THE REEB GRAPH OF A MAP GERM FROM $\mathbb{R}^3$ TO $\mathbb{R}^2$ WITH ISOLATED ZEROS

ERICA BOIZAN BATISTA, JOÃO CARLOS FERREIRA COSTA AND JUAN J. NUÑO-BALLESTEROS

Abstract. We consider finitely determined map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ with $f^{-1}(0) = \{0\}$ and we look at the classification of this kind of germs with respect to topological equivalence. By Fukuda's cone structure theorem, the topological type of f can be determined by the topological type of its associated link, which is a stable map from S^2 to S^1. We define a generalized version of the Reeb graph for stable maps $\gamma : S^2 \to S^1$ which turns out to be a complete topological invariant. If f has corank 1, then f can be seen as a stabilization of a function $h_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and we show that the Reeb graph is the sum of the partial trees of the positive and negative stabilizations of h_0. Finally, we apply this to give a complete topological description of all map germs with Boardman symbol $\Sigma^{2,1}$.

1. Introduction

The classification problem of singular points of smooth map germs is one of the most important problems in Singularity theory. The classical classification is done via $\mathcal{A}$-equivalence, where we take diffeomorphisms in the source and the target. However, this is a difficult problem and it presents a lot of rigidity. Then it seems natural to investigate the classification of mappings up to weaker equivalence relations. Here we consider topological equivalence or C^0-$\mathcal{A}$-equivalence, where the changes of coordinates are homeomorphisms instead of diffeomorphisms. Even that, Nakai showed in [19] that they appear moduli in the topological classification of polynomial map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$.

This paper is devoted to the topological classification of smooth map germs from $\mathbb{R}^3$ to $\mathbb{R}^2$ which are finitely determined. Finite determinacy is a key notion in Singularity theory because if $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is finitely determined, then it may be assumed polynomial. Restricted to the class of finitely determined map germs from $\mathbb{R}^3$ to $\mathbb{R}^2$ of a given degree, it follows from Thom or Nishimura's works (cf. [20, 25]) that the number of topological types is finite. In other words, this problem is tame in the sense that it does not have topological moduli.

The topological structure of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is given by the so-called link of f (cf. [7]). The link of f is obtained by taking a small enough representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ and the intersection of its image with a small enough sphere S^1_δ centered at the origin in $\mathbb{R}^2$. When f has isolated zeros (i.e., $f^{-1}(0) = \{0\}$), the link is a stable map $\gamma : S^2 \to S^1$ and f is topologically equivalent to the cone of γ. As a consequence, two finitely determined map germs $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ are

2010 *Mathematics Subject Classification.* 58K15, 58K40, 58K65.

Key words and phrases. topological equivalence, classification, link, Reeb graph.

The first named author has been partially supported by CAPES, Brasília-Brazil. The second named author has been partially supported by CAPES and FAPESP. The third named author has been partially supported by DGICYT Grant MTM2012–33073.

topologically equivalent if their associated links are topologically equivalent. Then, some open questions arise in a natural way related to our classification problem:

(1) Find a good combinatorial model to describe the topology of stable maps from S^2 to S^1.
(2) Show that if f, g are topologically equivalent then their associated links are also topologically equivalent.
(3) Find relations between the analytic invariants of f (e.g. corank, Boardman symbol, etc.) and the topological invariants of the link.
(4) Characterize all the stable maps which can be realized as the link of a finitely determined map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$.

Inspired by the works of Arnold, Prishlyak or Sharko (see [1, 21, 24]) we introduce in Section 3 an adapted version of the Reeb graph to answer question (1). The classical Reeb graph is defined for a Morse function $\gamma : M \to \mathbb{R}$, but here we have to extend it to the case that the map takes values on S^1 instead of $\mathbb{R}$. Then, our generalized version of the Reeb graph turns out to be a complete topological invariant for stable maps $\gamma : S^2 \to S^1$ (see Corollary 3.9). Moreover, the Reeb graph is also the key tool which gives the answer to question (2) (Corollary 3.12).

In Section 4 we take special attention to the case that f has corank 1. In this case, f can be written as $f(x, y, z) = (x, h_x(y, z))$ and gives a stabilization of $h_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The topology of f is now determined by the two stabilizations h_x^+, with $x > 0$ and h_x^-, with $x < 0$. We introduce the notion of partial trees associated to h_x^+ and h_x^- and show that the sum of the partial trees is equivalent to the Reeb graph of the link of f (Theorem 4.10). In the last part we give a complete description of those map germs with Boardman symbol $\Sigma^{2,1}$ and provide a complete topological classification of this type of map germs up to multiplicity 6 (Theorem 4.13). This partially answers the questions (3) and (4).

The case where f has non isolated zeros (i.e., $f^{-1}(0) \neq \{0\}$) is more complicated. In that case, the link is a stable map $\gamma : M \to S^1$, where now M is a compact surface with boundary and genus zero. However, we need a generalized version of the cone to describe the topology of f (see [3]). The topological classification of map germs with non isolated zeros will be considered in a forthcoming paper [2].

Some recent papers treat the topological classification of finitely determined map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by looking at the topological type of the link (see, for instance, [3, 13, 16, 17, 18]). However, as far as we know, this is the first time where it is considered the case $n > p$.

All map germs considered are real analytic except otherwise stated. We adopt the usual notation and basic definitions that are usual in Singularity theory (e.g., $\mathcal{A}$-equivalence, finite determinacy, stability, bifurcation set, etc.) as the reader can find in Wall's survey paper [26].

2. Finite determinacy and the link of a map germ

Two smooth map germs $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ and $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = \phi \circ g \circ \psi$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f and g are *topologically equivalent* (or *C^0-$\mathcal{A}$-equivalent*).

We say that $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathcal{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to \mathbb{R}^2$ be a smooth map, where $U \subset \mathbb{R}^3$ is an open subset. We denote by $S(f) = \{p \in U \mid Jf(p)$ does not have rank $2\}$ the *singular set* of f, where $Jf(p)$ is the Jacobian matrix of f. We also denote the *discriminant set* of f by $\Delta(f) = f(S(f))$.

When we start a classification of generic singularities, the first step is know the stable singularities. The characterization of stable singularities of maps from $\mathbb{R}^3$ to $\mathbb{R}^2$ is well known (cf. [8]) and it is given by:

Proposition 2.1. *A smooth map $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ is stable if only if the following conditions are satisfied:*

(1) *For every $p \in U$, the germ of f at p is $\mathcal{A}$-equivalent to one of the following germs:*
- (x, y) *if p is a regular point;*
- $(x, y^2 + z^2)$ *if p is a definite fold point D;*
- $(x, y^2 - z^2)$ *if p is an indefinite fold point I;*
- $(x, y^3 + xy + z^2)$ *if p is a cusp point.*

(2) *For every $q \in \Delta(f)$, $f^{-1}(q) \cap S(f)$ consists of at most two points. Moreover, if $f^{-1}(q) \cap S(f)$ consists of two points, then the multi-germ of f at $f^{-1}(q) \cap S(f)$ is $\mathcal{A}$-equivalent to one of the following three multi-germs:*
- $(x_1, y_1^2 + z_1^2), (y_2^2 + z_2^2, x_2)$ *called nodefold D&D;*
- $(x_1, y_1^2 + z_1^2), (y_2^2 - z_2^2, x_2)$ *called nodefold D&I;*
- $(x_1, y_1^2 - z_1^2), (y_2^2 - z_2^2, x_2)$ *called nodefold I&I.*

From the global point of view, it is a consequence of Levine's work [12] that f is stable if and only if:

(G1) If $p \in U$ is a cusp point, then $f^{-1}(f(p)) \cap S(f) = \{p\}$,

(G2) $f|(S(f) - \{\text{cusp points}\})$ is an immersion with normal crossings.

When $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ is not stable but it is finitely determined, then roughly speaking, f has an isolated instability at the origin. This is known as Mather-Gaffney finite determinacy criterion [26]. In fact, the Mather-Gaffney criterion is valid for holomorphic map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, but we can obtain some consequences of this criterion in the real case as follows.

Theorem 2.2. *A holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is finitely determined if and only if there is a representative $f : U \subset \mathbb{C}^n \to \mathbb{C}^p$ such that*

i) $S(f) \cap f^{-1}(0) = \{0\}$,
ii) *the restriction $f|U - \{0\}$ is stable.*

Since the case of our interest is $n = 3$ and $p = 2$, from the condition ii), the cusps are isolated points in $U - \{0\}$. Then we can shrink the neighborhood U if necessary in Theorem 2.2 to get a representative $f : U \subset \mathbb{C}^3 \to \mathbb{C}^2$ such that the restriction $f|U - \{0\}$ is stable with only simple folds. The word simple here means that the folds are not double points.

Coming back to real case, we consider now an analytic map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$. If $f_{\mathbb{C}} : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ is the complexification of f, it follows from Wall's survey paper [26] that f is finitely determined if and only if its complexification $f_{\mathbb{C}}$ is finitely determined. Then we have as consequence of the Theorem 2.2 the following real criterion:

Corollary 2.3. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ. Then there exists a representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ such that*

i) $S(f) \cap f^{-1}(0) = \{0\}$,

ii) *the restriction* $f|U-\{0\}$ *is stable with only definite and indefinite simple folds.*

If f is finitely determined, then its discriminant $\Delta(f)$ is a plane curve with an isolated singularity at the origin. The number of half branches of $\Delta(f)$ will play a crucial role in the analysis of the Reeb graph associated to link of f and consequently, in the topological classification of f.

Denote by $J^r(n,p)$ the r-jet space from $(\mathbb{R}^n,0)$ to $(\mathbb{R}^p,0)$. For positive integers r and s with $s\geq r$, let $\pi^s_r : J^s(n,p)\to J^r(n,p)$ be the canonical projection defined by $\pi^s_r(j^sf(0)) = j^rf(0)$. For a positive number $\epsilon>0$ we set

$$D^n_\epsilon=\{x\in\mathbb{R}^n \mid \|x\|^2\leq\epsilon\},\ B^n_\epsilon=\{x\in\mathbb{R}^n \mid \|x\|^2<\epsilon\} \text{ and } S^{n-1}_\epsilon=\{x\in\mathbb{R}^n \mid \|x\|^2=\epsilon\}.$$

We denote D^n, B^n and S^{n-1} the standard disk, ball and sphere of radius 1, respectively.

T. Fukuda has proved the following cone structure theorem in his papers [6, 7]:

Theorem 2.4. *For any semialgebraic subset* W *of* $J^r(n,p)$*, there exist an integer* s *(*$s\geq r$*) depending only* n,p *and* r*, and there exists a closed semialgebraic subset* Σ_W *of* $(\pi^s_r)^{-1}(W)$ *having codimension* ≥ 1 *such that for any* C^∞ *mapping* $f:\mathbb{R}^n\to\mathbb{R}^p$ *with* $j^sf(0)$ *belonging to* $(\pi^s_r)^{-1}(W)\setminus\Sigma_W$ *we have the following properties:*

(A) $f^{-1}(0)=\{0\}$ *there is* $\epsilon_0>0$ *such that for any number* ϵ *with* $0<\epsilon\leq\epsilon_0$ *we have:*

- (A-i) *the set* $\tilde{S}^{n-1}_\epsilon = f^{-1}(S^{p-1}_\epsilon)$ *is a* C^∞ *submanifold without boundary, which is diffeomorphic to the standard unit sphere* S^{n-1}.
- (A-ii) *The restricted mapping* $f|\tilde{S}^{n-1}_\epsilon : \tilde{S}^{n-1}_\epsilon\to S^{p-1}_\epsilon$ *is topologically stable (*C^∞ *stable if* (n,p) *is a nice pair in Mather's sense).*
- (A-iii) *If* $\tilde{D}^{n-1}_\epsilon = f^{-1}(D^{p-1}_\epsilon)$*, then the restricted mapping* $f|\tilde{D}^{n-1}_\epsilon : \tilde{D}^{n-1}_\epsilon\to D^p_\epsilon$ *is topologically equivalent to the cone of* $f|\tilde{S}^{n-1}_\epsilon$.

(B) $f^{-1}(0)\neq\{0\}$ *for any sufficiently small positive numbers* ϵ *and* δ*, the upper bound of* ϵ *depending of* f *and the upper bound of* δ *depending of* ϵ *and* f*, we have:*

- (B-i) $f^{-1}(0)\cap S^{n-1}_\epsilon$ *is an* $(n-p-1)$*-dimensional manifold and it is diffeomorphic to* $f^{-1}(0)\cap S^{n-1}_{\epsilon_0}$.
- (B-ii) $D^n_\epsilon\cap f^{-1}(S^{p-1}_\delta)$ *is a* C^∞ *manifold, in general with boundary and it is diffeomorphic to* $D^n_{\epsilon_0}\cap f^{-1}(S^{p-1}_{\delta_0})$.
- (B-iii) *the restriction* $f|D^n_\epsilon\cap f^{-1}(S^{p-1}_\delta) : D^n_\epsilon\cap f^{-1}(S^{p-1}_\delta)\to S^{p-1}_\delta$ *is a topologically stable map (*C^∞ *stable if* (n,p) *is a nice pair in Mather's sense) and its topological class is independent of* ϵ *and* δ.

Assuming that f is r-determined for some r and taking $W=\{j^rf(0)\}$, we can apply Theorem 2.4 to obtain a representative of f satisfying (A) or (B), depending on if $f^{-1}(0)=\{0\}$ or $f^{-1}(0)\neq\{0\}$. Note that when $n\leq p$ we always have $f^{-1}(0)=\{0\}$ but when $n>p$ we may have the two possibilities.

Definition 2.5. Let $f:(\mathbb{R}^3,0)\to(\mathbb{R}^2,0)$ be a finitely determined map germ such that $f^{-1}(0)=\{0\}$. We say that the stable map $f|\tilde{S}^2_\epsilon:\tilde{S}^2_\epsilon\to S^1_\epsilon$ is the *link* of f, where f is a representative that satisfies the Fukuda's conditions (A) of Theorem 2.4 adapted for case $n=3$ and $p=2$.

It follows from the definition that the link of f is a stable map $\gamma:S^2\to S^1$, that is, γ has only Morse singularities with distinct critical values. Moreover, the link is well defined up to $\mathcal{A}$-equivalence and f is topologically equivalent to the cone of γ. Hence, we have the following immediate consequence.

Corollary 2.6. *Two finitely determined map germs $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ with $f^{-1}(0) = \{0\} = g^{-1}(0)$ are topologically equivalent if their associated links are topologically equivalent.*

Remark 2.7. When $f^{-1}(0) \neq \{0\}$, it is also common to call the link of f to the stable map $f|D^3_\epsilon \cap f^{-1}(S^1_\delta) : D^3_\epsilon \cap f^{-1}(S^1_\delta) \to S^1_\delta$, where f is a representative that satisfies the Fukuda's conditions (B) of Theorem 2.4 adapted for case $n = 3$ and $p = 2$. However, in this case, f is not topologically equivalent to the cone of the link in the classical sense. Instead of this, we have to consider a generalized version of the cone which turns out to be topologically equivalent to f (see [3] for details). The topological classification of this class of map germs will be considered in a forthcoming paper [2].

3. The generalized Reeb graph

The Reeb graph was introduced by Reeb in [22] and it is well known that it is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$ (see [1, 24]). In this section we extend the concept of Reeb graph for stable maps from S^2 to S^1.

Let $\gamma : S^2 \to S^1$ be a stable map. Consider the following equivalence relation on S^2: $x \sim y \Leftrightarrow \gamma(x) = \gamma(y)$ and x and y are in the same connected component of $\gamma^{-1}(\gamma(x))$.

Proposition 3.1. *Let $\gamma : S^2 \to S^1$ be a stable map. Then γ is not a regular map.*

Proof. Suppose γ is a regular map, then $\gamma(S^2) \subset S^1$ would be an open set. Since $\gamma(S^2)$ is also closed, we get $\gamma(S^2) = S^1$ and hence, γ is surjective. By Ehresmann's fibration theorem [4], f is a locally trivial fibration. In particular, if F is a fiber we have that

$$2 = \chi(S^2) = \chi(S^1)\chi(F) = 0,$$

what is an absurd. □

Proposition 3.2. *Let $\gamma : S^2 \to S^1$ be a stable map. Then the quotient space $S^2/\sim$ admits the structure of a connected graph in the following way:*

1. *the vertices are the connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a critical value;*
2. *each edge is formed by points that correspond to connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a regular value.*

Proof. Since γ is stable we have a finite number of critical values $v_1, \dots, v_r$ and for each $i = 1, \dots, r$, $\gamma^{-1}(v_i)$ has a finite number of connected components. Then,

$$\gamma|S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}) : S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}) \to S^1 - \{v_1, \dots, v_r\}$$

is regular, and the induced map

$$\tilde{\gamma} : (S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}))/\sim \to S^1 - \{v_1, \dots, v_r\}$$

is a local homeomorphism. Each connected component of $S^1 - \{v_1, \dots, v_r\}$ is homeomorphic to an open interval, so each connected component of $(S^2 - \gamma^{-1}(\{v_1, \dots, v_r\}))/\sim$ is also homeomorphic to an open interval.

□

Each vertex of the graph can be of three types, depending on if the connected component has a maximum/minimum critical point, a saddle point or just regular points. Then, the possible incidence rules of edges and vertices are given in fig. 1.

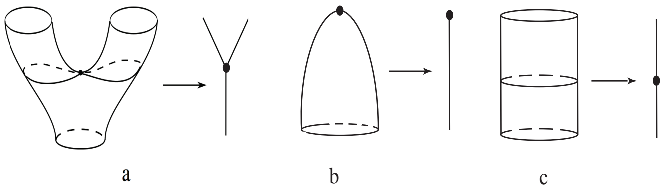


FIGURE 1

Let $v_1, \ldots, v_r \in S^1$ be the critical values of γ. Let us choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 \leq v_1 < \ldots < v_r$ and we label each vertex with the index $i \in \{1, \ldots, r\}$, if it corresponds to the critical value v_i.

Definition 3.3. The graph given by $S^2/\sim$ together with the the labels of the vertices, as previously defined, is said to be the *generalized Reeb graph* associated to $\gamma : S^2 \to S^1$.

For simplicity, from now on we will just call Reeb graph to the generalized Reeb graph, unless otherwise specified.

Proposition 3.4. *Let $\gamma : S^2 \to S^1$ be a stable map. Then the Reeb graph of γ is a tree.*

Proof. Let Γ be the Reeb graph of γ. Since Γ is connected, in order to show that Γ is a tree, we only need to prove that its Euler characteristic is $\chi(\Gamma) = 1$. We have that $\chi(\Gamma) = V - E$, where V, E denote the number of vertices and edges of Γ, respectively.

On one hand, $V = M + S + I$ where M, S, I denote the numbers of vertices of each type: maximum/minimum, saddle or regular, respectively. Note that $V \neq 0$ by Proposition 3.1.

On the other hand, by Euler's formula $E = \frac{1}{2}\sum \deg(v_i)$ where v_i are the vertices of Γ. Since γ is stable, the degree of each vertex of maximum/minimum type is 1, while of regular type is 2 and of saddle type is 3 (see fig. 1). Hence,

$$\chi(\Gamma) = V - E = M + S + I - \frac{1}{2}(M + 2I + 3S) = \frac{M - S}{2} = 1,$$

where the last equality follows from the Morse formula: $M - S = \chi(S^2) = 2$. □

Remark 3.5. The classical Reeb graph is defined in the same way, but the vertices are just the connected components of level curves $\gamma^{-1}(v)$ which contain a critical point. Hence, our generalized Reeb graph contains some extra vertices corresponding to the regular connected components of $\gamma^{-1}(v)$, where v is a critical value. Of course the classical Reeb graph can be obtained from the generalized one just by eliminating the extra vertices and joining the two adjacent edges. But in general, the generalized Reeb graph provides more information.

We present in fig. 2 two examples of stable maps $\gamma_1, \gamma_2 : S^2 \to S^1$ with their respective generalized Reeb graphs. Both examples share the same classical Reeb graph, but the generalized Reeb graphs are different. The example on the left hand side is a non-surjective map, whilst the map on the right hand side is surjective, therefore the maps are not topologically equivalent. This shows that the classical Reeb graph is not sufficient to distinguish between these two examples.

Notice that if $\gamma : S^2 \to S^1$ is not surjective, then we can look at γ as a Morse function from S^2 to $\mathbb{R}$ (via stereographic projection). In this case, the generalized Reeb graph can be deduced from the classical one just by adding the extra vertices each time that one passes through a critical value.

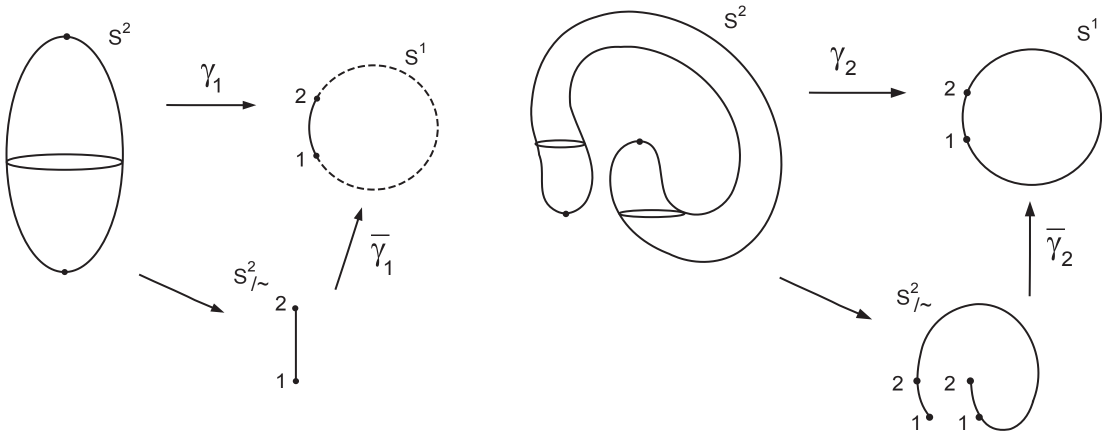


FIGURE 2

It is obvious that labeling of vertices of the Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base points on each S^1. Different choices will produce either a cyclic permutation or a reversion of the labeling in the Reeb graph. This leads us to the following definition of equivalent Reeb graphs.

Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. Let Γ_γ and Γ_δ be their respective Reeb graphs. Consider $\bar{\gamma} : \Gamma_\gamma \to S^1_\gamma$ and $\bar{\delta} : \Gamma_\delta \to S^1_\delta$ the induced quotient maps, where S^1_γ, S^1_δ denote S^1 with the graph structure whose vertices are the critical values of γ, δ respectively (as illustrated in fig. 2).

Definition 3.6. We say that Γ_γ *is equivalent to* Γ_δ and we denote it by $\Gamma_\gamma \sim \Gamma_\delta$, if there exist graph isomorphisms $j : \Gamma_\gamma \to \Gamma_\delta$ and $l : S^1_\gamma \to S^1_\delta$, such that the following diagram is commutative:

$$\begin{array}{ccc} V_\gamma & \xrightarrow{\bar{\gamma}|V_\gamma} & \Delta_\gamma \\ {\scriptstyle j|V_\gamma}\big\downarrow & & \big\downarrow{\scriptstyle l|\Delta_\gamma} \\ V_\delta & \xrightarrow{\bar{\delta}|V_\delta} & \Delta_\delta \end{array}$$

where $V_\gamma = \{\text{vertices of } \Gamma_\gamma\}$, $V_\delta = \{\text{vertices of } \Gamma_\delta\}$ and Δ_γ and Δ_δ are their respective discriminant sets.

Theorem 3.7. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. If γ and δ are topologically equivalent then their respective Reeb graphs are equivalent.*

Proof. Since γ and δ are topologically equivalent there exist $h : S^2 \to S^2$ and $k : S^1 \to S^1$ homeomorphisms such that $k \circ \gamma \circ h = \delta$. Then h takes critical points into critical points and k takes critical values into critical values. Hence h induces a graph isomorphism from Γ_γ to Γ_δ and k induces a graph isomorphism from S^1_γ to S^1_δ which give the equivalence between the Reeb graphs. □

Theorem 3.8. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps such that $\Gamma_\gamma \sim \Gamma_\delta$. Then γ is $\mathcal{A}$-equivalent to δ.*

Proof. This is an adaptation of the proof of [10, Theorem 4.1]. Since $\Gamma_\gamma \sim \Gamma_\delta$, there exist graph isomorphisms $j : \Gamma_\gamma \to \Gamma_\delta$ and $l : S^1_\gamma \to S^1_\delta$ as in Definition 3.6. We choose a homeomorphism $h : \Gamma_\gamma \to \Gamma_\delta$ and a diffeomorphism $k : S^1_\gamma \to S^1_\delta$ which realize the graph isomorphisms j, l respectively and such that $\bar{\delta} \circ h = k \circ \bar{\gamma}$.

Since $k \circ \gamma$ is $\mathcal{A}$-equivalent to γ then by Theorem 3.7 we have $\Gamma_{k\circ\gamma} \sim \Gamma_\gamma$. Moreover, these graphs are the same because $k \circ \bar{\gamma} = \overline{k \circ \gamma}$. In other words the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_\delta & \xrightarrow{\bar{\delta}} & S^1 \\ \uparrow h & \nearrow_{\overline{k\circ\gamma}} & \\ \Gamma_\gamma & & \end{array}$$

For simplicity, we write simply γ instead of $k \circ \gamma$. By construction $h(V_\gamma) = V_\delta$, but now γ and δ have the same critical values $v_1, \dots, v_n \in S^1$. We choose a base point and an orientation in S^1 and assume that

$$v_1 < v_2 < \ldots < v_n.$$

Denote by $\operatorname{arc}(a, b)$ the oriented arc from a to b in S^1, and by $\overline{\operatorname{arc}(a, b)}$ its closure. Let w_i be the middle point of $\operatorname{arc}(v_i, v_{i+1})$, for $i = 1, \dots, n$ with $v_{n+1} = v_1$ and let $\xi : S^1 \setminus \{w_n\} \to \mathbb{R}$ be an orientation preserving diffeomorphism.

For each critical value v_i with $i = 1, \dots, n$, we can choose $\epsilon_i > 0$ as in Definition A.5, and by Theorem A.6, there exists a diffeomorphism

$$h_i : (\xi \circ \gamma)^{-1}[\xi(v_i) - 2\epsilon_i^2, \xi(v_i) + 2\epsilon_i^2] \to (\xi \circ \delta)^{-1}[\xi(v_i) - 2\epsilon_i^2, \xi(v_i) + 2\epsilon_i^2]$$

such that $\xi \circ \gamma = \xi \circ \delta \circ h_i$. Since ξ is a diffeomorphism, it follows that $\gamma = \delta \circ h_i$ when restricted to

$$\gamma^{-1}(\operatorname{arc}(\xi^{-1}(\xi(v_i) - 2\epsilon_i^2), \xi^{-1}(\xi(v_i) + 2\epsilon_i^2))).$$

Let $a_i, a_i^-, b_i, b_i^- \in S^1$ be given by

$$a_i = \xi^{-1}(\xi(v_i) + 2\epsilon_i^2), \qquad a_i^- = \xi^{-1}(\xi(v_i) - 2\epsilon_i^2),$$
$$b_i = \xi^{-1}(\xi(v_i) + \epsilon_i^2), \qquad b_i^- = \xi^{-1}(\xi(v_i) - \epsilon_i^2).$$

Since ξ is orientation preserving,

$$w_i < a_i^- < b_i^- < v_i < b_i < a_i < w_{i+1}.$$

Denote by

$$A_i = \gamma^{-1}(\overline{\operatorname{arc}(a_i^-, a_i)}), \qquad A'_i = \delta^{-1}(\overline{\operatorname{arc}(a_i^-, a_i)}),$$
$$B_i = \gamma^{-1}(\overline{\operatorname{arc}(b_i, b_{i+1}^-)}), \qquad B'_i = \delta^{-1}(\overline{\operatorname{arc}(b_i, b_{i+1}^-)}),$$

for $i = 1, \dots, n$ with $b_{n+1} = b_1$. With this notation, $h_i : \operatorname{Int}(A_i) \to \operatorname{Int}(A'_i)$ is a diffeomorphism such that $\gamma = \delta \circ h_i$ on $\operatorname{Int}(A_i)$, $\forall i = 1, \dots, n$.

Notice that $\gamma|B_i$ and $\delta|B'_i$ are regular maps, for all $i = 1, \dots, n$. Then by Theorem A.4 there exist diffeomorphisms ϕ_i and ψ_i such that the following diagrams are commutative:

$$\begin{array}{ccc} \gamma^{-1}(b_i) \times \operatorname{arc}(b_i, b_{i+1}^-) & \xrightarrow{p} & \operatorname{arc}(b_i, b_{i+1}^-) \\ \uparrow \phi_i & \nearrow_{\gamma|B_i} & \\ B_i & & \end{array} \qquad \begin{array}{ccc} \delta^{-1}(b_i) \times \operatorname{arc}(b_i, b_{i+1}^-) & \xrightarrow{\tilde{p}} & \operatorname{arc}(b_i, b_{i+1}^-) \\ \uparrow \psi_i & \nearrow_{\delta|B'_i} & \\ B'_i & & \end{array}$$

where p and $\tilde{p}$ are the projections in the second coordinate.

Since the Reeb graphs of γ and δ are equivalent, it follows that $\gamma^{-1}(b_i)$ is diffeomorphic to $\delta^{-1}(b_i)$. Consequently, B_i is diffeomorphic to B'_i via a diffeomorphism which gives the $\mathcal{A}$-equivalence between $\gamma|B_i$ and $\delta|B'_i$.

Notice that the boundary of A_i is diffeomorphic to a finite union of circles S^1. Then the diffeomorphisms h_i when restricted to the boundary of A_i may be assumed orientation preserving. Hence $h_i|\gamma^{-1}(b_i)$ and $h_{i+1}|\gamma^{-1}(b^-_{i+1})$ are isotopic because both are isotopic to the identity. Let $F_i : \gamma^{-1}(b_i)\times\overline{\text{arc}(a_i, a^-_{i+1})} \to \delta^{-1}(b_i)\times\overline{\text{arc}(a_i, a^-_{i+1})}$ be an isotopy between $h_i|\gamma^{-1}(b_i)$ and $h_{i+1}|\gamma^{-1}(b^-_{i+1})$, for $i = 1, \dots, n$.

Define $\beta_i : \gamma^{-1}(b_i) \times \overline{\text{arc}(b_i, b^-_{i+1})} \to \delta^{-1}(b_i) \times \overline{\text{arc}(b_i, b^-_{i+1})}$ by

$$\beta_i(x,t) = \begin{cases} (h_i(x), t), & \text{if } b_i < t \leq a_i, \\ (F_i(x,t), t), & \text{if } a_i \leq t \leq a^-_{i+1}, \\ (h_{i+1}(x), t), & \text{if } a^-_{i+1} < t \leq b^-_{i+1}, \end{cases}$$

and let $\alpha_i : \text{Int}(B_i) \to \text{Int}(B'_i)$ be given by $\alpha_i = \psi_i^{-1} \circ \beta_i \circ \phi_i$, with $i = 1, \dots, n$.

Since each β_i is a diffeomorphism, it follows that α_i is also a diffeomorphism. Moreover, $\delta \circ \alpha_i = \gamma$ on $\text{Int}(B_i)$, because:

$$\delta \circ \alpha_i = \delta \circ \psi_i^{-1} \circ \beta_i \circ \phi_i = \tilde{p} \circ \beta_i \circ \phi_i = p \circ \phi_i = \gamma.$$

We now define a map $H : S^2 \to S^2$ given by

$$H(x) = \begin{cases} h_i(x), & \text{if } x \in \text{Int}(A_i),\ i = 1, \dots, n, \\ \alpha_i(x), & \text{if } x \in \text{Int}(B_i),\ i = 1, \dots, n. \end{cases}$$

By construction, $h_i = \alpha_i$ on $\text{Int}(A_i) \cap \text{Int}(B_i)$ and $\alpha_i = h_{i+1}$ on $\text{Int}(B_i) \cap \text{Int}(A_{i+1})$, $\forall i = 1, \dots, n$. Therefore, H is well defined and smooth. Moreover, $H : S^2 \to S^2$ is a diffeomorphism such that $\gamma = \delta \circ H$.

□

The two theorems 3.7 and 3.8 together give that the Reeb graph is a complete topological invariant for stable maps from S^2 to S^1. In fact, we have a little bit more, as we can see in the following corollary.

Corollary 3.9. *Let $\gamma, \delta : S^2 \to S^1$ be two stable maps. Then the following statements are equivalent:*

(1) *γ, δ are $\mathcal{A}$-equivalent,*
(2) *γ, δ are topologically equivalent,*
(3) *$\Gamma_\gamma \sim \Gamma_\delta$.*

In the last part of this section, we consider the Reeb graph of the link of a finitely determined map germ with isolated zeros.

Remark 3.10. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a finitely determined map germ with $f^{-1}(0) = \{0\}$ and let $\gamma_f : \tilde{S}^2_\epsilon \to S^1_\epsilon$ be the link of f. The critical values of γ_f are given by $S^1_\epsilon \cap \Delta(f)$. In fact, if we denote by $A_1, \dots, A_r$ the half branches of $\Delta(f)$, then by the cone structure theorem each half branch of A_i intersects S^1_ϵ in a unique critical value v_i of γ_f. Analogously, the edges of Γ_{γ_f} correspond to the connected components of $f^{-1}(\alpha_j)$, where $\alpha_1, \dots, \alpha_r$ are the arcs of S^1_ϵ limited by two consecutive half branches of $\Delta(f)$.

Theorem 3.11. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be two finitely determined map germs such that $f^{-1}(0) = \{0\} = g^{-1}(0)$. If f and g are topologically equivalent then the Reeb graphs of their links are equivalent.*

Proof. By hypothesis, there exist two homeomorphisms germs h, k such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^3,0) & \xrightarrow{f} & (\mathbb{R}^2,0) \\ h\downarrow & & \downarrow k \\ (\mathbb{R}^3,0) & \xrightarrow{g} & (\mathbb{R}^2,0) \end{array} \tag{1}$$

We take representatives of f, g, h and k and for any small enough $\epsilon > 0$, the next diagram is also commutative:

$$\begin{array}{ccc} \tilde{S}^2_\epsilon & \xrightarrow{\gamma_f} & S^1_\epsilon \\ h\downarrow & & \downarrow k \\ M_\epsilon & \xrightarrow{g|M_\epsilon} & P_\epsilon \end{array} \tag{2}$$

where $M_\epsilon = h(\tilde{S}^2_\epsilon)$ and $P_\epsilon = k(S^1_\epsilon)$.

Choose $\epsilon_0, \epsilon_1 > 0$ such that $\gamma_f : \tilde{S}^2_{\epsilon_0} \to S^1_{\epsilon_0}$ and $\gamma_g : \tilde{S}^2_{\epsilon_1} \to S^1_{\epsilon_1}$ are the links of f and g, respectively, and $S^1_{\epsilon_1} \subset k(D^2_{\epsilon_0})$. From the commutativity of diagram (2) we can associate a Reeb graph $\Gamma_{g|M_{\epsilon_0}}$ for the map $g|M_{\epsilon_0}$ induced by the Reeb graph Γ_{γ_f} of γ_f. Furthermore, $\Gamma_{g|M_{\epsilon_0}} \sim \Gamma_{\gamma_f}$ in the sense of Definition 3.6.

Consider $A_1, \dots, A_n$ the half branches of the discriminant $\Delta(g)$ ordered in the anticlockwise orientation. By the cone structure of f (see Theorem 2.4), each half branch A_i intersects P_{ϵ_0} in a unique point v_i so that $v_1, \dots, v_n$ are the critical points of $g|M_{\epsilon_0}$. Analogously, each A_i intersects $S^1_{\epsilon_1}$ in a unique point w_i, where now $w_1, \dots, w_n$ are the critical points of γ_g. We have a graph isomorphism $l : P_{\epsilon_0} \to S^1_{\epsilon_1}$ given by $l(v_i) = w_i$, $\forall i = 1, \dots, n$.

Let $C_1, \dots, C_r$ be the connected components of $g^{-1}(\Delta(g)) - \{0\} = \cup_{i=1}^n g^{-1}(A_i)$. Again by the cone structure of f, each connected component C_j intersects M_{ϵ_0} in a unique connected component V_j of some $g^{-1}(v_i)$, so that $V_1, \dots, V_r$ are the vertices of $\Gamma_{g|M_{\epsilon_0}}$. Finally, each C_j intersects $\tilde{S}^2_{\epsilon_1}$ in a unique connected component W_j of $g^{-1}(w_i)$, in such a way that $W_1, \dots, W_r$ are now the vertices of Γ_{γ_g}. We have a bijection φ defined by $\varphi(V_j) = W_j$, $\forall j = 1, \dots, r$. In order to have a graph isomorphism between $\Gamma_{g|M_{\epsilon_0}}$ and Γ_{γ_g} we need to show that φ is edge preserving.

Consider $U = k(D^2_{\epsilon_0}) - (\Delta(g) \cup B^2_{\epsilon_1})$, and let Y_i be one of its connected components limited by two consecutive half branches A_i and A_{i+1}. We denote by α_i and β_i the arcs of $S^1_{\epsilon_1}$ and P_{ϵ_0} respectively, which bound Y_i, $\forall i = 1, \dots, n$ (see fig. 3). As pointed out in Remark 3.10, the connected components of $g^{-1}(\alpha_i)$ and $g^{-1}(\beta_i)$ give all the edges of the graphs Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0}}$, respectively.

Take X any connected component of $f^{-1}(Y_i)$, for some $1 \le i \le n$. Since $g|X : X \to Y_i$ is regular, the induced map $\tilde{g} : X/\sim \to Y_i$ is a local homeomorphism and hence, a covering space. But Y_i is simply connected, so $\tilde{g}$ is in fact a homeomorphism. We deduce that the boundary of $X/\sim$ has two components: one is an edge of Γ_{γ_g} given by the quotient of $X \cap g^{-1}(\alpha_i)$ and the other is an edge of $\Gamma_{g|M_{\epsilon_0}}$ given by the quotient of $X \cap g^{-1}(\beta_i)$.

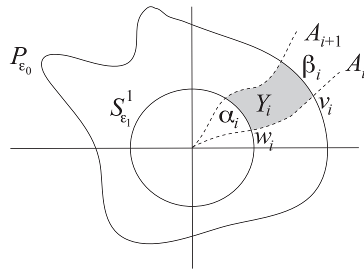


FIGURE 3

Notice that all the edges of Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0}}$ can be obtained in this way, hence we have a bijection between the edges of Γ_{γ_g} and $\Gamma_{g|M_{\epsilon_0}}$ which is compatible with the above bijection φ defined between the vertices.

□

Again, Theorem 3.11 together with Corollary 2.6 show that the Reeb graph is a complete topological invariant for map germs from with isolated zeros.

Corollary 3.12. *Let $f, g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be finitely determined map germs such that $f^{-1}(0) = \{0\} = g^{-1}(0)$. Then the following statements are equivalent:*

(1) *f, g are topologically equivalent,*
(2) *the Reeb graphs of the links of f, g are equivalent,*
(3) *the links of f, g are topologically equivalent.*

4. Topological classification of corank 1 map germs with $f^{-1}(0) = \{0\}$

Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ. After appropriate change of coordinates in the source and the target we can write f as $f(x, y, z) = (x, h_x(y, z))$. In other words, f can be seen as an unfolding of the map germ $h_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. In the case that $f^{-1}(0) = \{0\}$, this also implies that $h_0^{-1}(0) = \{0\}$.

Lemma 4.1. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ given by $f(x, y, z) = (x, h_x(y, z))$. Then $h_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a finitely determined map germ.*

Proof. Since f is finitely determined, then its complexification $f_{\mathbb{C}}$ is also finitely determined and vice-versa and by the Mather-Gaffney criterion $S(f_{\mathbb{C}}) \cap f_{\mathbb{C}}^{-1}(0) = \{0\}$ (see [26]). This implies that $S((h_0)_{\mathbb{C}}) \cap (h_0)_{\mathbb{C}}^{-1}(0) = \{0\}$ and hence h_0 is finitely determined for the contact group $\mathcal{K}$. But for function germs, is well known that the finite determinacy with respect the contact group $\mathcal{K}$ is equivalent to the finite determinacy with respect to the group $\mathcal{A}$ (see again [26]). □

We get a first important consequence of this lemma in the case that $f^{-1}(0) = \{0\}$.

Theorem 4.2. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ with $f^{-1}(0) = \{0\}$. Then the associated link of f is not surjective.*

Proof. Consider f written by $f(x, y, z) = (x, h_x(y, z))$, where h_0 is also finitely determined and $h_0^{-1}(0) = \{0\}$. By Fukuda's theorem 2.4, $h_0^{-1}(S^0_\epsilon)$ is diffeomorphic to S^1, for small enough $\epsilon > 0$.

Suppose that associated link of f is surjective. Then $(0,\epsilon)$ and $(0,-\epsilon)$ belong to image of the map $\gamma_f : f^{-1}(S^1_\epsilon) \to S^1_\epsilon$. But

$$\gamma_f^{-1}(\{(0,\epsilon),(0,-\epsilon)\}) = f^{-1}(\{(0,\epsilon),(0,-\epsilon)\}) \simeq h_0^{-1}(\{\epsilon,-\epsilon\}) \simeq S^1,$$

where $\simeq$ indicates homeomorphism of sets. This gives a contradiction because S^1 is connected, $\{(0,\epsilon),(0,-\epsilon)\}$ is not connected and γ_f is a continuous map. □

Remark 4.3.
(1) It follows from Theorem 4.2 that the stable map $\gamma : S^2 \to S^1$ presented in the right hand side of fig. 2 cannot be realized as the link of a corank 1 finitely determined map germ $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$. Up to this moment, we do not know if in fact, this stable map can be realized or not as the link of a corank 2 map germ.
(2) Another consequence of Theorem 4.2 is that if f has corank 1 and $f^{-1}(0) = \{0\}$, then the generalized Reeb graph can obtained from the classical one, since the link is not surjective (see Remark 3.5). From now on in this section, the Reeb graph will be referred to the classical version, unless otherwise specified.

Given $f(x,y,z) = (x, h_x(y,z))$, we say that f is a stabilization of h_0 if there is a representative $f : U = (-\epsilon,\epsilon) \times V \to \mathbb{R}^2$ such that for any x, with $0 < |x| < \epsilon$, $h_x : V \subset \mathbb{R}^2 \to \mathbb{R}$ is stable (i.e., it is a Morse function with distinct critical values).

Proposition 4.4. *Let $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ be a finitely determined map germ given by $f(x,y,z) = (x, h_x(y,z))$. Then, f is a stabilization of h_0.*

Proof. By Corollary 2.3, if f is finitely determined we can choose a representative $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ such that $S(f) \cap f^{-1}(0) = \{0\}$ and the restriction $f|_{U-\{0\}}$ is stable with only simple definite and indefinite folds. By shrinking U is necessary, we can assume $U = (-\epsilon,\epsilon) \times V$, where V a neighborhood of 0 in $\mathbb{R}^2$ and $\epsilon > 0$. Let us take $x_0 \in (-\epsilon,\epsilon)$, $x_0 \neq 0$.

Suppose that h_{x_0} has a degenerate singularity at $p \in V$, then the Hessian determinant of h_{x_0} at p is equal to 0. Since $p \in S(h_{x_0})$, then $(x_0,p) \in S(f)$ and (x_0,p) is not a fold of f in $U - \{0\}$. Analogously, if h_{x_0} is singular at two distinct points $p_0, p_1 \in V$, such that $h_{x_0}(p_0) = h_{x_0}(p_1)$, then $(x_0,p_0),(x_0,p_1) \in S(f)$ and f should have a double fold at $(x_0,p_0),(x_0,p_1) \in U - \{0\}$. □

Let $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ be a finitely determined map germ given by $f(x,y,z) = (x, h_x(y,z))$. We take a representative $f : U = (-\epsilon,\epsilon) \times V \to \mathbb{R}^2$ such that for any x, with $0 < |x| < \epsilon$, $h_x : V \subset \mathbb{R}^2 \to \mathbb{R}$ is stable. By Lemma 4.1, h_0 has isolated singularity. By shrinking U if necessary, we can also assume that h_0 is regular in $V - \{0\}$. Moreover, in the case that f has isolated zero, we also impose that $f^{-1}(0) = \{0\}$ on U and hence, $h_0^{-1}(0) = \{0\}$ on V.

Because of stability, all the functions $h_x : V \subset \mathbb{R}^2 \to \mathbb{R}$ are $\mathcal{A}$-equivalent if $-\epsilon < x < 0$ and we will denote by h_x^- one of these functions. Analogously, all functions $h_x : V \subset \mathbb{R}^2 \to \mathbb{R}$ are $\mathcal{A}$-equivalent if $0 < x < \epsilon$ and we will denote by h_x^+ one of these functions.

Given a finitely determined map germ $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$, we denote by $X(f)$ the set germ in $(\mathbb{R}^3,0)$ defined by the closure of $f^{-1}(\Delta(f)) - S(f)$. By Corollary 2.3, since f has only folds outside the origin, f is transverse to $\Delta(f)$ and hence, $X(f)$ is a smooth surface outside the origin.

Lemma 4.5. *Let $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ be a finitely determined corank 1 map germ given by $f(x,y,z) = (x, h_x(y,z))$. Then $S(f), X(f)$ and $\Delta(f)$ are transverse to the planes $\{x\} \times \mathbb{R}^2$ and to the lines $\{x\} \times \mathbb{R}$, respectively, with $0 < |x| < \epsilon$ and ϵ small enough.*

Proof. It follows from Proposition 4.4 that there exists $\epsilon > 0$ small enough and $V \subset \mathbb{R}^2$ an open neighborhood of 0 such that $h_x : V \to \mathbb{R}$ is stable for all x, with $0 < |x| < \epsilon$.

Suppose $(x_0, y_0, z_0) \in S(f) \cap \{x_0\} \times \mathbb{R}^2$ and consider a parametrization of $S(f)$ near (x_0, y_0, z_0) given by $\alpha(t) = (x(t), y(t), z(t))$. We only need to show that $x'(t) \neq 0$.

For simplicity we write $H(x, y, z) = h_x(y, z)$. Then $S(f)$ is given by the implicit equations $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial z} = 0$. By taking partial derivatives of these equations:

$$x'\frac{\partial^2 H}{\partial x \partial y} + y'\frac{\partial^2 H}{\partial y^2} + z'\frac{\partial^2 H}{\partial y \partial z} = 0, \quad x'\frac{\partial^2 H}{\partial x \partial z} + y'\frac{\partial^2 H}{\partial y \partial z} + z'\frac{\partial^2 H}{\partial z^2} = 0.$$

If $x' = 0$, since $(y', z') \neq (0, 0)$ we get that

$$\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial z^2} - \left(\frac{\partial^2 H}{\partial y \partial z}\right)^2 = 0.$$

But this is the hessian of h_x at the singular point (y, z), which contradicts the fact that h_x is a Morse function.

Note that $\Delta(f)$ is parametrized by $f(\alpha(t)) = (x(t), H(x(t), y(t), z(t))$ near $f(x_0, y_0, z_0)$. Since $x'(t) \neq 0$, we also have that $\Delta(f)$ is transverse to $\{x_0\} \times \mathbb{R}$ at $f(x_0, y_0, z_0)$.

Finally, let $(x_0, y_0', z_0') \in X(f) \cap \{x_0\} \times \mathbb{R}^2$ be a point such that $f(x_0, y_0, z_0) = f(x_0, y_0', z_0')$. Then the transversality between $X(f)$ to $\{x_0\} \times \mathbb{R}^2$ is a consequence that f is transverse to $\Delta(f)$ and that $X(f) = f^{-1}(\Delta(f))$ and $\{x_0\} \times \mathbb{R}^2 = f^{-1}(\{x_0\} \times \mathbb{R})$ near that point. □

Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ with $f^{-1}(0) = \{0\}$, given by $f(x, y, z) = (x, h_x(y, z))$. By Lemmas 4.1 and 4.5, we consider small enough representatives $f : (-\epsilon, \epsilon) \times V \to \mathbb{R}^2$ such that for any $0 < |x| < \epsilon$, $h_x : V \to$ is stable and moreover $S(f), X(f), \Delta(f)$ are transverse to $\{x\} \times \mathbb{R}^2, \{x\} \times \mathbb{R}$, respectively.

We fix $x_0 \in \mathbb{R}$ such that $0 < |x_0| < \epsilon$ and take $\delta > 0$ small enough such that $(h_{x_0})^{-1}([-\delta, \delta]) \subset V$ and $[-\delta, \delta]$ intersects all the positive (resp. negative) half branches of $\Delta(f)$ if $x_0 > 0$ (resp. if $x_0 < 0$).

Consider the following equivalence relation on $(h_{x_0})^{-1}([-\delta, \delta])$: $v \sim w$ if and only if $h_{x_0}(v) = h_{x_0}(w)$ with v and w in the same connected component of $h_{x_0}^{-1}(h_{x_0}(v))$. Then the quotient $(h_{x_0})^{-1}([-\delta, \delta])/\sim$ has a graph structure whose the vertices are:

(1) The connected components of $h_{x_0}^{-1}(v)$, where v is any critical value of h_{x_0}.
(2) The connected components of the boundary of $(h_{x_0})^{-1}([-\delta, \delta])$. This type of vertex will be called the boundary vertex and will be denoted by the symbol "$\circ$".

Moreover, we denote by $v_1 < \cdots < v_n$ the ordered set of critical values of h_{x_0} together with the value corresponding to the boundary vertex. We assign to each vertex the label $i \in \{1, \ldots, n\}$ if it has value v_i. The graph $(h_{x_0})^{-1}([-\delta, \delta])/\sim$ together with the labels of the vertices is called the Reeb graph of h_{x_0}.

Definition 4.6. We define the *partial tree* of h_x^+ as being the Reeb graph of h_{x_0} if $x_0 > 0$ and the *partial tree* of h_x^- as being the Reeb graph of $-h_{x_0}$ if $x_0 < 0$.

Example 4.7. Consider the map germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by $f(x, y, z) = (x, h_x(y, z))$, where $h_x(y, z) = y^4 + xy^2 + 3x^5 + z^2$. Here h_x has 3 critical values for $x < 0$, but only 1 critical value for $x > 0$. The partial trees of h_x^+ and h_x^- are shown in fig. 4.

We remark that the partial trees h_x^+ and h_x^- do not depend on of the choice of the representatives, the choice of x_0 nor the choice of the interval $[-\delta, \delta]$. This follows from

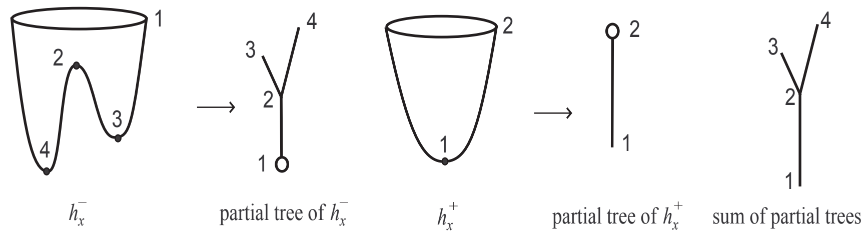


FIGURE 4

the fact that the functions $h_x : V \to \mathbb{R}$ are all $\mathcal{A}$-equivalent if either $-\epsilon < x < 0$ or $0 < x < \epsilon$. Then we can use the same arguments of that of the proof of Theorem 3.7.

Consider the partial trees of h_x^+ and h_x^-. Assume that $u_1 < \ldots < u_r$ and $v_1 < \ldots < v_s$ are the critical values of h_x^+ and h_x^-, respectively. Since $f^{-1}(0) = \{0\}$, the link γ_f is not surjective and, without loss of generality, we can assume that $(0, \epsilon)$ is a regular value which belongs to the image of the link. Consequently, u_r and v_s correspond to the boundary vertices of h_x^+ and h_x^-, respectively.

Definition 4.8. Let $\Gamma_{x>0}$ and $\Gamma_{x<0}$ be the graphs corresponding to the partial trees of h_x^+ and h_x^-, respectively. Consider Γ the graph obtained by connecting the upper edge of $\Gamma_{x>0} - \{u_r\}$ to the lower edge of $\Gamma_{x<0} - \{v_s\}$. We relabel each vertex v_{s-i} by $u_{r+(i-1)}$, where $i = 1, \ldots, s-1$. We say that Γ is the *sum of partial trees of* h_x^+ and h_x^-.

Example 4.9. The sum of the partial trees of the map germ in Example 4.7 is also shown in the right hand side of fig. 4.

The main result of this Section is the following:

Theorem 4.10. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ with $f^{-1}(0) = \{0\}$, given by $f(x, y, z) = (x, h_x(y, z))$. Then, the sum of partial trees of h_x^+ and h_x^- is equivalent to the Reeb graph of the associated link of f.*

Proof. Take $\epsilon > \delta > 0$ small enough and $V \subset \mathbb{R}^2$ a neighborhood of origin such that the following four conditions are satisfied:

i) $\gamma_f : \tilde{S}^2_\delta \to S^1_\delta$ is the link of f;
ii) The function $h_x|V : V \to \mathbb{R}$ is stable for all $x \in (-\epsilon, \epsilon)$, $x \neq 0$;
iii) $\{x\} \times V$ intercepts all half branches of $S(f)$ with the same sign of x;
iv) $\tilde{S}^2_\delta \subset (-\epsilon, \epsilon) \times V$.
v) $h_0^{-1}(0) = \{0\}$ and h_0 is regular on $V - \{0\}$.

We have from v) that $S(f) \cap (\{0\} \times \mathbb{R}^2) = \{0\}$ and $\Delta(f) \cap (\{0\} \times \mathbb{R}) = \{0\}$. Hence $(0, \delta)$ and $(0, -\delta)$ are regular values of $\gamma_f : \tilde{S}^2_\delta \to S^1_\delta$. Moreover, since the link of f is not surjective just one of the points $(0, -\delta)$, $(0, \delta)$ belongs to the image of link. We assume here that $(0, \delta) \in \mathrm{Im}(\gamma_f)$.

Let $A_1, \ldots, A_n$ be the half branches of $\Delta(f)$ considered in the anti-clockwise orientation and such that $(0, -\delta)$ is the base point. We also assume that $A_1, \ldots, A_r$ are on the half plane $x > 0$ and that $A_{r+1}, \ldots, A_n$ are on the half plane $x < 0$.

By the cone structure of f, each half branch A_i intersects S^1_δ in a unique point v_i, so that $v_1 < \cdots < v_n$ are the critical points of γ_f in the chosen orientation. By the transversality of $\Delta(f)$ to the vertical lines $\{x\} \times \{\mathbb{R}\}$, given $\delta < x < \epsilon$ we have that each half branch A_i also intersects $\{x\} \times \{\mathbb{R}\}$ in a unique point w_i. But now $w_1 < \cdots < w_r$ are the critical values of h_x^+ and $w_n < \cdots < w_{r+1}$ the critical values of h_x^-.

Since we are considering the classical version of the Reeb graph, each critical value corresponds to a unique vertex. Thus, there is a bijection given by $\varphi(v_i) = w_i$ for $i \in \{1, \ldots, n\}$ between the vertices of Γ_{γ_f} and the vertices of Γ, the sum of the partial trees of h_x^+ and h_x^-. Moreover, the bijection is compatible with the labels of the vertices as defined in Definition 4.8.

To finish the proof, we only need to show that there is also a bijection between the edges compatible with φ. Consider the following sets (fig. 5):

- U_i the set of points limited by A_i, A_{i+1}, S^1_δ and $\{x\} \times \mathbb{R}$;
- α_i the arc of S^1_δ limited by A_i and A_{i+1};
- β_i the segment of line of $\{x\} \times \mathbb{R}$ limited by A_i and A_{i+1};
- $Y_i = U_i \cup \alpha_i \cup \beta_i$

with $\delta < x < \epsilon$ if $1 \leq i < r$ and $-\epsilon < x < -\delta$ if $r + 1 \leq i < n$.

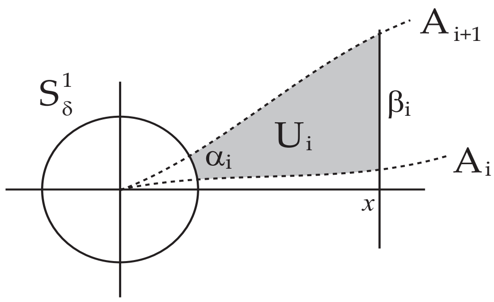


FIGURE 5

Each one of the connected components of $f^{-1}(\alpha_i)$ and $f^{-1}(\beta_i)$ gives an edge for the graphs Γ_{γ_f} and Γ, respectively.

Let X be any connected component of $f^{-1}(Y_i)$. Notice that $f|X : X \to Y_i$ is regular. So, the induced map $\tilde{f} : X/\sim \to Y_i$ is a local homeomorphism and hence, a covering map. Since Y_i is simply connected and X is connected we have that $\tilde{f}$ is a homeomorphism. Hence, $X/\sim$ contains only one edge of Γ_{γ_f} corresponding to $X \cap f^{-1}(\alpha_i)$, and also only one edge of Γ corresponding to $X \cap f^{-1}(\beta_i)$.

Moreover, since $f^{-1}(0, \delta)$ is diffeomorphic to S^1, the arc of S^1_δ delimited by A_s and A_{s+1} corresponds to a unique edge of Γ_{γ_f}. We associate this edge with the edge of Γ used to join the partial trees of h_x^+ and h_x^-.

In this way, we can define a bijection ϕ between the edges of Γ_{γ_f} and the edges of Γ, which is compatible with φ. Hence the graphs Γ_{γ_f} and Γ are equivalent. □

4.1. **Classification of germs with Boardman symbol $\Sigma^{2,1}$.**

Next, we state a result due Rieger and Ruas ([23]) which gives a classification of corank 1 map germs according to its 2-jet. We denote by $\Sigma^1 J^2(3,2)$ the space of 2-jets of corank 1 map germs from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^2, 0)$ and $\mathcal{A}^2$ denotes the space of 2-jets of diffeomorphisms in the source and target.

Lemma 4.11. *There exist the following orbits in $\Sigma^1 J^2(3,2)$ under the action of $\mathcal{A}^2$:*

$$(x, y^2+z^2),\quad (x, y^2-z^2),\quad (x, xy+z^2),\quad (x, xy-z^2),\quad (x, z^2),\quad (x,0)$$

The germ $f(x,y,z) = (x, y^2 \pm z^2)$ is 2-$\mathcal{A}$-determined. Thus, if a map germ has 2-jet equivalent to $(x, y^2 \pm z^2)$ then it is in fact $\mathcal{A}$-equivalent to the definite or indefinite fold. Hence, we do not need consider this case. The orbits distinct to $(x,0)$ have Boardman symbol $\Sigma^{2,1}$.

Now, we center our attention in corank 1 finitely determined map germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, with $f^{-1}(0) = \{0\}$ and Boardman symbol $\Sigma^{2,1}$. By Splitting Lemma [23], we can choose coordinates in the source an the target such that f is given by $f(x,y,z) = (x, \tilde{h}_x(y) + z^2)$. Moreover, $\tilde{h}_0$ is $\mathcal{A}$-equivalent to y^k, for some k even and by using the versal unfolding of y^k we can assume that

$$\tilde{h}_x(y) = y^k + a_{k-2}(x)y^{k-2} + ... + a_1(x)y.$$

Notice that k is the multiplicity of $\tilde{h}_0$.

We want to construct the partial trees of h_x^+ and h_x^-, where $h_x(y,z) = \tilde{h}_x(y) + z^2$. The Jacobian and Hessian matrices of $h_x(y,z)$ are, respectively:

$$J = \begin{pmatrix} \tilde{h}'_x(y) & 2z \end{pmatrix}, \quad H = \begin{pmatrix} \tilde{h}''_x(y) & 0 \\ 0 & 2 \end{pmatrix}.$$

Hence the critical points of h_x are those of the form $(y,0)$, where y is a critical point of $\tilde{h}_x$. Moreover, $(y,0)$ is a saddle point of h_x if an only if y is a maximum of $\tilde{h}_x$ and $(y,0)$ is a maximum or minimum of h_x if an only if y is a minimum of $\tilde{h}_x$.

Example 4.12. Let $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ be a corank 1 finitely determined map germ with $f^{-1}(0) = \{0\}$ with Boardman symbol $\Sigma^{2,1}$ and multiplicity 4. After change of coordinates in the source and target, we can assume f is given by

$$f(x,y,z) = (x, y^4 + a(x)y^2 + b(x)y + z^2).$$

Notice that the bifurcation set $\mathcal{B}$ of the versal unfolding of h_0 in this case is given in the (a,b)-plane by by $b(-4a^3b - 27b^3) = 0$ (see fig. 6), which permits us to choose appropriate functions $a(x)$ and $b(x)$ such that we can obtain all types of possible trees.

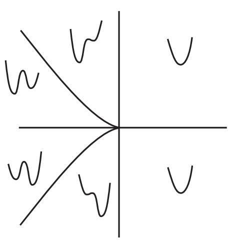


FIGURE 6

Then, there are 3 possibilities to the Reeb graph of the link of f, according to the number of saddles:

- 0 saddle, f is topologically equivalent to $(x, y^4 + x^2y + z^2)$ (see fig. 7);
- 1 saddle, f is topologically equivalent to $(x, y^4 + xy^2 + 3x^5y + z^2)$ (see fig. 8);
- 2 saddles, f is topologically equivalent to $(x, y^4 - x^2y^2 + x^5y + z^2)$. (see fig. 9);

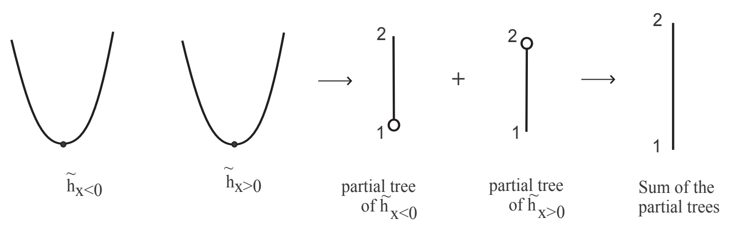


FIGURE 7

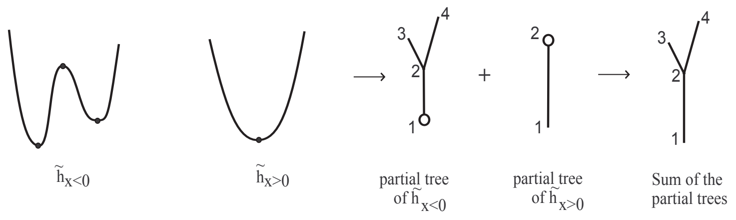


FIGURE 8

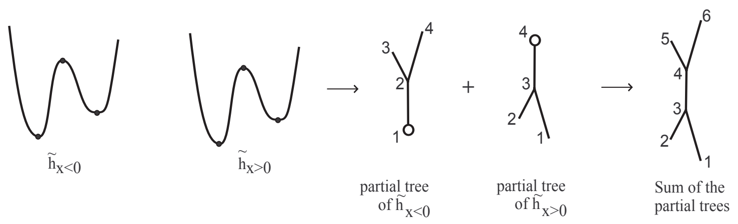


FIGURE 9

Theorem 4.13. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be a corank 1 finitely determined map germ, $f^{-1}(0) = \{0\}$ with Boardman symbol $\Sigma^{2,1}$ and multiplicity $\leq$ 6. Then all the possibilities for the Reeb graph of the link of f are realized and are presented in Table 1.*

Proof. Assume that f is given by

$$f(x, y) = (x, y^6 + a(x)y^4 + b(x)y^3 + c(x)y^2 + d(x)y + z^2).$$

Notice that $\tilde{h}_x$ may have 0, 1 or 2 saddles as shown in fig. 10. All the possibilities for the

FIGURE 10

partial trees of the link of f are given in fig. 11.

In this way, all the Reeb graphs of the link of f can be obtained by taking all possible combinations among these six models of partial trees. Note that $(a) + (a)$ is equivalent to

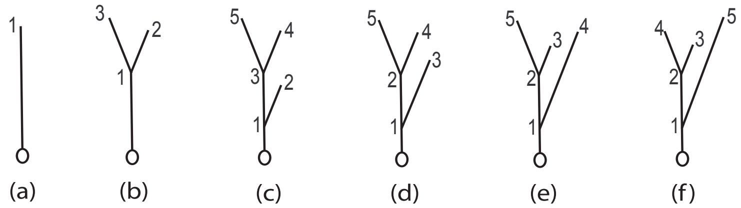
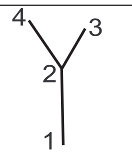
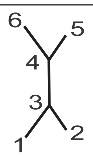
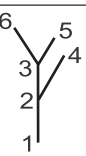
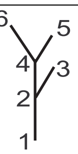
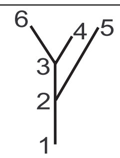
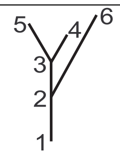
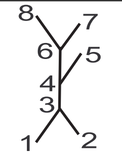
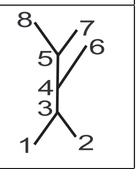
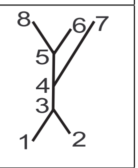
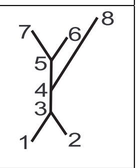
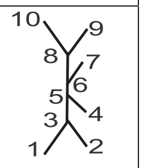
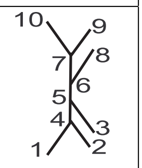
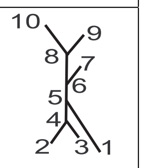
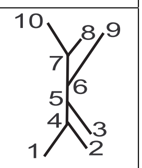
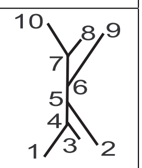
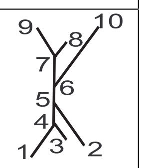
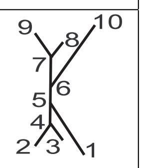


FIGURE 11

the Reeb graph of $(x, y^2 + z^2)$; $(a) + (b)$ and $(b) + (b)$ are equivalent to the Reeb graphs given in the Example 4.12.

Germ	Associated Tree
$(x, y^2 + z^2)$	
$(x, y^4 + xy^2 + 3x^5y + z^2)$	
$(x, y^4 - x^2y^2 + x^5y + z^2)$	
$(x, y^6 + 2xy^4 + x^2y^2 + x^4y + z^2)$	
$(x, y^6 + 2xy^4 + x^3y^3 - x^2y^3 - x^4y^2 + \frac{5}{4}x^2y^2 + x^4y + z^2)$	
$(x, y^6 + xy^4 + x^3y^3 + x^4y^2 + x^7y + z^2)$	
$(x, y^6 + x^3y^4 + \frac{1}{9}xy^4 + x^3y^3 + \frac{1}{9}x^4y^2 + x^6y + z^2)$	

$(x, y^6 - \frac{3}{10}x^2y^4 - \frac{1}{15}x^3y^3 - \frac{1}{2}x^5y^2 - \frac{1}{5}x^6y + z^2)$	
$(x, y^6 + 6x^3y^4 + 9x^6y^2 + 9x^9y + z^2)$	
$(x, y^6 - 4x^2y^4 + x^4y^3 - 3x^5y^2 + z^2)$	
$(x, y^6 - 6x^2y^4 + xy^4 + x^4y^3 - 6x^3y^2 - 6x^6y + z^2)$	
$(x, y^6 - 4x^4y^4 + 4x^8y^2 - 2x^{10}y + z^2)$	
$(x, y^6 - \frac{93}{20}x^4y^4 + 4x^8y^2 - 2x^{10}y + z^2)$	
$(x, y^6 + \frac{1}{2}xy^5 + \frac{1}{16}x^2y^4 + \frac{1}{12}x^4y^3 - \frac{1}{8}x^7y^2 + z^2)$	
$(x, y^6 - \frac{1}{10}xy^5 - \frac{23}{40}x^3y^4 - \frac{35}{32}x^5y^3 - \frac{441}{640}x^7y^2 + z^2)$	
$(x, y^6 - x^2y^4 + x^4y^3 + x^6y^2 + z^2)$	
$(x, y^6 + \frac{1}{45}x^2y^4 - \frac{1}{15}x^4y^3 - \frac{1}{20}x^6y^2 + \frac{1}{15}x^9y + z^2)$	
$(x, y^6 - \frac{3}{6}x^2y^4 + \frac{1}{3}x^5y^3 + 3x^6y^2 - x^9y + z^2)$	

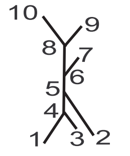
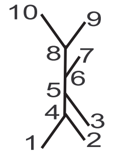
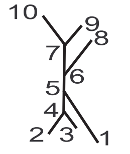
$(x, y^6 - 6x^2y^5 - \frac{4}{5}xy^5 + 4x^3y^4 - 5x^8y^3 + 15x^8y^2 + z^2)$	
$(x, y^6 + 6xy^5 + 16x^3y^4 + 14x^5y^3 + 4x^7y^2 + z^2)$	
$(x, y^6 - \frac{27}{10}xy^5 - \frac{9}{5}x^3y^4 + \frac{33}{160}x^5y^3 + \frac{81}{320}x^7y^2 + \frac{81}{80}x^{10}y + z^2)$	

Table 1

□

Appendix A. Morse functions and Cobordism

In this Appendix we will describe some results about Morse function theory and cobordism theory given by V.I. Arnold, J. Milnor and S.A. Izar (cf. [1, 9, 10, 11, 15]). We adopt the notation and basic definitions that are usual in Morse theory and cobordism theory. The reader can use [14, 15] as basic references.

Definition A.1. We say that $(M; V_0, V_1)$ is a *smooth triad* if M is a smooth compact manifold with boundary and ∂M is the disjoint union of two closed submanifolds V_0 and V_1 (see fig. 12).

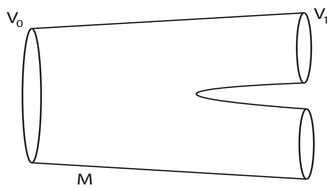


Figure 12

Definition A.2. A *Morse function on a smooth triad* $(M; V_0, V_1)$ is a smooth function $f : M \to [a, b]$ such that

i) $f^{-1}(a) = V_0$ and $f^{-1}(b) = V_1$;
ii) all critical points of f are interior (lie in $M - \partial M$) and non-degenerated;
iii) f is injective when restricted to the set of its critical points.

Roughly speaking, by using Morse functions it is possible to express any complicated cobordism as a composition of simpler cobordisms.

Theorem A.3. *([15]) For every Morse function f on a triad $(M; V_0, V_1)$, there exists a gradient-like vector field ξ for f.*

Theorem A.4. *([15]) If a triad $(M; V_0, V_1)$ admits a function without critical points, then it is a product cobordism, i.e., it is diffeomorphic to the triad $(V_0 \times [0,1], V_0 \times \{0\}, V_0 \times \{1\})$.*

Definition A.5. ([15] Characteristic embedding) Let $(M; V_0, V_1)$ be a triad with a Morse function $f : M \to \mathbb{R}$ and a gradient-like vector field ξ for f. Suppose $p \in M$ is a critical point of f and let $V_0 = f^{-1}(c_0)$ and $V_1 = f^{-1}(c_1)$ be the levels such that $c_0 < c = f(p) < c_1$, where c is the unique critical value of f in $[c_0, c_1]$.

Since ξ is a gradient-like vector field for f, there exists a neighborhood U of p in M and a parametrization $\alpha : B^n_{2\epsilon} \to U$ such that $f \circ \alpha(x, y) = f(p) - |x|^2 + |y|^2$ and so that ξ has coordinates $(-x, y)$ through U, where $x = (x_1, \ldots, x_\lambda)$, $y = (x_{\lambda+1}, \ldots, x_n)$ for some $0 \le \lambda \le n$ and $\epsilon > 0$. Set $V_\epsilon = f^{-1}(c + \epsilon^2)$ and $V_{-\epsilon} = f^{-1}(c - \epsilon^2)$. We may assume $4\epsilon^2 < \min\{|c - c_0|, |c - c_1|\}$, so that $V_{-\epsilon}$ lies between V_0 and $f^{-1}(c)$ and V_ϵ lies between $f^{-1}(c)$ and V_1. The situation is represented schematically in fig. 13:

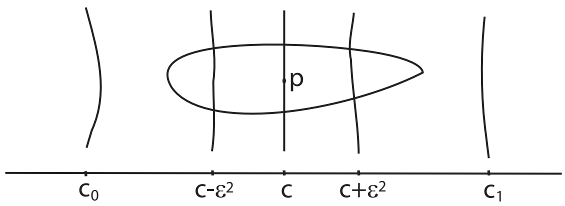


FIGURE 13

The *left characteristic embedding* of p is a map $\phi_L : S^{\lambda-1} \times B^{n-\lambda} \to V_0$ obtained as follows. First define an embedding $\phi : S^{\lambda-1} \times B^{n-\lambda} \to V_{-\epsilon}$ by

$$\phi(u, \theta v) = \alpha(\epsilon u \cosh(\theta), \epsilon v \sinh(\theta)), \quad u \in S^{\lambda-1}, v \in S^{n-\lambda-1}, 0 \le \theta < 1.$$

Starting at the point $\phi(u, \theta v)$ in $V_{-\epsilon}$ the integral curve of ξ is a non-singular curve which leads from $\phi(u, \theta v)$ back to some well-defined point $\phi_L(u, \theta v)$ in V_0. Define the left-hand sphere S_L of p in V_0 to be the image $\phi_L(S^{\lambda-1} \times \{0\})$. Notice that S_L is just the intersection of V_0 with all integral curves of ξ leading to the critical point p . The left hand-disk D_L is a smoothly embedded disk with boundary S_L, defined to be the union of all segments of these integral curves beginning in S_L and ending at p (see fig. 14).

Similarly the *right characteristic embedding* $\phi_R : B^\lambda \times S^{n-\lambda-1} \to V_1$ is obtained by embedding $\phi : B^\lambda \times S^{n-\lambda-1} \to V_\epsilon$ by

$$(\theta u, v) \mapsto \alpha(\epsilon u \sinh(\theta), \epsilon v \cosh(\theta)),$$

and then translating the image to V_1. The right-hand sphere S_R of p in V_1 is defined to be $\phi_R(\{0\} \times S^{n-\lambda-1})$. It is the boundary of the right-hand disk D_R, defined as the union of segments of integral curves of ξ beginning at p and ending in S_R.

Theorem A.6. *([10]) Let $(M; V_0, V_1)$ and $(M'; V_0', V_1')$ be two triads with Morse functions $f : M \to [c_0, c_1]$ and $g : M' \to [c_0, c_1]$, where M and M' are compact 2-manifold. Suppose $c_0 < c < c_1$ be the unique critical value of f and g. Moreover, suppose that it corresponds to a unique critical point $p \in f^{-1}(c)$ and $q \in g^{-1}(c)$ such that the index of f in p is equal to index of g in q. Assume that $f^{-1}(c_i) \approx g^{-1}(c_i)$, $i = 0, 1$. Then there exists a diffeomorphism $h : M \to M'$ such that $f = g \circ h$.*

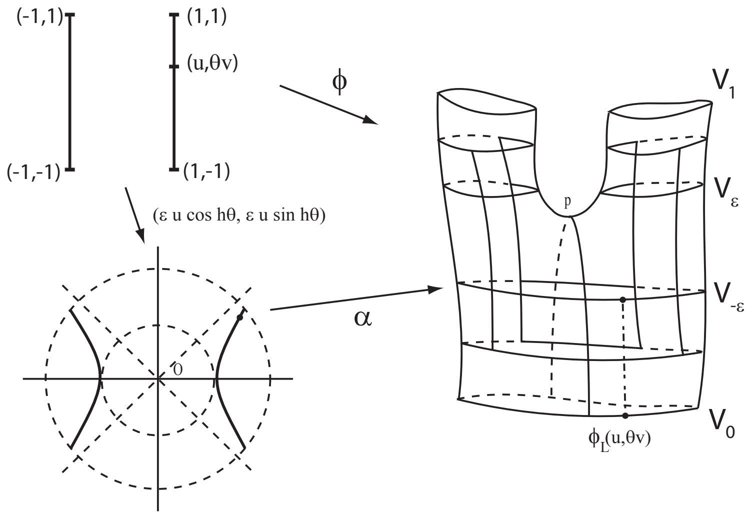


FIGURE 14

REFERENCES

[1] V.I. Arnold, *Topological classification of Morse functions and generalisations of Hilbert's 16-th problem*, Math. Phys. Anal. Geom. Vol. 10 (2007) 227–236.

[2] E.B. Batista, J.C.F. Costa, J.J. Nuño-Ballesteros, *The Reeb graph of a map germ from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ *with non isolated zeros*, in preparation (2013).

[3] J.C.F. Costa, J.J. Nuño-Ballesteros, *Topological* $\mathcal{K}$*-classification of finitely determined map germs*, Geom. Dedicata 166 (2013) 147–162.

[4] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Colloque de Topologie, Bruxelles (1950) 29–55.

[5] T. Fukuda, *Types topologiques des polynômes*, Publ. Math. I.H.E.S. no. 46 (1976) 87–106.

[6] T. Fukuda, *Local Topological Properties of Differentiable Mappings I*, Invent. Math. 65 no. 2 (1981/82) 227–250.

[7] T. Fukuda, *Local Topological Properties of Differentiable Mappings II*, Tokyo J. Math. 8, no.2 (1985) 501–520.

[8] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer, New York, 1973.

[9] S.A. Izar, *Funções de Morse e topologia das superfícies I: O grafo de Reeb de* $f : M \to \mathbb{R}$, Métrica no. 31, Estudo e Pesquisas em Matemática, IBILCE, Brazil, 1988. (Available at `http://www.ibilce.unesp.br/Home/Departamentos/Matematica/metrica-31.pdf`)

[10] S.A. Izar, *Funções de Morse e topologia das superfícies II: Classificação das funções de Morse estáveis sobre superfícies*, Métrica no. 35, Estudo e Pesquisas em Matemática, IBILCE, Brazil, 1989. (Available at `http://www.ibilce.unesp.br/Home/Departamentos/Matematica/metrica-35.pdf`)

[11] S.A. Izar, *Funções de Morse e topologia das superfícies III: Campos pseudo-gradientes de uma função de Morse sobre uma superfície*, Métrica no. 44, Estudo e Pesquisas em Matemática, IBILCE, Brazil, 1992. (Available at `http://www.ibilce.unesp.br/Home/Departamentos/Matematica/metrica-44.pdf`)

[12] H. Levine, *Elimination of cusps*, Topology 3 (suppl. 2) (1965) 263–296.

[13] W.L. Marar, J.J. Nuño-Ballesteros, *The doodle of a finitely determined map germ from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$, Adv. Math. 221 (2009) 1281–1301.

[14] J.W. Milnor, *Morse Theory*, Annals of Math. Studies 51, Princeton University Press, Princeton, N.J. 1963.

[15] J.W. Milnor, *Lectures on the h-cobordism theory*, Math. Notes, Princeton Univ. Press 1965.

[16] J.A. Moya-Pérez, J.J. Nuño-Ballesteros, *The link of finitely determined map germ from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, J. Math. Soc. Japan 62, no. 4 (2010) 1069–1092.

[17] J.A. Moya-Pérez, J.J. Nuño-Ballesteros, *Gauss words and the topology of map germs from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$, Preprint (2013), available at http://www.uv.es/nuno
[18] J.A. Moya-Pérez, J.J. Nuño-Ballesteros, *Topological classification of corank 1 map germs from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$, to appear in Rev. Mat. Complut. (2013), doi:10.1007/s13163-013-0137-z
[19] I. Nakai, *On topological types of polynomial mappings*, Topology 23, no. 1 (1984) 45–66.
[20] T. Nishimura, *Topological types of finitely* C^0*-*$\mathcal{K}$*-determined map germs*, Trans. Amer. Math. Soc. 312, no. 2 (1989) 621–639.
[21] A.O. Prishlyak, *Topological equivalence of smooth functions with isolated critical points on a closed surface*, Topology and its Applications 119 (2002) 257–267.
[22] G. Reeb, *Sur les points singuliers d'une forme de Pfaff completement intégrable ou d'une fonction numérique*, C. R. Acad. Sci. Paris 222 (1946) 847–849.
[23] J.H. Rieger and M.A.S. Ruas, *Classification of A-simple germs from* k^n *to* k^2, Compositio Math. 79 (1991) 99–108.
[24] V.V. Sharko, *Smooth and topological equivalence of functions on surfaces*, Ukrainian Math. J. 55 (2003) 832–846.
[25] R. Thom, *Local topological properties of differentiable mappings*, Colloquium on Differential Analysis (Tata Inst.), Oxford Univ. Press, Oxford (1964) 191–202.
[26] C.T.C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. 13, no. 6 (1981) 481–539.

Departamento de Matemática, IBILCE-UNESP, Campus de São José do Rio Preto-SP, Brazil
E-mail address: ericabbatista@gmail.com

Departamento de Matemática, IBILCE-UNESP, Campus de São José do Rio Preto-SP, Brazil
E-mail address: jcosta@ibilce.unesp.br

Departament de Geometria i Topologia, Universitat de València, Campus de Burjassot 46100 Spain
E-mail address: Juan.Nuno@uv.es