A NOTE ON FINITE DETERMINACY FOR CORANK 2 MAP GERMS FROM SURFACES TO 3-SPACE

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Abstract. We study properties of finitely determined corank 2 quasihomogeneous map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Examples and counter examples of such map germs are presented.

1. Introduction

The mere existence of corank 2 finitely determined quasihomogeneous map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can be quite a surprise. Indeed, the three adjectives create tremendous restrictions and examples seems hard to find.

In [7], D. Mond gives formulas for the analytic invariants of finitely determined quasihomogeneous map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ in terms of the weights and degrees. However, when you try to apply these formulas in the corank 2 case, it is not easy to find examples. Even for homogeneous map germs, we find some examples in [2] or [3], but it becomes clear that there are strong restrictions.

In this paper, we study several types of quasihomogeneous map germs and discuss when they can be finitely determined or not. Firstly we consider the class of double fold map germs; that is, map germs of the form $f(x, y) = (x^2, y^2, g(x, y))$. We prove that there exist homogeneous double fold finitely determined map germs, but there are no quasihomogeneous ones (with distinct weights).

Then we present a study of finite determinacy of a general quasihomogeneous map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ whose weights of the variables $(x, y) \in \mathbb{C}^2$ are respectively $\omega_x = 2$ and $\omega_y = 1$. Finally, we turn our attention to double cusp map germs; that is, map germs of the form $f(x, y) = (x^2, xy + y^3, g(x, y))$. For this kind of map germs we find examples of finitely determined quasihomogeneous ones, when the weighted degree of g is congruent to $1, 2, 5$ or 10 modulo 12. In fact, we show this is a necessary condition for f to be finitely determined.

2. Generalities

All map germs considered are holomorphic except otherwise stated and we adopt the notation that is usual in singularity theory, as the reader can find in Wall's paper [9]. Throughout, finite determinacy of map germs means finite determinacy under the equivalence given by changes of coordinates in the source and target, that is, $\mathcal{A}$-finite determinacy.

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2.1. **Finite determinacy criteria.** David Mond in his seminal work [6] on the geometry of map germs from surfaces to 3-space shows that finite determinacy of $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is equivalent to the finiteness of three analytic invariants, namely $C(f)$ the number of pinch-points, $T(f)$ the number of triple points and $N(f)$ that measures, in some sense (sic) the non transverse self-intersections. This is a very useful result to check finite determinacy of map germs, for the alternative would involve quite lengthy calculations of tangent spaces to the orbit of f, under the action of the group $\mathcal{A}$. When f is of corank 1 at the origin we write $f(x, y) = (x, p(x, y), q(x, y))$ and consider $\mathcal{I}_2(f)$ the $\mathcal{O}_3$-ideal generated by the divided differences

$$\mathcal{I}_2(f) = \left\langle \frac{p(x, y) - p(x, u)}{y - u}, \frac{q(x, y) - q(x, u)}{y - u} \right\rangle.$$

In [4] the number $N(f)$ is replaced by $\mu(\widetilde{D}^2(f)/\mathbb{Z}_2)$, the Milnor number of the curve $\widetilde{D}^2(f)/\mathbb{Z}_2$, where $\widetilde{D}^2(f) = V(\mathcal{I}_2(f))$ is the double point scheme of f. There it is proved that

$$\mu(D^2(f)) = 6T(f) + C(f) + 2\mu(\widetilde{D}^2(f)/\mathbb{Z}_2) - 1,$$

where $\mu(D^2(f))$ is the Milnor number of a defining equation of the plane curve singularity $D^2(f) \subset \mathbb{C}^2$, the double point curve of f. As a corollary ([4] 3.5), $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely determined iff $\mu(D^2(f)) < \infty$.

So, finiteness of the single invariant $\mu(D^2(f))$ is equivalent to finite determinacy of $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and we shall refer to this number as the *Mond number* of the map germ f. While in [4] all is done for the corank 1 case, few adaptations extend the results to the corank 2 case, as well. We use this criteria, namely, finiteness of the Mond number of f to study finite determinacy of some classes of corank two map-germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

Given a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, there are some recipes to obtain a defining equation for the double point curve $D^2(f) \subset \mathbb{C}^2$ of f ([8], [2]). In [8] $D^2(f)$ is defined as $f^{-1}(V(\mathcal{F}_1))$, where $\mathcal{F}_1$ is the first Fitting ideal of the presentation matrix of the push-forward $f_*\mathcal{O}_2$ as $\mathcal{O}_3$-module. Alternatively, an efficient method to obtain a defining equation of the double point curve $D^2(f) \subset \mathbb{C}^2$ is achieved by comparing a defining equation of the image of f with its parametrization.

Proposition 2.1 ([2]). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a finite map germ, so that the image of f is a hypersurface with defining equation given by the zeros of $F : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Suppose further that f is an immersion off a subset of $\mathbb{C}^n$ of codimension 2. Then for some holomorphic function germ $\lambda : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ we have $\lambda\Delta_i = \pm\frac{\partial F}{\partial x_i}(f)$, where Δ_i is the determinant of the matrix obtained by deleting the i^{th} row from the Jacobian matrix of f. Moreover, $\lambda(x, y) = 0$ is a defining equation of the double point curve $D^2(f) \subset \mathbb{C}^n$.*

In particular, given a finite map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, $f(x, y) = (X(x, y), Y(x, y), Z(x, y))$, then a defining equation of $D^2(f) \subset \mathbb{C}^2$ is obtained as the zeros of $\lambda(x, y) = F_Z^*/\eta$, where F_Z^* is the partial derivative of $F(X, Y, Z)$ with respect to the variable Z restricted to the image of f, that is, $F_Z^* = F_Z(X(x, y), Y(x, y), Z(x, y))$ and η is the 2×2 determinant of the partial derivatives

$$\eta = \begin{vmatrix} X_x & X_y \\ Y_x & Y_y \end{vmatrix}.$$

2.2. **Quasihomogeneous and semi-quasihomogeneous functions.**

Definition 2.2. A polynomial $p(x_1,\ldots,x_n)$ is *quasihomogeneous* if there are positive integers $\omega_1,\ldots,\omega_n$, with no common factor and an integer d such that $p(t^{\omega_1}x_1,\ldots,t^{\omega_n}x_n) = t^d p(x_1,\ldots,x_n)$. The number ω_i is called the weight of the variable x_i and d is called the weighted degree of p. In this case, we say p is of type $(d;\omega_1,\ldots,\omega_n)$.

Remark 2.3. By allowing the weights of the variables to be positive rational numbers we can make the weighted degree equals to one. This yields an equivalent definition, namely, a polynomial $p(x_1,\ldots,x_n)$ is *quasihomogeneous* of type $(r_1,\ldots,r_n)$ if p can be expressed as a linear combination of monomials $x_1^{i_1}\ldots x_n^{i_n}$ for which $i_1/r_1+\ldots+i_n/r_n = 1$, where $r_1,\ldots,r_n$ are positive rational numbers. The equality $dr_i = \omega_i$ makes it explicit the relation between the two definitions.

If a quasihomogeneous polynomial $p : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ has isolated singularity at the origin then its Milnor number $\mu(p)$ can be expressed by the weights of the variables.

Theorem 2.4 (Milnor-Orlik [5])**.** *Assume $p : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ has isolated singularity at the origin. If $p(x_1,\ldots,x_n)$ is quasihomogeneous of type $(r_1,\ldots,r_n)$ then*

$$\mu(p) = (r_1-1)\ldots(r_n-1).$$

Definition 2.5. A polynomial $p(x_1,\ldots,x_n)$ is *semiquasihomogeneous* of type $(d;\omega_1,\ldots,\omega_n)$ if $p(x_1,\ldots,x_n) = p_0(x_1,\ldots,x_n) + p_1(x_1,\ldots,x_n)$, with p_0 quasihomogeneous of type $(d;\omega_1,\ldots,\omega_n)$ and non-degenerated (that is, $\mu(p_0) < \infty$) and all monomials of p_1 have weighted degrees greater than d.

Theorem 2.6 (Arnol'd [1])**.** *If $p = p_0 + p_1$ is semi-quasihomogeneous then $\mu(p) = \mu(p_0)$.*

These definitions extend to polynomial map germs $f : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ by just requiring each coordinate function f_i to be quasihomogeneous of type $(d_i;\omega_1,\ldots,\omega_n)$. In particular, we say that $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ is quasihomogeneous of type $(d_1,d_2,d_3;\omega_1,\omega_2)$ or, in short, f is of type (ω_x,ω_y).

A result for quasihomogeneous finitely $\mathcal{A}$-determined map germs $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, analogous to that of Milnor/Orlik for isolated singularity function germ, has been obtained by David Mond.

Theorem 2.7 (Mond [7])**.** *Let $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ be a finitely $\mathcal{A}$-determined quasi homogeneous map germ of type $(d_1,d_2,d_3;\omega_1,\omega_2)$. If $\epsilon = d_1+d_2+d_3-\omega_1-\omega_2$ and $\delta = d_1d_2d_3/(\omega_1\omega_2)$, then the number $C(f)$ of cross-caps, the number $T(f)$ of triple points, the Milnor number $\mu(D^2(f))$ of the double point curve and the $\mathcal{A}_e$-codimension of f are given in terms of the weights and degrees, as follows:*

$$C(f) = \frac{1}{\omega_1\omega_2}((d_2-\omega_1)(d_3-\omega_2)+(d_1-\omega_2)(d_3-\omega_2)+(d_1-\omega_1)(d_2-\omega_1)),$$

$$T(f) = \frac{1}{6\omega_1\omega_2}(\delta-\epsilon)(\delta-2\epsilon)+\frac{C(f)}{3}, \qquad \mu(D^2(f)) = \frac{1}{\omega_1\omega_2}(\delta-\epsilon-\omega_1)(\delta-\epsilon-\omega_2),$$

$$\mathcal{A}_e - cod(f) = \frac{1}{2}(\mu(D^2(f)) - 4T(f) + C(f) - 1).$$

3. Double fold maps

By means of changes of coordinates in the source and in the target, all corank 1 map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ can be written in the form $f(x, y) = (x, p(x, y), q(x, y))$. Thus, a multiple point problem for a corank 1 map germ is reduced to a one variable problem. This is not the case for corank 2 map germs, since no obvious normal forms are available. However, for some special classes of corank 2 map germs some reduced forms are obtained; for instance, the *double fold maps*, that is map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of the form $f(x, y) = (x^2, y^2, p(x, y))$. Indeed, by taking again coordinate changes and using the Malgrange preparation theorem, any double fold map germ can be written in the form

$$f(x, y) = (x^2, y^2, xp_1(x^2, y^2) + yp_2(x^2, y^2) + xyp_3(x^2, y^2)).$$

Proposition 3.1. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a double fold map germ. If f is written in the form $f(x, y) = (x^2, y^2, xp_1(x^2, y^2) + yp_2(x^2, y^2) + xyp_3(x^2, y^2))$, then:*

(i) *The double point curve $D^2(f)$ has defining equation*

$$\lambda(x, y) = (yp_3 + p_1)(xp_3 + p_2)(xp_1 + yp_2),$$

with $p_i = p_i(x^2, y^2)$.

(ii) *If f is finitely determined then both p_1 and p_2 must be different from zero.*

(iii) *If f is homogeneous and d_i is the degree of the polynomial p_i, $i = 1, 2, 3$ then $d_1 = d_2 = d$ and $p_3 = 0$. Furthermore, if p_1, p_2 and the product $p_1 p_2$ all have isolated singularity at zero then the Mond number of f is $36d^2$.*

Proof. (i) The image of f has defining equation $F(X, Y, Z) = 0$, where F generates $\mathcal{F}_0$, the zero Fitting ideal of the presentation matrix of f; that is, the determinant of the presentation matrix of the push forward $f_*(\mathcal{O}_2)$ (see [8]). In this case, the presentation matrix is

$$\begin{pmatrix} -Z & p_1 & p_2 & p_3 \\ Xp_1 & -Z & Xp_3 & p_2 \\ Yp_2 & Yp_3 & -Z & p_1 \\ XYp_3 & Yp_2 & Xp_1 & -Z \end{pmatrix},$$

where $p_i = p_i(X, Y)$. So, $F(X, Y, Z) = Z^4 + q_2 Z^2 + q_1 Z + q_0$, with

$$q_2 = -2Xp_1^2 - 2Yp_2^2 - 2XYp_3^2,$$
$$q_1 = -8XYp_1p_2p_3 \text{ and}$$
$$q_0 = X^2(p_1^2 - Yp_3^2)^2 - 2XYp_2^2(p_1^2 + Yp_3^2) + Y^2p_2^4.$$

This yields

$$F_Z^* = 4(xp_1 + yp_2 + xyp_3)^3 + 2(-2x^2p_1^2 - 2y^2p_2^2 - 2x^2y^2p_3^2)(xp_1 + yp_2 + xyp_3) - 8x^2y^2p_1p_2p_3.$$

Hence, using proposition 2.1 above, $D^2(f)$ has defining equation

$$\lambda(x, y) = F_Z^*/\eta = 2(yp_3 + p_1)(xp_3 + p_2)(xp_1 + yp_2),$$

with $p_i = p_i(x^2, y^2)$.

(ii) If $p_1 = 0$ (resp. $p_2 = 0$) then the defining equation of $D^2(f)$ is $\lambda(x, y) = y^2p_2p_3(xp_3 + p_2)$ (resp. $\lambda(x, y) = x^2p_1p_3(yp_3 + p_1)$). So, clearly the Mond number $\mu(D^2(f))$ is not finite in both cases. Hence, finite determinacy of f fails.

(iii) The homogeneity of f is equivalent to that of $p(x, y) = xp_1(x^2, y^2) + yp_2(x^2, y^2) + xyp_3(x^2, y^2)$. So, we must have $1 + 2d_1 = 1 + 2d_2 = 2 + 2d_3$. On one hand the first equality implies $d_1 = d_2$, on the other hand the second equality can never be satisfied. So, the homogeneity of p implies p_3 must vanishes. Furthermore, by part (i) the defining equation of $D^2(f)$ is reduced to $\lambda(x,y) = p_1p_2(xp_1+yp_2)$ which in its turn is homogeneous of degree $2d+2d+1+2d = 6d+1$. Since the partial derivatives λ_x and λ_y, generators of the jacobian ideal of λ, are homogeneous of degree $6d$, it follows that $\mu(D^2(f)) = (6d)^2$. □

Remark 3.2. For any double fold map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, the expression for the defining equation $\lambda(x,y) = (yp_3 + p_1)(xp_3 + p_2)(xp_1 + yp_2)$, tell us that the double point curve $D^2(f)$ has at least 3 branches. Three is also the minimal number of pinch-points of any double fold map germ $f(x,y) = (x^2, y^2, g(x,y))$.

Example 3.3. Consider the double fold corank 2 map germ

$$f(x,y) = (x^2, y^2, x(x^{2a} + y^{2b}) + y(x^{2c} - y^{2d})),$$

with $a, b, c, d \geq 0$. Suppose f is quasihomogeneous of type (ω_x, ω_y), $\omega_x \neq \omega_y$, that is, $p(x,y) = x^{2a+1} + xy^{2b} + x^{2c}y - y^{2d+1}$ is quasihomogeneous of type (ω_x, ω_y). By the definition of quasihomogeneity we have four equalities:

$$(i)\ \frac{2a+1}{\omega_x} = 1, \quad (ii)\ \frac{1}{\omega_x} + \frac{2b}{\omega_y} = 1, \quad (iii)\ \frac{2c}{\omega_x} + \frac{1}{\omega_y} = 1, \quad (iv)\ \frac{2d+1}{\omega_y} = 1.$$

We will distinguish two cases:

(1) $a \neq 0$. From equalities (i) and (ii) it follows that, if $a \neq 0$ then $b \neq 0$. Then we can write $\omega_x = 2a+1$, $\omega_y = b(2a+1)/a$. This and equalities (iii) and (iv) give us $c = a - \frac{a}{2b} + \frac{1}{2}$ and $d = b + \frac{b}{2a} - \frac{1}{2}$. Since c and d must be integers then both $\frac{b}{a}$ and $\frac{a}{b}$ must also be integers. That is, $a = \pm b$. But that is impossible, for if $a = b$ then $\omega_x = \omega_y$ and if $a = -b$ then $a = 0 = b$, contrary to the hypothesis.

(2) $a = 0$. If $a = 0$ then $f(x,y) = (x^2, y^2, x(y^{2b}) + y(x^{2c} - y^{2d}))$ and $D^2(f)$ is given by $y^{2b}(x^{2c} - y^{2d})(xy^{2b} + yx^{2c} - y^{2d+1}) = 0$. So, if $b \neq 0$ then f is not finitely determined and if $b = 0$ then, by the proposition 2.1(ii), $f(x,y) = (x^2, y^2, y(x^{2c} - y^{2d}))$ is not finitely determined as well.

In conclusion, there is no finitely determined quasihomogeneous map germ of the form $f(x,y) = (x^2, y^2, x(x^{2a} + y^{2b}) + y(x^{2c} - y^{2d}))$. In other words, a quasihomogeneous map germ $f(x,y) = (x^2, y^2, x(x^{2a} + y^{2b}) + y(x^{2c} - y^{2d}))$, of type (ω_x, ω_y) is finitely determined iff $\omega_x = \omega_y$ (a homogeneous map germ) iff $a = b = c = d$.

The homogeneous map germ $f(x,y) = (x^2, y^2, x(x^{2k} + y^{2k}) + y(x^{2k} - y^{2k}))$ has Mond number equals to $36k^2$.

Theorem 3.4. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a quasihomogeneous double fold map germ given by $f(x,y) = (x^2, y^2, p(x,y))$. If f is finitely determined then f is homogeneous and the degree of $p(x,y)$ is odd.*

Proof. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a double fold map germ and assume that it is finitely determined and quasihomogeneous of type (ω_1, ω_2), with $\omega_1 \neq \omega_2$. Write $f(x,y) = (x^2, y^2, p(x,y))$, with $p(x,y) = xp_1(x^2, y^2) + yp_2(x^2, y^2) + xyp_3(x^2, y^2)$ having weighted degree d. We distinguish two cases according to the parity of ω_1, ω_2. Since, by definition, ω_1 and ω_2 have no common factor, we do not consider the case where both are even. In order to simplify the notation, given an integer n, we will denote its class modulo 2 by $\overline{n}$.

Assume, for instance, that $\overline{\omega_1} = 0$ and $\overline{\omega_2} = 1$. For any monomial x^iy^j of $p(x,y)$, we have $i\omega_1 + j\omega_2 = d$. This implies $\overline{j} = \overline{d}$. Hence, either $p_1 = 0$ if $\overline{d} = 1$, or $p_2 = p_3 = 0$ if

$\overline{d} = 0$, for the polynomials p_i are even in y. Both cases give us a contradiction with finite determinacy of f by proposition 3.1(ii).

Suppose now $\overline{\omega_1} = \overline{\omega_2} = 1$. Since $\omega_1 \neq \omega_2$, we can assume, for instance, that $\omega_2 > 1$. In this case, for any monomial $x^i y^j$ of $p(x,y)$, we have again $i\omega_1 + j\omega_2 = d$, which gives $\overline{i} + \overline{j} = \overline{d}$. If $\overline{d} = 0$, then $\overline{i} = \overline{j}$ and necessarily $p_1 = p_2 = 0$. This is not possible again by proposition 3.1(ii).

It only remains to consider the subcase $\overline{d} = 1$. This corresponds to $\overline{i} \neq \overline{j}$ and $p_3 = 0$. We analyze the general form of a quasihomogeneous polynomial $p(x,y)$:

$$x^k y^l \big(a_r (x^{\omega_2})^r + a_{r-1}(x^{\omega_2})^{r-1} y^{\omega_1} + \cdots + a_1 x^{\omega_2}(y^{\omega_1})^{r-1} + a_0 (y^{\omega_1})^r\big),$$

for some k, l, r such that $r\omega_1\omega_2 + k\omega_1 + l\omega_2 = d$ and coefficients $a_r, \ldots, a_0 \in \mathbb{C}$.

We recall from the proof of proposition 3.1(ii) that the defining equation of $D^2(f)$ is $\lambda = p_1 p_2 (x p_1 + y p_2)$, when $p_3 = 0$. Since f is finitely determined, p_1 and p_2 are both reduced and have no common factor. In particular, we must have $k, l \leq 1$. Let us consider all possibilities:

(1) $k = l = 0$. We have $\overline{r} = 1$, moreover

$$\begin{aligned} xp_1 &= a_r (x^{\omega_2})^r + \cdots + a_1 x^{\omega_2} (y^{\omega_1})^{r-1}, \\ yp_2 &= a_{r-1} (x^{\omega_2})^{r-1} y^{\omega_1} + \cdots + a_0 (y^{\omega_1})^r. \end{aligned}$$

Therefore, $x^{\omega_2 - 1}$ divides p_1 and $y^{\omega_1 - 1}$ divides p_2. Since we have assumed $w_2 \geq 3$, p_1 is not reduced.

(2) $k = 0$ and $l = 1$. We have $\overline{r} = 0$ and

$$\begin{aligned} xp_1 &= y\big(a_{r-1} (x^{\omega_2})^{r-1} y^{\omega_1} + \cdots + a_1 x^{\omega_2} (y^{\omega_1})^{r-1}\big), \\ yp_2 &= y\big(a_r (x^{\omega_2})^r + \cdots + a_0 (y^{\omega_1})^r\big). \end{aligned}$$

In this case, we have that $x^{\omega_2 - 1} y^{\omega_1 + 1}$ divides p_1 and it is not reduced.

(3) $k = 1$ and $l = 0$. We have $\overline{r} = 0$ and

$$\begin{aligned} xp_1 &= x\big(a_r (x^{\omega_2})^r + \cdots + a_0 (y^{\omega_1})^r\big), \\ yp_2 &= x\big(a_{r-1} (x^{\omega_2})^{r-1} y^{\omega_1} + \cdots + a_1 x^{\omega_2} (y^{\omega_1})^{r-1}\big). \end{aligned}$$

Analogously, $x^{\omega_2 + 1} y^{\omega_1 - 1}$ divides p_2 and it is not reduced.

(4) $k = 1$ and $l = 1$. We have $\overline{r} = 1$ and

$$\begin{aligned} xp_1 &= xy\big(a_r (x^{\omega_2})^r + \cdots + a_1 x^{\omega_2} (y^{\omega_1})^{r-1}\big), \\ yp_2 &= xy\big(a_{r-1} (x^{\omega_2})^{r-1} y^{\omega_1} + \cdots + a_0 (y^{\omega_1})^r\big). \end{aligned}$$

Therefore, $x^{\omega_2} y$ divides p_1 and $x y^{\omega_1}$ divides p_2. In particular, they have a common factor.

Finally, the value of the $\mathcal{A}_e - cod(f)$ is necessarily an integer. A simple calculation shows that this can only be achieved if the degree of the coordinate function $p(x,y)$ is odd. □

Remark 3.5. If $f(x,y) = (x^2, y^2, p(x,y))$ is a finitely determined homogeneous map germ then the Milnor number $\mu(p)$ of the coordinate function $p(x,y)$ is equal to $T(f)$, the number of triple points of f. Indeed, it is just a matter of comparing the value of $T(f)$ obtained from theorem 2.7 and $\mu(p)$ from theorem 2.4.

Example 3.6. Given any odd integer $d = 2k + 1$, we consider

$$f(x, y) = (x^2, y^2, x(x^{2k} + y^{2k}) + y(x^{2k} - y^{2k}).$$

This is an example of finitely determined homogeneous map germ such that degree of $p(x, y)$ is d.

On the other hand, we know from theorem 3.3 that there is no finitely determined quasihomogeneous double fold map germ $f(x, y) = (x^2, y^2, g(x, y))$ of type (ω_1, ω_2), with $\omega_1 \neq \omega_2$. However, we can choose degrees and weights $(d_1, d_2, d_3; \omega_1, \omega_2)$ such that the formula for the $\mathcal{A}_e - cod(f)$ gives an integer number. For instance, taking $(d_1, d_2, d_3; \omega_1, \omega_2) = (2, 4, 3; 1, 2)$, we should obtain $\mathcal{A}_e - cod(f) = 4$. In other words, the integrality of $\mathcal{A}_e - cod(f)$ is a necessary but not sufficient finite determinacy condition.

3.1. **Double fold map germs with** $p_3 = 1$**.** This is a special kind of double fold map germ. Although, $f(x, y) = (x^2, y^2, xp_1 + yp_2 + xy)$, with $p_i = p_i(x^2, y^2)$, is not quasihomogeneous, its double point curve can be semi-quasihomogeneous.

To see this, recall that, for a double fold map germ $f(x, y) = (x^2, y^2, xp_1 + yp_2 + xyp_3)$, the double point curve is defined by $\lambda(x, y) = (yp_3 + p_1)(xp_3 + p_2)(xp_1 + yp_2)$. Now, it can be rewritten as

$$\lambda(x, y) = p_3(xp_1 + yp_2)^2 + p_1p_2(xp_1 + yp_2) + xyp_3^2(xp_1 + yp_2).$$

If $p_3 = 1$ and if $(xp_1 + yp_2)$ is non-degenerate quasihomogeneous of type $(d; w_x, w_y)$ then $\lambda(x, y)$ is semi-quasihomogeneous with quasihomogeneous part $xy(xp_1 + yp_2)$, which is clearly non-degenerate. Moreover, by Arnold's theorem (2.6) coupled with Milnor/Orlik's (2.6), we get

$$\mu(\lambda) = \frac{(d + w_x)(d + w_y)}{w_x w_y}.$$

Let us consider, for example, $f(x, y) = (x^2, y^2, x^{2a+1} + y^{2b+1} + xy)$, where $p_1(x^2, y^2) = x^{2a}$ and $p_2(x^2, y^2) = y^{2b}$. This gives $\lambda = yx^{2a+2} + xy^{2b+2}$, which is quasihomogeneous of type

$$((2a + 1)(2b + 1) + (2a + 1) + (2b + 1); (2b + 1), (2a + 1)).$$

Since λ is clearly non-degenerate, f is finitely determined. Moreover, it has Mond number $\mu(\lambda) = (2a + 2)(2b + 2)$.

4. Quasihomogeneous maps of type $(d_1, d_2, d_3; 2, 1)$

In this last section we consider quasihomogeneous map germs of type $(d_1, d_2, d_3; 2, 1)$. We see that in this case it is possible to produce finitely determined examples, although there are restrictions on the degrees.

Lemma 4.1. *Any quasihomogeneous polynomial $h(x, y)$ of type $(d; 2, 1)$ can be written in the form $h(x, y) = p(x, y^2)$ or $h(x, y) = yp(x, y^2)$, where $p(x, y)$ is a homogenous polynomial.*

Proof. Assume d is even and write $d = 2r$. Then the only monomials that can appear in h are: $x^r, x^{r-1}y^2, \ldots, xy^{2r-2}, y^{2r}$. This means that h has the form

$$h(x, y) = a_r x^r + a_{r-1}x^{r-1}y^2 + \cdots + a_1 xy^{2r-2} + a_0 y^{2r},$$

for some $a_i \in \mathbb{C}$. Therefore, we put $p(x, y) = a_r x^r + a_{r-1}x^{r-1}y + \cdots + a_1 xy^{r-1} + a_0 y^r$. If d is odd we just multiply everything by y. ☐

Therefore, quasihomogeneous map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of type $(d_1, d_2, d_3; 2, 1)$ can take one of the following four forms, up to permutation of the three coordinate functions:

(I) $f(x, y) = (p_1(x, y^2), p_2(x, y^2), p_3(x, y^2))$,
(II) $f(x, y) = (p_1(x, y^2), p_2(x, y^2), yp_3(x, y^2))$,
(III) $f(x, y) = (p_1(x, y^2), yp_2(x, y^2), yp_3(x, y^2))$ and
(IV) $f(x, y) = (yp_1(x, y^2), yp_2(x, y^2), yp_3(x, y^2))$.

Corollary 4.2. *There is no finitely determined quasihomogeneous map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of type $(d_1, d_2, d_3; 2, 1)$ and of the form I or IV.*

Proof. If $f(x, y) = (p_1(x, y^2), p_2(x, y^2), p_3(x, y^2))$ then $f(x, y) = f(x, -y)$; that is $D^2(f)$ is the whole plane. If $f(x, y) = (yp_1(x, y^2), yp_2(x, y^2), yp_3(x, y^2))$ then all point $(x, 0)$ share the same image; that is, f is not finite. □

As a consequence, it only remains to check forms II and III for possible examples of corank 2 finitely determined quasihomogeneous map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of type $(d_1, d_2, d_3; 2, 1)$.

4.1. **An example.** We show that finitely determined quasihomogeneous map germs of type $(d_1, d_2, d_3; 2, 1)$ and of the form III can occur. Indeed, let us consider the quasihomogeneous map germ $f(x, y) = (x^2, xy + y^3, x^2y + y^5)$ of type $(4, 3, 5; 2, 1)$.

We are going to show that its Mond number is finite. To do so, we first obtain a reduced defining equation $F(X, Y, Z) = 0$ for the image, say

$$F(X, Y, Z) = Y^{10} - 4ZXY^7 - 8X^3Y^6 - 2Z^3Y^5 + 44Z^2X^2Y^4 + 16ZX^4Y^3 + 16X^6Y^2 - 21Z^4XY^2 - 48Z^3X^3Y - 16Z^2X^5 + Z^6.$$

Taking the derivative of F with respect to Z, substituting $X = x^2$, $Y = xy + y^3$ and $Z = x^2y + y^5$ and finally dividing by $\eta(x, y) = 2x^2 + 6xy^2$ we obtain the defining equation $\lambda(x, y) = 0$ of $D^2(f)$, namely,

$$\lambda(x, y) = (4x^6 + x^5y^2 - 3x^4y^4 - 2x^3y^6 + 6y^8x^2 + 5y^{10}x + 5y^{12})(y^4 + 3xy^2 + 4x^2)(y^4 + x^2)y.$$

This is a quasihomoheneous function of type $(20; 2, 1)$. Moreover, it is not difficult to check with the aid of a computer that it has isolated singularity. This implies that f is finitely determined. By Milnor/Orlik formula (2.4), the Milnor number of λ is 190 and according to Mond's formula (2.7) the $\mathcal{A}_e$-codimension of f is 51.

4.2. **Double cusp map germs.** We call the map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of the form $f(x, y) = (x^2, xy + y^3, g(x, y))$ a *double cusp* map germ.

Proposition 4.3. *Let $g(x, y)$ be a quasihomogeneous polynomial of type $(d; 2, 1)$. If the double cusp map germ $f(x, y) = (x^2, xy + y^3, g(x, y))$ is finitely determined then d is congruent to $1, 2, 5$ or 10 modulo 12.*

Proof. The formula of theorem 2.7 for the $\mathcal{A}_e$-codimension of f gives

$$\mathcal{A}_e - cod(f) = \frac{1}{12}(d - 1)(35d - 22).$$

It is an easy exercise to check that this is an integer number if and only if d is congruent to $1, 2, 5$ or 10 modulo 12. □

Example 4.4. We present in table 1 examples of finitely determined map germs of the form $f(x,y) = (x^2, xy + y^3, g(x,y))$, where $g(x,y)$ is a quasihomogeneous polynomial of type $(d; 2, 1)$ and d is congruent to $1, 2, 5$ or 10 modulo 12. Note that the case $d = 5$ is in fact the example given in subsection 4.1.

d	$g(x,y)$
5	$x^2y + y^5$
10	$x^5 + 2x^4y^2 + y^{10}$
13	$x^6y + y^{13}$
14	$x^7 + 2x^6y^2 + y^{14}$

Table 1

It is not difficult to check, with the aid of a computer, that all of these map germs are finitely determined, by just following the same routine calculation used in 4.1. In fact, it is also possible to produce examples for other higher values of d congruent to $1, 2, 5, 10$ modulo 12.

References

[1] V.I. Arnold, Singularités des applications différentiables, vol. I, Editions Mir, 1986.
[2] J.W. Bruce and W.L. Marar, Images and varieties, *Topology, 3. J. Math. Sci.* **82** (1996), 3633–3641.
[3] J.W. Bruce, M.A.S. Ruas and M. Saia, A note on determinacy, *Proc. Amer. Math. Soc.*, **115**, no. 3 (1992), 865–871.
[4] W.L. Marar and D. Mond, Multiple point schemes for corank 1 maps, *J. London Math. Soc.*, **39** (1989), 553–567.
[5] J. Milnor and P. Orlik, Isolated singularities defined by weighted homogeneous polynomials, *Topology* **9** (1970), 385–393.
[6] D. Mond, Some remarks on the geometry and classification of germs of maps from surfaces to 3-space, *Topology* **26** (1987), 361–383.
[7] D. Mond, The number of vanishing cycles for a quasihomogeneous mapping from $\mathbb{C}^2$ to $\mathbb{C}^3$, *Quart. J. Math. Oxford* (2), **42** (1991), 335–345.
[8] D. Mond and R. Pellikaan, Fitting ideals and multiple points of analytic mappings, *Lecture Notes in Math.*, **1414**, Springer (1989) 107–161.
[9] C.T.C. Wall, Finite determinacy of smooth map-germs, *Bull. London. Math. Soc.*, **13** (1981), 481–539.

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