{0\}$ on U. We consider $g : \mathbb{R}^n \to \mathbb{R}$ given by $g = \|f\|^2$. By the Curve Selection Lemma, g has a finite number of critical values. There exists $\epsilon_0 > 0$ such that for all ϵ, $0 < \epsilon \leq \epsilon_0$, ϵ is a regular value of g. This implies that f is transverse to S^{p-1}_ϵ. Hence, $\tilde{S}^{n-1}_\epsilon = f^{-1}(S^{p-1}_\epsilon)$ is a closed hypersurface in U and $f|\tilde{S}^n_\epsilon$ is a smooth map.

To prove (1), we use Reeb Theorem [10]: since $f^{-1}(0) = \{0\}$ and 0 is an isolated minimum of g. Then, $\tilde{D}^n_\epsilon = g^{-1}([0, \epsilon])$ is homeomorphic to D^n. Therefore, $\tilde{S}^{n-1}_\epsilon = \partial \tilde{D}^n_\epsilon$ is homeomorphic (and thus diffeomorphic) to S^{n-1}.

Let us see (2). Take $\alpha > 0$ small enough such that $\epsilon < \epsilon + \alpha < \epsilon_0$ and denote $I = (0, \epsilon + \alpha)$. Consider the following diffeomorphisms:

$$\begin{aligned} \Phi : \tilde{D}^n_{\epsilon+\alpha} \setminus \{0\} &\longrightarrow I \times \tilde{S}^{n-1}_\epsilon, & \Psi : D^p_{\epsilon+\alpha} \setminus \{0\} &\longrightarrow I \times S^{p-1}_\epsilon, \\ x &\longmapsto (g(x), \phi(x)), & y &\longmapsto (\|y\|^2, \sqrt{\epsilon}\frac{y}{\|y\|}), \end{aligned}$$

where $\phi(x)$ is the point of $\tilde{S}^{n-1}_\epsilon$ where the integral curve of the gradient of g passing through x meets $\tilde{S}^{n-1}_\epsilon$.

Define $F : I \times \tilde{S}^{n-1}_\epsilon \to I \times S^{p-1}_\epsilon$ by $F = \Psi \circ f \circ \Phi^{-1}$. By construction, $F(\{t\} \times \tilde{S}^{n-1}_\epsilon) \subset \{t\} \times S^{p-1}_\epsilon$, for all $t \in I$. Hence, we can write F in the form $F(t, x) = (t, f_t(x))$, with $f_t : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ and $t \in I$. Since all the maps f_t are homotopic, we have (2).

To prove (3), we consider the following map

$$\begin{array}{rccc} \hat{H} : & (I \times \tilde{S}^{n-1}_\epsilon) \times [0,1] & \to & I \times S^{p-1}_\epsilon \\ & ((t, x), s) & \mapsto & (t, f_{st+(1-s)\epsilon}(x)), \end{array}$$

with $\hat{H}_0 = id \times f_\epsilon$ and $\hat{H}_1 = F$. By adding the origin, this induces a C^0-deformation in the cones

$$H : C(\tilde{S}^{n-1}_\epsilon) \times [0,1] \to C(S^{p-1}_\epsilon),$$

such that $H_s^{-1}(0) = \{0\}$, for all $s \in [0,1]$, where H_0 is the cone of f_ϵ and H_1 is a map C^0-$\mathcal{A}$-equivalent to f. By Theorem 2.2, H_0 and H_1 are C^0-$\mathcal{K}$-equivalent. Hence, f is also C^0-$\mathcal{K}$-equivalent to the cone of f_ϵ. □

Definition 3.3. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ such that $f^{-1}(0) = \{0\}$. We denote by $f|\tilde{S}^{n-1}_\epsilon : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ the *link* of f, where ϵ is given in Theorem 3.2. We have the following properties:

(i) The link is a smooth map between spheres,
(ii) it is well defined up to homotopy $\mathcal{A}$-equivalence,
(iii) f is C^0-$\mathcal{K}$-equivalent to the cone of its link.

Remark 3.4. When f is finitely C^0-$\mathcal{A}$-determined and $f^{-1}(0) = \{0\}$, by Theorem 2.3 the link $f|\tilde{S}^{n-1}_\epsilon : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ is a topologically stable map between spheres which is well defined up to C^0-$\mathcal{A}$-equivalence. This is a crucial difference with the finitely C^0-$\mathcal{K}$-determined case, since the link is not topologically stable and it is not well defined up to C^0-$\mathcal{A}$-equivalence. Moreover, if f is finitely C^0-$\mathcal{A}$-determined, then f is C^0-$\mathcal{A}$-equivalent to the cone of the link.

Notice that condition $f^{-1}(0) = \{0\}$ is always satisfied for finitely C^0-$\mathcal{A}$-determined map germs when $n \leq p$. In this paper we want to consider even a more general condition about f, which includes the case that f is finitely C^0-$\mathcal{K}$-determined.

Lemma 3.5. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ such that $f^{-1}(0) = \{0\}$. We take $U \subset \mathbb{R}^n$ an open neighbourhood of the origin such that $f^{-1}(0) = \{0\}$ on U. If $B \subset U$ is homeomorphic to D^n and $0 \in \operatorname{int}(B)$, the interior of B, then the map*

$$\gamma := \frac{f}{\parallel f \parallel} : \partial B \to S^{p-1}$$

is homotopically $\mathcal{A}$-equivalent to the link of f.

Proof. Let $\epsilon > 0$ such that $\tilde{D}^n_\epsilon \subset \operatorname{int}(B)$ and such that the restriction $f|\tilde{S}^{n-1}_\epsilon : \tilde{S}^{n-1}_\epsilon \to S^{p-1}_\epsilon$ is the link of f. By the annulus theorem, the set $B \setminus \operatorname{int}(\tilde{D}^n_\epsilon)$ is homeomorphic to the cylinder $S^{n-1} \times I$, $I = [0,1]$. We choose a homeomorphism

$$\Phi : B \setminus \operatorname{int}(\tilde{D}^n_\epsilon) \to S^{n-1} \times I$$

such that $\Phi(\tilde{S}^{n-1}_\epsilon) = S^{n-1} \times \{0\}$ and $\Phi(\partial B) = S^{n-1} \times \{1\}$. We denote the induced homeomorphisms by $\phi_0 : \tilde{S}^{n-1}_\epsilon \to S^{n-1}$ and $\phi_1 : \partial B \to S^{n-1}$.

Let $H : S^{n-1} \times I \to S^{p-1}$ given by the composition

$$S^{n-1} \times I \xrightarrow{\Phi^{-1}} B \setminus \operatorname{int}(\tilde{D}^n_\epsilon) \subset U - \{0\} \xrightarrow{\gamma} S^{p-1},$$

i.e.,

$$H(x,t) = \frac{f(\Phi^{-1}(x,t))}{\|f(\Phi^{-1}(x,t))\|}.$$

By construction, H_0 is C^0-$\mathcal{A}$-equivalent to $f|\tilde{S}^{n-1}_\epsilon$ and H_1 is C^0-$\mathcal{A}$-equivalent to γ. Hence, γ is homotopically $\mathcal{A}$-equivalent to $f|\tilde{S}^{n-1}_\epsilon$.

□

In particular, if $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a polynomial map germ such that $f^{-1}(0) = \{0\}$, then for any $\epsilon > 0$ small enough, the map

$$\gamma := \frac{f}{\parallel f \parallel} : S^{n-1}_\epsilon \to S^{p-1}$$

is homotopically $\mathcal{A}$-equivalent to the link of f.

Lemma 3.6. *Let* $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a continuous map germ such that* $h^{-1}(0) = \{0\}$ *and* $\deg(h) = 1$. *Then there is* $U \subset \mathbb{R}^n$ *open neighbourhood of the origin such that* $h : U \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ *is homotopic to the identity.*

Proof. We take a representative $h : V \to \mathbb{R}^n$ where $V \subset \mathbb{R}^n$ is an open neighbourhood of the origin such that $h^{-1}(0) = \{0\}$ on V. We also fix $\epsilon > 0$ such that $D^n_\epsilon \subset V$ and consider $h : D^n_\epsilon \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$. We divide the proof into three steps:

(1) There is a continuous map $h_1 : D^n_\epsilon \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ homotopic to h, such that $h_1(D^n_\epsilon \setminus \{0\}) \subset D^n_\epsilon \setminus \{0\}$ and $h_1(S^{n-1}_\epsilon) \subset S^{n-1}_\epsilon$.

In fact, h_1 is the map defined by

$$h_1(x) = \frac{\|x\| h(x)}{\|h(x)\|},$$

and the homotopy $H : (D^n_\epsilon \setminus \{0\}) \times I \to \mathbb{R}^n \setminus \{0\}$ is given by

$$H(x,t) = (1-t)h(x) + t h_1(x).$$

Note that H is well defined since for any $x \neq 0$, $h_1(x) = \lambda(x)h(x)$, with $\lambda(x) > 0$.

(2) The map h_1 is homotopic to the continuous map $h_2 : D^n_\epsilon \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by

$$h_2(x) = \frac{1}{\sqrt{\epsilon}} \|x\| h_1 \left(\frac{x}{\|x\|} \sqrt{\epsilon} \right).$$

The homotopy $G : (D^n_\epsilon \setminus \{0\}) \times I \to \mathbb{R}^n \setminus \{0\}$ between h_1 and h_2 is defined by

$$G(x,t) = \begin{cases} \frac{1}{\sqrt{\epsilon t}} \|x\| h_1 \left(\frac{x}{\|x\|} \sqrt{\epsilon t} \right), & \text{if } \|x\| \leq \sqrt{\epsilon t}, \\ h_1(x), & \text{if } \|x\| \geq \sqrt{\epsilon t}. \end{cases}$$

(3) Finally, the map h_2 is homotopic to id.

In fact, since $\deg(h) = 1$, the restriction $h_2|S^{n-1}_\epsilon : S^{n-1}_\epsilon \to S^{n-1}_\epsilon$ has degree 1. By the Brouwer Degree Theorem, it is homotopic to the identity, that is, there is a homotopy $F : S^{n-1}_\epsilon \times I \to S^{n-1}_\epsilon$ between $h_2|S^{n-1}_\epsilon$ and id. We can extend this to a homotopy between h_2 and id, $\tilde{F} : (D^n_\epsilon \setminus \{0\}) \times I \to \mathbb{R}^n \setminus \{0\}$ in the obvious way:

$$\tilde{F}(x,t) = \frac{1}{\sqrt{\epsilon}} \|x\| F \left(\frac{x}{\|x\|} \sqrt{\epsilon}, t \right).$$

□

Lemma 3.7. *If* $\gamma_0, \gamma_1 : S^{n-1} \to S^{p-1}$ *are homotopically* $\mathcal{A}$*-equivalent, then their cones are* C^0*-*$\mathcal{K}$*-equivalent.*

Proof. By hypothesis, there exist homeomorphisms $\phi : S^{n-1} \to S^{n-1}$ and $\psi : S^{p-1} \to S^{p-1}$ such that $\psi \circ \gamma_0 \circ \phi$ is homotopic to γ_1. Let $H : S^{n-1} \times [0,1] \to S^{p-1}$ such homotopy. That is, $H_0 = \psi \circ \gamma_0 \circ \phi$ and $H_1 = \gamma_1$.

The homotopy H_t induces a natural C^0-deformation in the cones of γ_i given by

$$c(H_t) : c(S^{n-1}) \times [0,1] \to c(S^{p-1}),$$

such that $c(H_t)(x,s) = (H_t(x), s)$, $c(H_0) = c(\psi \circ \gamma_0 \circ \phi)$ and $c(H_1) = c(\gamma_1)$.

Also, $c(H_t)_s^{-1}(0) = \{0\}$, for all $s \in [0,1]$. Then, by Theorem 2.2, $c(\psi \circ \gamma_0 \circ \phi)$ and $c(\gamma_1)$ are C^0-$\mathcal{K}$-equivalent. But $c(\psi \circ \gamma_0 \circ \phi)$ is C^0-$\mathcal{A}$-equivalent to $c(\gamma_0)$ since $\psi \circ \gamma_0 \circ \phi$ is C^0-$\mathcal{A}$-equivalent to γ_0.

Hence, the cones $c(\gamma_0)$ and $c(\gamma_1)$ are C^0-$\mathcal{K}$-equivalent.

□

Theorem 3.8. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be polynomial map germs such that $f^{-1}(0) = g^{-1}(0) = \{0\}$. Then f and g are C^0-$\mathcal{K}$-equivalent if and only of their links are homotopically $\mathcal{A}$-equivalent.*

Proof. Assume that f, g are C^0-$\mathcal{K}$-equivalent, we have:

$$H_x(f(x)) = g(h(x)), \quad \forall x \in \mathbb{R}^n,$$

where $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a homeomorphism and $H_x : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ is a continuous family of homeomorphisms.

We can assume without loss of generality that both h and H_0 are orientation preserving homeomorphisms. If not, we could change g by $\overline{g}$ or f by $\hat{f}$ respectively, where

$$\hat{f}(x_1, \ldots, x_n) = f(-x_1, x_2, \ldots, x_n), \quad \overline{g} = (-g_1, g_2, \ldots, g_p).$$

It is obvious that the links of f and $\hat{f}$ are homotopically $\mathcal{A}$-equivalent and the same is true for g and $\overline{g}$.

Then, h^{-1} and H_0^{-1} have local degree 1 and by Lemma 3.6, there are U, V open neighbourhoods of the origin in $\mathbb{R}^n, \mathbb{R}^p$ respectively and homotopies:

(1) $\phi : (U \setminus \{0\}) \times I \to \mathbb{R}^n \setminus \{0\}$ such that $\phi_0 = h^{-1}$ and $\phi_1 = \mathrm{id}$;
(2) $\psi : (V \setminus \{0\}) \times I \to \mathbb{R}^p \setminus \{0\}$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 = H_0^{-1}$.

Let $W \subset \mathbb{R}^n$ be an open neighbourhood of the origin such that we can define a homotopy $F : (W \setminus \{0\}) \times I \to \mathbb{R}^p \setminus \{0\}$ given by

$$F(x,t) = \psi_t(H_{(1-t)h^{-1}(x)}(f(\phi_t(x)))).$$

We have $F(x,t) \neq 0$ if $x \neq 0$, hence it is well defined, it is continuous and gives and homotopy between $F_0 = g$ and $F_1 = f$.

We take $\epsilon > 0$ such that $D_\epsilon^n \subset W$. We consider the continuous map

$$\frac{F}{\|F\|} : S_\epsilon^{n-1} \times I \to S^{p-1},$$

which defines a homotopy between the links of f and g by Lemma 3.5.

The converse follows directly from Lemma 3.7. If $f|\tilde{S}_\epsilon^{n-1}$ and $g|\tilde{S}_\epsilon^{n-1}$ are homotopically $\mathcal{A}$-equivalent, then their cones are C^0-$\mathcal{K}$-equivalent. By Theorem 3.2, f and g are C^0-$\mathcal{K}$-equivalent.

□

If $n = p$, the link of f is a map $\gamma : S^{n-1} \to S^{n-1}$. By the Brouwer Degree Theorem, two such maps γ and δ are homotopic if and only if $\deg(f) = \deg(g)$, where deg means the topological degree. Hence, they are homotopically $\mathcal{A}$-equivalent if and only if $|\deg(f)| = |\deg(g)|$. As a consequence we get another proof of Nishimura's result contained in [12]:

Corollary 3.9. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be finitely C^0-$\mathcal{K}$determined map germs. Then f and g are C^0-$\mathcal{K}$-equivalent if and only if $|\deg(f)| = |\deg(g)|$.*

If $n < p$, then two maps $\gamma, \delta : S^{n-1} \to S^{p-1}$ are always homotopic, since the homotopy group $\pi_{n-1}(S^{p-1})$ is trivial. Hence, we also recover the following result of Costa [4]:

Corollary 3.10. *Let* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *be finitely* C^0*-*$\mathcal{K}$*-determined map germs such that* $n < p$*. Then* f *and* g *are* C^0*-*$\mathcal{K}$*-equivalent.*

Notice that the hypothesis of finite C^0-$\mathcal{K}$-determinacy in both above corollaries implies that $f^{-1}(0) = \{0\}$ and we can apply our results.

If $p = 1$, given functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $f^{-1}(0) = g^{-1}(0) = \{0\}$, then it is obvious that they are C^0-$\mathcal{K}$-equivalent.

For $p = 2$ and $n > 2$, the link is a map $\gamma : S^{n-1} \to S^1$ and two such maps are always homotopic, since the homotopy group $\pi_{n-2}(S^1)$ is again trivial. Thus, we have:

Corollary 3.11. *Two map germs* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ *such that* $f^{-1}(0) = g^{-1}(0) = \{0\}$ *are always* C^0*-*$\mathcal{K}$*-equivalent.*

For $p = 3$ and $n > 3$, the link is a map $\gamma : S^{n-1} \to S^2$ and now the homotopy group $\pi_{n-2}(S^2)$ is not trivial. If $n = 4$ we have, for instance, the Hopf fibration $S^3 \to S^2$ which is a generator of $\pi_3(S^2) = \mathbb{Z}$. Two maps $\gamma, \delta : S^3 \to S^2$ are homotopically $\mathcal{A}$-equivalent if and only if $|h(\gamma)| = |h(\delta)|$, where $h(\cdot)$ denotes the Hopf invariant (see [13]).

Corollary 3.12. *Two map germs* $f, g : (\mathbb{R}^4, 0) \to (\mathbb{R}^3, 0)$ *such that* $f^{-1}(0) = g^{-1}(0) = \{0\}$ *are* C^0*-*$\mathcal{K}$*-equivalent if and only if* $|h(f)| = |h(g)|$*, where now* $h(\cdot)$ *denotes the corresponding local Hopf invariant for map germs.*

The Hopf fibration (with Hopf invariant 1) is the link of the map germ $f : (\mathbb{R}^4, 0) \to (\mathbb{R}^3, 0)$ given in complex coordinates by

$$f(u, v) = (2uv; |u|^2 - |v|^2), \quad (u, v) \in \mathbb{C}^2.$$

4. THE CASE $f^{-1}(0) \neq \{0\}$

Milnor [10] introduced in 1968 a real fibration for a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with isolated critical point ($n \geq p \geq 2$) (see fig. 1).

Theorem 4.1. *For any* $\epsilon \gg \delta > 0$ *small enough, the set* $N_{\epsilon,\delta} := D_n^\epsilon \cap f^{-1}(S_\delta^{p-1})$ *is a* C^∞*-manifold with boundary and the restriction*

$$f|N_{\epsilon,\delta} : N_{\epsilon,\delta} \to S_\delta^{p-1}$$

is a C^∞*-fibre bundle.*

The manifold $N_{\epsilon,\delta}$ is usually called *Milnor tube* for f. Notice that both the Milnor tube $N_{\epsilon,\delta}$ and the Milnor fibration $f|N_{\epsilon,\delta}$ are independent of ϵ and δ up to diffeomorphisms. However, the condition of isolated critical point is very strong.

Later, Fukuda (see [5, 6]) considered a generic condition which includes the case that f is finitely C^0-$\mathcal{A}$-determined. In [6] Fukuda extends the conic structure theorem of his previous paper [5] to a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $n > p$. However an essential difference when $n > p$ and $f^{-1}(0) \neq \{0\}$ is that the Milnor tube $N_{\epsilon,\delta}$ is not diffeomorphic to a sphere. Notice also that the restriction $f|N_{\epsilon,\delta}$ is not a fibration anymore as in Milnor's work. But it is a topologically $\mathcal{A}$-stable (or $\mathcal{A}$-stable if (n, p) are in the nice dimensions) and is independent of ϵ and δ up to C^0-$\mathcal{A}$-equivalence. Moreover, by Fukuda we also have that the topological type of $f|N_{\epsilon,\delta}$ determines the topological type of the germ of f at the origin of $\mathbb{R}^n$.

Now, if f is finitely C^0-$\mathcal{K}$-determined, we have the following theorem:

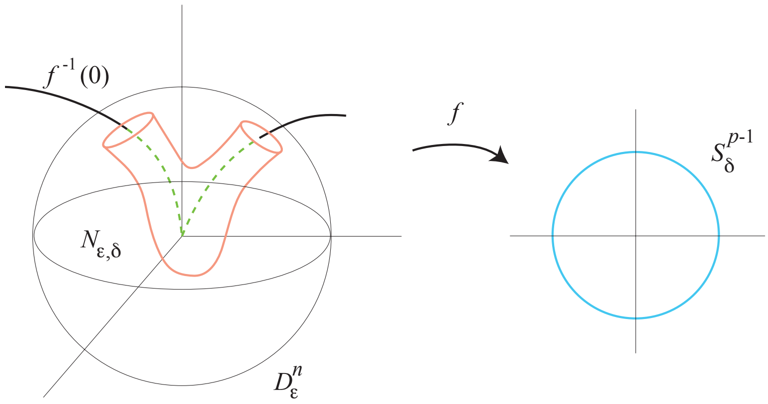

FIGURE 1. The map $f|N_{\epsilon,\delta}$.

Theorem 4.2. *Let* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *be polynomial such that* $f^{-1}(0) \neq \{0\}$ *and* $\Sigma(f) \cap f^{-1}(0) = \{0\}$. *There are sufficiently small positive numbers* ϵ_0 *and* δ, *with* $0 < \epsilon \leq \epsilon_0$ *and* $0 < \delta < \delta(\epsilon)$ *such that the following properties hold:*

(i) $N_{\epsilon,\delta} := D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ *is a smooth manifold with boundary diffeomorphic to* $N_{\epsilon_0,\delta(\epsilon_0)}$.

(ii) *The restriction* $f|N_{\epsilon,\delta} : N_{\epsilon,\delta} \to S^{p-1}_\delta$ *is a smooth map and its homotopy* $\mathcal{A}$-*equivalence type is independent of* ϵ *and* δ.

The proof of property (i) is equal to that given in Theorem 2.3. The proof of (ii) is analogous to the proof of item (2) of Theorem 3.2 and then we will not give it here.

Let X, Y be topological spaces and $A \subset X$. Given a continuous function $f : A \to Y$ we denote the attachment by

$$X \cup_f Y := \frac{X \sqcup Y}{x \sim f(x)\ :\ x \in A}$$

where $\sqcup$ means disjoint union and $\sim$ indicates that all points of A are identified with its images.

Definition 4.3. A *link diagram* is a diagram of the form

$$V \xleftarrow{r} N \xrightarrow{\gamma} S^{p-1},$$

where N is a manifold with boundary, γ is a continuous map, $V \subset \mathbb{R}^n$ and r is a continuous surjective map such that the attachment $(N \times I) \cup_r V$ is homeomorphic to the closed disk D^n (here we identify $N \equiv N \times \{0\} \subset N \times I$).

Definition 4.4. Given a link diagram $V \xleftarrow{r} N \xrightarrow{\gamma} S^{p-1}$, the *generalized cone of a link diagram* is the induced map

$$C(\gamma, r) : (N \times I) \cup_r V \to c(S^{p-1}),$$

defined in the obvious way (that is, $[x, t] \mapsto [\gamma(x), t]$ if $(x, t) \in N \times I$ and $[y] \mapsto 0$ if $y \in V$).

Notice that here we are using the small letter c to the usual notion of cone and the capital letter C to indicates the generalized cone. When $f^{-1}(0) = \{0\}$ the notion of generalized cone of a link diagram is essentially the usual cone notion.

Definition 4.5. We say that two link diagrams

$$V_0 \xleftarrow{r_0} N_0 \xrightarrow{\gamma_0} S^{p-1}, \quad V_1 \xleftarrow{r_1} N_1 \xrightarrow{\gamma_1} S^{p-1}$$

are *homotopically $\mathcal{A}$-equivalent* if there are homeomorphisms $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and γ_1 is homotopic to $\psi \circ \gamma_0 \circ \phi^{-1}$.

Theorem 4.6. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be polynomial such that $f^{-1}(0) \neq \{0\}$ and $\Sigma(f) \cap f^{-1}(0) = \{0\}$. For each $\epsilon \gg \delta > 0$ small enough, there is a continuous surjective map $r : N_{\epsilon,\delta} \to V_\epsilon$, where $V_\epsilon = D^n_\epsilon \cap f^{-1}(0)$, with the following properties:*

1. *The link diagram*
$$V_\epsilon \xleftarrow{r} N_{\epsilon,\delta} \xrightarrow{f|N_{\epsilon,\delta}} S^{p-1}_\delta,$$
is independent of ϵ, δ, up to homotopy $\mathcal{A}$-equivalence.
2. *$f|D^n_\epsilon \cap f^{-1}(D^p_\delta)$ is C^0-$\mathcal{K}$-equivalent to the generalized cone:*
$$C(f|N_{\epsilon,\delta}, r) : (N_{\epsilon,\delta} \times I) \cup_r V_\epsilon \to c(S^{p-1}_\delta),$$
where $I = [0, \delta]$.

Proof. We denote

$$U = D^n_\epsilon \cap f^{-1}(D^p_\delta), \quad \dot{U} = U - f^{-1}(0) \quad \text{and} \quad \dot{f} : \dot{U} \to D^p_\delta - \{0\}.$$

We consider any smooth vector field $v(x)$ defined on $D^n_\epsilon \setminus V_\epsilon$ and such that

$$\langle v(x), \nabla \|f(x)\|^2 \rangle > 0, \quad \forall x \in D^n_\epsilon \setminus V_\epsilon.$$

Let $r : N_{\epsilon,\delta} \to V_\epsilon$ be the map such that $r(x)$ is the point of $f^{-1}(0)$ where the integral curve of $v(x)$ passing through x meets $f^{-1}(0)$ (see fig. 2).

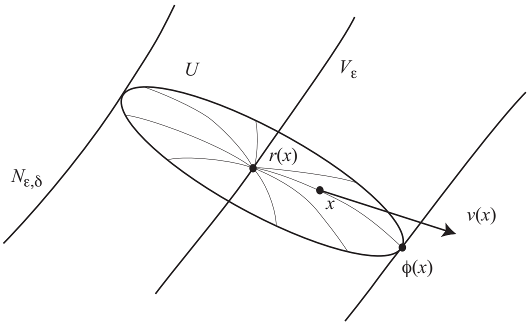

FIGURE 2. The maps r and ϕ.

By construction, the function r is continuous and surjective. Moreover, part (1) follows easily from Theorem 4.2 and the definition of r. Observe that the set $(N_{\epsilon,\delta} \times I) \cup_r V_\epsilon$ is homeomorphic to U and $c(S^{p-1}_\delta)$ is hoemomorphic to D^p_δ.

Consider the following homeomorphisms:

$$\Phi : \dot{U} \longrightarrow N_{\epsilon,\delta} \times I, \qquad \Psi : D^p_\delta \setminus \{0\} \longrightarrow S^{p-1}_\delta \times I,$$

$$x \longmapsto (\phi(x), \|f(x)\|^2), \qquad y \longmapsto (\sqrt{\delta}\frac{y}{\|y\|}, \|y\|^2).$$

where $\phi(x)$ is the point of $N_{\epsilon,\delta}$ where the integral curve of $v(x)$ passing through x meets $N_{\epsilon,\delta}$ (see fig. 2).

Define $\dot{F} : N_{\epsilon,\delta} \times I \to S^{p-1}_{\delta} \times I$ by $\dot{F} = \Psi \circ \dot{f} \circ \Phi^{-1}$. By construction, $\dot{F}(\{t\} \times N_{\epsilon,\delta}) \subset \{t\} \times S^{p-1}_{\delta}$, for all $t \in I$. Hence, we can write $\dot{F}$ in the form $\dot{F}(t,x) = (t, \dot{f}_t(x))$, with $\dot{f}_t : N_{\epsilon,\delta} \to S^{p-1}_{\delta}$ and $t \in I$.

We consider the following map

$$\begin{array}{rccc} \hat{H}: & (N_{\epsilon,\delta} \times I) \times [0,1] & \longrightarrow & S^{p-1}_{\epsilon} \times I \\ & ((x,t),s) & \longmapsto & (\dot{f}_{st+(1-s)\epsilon}(x), t), \end{array}$$

with $\hat{H}_0 = \dot{f}_{\epsilon} \times id$ and $\hat{H}_1 = \dot{F}$. By adding the set $f^{-1}(0)$ in $\dot{U}$, we obtain U and we can extend the homeomorphisms Φ and Ψ in a natural way. In other words, we can define $F : (N_{\epsilon,\delta} \times I) \cup_r V_{\epsilon} \to c(S^{p-1}_{\delta})$ by $F = \tilde{\Psi} \circ f \circ \tilde{\Phi}^{-1}$, where $\tilde{\Phi}$ and $\tilde{\Psi}$ are the respective extensions of Φ and Ψ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\tilde{\Phi}"'] & D^p_{\delta} \arrow[d, "\tilde{\Psi}"'] \\ (N_{\epsilon,\delta} \times I) \cup_r V_{\epsilon} \arrow[r, "F"] & c(S^{p-1}_{\delta}). \end{tikzcd}$$

On the other hand, by adding $f^{-1}(0)$ in the homotopy $\hat{H}$ this induces a C^0-deformation

$$H : (N_{\epsilon,\delta} \times I) \cup_r V_{\epsilon} \times [0,1] \to c(S^{p-1}_{\epsilon}),$$

such that $H_0 = C(f|N_{\epsilon,\delta}, r)$, $H_1 = F$ and $H_t^{-1}(0) = V_{\epsilon}$, for all $t \in [0,1]$.

By the Theorem 2.2, H_0 and H_1 are C^0-$\mathcal{K}$-equivalent. Since $H_1 = F$ is C^0-$\mathcal{A}$-equivalent to f, we conclude that f is also C^0-$\mathcal{K}$-equivalent to $C(f|N_{\epsilon,\delta}, r)$.

□

Definition 4.7. Given $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ polynomial such that $f^{-1}(0) \neq \{0\}$ and $\Sigma(f) \cap f^{-1}(0) = \{0\}$, we call the *link diagram of* f to the link diagram:

$$V_{\epsilon} \xleftarrow{\ r\ } N_{\epsilon,\delta} \xrightarrow{f|N_{\epsilon,\delta}} S^{p-1}_{\delta},$$

constructed in Theorem 4.6 for $\epsilon \gg \delta > 0$ small enough.

Lemma 4.8. *If two link diagrams are homotopically $\mathcal{A}$-equivalent, then their generalized cones are C^0-$\mathcal{K}$-equivalent.*

Proof. Suppose that the two link diagrams

$$V_0 \xleftarrow{\ r_0\ } N_0 \xrightarrow{\ \gamma_0\ } S^{p-1}, \quad V_1 \xleftarrow{\ r_1\ } N_1 \xrightarrow{\ \gamma_1\ } S^{p-1}$$

are homotopically $\mathcal{A}$-equivalent. Then there are homeomorphisms $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and there is a homotopy $H : N_1 \times I \to V_1$ between γ_1 and $\bar{\gamma}_0 = \psi \circ \gamma_0 \circ \phi^{-1}$.

On one hand, we have an induced C^0-$\mathcal{A}$-equivalence between the generalized cones $C(\gamma_0, r_0)$ and $C(\bar{\gamma}_0, r_1)$:

$$\begin{tikzcd} (N_0 \times I) \cup_{r_0} V_0 \arrow[r, "C(\gamma_0,r_0)"] \arrow[d, "\tilde{\Phi}"'] & c(S^{p-1}) \arrow[d, "c(\psi)"'] & \\ & & . \\ (N_1 \times I) \cup_{r_1} V_1 \arrow[r, "C(\bar{\gamma}_0,r_1)"] & c(S^{p-1}) & \end{tikzcd}$$

Here, $\tilde{\Phi}$ is the induced homeomorphism from α, ϕ in the obvious way (that is, $\tilde{\Phi}([x,t]) = [\phi(x), t]$ if $x \in N_0$ and $\tilde{\Phi}([y]) = [\alpha(y)]$ if $y \in V_0$) and $c(\psi)$ is the usual cone mapping induced by ψ.

On the other hand, we have an induced homotopy:

$$\tilde{H} : ((N_1 \times I) \cup_{r_1} V_1) \times I \to c(S^{p-1}).$$

This is defined in such a way that for each $t \in I$, $\tilde{H}_t = C(H_t, r_1)$. Hence, $\tilde{H}$ is a homotopy between $C(\bar{\gamma}_0, r_1)$ and $C(\gamma_1, r_1)$. Since $\tilde{H}_t^{-1}(0) = V_1$, it follows from Theorem 2.2 that $\tilde{H}$ is C^0-$\mathcal{K}$-trivial.

□

The following theorem follows directly from Theorem 4.6 and Lemma 4.8.

Theorem 4.9. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be polynomial map germs such that $\Sigma(f) \cap f^{-1}(0) = \Sigma(g) \cap g^{-1}(0) = \{0\}$. If the link diagrams of f and g are homotopically $\mathcal{A}$-equivalent then f and g are C^0-$\mathcal{K}$-equivalent.*

In definition of C^0-$\mathcal{K}$-equivalence, when the homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the identity, we denote the C^0-$\mathcal{K}$-equivalence by C^0-$\mathcal{C}$-equivalence. I.e., two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be *C^0-$\mathcal{C}$-equivalent* if there exist a homeomorphism $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ such that

$$H(x, y) = (x, \theta(x, y)), \quad \theta(x, 0) = 0 \text{ and } H(x, f(x)) = (x, g(x)),$$

$(x, y) \in \mathbb{R}^n \times \mathbb{R}^p$.

Theorem 4.10. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be polynomial map germs such that $\Sigma(f) \cap f^{-1}(0) = \Sigma(g) \cap g^{-1}(0) = \{0\}$. If f and g are C^0-$\mathcal{C}$-equivalent then their link diagrams are homotopically $\mathcal{A}$-equivalent.*

Proof. If f and g are C^0-$\mathcal{C}$-equivalent then there exist a family of homeomorphism $H_x : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ $(x \in \mathbb{R}^n)$ such that

$$H_x(f(x)) = g(x).$$

As in the proof of Theorem 3.8, we can assume without loss of generality that H_0 is orientation preserving homeomorphism. Then, H_0^{-1} has local degree 1 and by Lemma 3.6, there is U an open neighbourhood of the origin in $\mathbb{R}^p$ and a homotopy $\psi : (U \setminus \{0\}) \times I \to \mathbb{R}^p \setminus \{0\}$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 = H_0^{-1}$.

Since f, g are C^0-$\mathcal{C}$-equivalent, we have $f^{-1}(0) = g^{-1}(0)$. We choose $\epsilon > 0$ small enough and denote

$$V_\epsilon := f^{-1}(0) \cap D_\epsilon^n = g^{-1}(0) \cap D_\epsilon^n.$$

We take $v(x)$ a smooth vector field on $D_\epsilon^n \setminus V_\epsilon$ such that

$$\langle v(x), \nabla \|f(x)\|^2 \rangle > 0, \quad \langle v(x), \nabla \|g(x)\|^2 \rangle > 0, \quad \forall x \in D_\epsilon^n \setminus V_\epsilon.$$

It is possible to choose such a vector field since $\nabla\|f(x)\|^2$ and $\nabla\|g(x)\|^2$ are non-zero through $D_\epsilon^n \setminus V_\epsilon$ and cannot point in opposite directions by [10, 3.4].

Given $\delta, \delta' > 0$ we also denote

$$N_0 := D_\epsilon^n \cap f^{-1}(S_\delta^{p-1}), \; N_1 := D_\epsilon^n \cap g^{-1}(S_{\delta'}^{p-1}).$$

By shrinking ϵ if necessary, we can choose $\delta, \delta' > 0$ small enough such that the link diagrams of f, g are respectively

$$V_\epsilon \xleftarrow{r_0} N_0 \xrightarrow{f|N_0} S_\delta^{p-1}, \quad V_\epsilon \xleftarrow{r_1} N_1 \xrightarrow{g|N_1} S_{\delta'}^{p-1},$$

where r_0, r_1 are continuous surjective maps constructed as in the proof of Theorem 4.6 by means of the flow of $v(x)$.

Moreover, we also assume that $N_1 \subset \operatorname{int}(N_0)$. Then, we can use again the flow of $v(x)$ and define a continuous map $\phi : N_1 \times [0,1] \to D^n_\epsilon$ such that

(1) $\phi_0(N_1) = N_1$ and the restriction $\phi_0 : N_1 \to N_1$ is the identity map.
(2) $\phi_1(N_1) = N_0$ and the restriction $\phi_1 : N_1 \to N_0$ is a homeomorphism with the property that $r_0 \circ \phi_1 = r_1$.

Finally, we denote by $\omega : \mathbb{R}^p \setminus \{0\} \to S^{p-1}_{\delta'}$ the continuous map defined by

$$\omega(y) = \delta' \frac{y}{\|y\|}.$$

With all these ingredients we construct a homotopy $F : N_1 \times [0,1] \to S^{p-1}_{\delta'}$ in the following way:

$$F(x,t) = \omega \circ \psi_t \circ H_{(1-t)\phi_t(x)} \circ f \circ \phi_t(x).$$

For $t = 0$, we have $F_0(x) = H_x(f(x)) = g(x)$, while for $t = 1$ we have

$$F_1(x) = \omega \circ H_0^{-1} \circ H_0 \circ f \circ \phi_1(x) = \omega \circ f \circ \phi_1(x).$$

Thus, we have that the link diagrams of f, g are homotopically $\mathcal{A}$-equivalent.

□

For a finitely C^0-$\mathcal{K}$-determined smooth function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, the link is a map $\gamma : N_{\epsilon,\delta} \to S^0 = \{-1, 1\}$. Moreover, it is constant on each connected component of $N_{\epsilon,\delta}$ and two adjacent connected component must have alternate signs. Then as a consequence of Theorem 4.9 we have:

Corollary 4.11. *(*[12] *or* [1]*) Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be finitely C^0-$\mathcal{K}$-determined such that $f^{-1}(0) = g^{-1}(0) \neq \{0\}$. Then f and g are C^0-$\mathcal{K}$-equivalent.*

It was conjectured by R. Thom that C^0-$\mathcal{V}$-equivalence of functions implies the C^0-$\mathcal{A}$-equivalence, but H. King [7] obtained counterexamples of this conjecture. However when $n = 2$, for the class of analytic finitely determined map germs, the notions of C^0-$\mathcal{A}$, C^0-$\mathcal{K}$ and C^0-$\mathcal{V}$ equivalences are all equivalents. This fact can be deduced from the works [1, 12, 14].

To finish this section, we apply the Theorem 4.9 to compare the C^0-$\mathcal{V}$ and C^0-$\mathcal{K}$ equivalences when the target dimension is $p = n - 1$.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n-1}, 0)$ be a polynomial map germ such that $f^{-1}(0) \neq \{0\}$ and $\Sigma(f) \cap f^{-1}(0) = \{0\}$. In this case, if $\epsilon > 0$ is small enough, the set $D^n_\epsilon \cap f^{-1}(0) - \{0\}$ is a union of smooth curves, which we call *half-branches* of $f^{-1}(0)$ at 0. Then, the set $K_\epsilon := S^{n-1}_\epsilon \cap f^{-1}(0)$ is a finite number of points $\{x_1, \dots, x_{2r}\}$, one for each half-branch of $f^{-1}(0)$. The Milnor tube $N_{\epsilon,\delta}$ is diffeomorphic to $S^{n-1} - (e_1 \sqcup \cdots \sqcup e_{2r})$, where each e_i is an open disk. It has the homotopy type of a bouquet of $2r - 1$ spheres of dimension $n - 2$.

Theorem 4.12. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n-1}, 0)$ be finitely C^0-$\mathcal{K}$-determined map germs. Then f and g are C^0-$\mathcal{K}$-equivalent if and only if the zero-sets of f and g have the same number of half-branches (provided this number is $\neq 0$).*

Proof. It is obvious that if f, g are C^0-$\mathcal{K}$-equivalent, then the zero sets have the same number of half-branches. To see the converse, we assume that $f^{-1}(0)$ and $g^{-1}(0)$ have $2r$ half-branches at 0.

Since f, g are finitely C^0-$\mathcal{K}$-determined, we can suppose, without loss of generality, that f, g are both polynomial map germs, that $\Sigma(f) \cap f^{-1}(0) = \Sigma(g) \cap g^{-1}(0) = \{0\}$. We also assume, for simplicity, that $f^{-1}(0) = g^{-1}(0)$.

Given $\epsilon, \delta, \delta' > 0$ we denote:

$$V_\epsilon := f^{-1}(0) \cap D^n_\epsilon = g^{-1}(0) \cap D^n_\epsilon, \quad N_0 := D^n_\epsilon \cap f^{-1}(S^{n-2}_\delta), \quad N_1 := D^n_\epsilon \cap g^{-1}(S^{n-2}_{\delta'}).$$

We choose these numbers small enough in such a way that

$$\begin{tikzcd} V_\epsilon & N_0 \arrow[l, "r_0"'] \arrow[r, "f|N_0"] & S^{n-2}_\delta, & V_\epsilon & N_1 \arrow[l, "r_1"'] \arrow[r, "g|N_1"] & S^{n-2}_{\delta'}, \end{tikzcd}$$

are the diagram links of f, g respectively. By using an argument similar to that of the proof of Theorem 4.10, we choose a convenient vector field $v(x)$. We use the flow of $v(x)$ to construct the maps r_0, r_1 and we have also a homeomorphism $\phi : N_1 \to N_0$ with the property that $r_0 \circ \phi = r_1$. Hence, we only need to show that $f|N_0$ and $\omega \circ g|N_1 \circ \phi$ are homotopic, where $\omega(y) = \sqrt{\delta/\delta'}y$.

Let us denote by $C_1, \ldots, C_{2r}$ the boundary components of N_0. By the Brouwer Degree Theorem, the homotopy type of $f|N_0$ is determined by the degree of each $f|C_i : C_i \to S^{n-2}_\delta$ for $i = 1, \ldots, 2r-1$. If $x_i \in V_\epsilon \cap S^{n-1}_\epsilon$ is the interior point of C_i, we have that for $\delta > 0$ small enough, the degree of $f|C_i$ is equal to the local degree of $f|S^{n-1}_\epsilon$ at x_i. Since f is regular at x_i and S^{n-1}_ϵ is transverse to V_ϵ, $f|S^{n-1}_\epsilon$ is also regular at x_i. Therefore $f|S^{n-1}_\epsilon$ has local degree ± 1 at x_i. Moreover, this degree is always $+1$ if we consider the induced orientation on C_i (see figure 3). Note that the same argument is valid for $g|N_1$, thus we deduce that $f|N_0$ and $\omega \circ g|N_1 \circ \phi$ are homotopic. $\square$

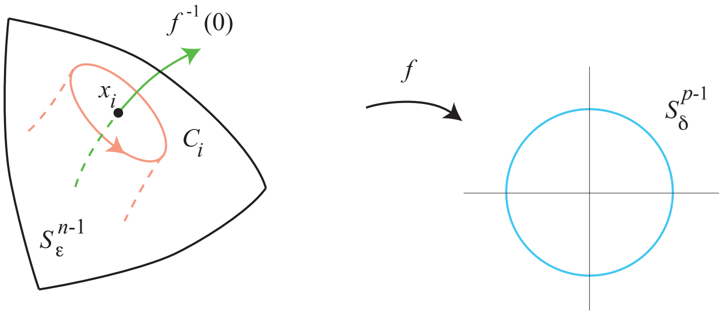

FIGURE 3. The map $f|C_i$.

Given $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n-1}, 0)$ be a finitely C^0-$\mathcal{K}$-determined analytic map germ such that $f^{-1}(0) - \{0\}$ have $2r$-half-branches, we define a notion of degree for f as follows:

$$|\deg(f)| := \sum_{i=1}^{2r-1} |\deg(f|C_i)| = 2r - 1,$$

where C_i, $i = 1, \ldots, 2r-1$ are the boundary components of Milnor tube of f, each one of them homeomorphic to S^{n-2}.

Hence $|\deg(f)|$ coincides with the number of half-branches of $f^{-1}(0)$ minus 1 which also coincides with the number of connected components of K_ϵ minus 1. Then Theorem 4.12 can be rewritten as the following corollary:

Corollary 4.13. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n-1}, 0)$ be finitely C^0-$\mathcal{K}$-determined map germs. Then f and g are C^0-$\mathcal{K}$-equivalent if and only if $|\deg(f)| = |\deg(g)|$.*

We can interpret the Corollary 4.13 as an analogous version of the beautiful Nishimura's result (Corollary 3.9) when $n = p - 1$.

Remark 4.14. In [2] the authors presented a generalization of the notion of degree of map germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ to the case of map germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $n > p > 1$ from a homological view point. When $p = n - 1$ the absolute value of homological degree of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n-1}, 0)$ is equal to the number of half-branches of $f^{-1}(0)$ minus 1 (see [2]). Hence the homological degree definition given in [2] coincides with our degree's definition for f when $p = n - 1$.

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