MEASURE THEORY

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Contents

Chapter 1

Abstract Measure

1.1 Basic notions on sets

Definition 1.1.1 *A non empty family of subsets of X, say $\mathcal{A} \subset \mathcal{P}(X)$, is called an* **algebra** *if*

(i) $\varnothing \in \mathcal{A}$,
(ii) If $A \in \mathcal{A}$ *then* $X \setminus A \in \mathcal{A}$,
(iii) If $A, B \in \mathcal{A}$ *then* $A \cup B \in \mathcal{A}$.

Definition 1.1.2 *A non empty family of subsets of X, say $\Sigma \subset \mathcal{P}(X)$, is called a* **$\sigma$-algebra** *if*

(i) $\varnothing \in \Sigma$,
(ii) If $A \in \Sigma$ *then* $X \setminus A \in \Sigma$,
(iii) If $A_n \in \Sigma$ *for all* $n \in \mathbb{N}$ *then* $\cup_n A_n \in \Sigma$.

Definition 1.1.3 *A non empty family of subsets of X, say $\mathcal{M} \subset \mathcal{P}(X)$, is called a* **monotone class** *if for all monotone sequence of sets $A_n \in \mathcal{M}$, increasing $A_n \subset A_{n+1}$ (respect. decreasing $A_{n+1} \subset A_n$), then $\cup_n A_n \in \mathcal{M}$ (respect. $\cap_n A_n \in \mathcal{M}$).*

Definition 1.1.4 *A non empty family of subsets of X, say $\mathcal{R} \subset \mathcal{P}(X)$, is called a* **ring** *if*

(i) $\varnothing \in \mathcal{R}$,
(ii) If $A, B \in \mathcal{R}$ *then* $A \setminus B \in \mathcal{R}$,
(iii) If $A, B \in \mathcal{R}$ *then* $A \cup B \in \mathcal{R}$.

Remark 1.1.1 *Let $\Sigma \subset \mathcal{P}(X)$. Σ is a σ-algebra if and only if Σ is a monotone class and an algebra.*

Let $\mathcal{A} \subset \mathcal{P}(X)$. $\mathcal{A}$ is an algebra if and only if $\mathcal{A}$ is a ring containing X.

Example 1.1.1 *(1) The trivial σ-algebras are $\mathcal{P}(X)$ and $\Sigma = \{\varnothing, X\}$.*

(2) Let $\mathcal{M} = \{A_n : n \in \mathbb{N}\} \cup X$ where $A_1 = \varnothing$, $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_n A_n = X$. This is a monotone class but not necessarily a ring.

(3) Let $X = [0, 1)$ and $\mathcal{A} = \{$finite unions of intervals $[a, b), 0 \leq a \leq b \leq 1\}$ is an algebra but not σ-algebra.

If the intervals in the previous family are assumed to have $0 \leq a \leq b < 1$ then it is a ring but not an algebra.

(4) Let X be non empty and numerable. $\mathcal{R} = \{A \subset X : card(A) < \infty\}$ is a ring but not algebra.

Definition 1.1.5 *Let $A_n \in \mathcal{P}(X)$ for $n \in \mathbb{N}$. The upper limit (respect. lower limit) of the sequence is defined by*

$$\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$$

(respect.

$$\liminf A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k.)$$

A sequence is said to have limit if $\limsup A_n = \liminf A_n$. Such a set is called $\lim A_n$.

Remark 1.1.2 *Any monotone sequence has a limit. If A_n is increasing (respect. decreasing) then $\lim A_n = \cup_n A_n$ (respect. $\lim A_n = \cap_n A_n$).*

Proposition 1.1.6 *Let $\mathcal{R}$ be a ring. If $A, B \in \mathcal{R}$ then $A \triangle B \in \mathcal{R}$ and $A \cap B \in \mathcal{R}$.*

PROOF: Note that $A \triangle B = ((A \cup B) \setminus A) \cup ((A \cup B) \setminus B)$ and $A \cap B = A \cup B \setminus (A \triangle B)$. ∎

Proposition 1.1.7 *Let Σ be a σ-algebra. If $A_n \in \Sigma$ for all $n \in \mathbb{N}$ then $\cap_n A_n \in \Sigma$, $\limsup A_n \in \Sigma$ and $\liminf A_n \in \Sigma$*

PROOF: Write $\cap_n A_n = X \setminus \cup_n (X \setminus A_n)$ and apply the properties of an σ-algebra. ∎

Definition 1.1.8 *A pair (X, Σ) given by a non-empty set X and a σ-algebra over X is called a measurable space.*

We shall give several methods of constructing measurable spaces. The proofs are rather straightforward and left to the reader.

Definition 1.1.9 *(Induced σ-algebra) Let (X, Σ) be a measurable space and let $Y \subset X$. Then $\Sigma_Y = \{A \cap Y : A \in \Sigma\}$ is a σ-algebra over Y.*

If $Y \in \Sigma$ then $\Sigma_Y = \{A \subset Y, A \in \Sigma\}$.

Definition 1.1.10 *(Image of a σ-algebra) Let (X, Σ) be a measurable space and let $f : X \to Y$ be a function. We define*

$$f(\Sigma) = \{B \subset Y : f^{-1}(B) \in \Sigma\}.$$

Then $(Y, f(\Sigma))$ is a measurable space.

Definition 1.1.11 *(σ-algebra generated by a family) Let $\mathcal{F} \subset \mathcal{P}(X)$. We denote $\sigma(\mathcal{F})$ the smallest σ-algebra containing $\mathcal{F}$, which is called the σ-algebra generated by $\mathcal{F}$.*

It is elementary to see that $\sigma(\mathcal{F}) = \cap\{\Sigma : \Sigma\ \sigma\text{-algebra}\ , \mathcal{F} \subset \Sigma\}$.

Remark 1.1.3 *(1) If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \sigma(\mathcal{F}_1)$ then $\sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_2)$.*

(2) $\mathcal{F}_1 \subset \sigma(\mathcal{F}_2)$ and $\mathcal{F}_2 \subset \sigma(\mathcal{F}_1)$ if and only if $\sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_2)$.

Definition 1.1.12 *(Borel σ-algebra) Let (X, τ) be a topological space and let $\mathcal{G}$ be the collection of open sets for the topology τ. The σ-algebra $\sigma(\mathcal{G})$ is called the Borel σ-algebra and denoted $\mathcal{B}(X)$. The elements in $\mathcal{B}(X)$ are called Borel sets.*

Remark 1.1.4 *Closed sets, G_δ sets (numerable intersection of open sets) or F_σ sets (numerable union of closed sets) are examples of Borel sets.*

Using Remark 1.1.3 one easily sees the following facts:

If $\mathcal{F}$ denotes the collection of closed sets in τ then $\mathcal{B}(X) = \sigma(\mathcal{F})$.

If (X, d) is a separable metric space (or a metric space where any open set is a numerable union of balls) and $\mathcal{E} = \{B(x, r) : x \in X, r > 0\}$ then $\mathcal{B}(X) = \sigma(\mathcal{E})$.

Proposition 1.1.13 *Let $X = \mathbb{R}$ and let us consider the following collections $\mathcal{E}_1 = \{(a, b) : a \leq b\}$, $\mathcal{E}_2 = \{(a, b] : a \leq b\}$, $\mathcal{E}_3 = \{[a, b) : a \leq b\}$ and $\mathcal{E}_4 = \{[a, b] : a \leq b\}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E}_i)$ for $i = 1, 2, 3, 4$.*

PROOF: Notice that any open set is a numerable union of open intervals. Hence Remark 1.1.3 shows that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E}_1)$.

Let us observe that $(a,b] = \cap_n (a, b+\frac{1}{n})$, $[a,b) = \cup_n (a-\frac{1}{n}, b)$ and $[a,b] = \cap_n (a-\frac{1}{n}, b+\frac{1}{n})$ to get the other cases. ∎

Proposition 1.1.14 *Let $n \in \mathbb{N}$ and $X = \mathbb{R}^n$ and consider $\mathcal{E} = \{(a_1, b_1] \times (a_2, b_2] \times ...(a_n, b_n] : a_i \leq b_i, i = 1, ..., n\}$. Then $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{E})$.*

PROOF: We sketch the proof for the case $n = 2$.

Consider $\mathcal{J}_0 = \{(n, n+1] \times (m, m+1] : n, m \in \mathbb{Z}\}$ and $\mathcal{J}_n$ the collection of intervals resulting of dividing each square of the previous family into four of the same area.

Now for each $x \in \mathbb{R}^2$ there exists a unique sequence of intervals $I_k(x) \in \mathcal{J}_k$ such that $x \in I_k(x)$ for all $k \in \mathbb{N}$, $I_{k+1}(x) \subset I_k(x)$ and $Area(I_k(x)) \to 0$ as $k \to \infty$.

Given an open set G we can consider the family $\mathcal{F} = \{J \in \cup_k \mathcal{J}_k : J \subset G\}$. Of course $G = \cup_{J \in \mathcal{F}} J$ and with a little effort it can be seen that only a numerable number of sets is needed. Since $\mathcal{F} \subset \mathcal{E}$ then the proof is finished. ∎

Definition 1.1.15 *Let $\mathcal{E} \subset \mathcal{P}(X)$. $\mathcal{M}(\mathcal{E})$ stands for the smallest monotone class containing $\mathcal{E}$, which is called the monotone class generated by $\mathcal{E}$.*

Theorem 1.1.16 *(The monotone class theorem) Let $\mathcal{A}$ be an algebra over X. Then $\sigma(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.*

PROOF: It suffices to see that $\mathcal{M}(\mathcal{A})$ is σ-algebra . Since $\mathcal{M}(\mathcal{A})$ is a monotone class we have only to show that it is an algebra.

Clearly $\varnothing \in \mathcal{A} \subset \mathcal{M}(\mathcal{A})$. Now given $A \in \mathcal{M}(A)$, to see that $X \setminus A \in \mathcal{M}(\mathcal{A})$ let us define $\Sigma = \{A \in \mathcal{M}(A) : X \setminus A \in \mathcal{M}(\mathcal{A})\}$ and show that $\Sigma = \mathcal{M}(\mathcal{A})$. For such a purpose we shall see that Σ is a monotone class and contains $\mathcal{A}$. Indeed, if $\mathcal{A}_n \in \Sigma$ is a monotone sequence then $\lim A_n \in \Sigma$. Clearly if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A} \subset \mathcal{M}(\mathcal{A})$. Therefore $\mathcal{A} \subset \Sigma$.

Given now $A, B \in \mathcal{M}(\mathcal{A})$, we need to show that $A \cup B \in \mathcal{M}(A)$. Let us define $\Gamma_A = \{B \in \mathcal{M}(A) : A \cup B \in \mathcal{M}(\mathcal{A})\}$. Note that Γ_A is a monotone class (since $(B_n)_n$ monotone sequence in Γ_A has limit in Γ_A).

Now let us deal first with the case $A \in \mathcal{A}$. In this case Γ_A contains $\mathcal{A}$ (since $B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A} \subset \mathcal{M}(A)$). Therefore $\Gamma_A = \mathcal{M}(\mathcal{A})$ for all $A \in \mathcal{A}$.

Now let us consider $A \in \mathcal{M}(A) \setminus \mathcal{A}$. To see that $\mathcal{A} \subset \Gamma_A$ observe that $B \in \mathcal{A}$ belongs to Γ_A since we have that $B \in \Gamma_A$ if and only if $A \in \Gamma_B$ and this was shown to be true in the previous case. ∎

1.2 Basic notions on set functions.

Definition 1.2.1 *Let $\mathcal{A}$ be an algebra over X, a set function $\mu : \mathcal{A} \to [0, \infty]$ is called additive (or finitely additive) if $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}$ such that $A \cap B = \varnothing$.*

μ is called a measure over $\mathcal{A}$ if $\mu(\varnothing) = 0$ and $\mu(\cup_n A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for any sequence (A_n) of pairwise disjoint sets in $\mathcal{A}$ such that $\cup_n A_n \in \mathcal{A}$.

μ is a finite measure if $\mu(X) < \infty$. μ is σ-finite if there exist $X_n \in \mathcal{A}$ such that $\mu(X_n) < \infty$ and $X = \cup_n X_n$.

Proposition 1.2.2 *Let $\mu : \mathcal{A} \to [0, \infty]$ be a additive set function.*

(1) If there exists $A \in \mathcal{A}$ with $\mu(A) \neq \infty$ then $\mu(\varnothing) = 0$.

(2) μ is monotone, i.e. if $A, B \in \mathcal{A}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$.

(3) If $A, B \in \mathcal{A}$ and $A \subset B$ with $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(4) If μ is a measure then μ is subadditive, i.e. if $(A_n) \subset \mathcal{A}$ and $\cup_n A_n \in \mathcal{A}$ then $\mu(\cup_n A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

(5) If μ is a measure and $(A_n) \subset \mathcal{A}$ is increasing and $\cup_n A_n \in \mathcal{A}$ then

$$\mu(\lim_n A_n) = \mu(\cup_n A_n) = \lim_n \mu(A_n).$$

(6) If μ is a measure and $(A_n) \subset \mathcal{A}$ is decreasing, $\mu(A_1) < \infty$ and $\cap_n A_n \in \mathcal{A}$ then

$$\mu(\lim_n A_n) = \mu(\cap_n A_n) = \lim_n \mu(A_n).$$

PROOF: (1) Observe that $\mu(A \cup \varnothing) = \mu(A) + \mu(\varnothing)$.

(2) and (3) Note that $\mu(B) = \mu(A) + \mu(B \setminus A)$.

(4) Define $B_1 = A_1$ and $B_n = A_n \setminus \cup_{k=1}^{n-1} A_k$ for $n \geq 2$. Now (B_n) is a sequence in $\mathcal{A}$ of pairwise sets such that $\cup_n A_n = \cup_n B_n$ and

$$\mu(\cup_n A_n) = \mu(\cup_n A_n) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

(5) Put $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Note that (B_n) is a sequence in $\mathcal{A}$ of pairwise sets such that $A_n = \cup_{k=1}^{n} B_k$. Hence

$$\mu(A_n) = \sum_{k=1}^{n} \mu(B_k) \to \sum_{n=1}^{\infty} \mu(B_n) = \mu(\cup_n B_n) = \mu(\cup_n A_n).$$

(6) Put $B_n = A_1 \setminus A_n$. Then (B_n) is an increasing sequence in $\mathcal{A}$, $\cup_n B_n = A_1 \setminus \cap_n A_n$ and $\mu(B_n) = \mu(A_1) - \mu(A_n)$. Now applying (5) we get the corresponding result. ∎

Definition 1.2.3 *A triplet (X, Σ, μ) given by a non-empty set X, a σ-algebra over X and a measure over Σ is called a measure space.*

It is said to be finite or σ-finite if the measure is finite or σ-finite over Σ.

It is said to be complete if for any $A \in \Sigma$ with $\mu(A) = 0$ and $B \subset A$ then $B \in \Sigma$ and hence $\mu(B) = 0$.

Remark 1.2.1 *If (X, Σ, μ) is a σ-finite measure space then X can be splited into a sequence (X_n) of either disjoint or increasing sets in Σ such that $X = \cup_n X_n$ and $\mu(X_n) < \infty$.*

Example 1.2.1 *(Counting measure) Let $X \neq \varnothing$, $\Sigma = \mathcal{P}(X)$ and $\nu(A) = card(A)$.*

Example 1.2.2 *(Dirac mass) Let $X \neq \varnothing$ and $a \in X$. Put $\Sigma = \mathcal{P}(X)$ and $\delta_a(A) = 1$ if $a \in A$ and $\delta_a(A) = 0$ if $a \notin A$.*

Example 1.2.3 *(Induced measure) If (X, Σ, μ) is a measure space and $Y \in \Sigma$, we define $\mu_Y(B) = \mu(B \cap Y)$ for all $B \in \Sigma$. Hence (Y, Σ_Y, μ_Y) is a measure space.*

Example 1.2.4 *(Image measure) If (X, Σ, μ) is a measure space and $f : X \to Y$ is a function, we define $f(\mu)(B) = \mu(f^{-1}(B))$ for all $B \in f(\Sigma)$. Hence $(Y, f(\Sigma), f(\mu))$ is a measure space.*

Theorem 1.2.4 *Let (X, Σ, μ) be measure space. Let us define*

$$\mathcal{N} = \{N \in \mathcal{P}(X) : N \subset B, B \in \Sigma, \mu(B) = 0\}.$$

Put $\bar{\Sigma} = \Sigma \cup \mathcal{N}$ and $\bar{\mu}(\bar{A}) = \mu(A)$ for $\bar{A} = A \cup N$ for some $A \in \Sigma$ and $N \in \mathcal{N}$. Then $(X, \bar{\Sigma}, \bar{\mu})$ is a complete measure space such that $\Sigma \subset \bar{\Sigma}$ and $\bar{\mu}(A) = \mu(A)$ for all $A \in \Sigma$.

$(X, \bar{\Sigma}, \bar{\mu})$ is called the completion of (X, Σ, μ).

Proof: To see that $\bar{\Sigma}$ is σ-algebra observe first that $\varnothing \in \mathcal{N}$. Now if $\bar{A} = A \cup N$ for some $A \in \Sigma$ and $N \in \mathcal{N}$ where $N \subset B$ for $B \in \Sigma$ and $\mu(B) = 0$ then we have

$$X \setminus \bar{A} = (X \setminus (A \cup B)) \cup (B \setminus (N \cup A)) \in \bar{\Sigma}.$$

If $\bar{A}_n = A_n \cup N_n$ where $A_n \in \Sigma$ and $N_n \in \mathcal{N}$ for all $n \in \mathbb{N}$. Hence $\cup_n \bar{A}_n = (\cup_n A_n) \cup (\cup_n N_n) \in \bar{\Sigma}$.

Let us now show that $\bar{\mu}$ is well defined on $\bar{\Sigma}$. Indeed, if $A \cup N = A' \cup N'$ where $A, A' \in \Sigma$ and $N, N' \in \mathcal{N}$ where $N \subset B$ and $N' \subset B'$ for $B, B' \in \Sigma$ and $\mu(B') = \mu(B) = 0$ then

$$\mu(A) \leq \mu(A' \cup B') \leq \mu(A') \text{ and } \mu(A') \leq \mu(A \cup B) \leq \mu(A).$$

It is elementary to see that $\bar{\mu}$ is a measure over $\bar{\Sigma}$. If $H \subset A \cup N$ with $\bar{\mu}(A \cup N) = \mu(A) = 0$ then $H \in \mathcal{N} \subset \bar{\Sigma}$. This shows the completeness and the proof is finished. ∎

Next proposition is immediate and left to the reader.

Proposition 1.2.5 *Let (X, Σ, μ) be a measure space. $A \in \bar{\Sigma}$ if and only if there exist $A_1, A_2 \in \Sigma$ such that $A_1 \subset A \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$.*

1.3 Outer measures.

We would like to be able to extend measures defined in an algebra $\mathcal{A}$ to a bigger family, for instance to the generated σ-algebra or even to $\mathcal{P}(X)$. Let us give a procedure which allows to get an extension preserving some properties.

Proposition 1.3.1 *Let (X, Σ, μ) be a measure space. We define*

$$\mu^*(A) = \inf\{\mu(E) : A \subset E, E \in \Sigma\}$$

for any $A \in \mathcal{P}(X)$. Then

(i) μ^* is an extension of μ,

(ii) μ^* is monotone, i.e. if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$ and

(iii) μ^* is subadditive, i.e. $\mu^*(\cup_n A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

Proof: (i) is immediate.

(ii) If $A \subset B$ and $E \in \Sigma$ with $B \subset E$ then $A \subset E$. Thus, $\mu^*(A) \leq \mu^*(B)$.

(iii) Given $(A_n) \subset \mathcal{P}(X)$ we may assume that $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$ there exist $E_n \in \Sigma$, $A_n \subset E_n$ such that $\mu^*(A_n) + \frac{\varepsilon}{2^n} > \mu(E_n) \geq \mu^*(A_n)$. Hence

$$\mu^*(\cup_n A_n) \leq \mu(\cup_n E_n) \leq \sum_n \mu(E_n) \leq \sum_n \mu^*(A_n) + \varepsilon.$$

∎

Motivated by these properties we give the following definition.

Definition 1.3.2 *Let $X \neq \varnothing$. A monotone and subadditive set function $\lambda : \mathcal{P}(X) \to [0,\infty]$ with $\lambda(\varnothing) = 0$ is called an outer measure.*

Proposition 1.3.3 *Let $\lambda : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. Then λ is a measure if and only if λ is additive.*

Proof: Assume λ is additive, and take (A_n) a sequence of pairwise disjoint sets. Since

$$\sum_{k=1}^{n} \lambda(A_k) = \lambda(\cup_{k=1}^{n} A_k) \leq \lambda(\cup_n A_n) \leq \sum_n \lambda(A_n),$$

passing to the limit we get the result. ∎

Let us now give a procedure of constructing outer measures from measures defined on algebras which generalize the method in Proposition 1.3.1.

Proposition 1.3.4 *Let $\mathcal{A}$ be an algebra over X and $\mu : \mathcal{A} \to [0,\infty]$ a measure on $\mathcal{A}$. Let us define*

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \mu(E_i) : A \subset \cup_i E_i, E_i \in \mathcal{A}, i \in \mathbb{N}\}.$$

Then μ^* is an outer measure which extends μ.

Proof: The facts $\mu^*(\varnothing) = 0$ and that μ^* is monotone are immediate.

Given $(A_n) \subset \mathcal{P}(X)$ we may assume that $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $E_{n,k} \in \Sigma$, $A_n \subset \cup_k E_{n,k}$ such that $\mu^*(A_n) + \frac{\varepsilon}{2^n} > \sum_k \mu(E_{n,k)} \geq \mu^*(A_n)$. Hence

$$\mu^*(\cup_n A_n) \leq \mu(\cup_{n,k} E_n) \leq \sum_{n,k} \mu(E_{n,k}) \leq \sum_n \mu^*(A_n) + \varepsilon.$$

To see that μ^* extends μ note first that if $A \in \mathcal{A}$ then $\mu^*(A) \leq \mu(A)$. On the other hand if $A \subset \cup_i E_i$ for $(E_i) \subset \mathcal{A}$ we have $A = \cup_i (A \cap E_i)$. Hence

$$\mu(A) \leq \sum_i \mu(A \cap E_i) \leq \sum_i \mu(E_i).$$

This gives that $\mu(A) \leq \mu^*(A)$ and the proof is finished. ∎

Remark 1.3.1 *Observe that in the case of $\mathcal{A}$ being a σ-algebra the procedures of Propositions 1.3.1 and 1.3.4 coincide.*

Remark 1.3.2 *The procedure in Proposition 1.3.4 does not need neither $\mathcal{A}$ to be an algebra nor μ to be a measure. This actually can be also done for $\mathcal{F}$ semiring, which is an family of sets with the properties*
(i) $\varnothing \in \mathcal{F}$,
(ii) If $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ and
(iii) If $A, B \in \mathcal{F}, A \subset B$ then $B \setminus A \subset \cup_{k=1}^n C_k$ for a finite family of pairwise disjoint sets $C_k \in \mathcal{F}$ and for premeasures $\mu : \mathcal{F} \to [0, \infty]$, which are set functions with the properties
(i) $\mu(\varnothing) = 0$,
(ii) If $A, B \in \mathcal{F}$ are disjoint then $\mu(A \cup B) = \mu(A) + \mu(B)$ and
(iii) If $A, A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ and $A \subset \cup_n A_n$ then $\mu(A) \leq \sum_n \mu(A_n)$.

Let us now give a procedure to obtain measures from outer measures.

Definition 1.3.5 *Let λ be an outer measure on X. A set A is called λ-measurable if, for all $E \in \mathcal{P}(X)$,*

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \cap (X \setminus A)),$$

(or equivalently $\lambda(E) \geq \lambda(E \cap A) + \lambda(E \setminus A)$.)
We denote by Σ_λ the family of all λ-measurable sets.

Theorem 1.3.6 *(Caratheodory's theorem) Let λ be an outer measure on X. Then $(X, \Sigma_\lambda, \mu_\lambda)$ is a complete measure space, where μ_λ denotes the restriction of λ to Σ_λ.*

Proof: We first prove that Σ_λ is an σ-algebra .
Obviously $\varnothing \in \Sigma_\lambda$ and if $A \in \Sigma_\lambda$ if and only if $X \setminus A \in \Sigma_\lambda$.

Assume $A, B \in \Sigma_\lambda$. Let us see that $A \cup B \in \Sigma_\lambda$. For each $E \in \mathcal{P}(X)$ we have, since $(A \cup B) \cap E = (A \cap E) \cup (E \cap (X \setminus A) \cap B)$, that

$$\lambda((A \cup B) \cap E) + \lambda((X \setminus (A \cup B)) \cap E) \leq$$
$$\lambda(A \cap E) + \lambda(E \cap (X \setminus A) \cap B) + \lambda(E \cap (X \setminus A) \cap (X \setminus B)) \leq$$
$$\lambda(A \cap E) + \lambda(E \cap (X \setminus A)) = \lambda(E).$$

Therefore Σ_λ is an algebra.

Before proving that Σ_λ is an σ-algebra let us observe that for any $E \in \mathcal{P}(X)$ we have that $\lambda_E(A) = \lambda(E \cap A)$ is a measure on Σ_λ.

Indeed, if A_1 and $A_2 \in \Sigma_\lambda$ and $A_1 \cap A_2 = \varnothing$ then, denoting $E' = (E \cap A_1) \cup (E \cap A_2)$, we have

$$\begin{aligned}
\lambda_E(A_1 \cup A_2) &= \lambda((E \cap A_1) \cup (E \cap A_2)) \\
&= \lambda(E' \cap A_1) + \lambda(E' \cap (X \setminus A_1)) \\
&= \lambda(E \cap A_1) + \lambda(E \cap A_2) \\
&= \lambda_E(A_1) + \lambda_E(A_2)
\end{aligned}$$

If (A_n) is a sequence of pairwise disjoint λ-measurable sets we have for all $n \in \mathbb{N}$

$$\lambda_E(\cup_n A_n) \geq \lambda_E(\cup_{k=1}^N A_k) = \sum_{k=1}^N \lambda_E(A_n).$$

Hence passing to the limit as $n \to \infty$ we get the result.

In particular, for $E = X$ we have that λ is countably additive on Σ_λ.

To see that Σ_λ is σ-algebra we need to show that countably union of pairwise disjoint of λ-measurable sets (since any union of elements in an algebra can be written as a union of pairwise disjoint sets in the algebra) is λ-measurable. Given now such a sequence (A_n) we have for each $N \in \mathbb{N}$

$$\begin{aligned}
\lambda(E) &= \lambda(E \cap (\cup_{k=1}^N A_k)) + \lambda(E \cap (X \setminus \cup_{k=1}^N A_k)) \\
&\geq \lambda(\cup_{k=1}^N (E \cap A_k)) + \lambda(E \cap (X \setminus \cup_{k=1}^\infty A_k)) \\
&= \sum_{k=1}^N \lambda(E \cap A_k) + \lambda(E \cap (X \setminus \cup_{k=1}^\infty A_k))
\end{aligned}$$

Taking limits as $N \to \infty$ we get

$$\begin{aligned}
\lambda(E) &\geq \sum_{k=1}^\infty \lambda(E \cap A_k) + \lambda(E \cap (X \setminus \cup_{k=1}^\infty A_k)) \\
&= \lambda(E \cap (\cup_{k=1}^\infty A_k)) + \lambda(E \cap (X \setminus \cup_{k=1}^\infty A_k)).
\end{aligned}$$

To finish the proof we need to show that μ_λ is complete. Note that $\lambda(A) = 0$ implies $A \in \Sigma_\lambda$. Hence the monotonicity of λ gives also the completeness. ∎

1.4 Extension of measures.

Theorem 1.4.1 *(Hahn's extension theorem) Let $\mathcal{A}$ be an algebra on X and μ a measure on $\mathcal{A}$. There exists a measure $\hat{\mu}$ defined on the σ-algebra $\sigma(\mathcal{A})$ which extends μ.*

Moreover if μ is σ-finite the extension is unique and σ-finite.

Proof: Let us consider the outer measure from Proposition 1.3.4, that is

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \mu(A_i) : A \subset \cup_i A_i, A_i \in \mathcal{A}, i \in \mathbb{N}\}.$$

We now apply Caratheodory's method to get the σ-algebra of Σ_{μ^*}. If we prove that $\sigma(\mathcal{A}) \subset \Sigma_{\mu^*}$ and we define $\hat{\mu}$ the restriction of μ^* to $\sigma(\mathcal{A})$ we have the result.

Actually we only need to show that $\mathcal{A} \subset \Sigma_{\mu^*}$. Let $A \in \mathcal{A}$ and $E \in \mathcal{P}(X)$. For each $\varepsilon > 0$ we have a sequence (A_n) in $\mathcal{A}$ such that $E \subset \cup_n A_n$ and

$$\mu^*(E) \leq \sum_n \mu(A_n) \leq \mu^*(E) + \varepsilon.$$

Hence

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)) &\leq \mu^*(\cup_n (A_n \cap A)) + \mu^*(\cup_n (A_n \cap (X \setminus A)) \\
&\leq \sum_n \mu^*(A_n \cap A) + \sum_n \mu^*(A_n \cap (X \setminus A)) \\
&= \sum_n \mu(A_n \cap A) + \sum_n \mu(A_n \cap (X \setminus A)) \\
&= \sum_n \mu(A_n) \leq \mu^*(E) + \varepsilon.
\end{aligned}$$

We show now that $\hat{\mu}$ es unique if μ is σ-finite on $\mathcal{A}$, say $X = \cup_n X_n$ where $X_n \in \mathcal{A}$, $X_n \subset X_{n+1}$ and $\mu_n(X_n) < \infty$. Assume that there are two measures μ_1 and μ_2 on $\sigma(\mathcal{A})$ which extend μ. Denote by $\mu_{1,n}(A) = \mu_1(A \cap X_n)$ and $\mu_{2,n}(A) = \mu_2(A \cap X_n)$ for any $A \in \sigma(\mathcal{A})$. For each $n \in \mathbb{N}$ denote

$$\mathcal{M}_n = \{A \in \sigma(\mathcal{A}) : \mu_{1,n}(A) = \mu_{2,n}(A)\}.$$

Using that $\mu_{i,n}$ are finite we have that $\mathcal{M}_n$ is a monotone class (see (5) and (6) in Proposition 1.2.2) which clearly contains $\mathcal{A}$. Therefore $\mu_{1,n} = \mu_{2,n}$ and hence $\mu_1 = \mu_2$ on $\sigma(\mathcal{A})$.

Now $\hat{\mu}$ is σ-finite because it extends μ. ∎

Remark 1.4.1 *Take $X = [0,1)$ and the algebra $\mathcal{A}$ given by finite unions of intervals $[a,b), 0 \le a \le b \le 1$. Consider μ the measure on $\mathcal{A}$ such that $\mu(\varnothing) = 0$ and $\mu(A) = \infty$ for any $\varnothing \neq A \in \mathcal{A}$.*

We can get two different extensions to $\sigma(\mathcal{A})$. The counting measure ν and the measure μ_1 defined by $\mu_1(\varnothing) = 0$ and $\mu_1(A) = \infty$ for any $\varnothing \neq A \in \sigma(\mathcal{A})$. They are different since $\nu(\{1/2\}) = 1$ and $\mu_1(\{1/2\}) = \infty$.

Theorem 1.4.2 *Let $\mathcal{A}$ be an algebra on X and $\mu : \sigma(\mathcal{A}) \to [0,\infty)$ a measure such that there exists $X_n \in \mathcal{A}$ of finite measure such that $X = \cup_n X_n$. Then for each $\varepsilon > 0$ and each $A \in \sigma(\mathcal{A})$ with $\mu(A) < \infty$ there exists $B \in \mathcal{A}$ such that $\mu(A \triangle B) < \varepsilon$.*

Proof: Using the uniqueness in Hahn's theorem we have that

$$\mu(A) = \inf\{\sum_n \mu(A_n) : A \subset \cup_n A_n, A_n \in \mathcal{A}\}$$

for all $A \in \sigma(\mathcal{A})$.

Now given $\varepsilon > 0$ and $A \in \sigma(\mathcal{A})$ with $\mu(A) < \infty$ there exists a sequence (A_n) in $\mathcal{A}$ such that $A \subset \cup_n A_n$ and $\mu(A) \le \sum_n \mu(A_n) < \mu(A) + \frac{\varepsilon}{2}$.

Take now $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \mu(A_k) \frac{\varepsilon}{2}$ and define $B = \cup_{k=1}^{N} A_k \in \mathcal{A}$.

Clearly $\mu(A \triangle B) \le \mu(\cup_{k=N+1}^{\infty} A_k) + \mu(\cup_n A_n \setminus A) \le \varepsilon$. ∎

Lemma 1.4.3 *Let μ be a measure on Σ and $\mu^*(A) = \inf\{\mu(E) : A \subset E, E \in \Sigma\}$. If $A \in \mathcal{P}(X)$ then there exists $E \in \Sigma$ such that $A \subset E$ and $\mu(E) = \mu^*(A)$.*

Proof: In the case $\mu^*(A) = \infty$ take $E = X$. If $\mu^*(A) < \infty$ we can select, for each $n \in \mathbb{N}$, $E_n \in \Sigma$ such that $\mu(E_n) < \mu^*(A) + \frac{1}{n}$. Then $E = \cap_n E_n \in \Sigma$ and $\mu(E) = \mu^*(A)$. ∎

Theorem 1.4.4 *Let (X, Σ, μ) be a σ finite measure space. Then the completion $(X, \bar{\Sigma}, \bar{\mu})$ coincides with $(X, \Sigma_{\mu^*}, \mu_{\mu^*})$.*

PROOF: Recall that from Proposition 1.3.1 we have that

$$\mu^*(A) = \inf\{\mu(E) : A \subset E, E \in \Sigma\}.$$

Let us see that $\Sigma \subset \Sigma_{\mu^*}$.

For each $A \in \Sigma$ and $E \in \mathcal{P}(X)$, select, using Lemma 1.4.3, $E_1 \in \Sigma$ with $E \subset E_1$ and $\mu(E_1) = \mu^*(E)$. Hence

$$\mu^*(E\cap A)+\mu^*(E\cap(X\setminus A)) \leq \mu(E_1\cap A)+\mu(E_1\cap(X\setminus A)) = \mu(E_1) = \mu^*(E).$$

On the other hand if $B \subset N$ for some $N \in \Sigma$ and $\mu^*(N) = \mu(N) = 0$ then $\mu^*(B) = 0$. Since any μ^*-nul set is μ^*-measurable then the set $\mathcal{N}$ associated to $(X; \Sigma, \mu)$ is contained in Σ_{μ^*}. This shows that $\bar{\Sigma} \subset \Sigma_{\mu^*}$.

Since μ^* restricted to Σ coincides with μ we get also that $\bar{\mu} = \mu^*$ on $\bar{\Sigma}$.

Conversely, assume first that $A \in \Sigma_{\mu^*}$ and $\mu^*(A) < \infty$. Take $E \in \Sigma$ such that $A \subset E$ and $\mu(E) = \mu^*(A)$. Hence

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)) \\ &= \mu^*(A) + \mu^*(E \setminus A) \\ &= \mu^*(E) + \mu^*(E \setminus A) \end{aligned}$$

Therefore $\mu^*(E \setminus A) = 0$. This allows to get $B \in \Sigma$ with $E \setminus A \subset B$ and $\mu(B) = 0$. So $E \setminus B \subset A \subset E$ and this gives that $A \in \bar{\Sigma}$.

The case $\mu^*(A) = \infty$ can be done by writting $A = \cup_n(A \cap X_n)$ where $X_n \in \Sigma$ and $\mu(X_n) < \infty$. Since $A \cap X_n \in \Sigma_{\mu^*}$ for all $n \in \mathbb{N}$ we obtain, from the previous case, that $A \in \bar{\Sigma}$. ∎

1.5 Borel-Stieltjes measures on $\mathbb{R}$.

Throughout this section I denotes an interval in $\mathbb{R}$ with not empty interior, $int(I) \neq \varnothing$, and let $x_0 \in int(T)$.

Definition 1.5.1 *Given a measure μ on $\mathcal{B}(I)$ which is finite on bounded intervals contained in I we define $F_\mu : I \to \mathbb{R}$ as the function $F_\mu(x) = \mu((x_0, x])$ for $x \in I, x \geq x_0$ and $F_\mu(x) = -\mu((x, x_0])$ for $x \in I, x < x_0$.*

Proposition 1.5.2 *If $\mu : \mathcal{B}(I) \to [0, \infty]$ is a measure such that $\mu(J) < \infty$ for all bounded interval $J \subset I$, then F_μ is increasing, right continuous and $F_\mu(x_0) = 0$.*

Moreover F_μ is continuous at x if and only if $\mu(\{x\}) = 0$.

PROOF: We shall see that if $x_1, x_2 \in I$ with $x_1 < x_2$ then $F_\mu(x_2) - F_\mu(x_1) = \mu((x_1, x_2]) \geq 0$. Hence F_μ is increasing.

Indeed, note that if $x_0 \leq x_1 < x_2$ then

$$F_\mu(x_2) - F_\mu(x_1) = \mu((x_0, x_2]) - \mu(x_0, x_1]) = \mu((x_1, x_2]),$$

if $x_1 \leq x_0 < x_2$ then

$$F_\mu(x_2) - F_\mu(x_1) = \mu((x_0, x_2]) + \mu(x_1, x_0]) = \mu((x_1, x_2]),$$

and if $x_1 < x_2 \leq x_0$ then

$$F_\mu(x_2) - F_\mu(x_1) = -\mu((x_2, x_0]) + \mu(x_1, x_0]) = \mu((x_1, x_2]).$$

To see that F_μ is right continuous we fix $x \in I$ and a sequence decreasing $(x_n) \subset I$ such that $x_n > x$ and $\lim_n x_n = x$. Since $\lim_n (x, x_n] = \varnothing$ then $\lim_n F_\mu(x_n) - F_\mu(x) = \lim_n \mu((x, x_n]) = 0$.

To see the last part, observe that for an increasing sequence $(x_n) \subset I$ such that $x_n \leq x$ and $\lim_n x_n = x$ we have $\lim_n F_\mu(x_n) - F_\mu(x) = \lim_n \mu((x, x_n]) = \mu(\{x\})$ ∎

Definition 1.5.3 *An increasing and right continuous function $F : I \to \mathbb{R}$ is called a distribution function over I. For such a function we define for each $A \subset I$ the set function*

$$\lambda_F(A) = \inf\{\sum_{k=1}^{\infty} F(b_k) - F(a_k) : A \subset \cup_k (a_k, b_k], a_k, b_k \in int(I)\}.$$

Our aim is, given a distribution function F to find a measure μ such that $F = F_\mu$. For such a purpose we first need the following lemma.

Proposition 1.5.4 *Let $F : I \to \mathbb{R}$ an increasing function. If we have$(a, b] \subset \cup_{k=1}^{n} (a_k, b_k] \subset I$ where $a_1 \leq a_2 \leq ...a_n$ and $a, b, a_k \in I$ for $k = 1, ..., n$, then*

$$F(b) - F(a) \leq \sum_{k=1}^{n} F(b_k) - F(a_k).$$

PROOF: We may assume that $(a, b] \cap (a_k, b_k] \neq \varnothing$ for all k and,since $(a, b]$ must be contained in one of its connected components, also assume that $\cup_{k=1}^{n} (a_k, b_k] = (a_{k_0}, b_{k_0}]$, .

We shall prove it using induction. The case $n = 1$ is obvious.

Let us assume the result is true for any family of m intervals for $m < n$.

Put $n_o = max\{k : a_k \leq a\}$ and $n_1 = min\{j : b \leq b_j\}$.

We may assume that $n_1 > n_0$, since $n_0 \geq n_1$ implies that $(a, b] \subset (a_{n_0}, b_{n_0}]$ and the result is clear.

We now have two possibilities:

If $(a_{n_0}, b_{n_0}] \cap (a_{n_1}, b_{n_1}] \neq \varnothing$ then $b_{n_0} > a_{n_1}$ what gives

$$F(b) - F(a) \leq F(b_{n_1}) - F(a_{n_0}) \leq F(b_{n_1}) - F(a_{n_1}) + F(b_{n_0}) - F(a_{n_0}).$$

If $(a_{n_0}, b_{n_0}] \cap (a_{n_1}, b_{n_1}] = \varnothing$ then $(b_{n_0}, a_{n_1}] \subset \cup_{k \neq n_o, n_1}(a_k, b_k]$. Using now the induction assumption we have

$$F(a_{n_1}) - F(b_{n_0}) \leq \sum_{k \neq n_0, n_1} F(b_k) - F(a_k)$$

and therefore

$$\begin{aligned} F(b) - F(a) &\leq F(b_{n_1}) - F(a_{n_0}) \\ &\leq F(b_{n_1}) - F(a_{n_1}) + F(a_{n_1}) - F(b_{n_0}) + F(b_{n_0}) - F(a_{n_0}) \\ &\leq \sum_{k=1}^{n} F(b_k) - F(a_k). \end{aligned}$$

■

Theorem 1.5.5 *If F is a distribution function over I then*
(i) λ_F is an outer measure,
(ii) $\lambda_F((a, b]) = F(b) - F(a)$ for all $a, b \in int(I)$ and
(iii) any borel set is λ_F-measurable.

PROOF: (i) Using that $\varnothing \subset (a, a]$ then $\lambda_F(\varnothing) = 0$.

Clearly $\lambda_F(A) \leq \lambda_F(B)$ if $A \subset B$.

Let $(A_n) \subset I$ with $\sum_n \lambda_F(A_n) < \infty$. For each $\varepsilon > 0$ we find $(a_k^n, b_k^n]$ such that $A_n \subset \cup_k (a_k^n, b_k^n]$ and

$$\sum_{k=1}^{\infty} F(b_k^n) - F(a_k^n) < \lambda_F(A_n) + \frac{\varepsilon}{2^n}.$$

Hence

$$\lambda_F(\cup_n A_n) \leq \sum_{k,n=1}^{\infty} F(b_k^n) - F(a_k^n) \leq \sum_{n=1}^{\infty} \lambda_F(A_n) + \varepsilon.$$

(ii) Clearly if $A = (a, b]$ then $\lambda_F(A) \leq F(b) - F(a)$. Assume now that $(a, b] \subset \cup_k (a_k, b_k]$ where $a_k, b_k \in int(I)$.

Given $\varepsilon > 0$, since F is right continuous there exists $a' > a$ so that $F(a') - F(a) < \frac{\varepsilon}{2}$ and there exist $b'_k > b_k$ such that $F(b'_k) - F(b_k) < \frac{\varepsilon}{2^{k+1}}$ for all $k \in .\mathbb{N}$.

Since $[a', b] \subset \cup_k (a_k, b'_k)$, using compactness, we have $[a', b] \subset \cup_{k=1}^N (a_k, b'_k)$ for some $N \in \mathbb{N}$. Now we can apply Proposition 1.5.4 to get

$$F(b) - F(a') \leq \sum_{k=1}^N F(b'_k) - F(a_k) \leq \sum_{k=1}^N F(b_k) - F(a_k) + \frac{\varepsilon}{2}.$$

Hence $F(b) - F(a) \leq \sum_{k=1}^N F(b_k) - F(a_k) + \varepsilon$.

(iii) It suffices to see that $(a, b]$ is λ_F-measurable.

Let $E \subset I$ with $\lambda_F(E) < \infty$ (the other case is obvious) and $\varepsilon > 0$. Let us take $E \subset \cup_k (a_k, b_k]$ where $a_k, b_k \in int(I)$ and $\sum_{k=1}^\infty F(b_k) - F(a_k) < \lambda_F(E) + \varepsilon$. Now

$$\lambda_F((a, b] \cap E) + \lambda_F((I \setminus (a, b]) \cap E)$$
$$\leq \sum_k \lambda_F((a, b] \cap (a_k, b_k]) + \lambda_F((I \setminus (a, b]) \cap (a_k, b_k]) = \sum_k A_k.$$

Observe now that if $(a, b] \cap (a_k, b_k] \neq \varnothing$ then coincides with the interval $(max(a, a_k), min(b, b_k)]$. We have four situations, namely

Case $a \leq a_k$ and $b \leq b_k$.

Hence $(a, b] \cap (a_k, b_k] = (a_k, b]$ and $(I \setminus (a, b]) \cap (a_k, b_k] = (b, b_k]$ what implies that

$$A_k = F(b) - F(a_k) + F(b_k) - F(b) = F(b_k) - F(a_k),$$

Case $a \leq a_k$ and $b_k < b$.

Hence $(a, b] \cap (a_k, b_k] = (a_k, b_k]$ and $(I \setminus (a, b]) \cap (a_k, b_k] = \varnothing$ what implies that

$$A_k = F(b_k) - F(a_k)$$

Case $a_k < a$ and $b \leq b_k$.

Hence $(a, b] \cap (a_k, b_k] = (a, b]$ and $(I \setminus (a, b]) \cap (a_k, b_k] = (a_k, a] \cup (b, b_k]$ what implies that

$$A_k = F(b) - F(a) + (F(a) - F(a_k) + F(b_k) - F(b)) = F(b_k) - F(a_k),$$

Case $a_k < a$ and $b_k < b$.

Hence $(a,b] \cap (a_k,b_k] = (a,b_k]$ and $(I \setminus (a,b]) \cap (a_k,b_k] = (a_k,a]$ what implies that

$$A_k = F(b_k) - F(a) + F(a) - F(a_k) = F(b_k) - F(a_k).$$

Therefore

$$\begin{aligned} \lambda_F((a,b] \cap E) &+ \lambda_F((I \setminus (a,b]) \cap E) \\ &\leq \sum_k F(b_k) - F(a_k) \\ &< \lambda_F(E) + \varepsilon \end{aligned}$$

■

Definition 1.5.6 *Given a distribution function F over I we denote by $\mathcal{M}_F$ and m_F the family of λ_F-measurable sets and the measure λ_F restricted to $\mathcal{M}_F$ obtained by using the Caratheodory method. These are called Lebesgue-Stieltjes measurable sets and the Lebesgue-Stieltjes measure respectively. The case $I = \mathbb{R}$ and $F(x) = x$ corresponds to the Lebesgue measure space and it is simply denoted $(\mathbb{R}, \mathcal{M}, m)$.*

Theorem 1.5.7 *Let $F : I \to \mathbb{R}$ be a distribution function. Then there exists a unique Borel measure $m_F : \mathcal{B}(I) \to [0,\infty]$ verifying that $m_F((a,b]) = F(b) - F(a)$ for all $a,b \in int(I)$. (This is called the Borel-Stieltjes measure associated to the distribution function F.)*

PROOF: Consider the outer measure λ_F and let μ be the measure λ_F restricted to $\mathcal{B}$. Note that any interval can be decomposed as $I = \cup_n A_n$ where $A_n = (a_n, b_n]$ (and eventually $A_0 = \{a\}$ in the case $I = [a,b]$ or $I = [a,\infty]$) and, of course $\mu(A_n) < \infty$. Therefore we can use the uniqueness of the Hahn theorem to conclude the result. ■

Remark 1.5.1 *Let us list some properties of m_F which are left to the reader.*

(1) F is continuous at x if and only if $m_F(\{x\}) = 0$.

(In particular $m(N) = 0$ for all numerable set in $\mathbb{R}$.)

(2) m_F is always σ-finite and m_F is finite if and only if F is bounded.

(3) For any $a,b \in int(I)$ we have $m_F((a,b)) = F(b^-) - F(a)$, $m_F([a,b)) = F(b^-) - F(a^-)$, and $m_F([a,b]) = F(b) - F(a^-)$, where $F(a^-) = \lim_{x\to a^-} F(x)$.

Let us now characterize the Legesgue measure on $\mathbb{R}$ by means of its invariance under translations.

Theorem 1.5.8 *Let μ be a Borel measure over $\mathbb{R}$ which is invariant under translations and finite over bounded intervals in $\mathbb{R}$. Then $\mu = cm$ where $c = \mu(0, 1])$ and m is the Lebesgue measure restricted to $\mathcal{B}$.*

PROOF: Let us first see that $\mu((a, b]) = c(b - a)$ where $c = \mu((0, 1])$.

Since $(a, b] = a + (0, b - a]$ it suffices to see that $\mu((0, x]) = cx$ for any $x > 0$. Since $x = \lim_n q_n$ where $q_n \in \mathbb{Q}$, and (q_n) is an increasing sequence, then it is enough to prove that $\mu((0, q]) = cq$ for any $q > 0$ and $q \in \mathbb{Q}$.

Now writing $q = \frac{m}{n}$ where $m, n \in \mathbb{N}$ and observing that

$$(0, \frac{m}{n}] = (0, \frac{1}{n}] \cup (\frac{1}{n}, \frac{2}{n}] \cup ... \cup (\frac{m-1}{n}, \frac{m}{n}]$$

we have that $\mu((0, \frac{m}{n}]) = \sum_{k=1}^{m} \mu((\frac{k-1}{n}, \frac{k}{n}]) = m\mu(0, \frac{1}{n}])$. Finally to see that $\mu(0, \frac{1}{n}]) = \frac{c}{n}$ simply note that $1 = \frac{n}{n}$ and then $\mu((0, 1]) = n\mu(0, \frac{1}{n}])$.

Now let $\mathcal{A}$ be the algebra generated by $\{(a, b] : a \leq b\}$ which is easily seen to coincide with finite unions of intervals $(a, b]$ or (a, ∞) where $-\infty \leq a \leq b < \infty$. Using that $\mathbb{R} = \cup(-n, n]$ and $\mu(-n, n] = 2nc$ we can apply the uniqueness of the Hanh theorem to conclude that $\mu = cm$ (because both measures coincide on $\mathcal{A}$). ∎

Let us now use general theory to get information on Lebesgue measurable sets. Recall that Theorem 1.4.4 one has that the complection of $(\mathbb{R}, \mathcal{B}, m_F)$ coincides with $(\mathbb{R}, \mathcal{M}, m_F)$. In particular, E is a Lebesgue measurable set if and only if there exist Borel sets A, B such that $A \subset E \subset B$ and $m(B \setminus A) = 0$. We can improve this result as follows.

Theorem 1.5.9 *Let $E \subset \mathbb{R}$, $F(x) = x$ and write $\lambda_F = \lambda$. The following are equivalent:*

(1) E is a Lebesgue measurable set.

(2) For any $\varepsilon > 0$ there exists an open set G with $E \subset G$ and $\lambda(G \backslash E) < \varepsilon$.

(3) For any $\varepsilon > 0$ there exists a closed set F with $F \subset E$ and $\lambda(E \backslash F) < \varepsilon$.

(4) There exist $A \in F_\sigma$(countable union of closed sets) and $B \in G_\delta$ (countable intersection of open sets) with $A \subset E \subset B$ and $m(B \setminus A) = 0$.

PROOF: (1)$\Longrightarrow$(2) Assume $m(E) = \lambda(E) < \infty$. There exist $(a_k, b_k]$ such that $E \subset \cup_k (a_k, b_k]$ and $\sum_k b_k - a_k < \lambda(E) + \frac{\varepsilon}{2}$.

Define $G = \cup_k (a_k, b_k + \frac{\varepsilon}{2^k})$. Hence

$$\lambda(G) = m(G) \leq \sum_k b_k - a_k + \frac{\varepsilon}{2} < m(E) + \varepsilon.$$

Therefore $\lambda(G \setminus E) = m(G \setminus E) \leq m(G) - m(E) < \varepsilon$.

Assume now $m(E) = \lambda(E) = \infty$. First consider $E \cap (-n, n)$ find open sets G_n with $E_n \subset G_n$ and $m(G_n \setminus E_n) < \frac{\varepsilon}{2^n}$ and then define $G = \cup_n G_n$ which satisfies $E \subset G$ and $m(G \setminus E) < \varepsilon$.

(2)$\Longrightarrow$(1) For each $n \in \mathbb{N}$, let G_n be an open set with $E \subset G_n$ and $m(G_n \setminus E) < \frac{1}{n}$. Define $G = \cap_n G_n$ and $N = G \setminus E$. Clearly $\lambda(N) = 0$ and $E \subset G$.

Therefore $F = \mathbb{R} \setminus G \in \mathcal{B}$ and $\mathbb{R} \setminus E = F \cup N$. Hence $\mathbb{R} \setminus E \in \mathcal{M}$ and also $E \in \mathcal{M}$.

(1)$\Longleftrightarrow$(3) It follows from the equivalence between (1) and (2)

(1)$\Longrightarrow$(4) Using that (2) and (3) holds we have a sequence of open sets G_n and closed sets F_n such that $F_n \subset E \subset G_n$, $m(G_n \setminus E) < \frac{1}{2n}$ and $m(E \setminus F_n) < \frac{1}{2n}$. Define $A = \cup_n F_n$ and $B = \cap_n G_n$. We have $m(B \setminus A) \leq m(F_n \setminus G_n) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

(4)$\Longrightarrow$(1) It follows from the fact that $\mathcal{M}$ is the complection of $\mathcal{B}$. ∎

1.6 Measurable and non-measurable sets.

Let us now construct some examples of sets in $\mathcal{B}$ and $\mathcal{M}$.

Example 1.6.1 *A non numerable Borel set with measure zero: The Cantor set in* $[0, 1]$.

Proof: Let us define the following family of open sets: First divide the interval $[0, 1]$ into three subintervals of the same measure. Select the open interval in the middle. Now divide each of the remaining intervals into three equal parts and select each open intervarl in the middle. Repeat this process.

Hence in the first step we select $J_1 = (\frac{1}{3}, \frac{2}{3})$, in the second step we select $J_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, and in the third step we get $J_3 = (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{17}{27}, \frac{18}{27}) \cup (\frac{25}{27}, \frac{26}{27})$

In this way we obtain a sequence of sets $J_k = \cup_{l=1}^{2^{k-1}} I_{l,k}$ where $I_{l,k}$ are disjoint open intervals of length 3^{-k}.

Now define the Cantor set by $C = [0,1] \setminus \cup_k J_k$. Another description of C is given by $C = \{x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3^n} : \varepsilon_n \in \{0,2\}\}$.

Let us list some of its properties:

(1) C is non empty and compact.

(2) $int(C) = \varnothing$.

(3) C is non numerable.

(4) $m(C) = 0$.

Clearly $\frac{1}{3} \in C$ and $[0,1] \setminus C$ is an open set contained into $[0,1]$ what gives (1).

To see (2) notice first that for each $m \in \mathbb{N}$ we have that $C \setminus \cup_{k=1}^{m} J_k$ is a union of closed intervals of length 3^{-m}. Now if $x \in int(C)$ then we would have $B(x,\varepsilon) \subset C$ for some $\varepsilon > 0$. Taking m such that $3^{-m} < \varepsilon$ we would contradict the first observation.

Using the description in terms of ternary decompositions one gets that $card(C) = card(2^{\mathbb{N}}) = \chi_0$. This shows (3).

Finally

$$m([0,1] \setminus C) = \sum_k m(J_k) = \sum_k \sum_{l=1}^{2^{k-1}} m(I_{k,l}) = \sum_k 2^{k-1} \frac{1}{3^k} = 1.$$

∎

Remark 1.6.1 *$card(\mathcal{B}) \leq 2^{\chi_0}$ and $card(\mathcal{M}) = card(\mathcal{P}(\mathbb{R})) = 2^{\chi_1}$*

Example 1.6.2 *A non Lebesgue measurable set: The Vitali set in $[0,1]$.*

PROOF: Let us introduce in $[0,1]$ the equivalence relation: $x\mathcal{R}t$ if and only if $x - y \in \mathbb{Q}$. Consider the set of equivalence classes $[0,1]/\mathcal{R}$.

Let us select a representant of each equivalence class and define the set $E = \{e \in [0,1] : e \text{ is a representant }\}$.

The set E verifies that

$$[0,1] \subset \cup_{r \in \mathbb{Q} \cap [-1,1]} (E + r) \subset [-1,2]$$

and $\cup_{r \in \mathbb{Q} \cap [-1,1]} (E + r)$ is a countable union of pairwise disjoint sets.

Indeed, on the one hand, if $x \in [0,1]$ there exists $e \in E$ such that $x - e \in \mathbb{Q} \cap [-1,1]$ and if $x \in E + r$ for some $r \in \mathbb{Q} \cap [-1,1]$ then $1 \leq x \leq 2$.

On the other hand $r_1 \neq r_2$ implies that $(E + r_1) \cap (E + r_2) \neq \varnothing$, since $e_1 + r_1 = e_2 + r_2$ gives $[e_1] = [e_2]$ and then $e_1 = e_2$.

Let us now show that E is not Lebesgue measurable. In case it is measurable one has

$$1 \leq \sum_{r \in \mathbb{Q} \cap [-1,1]} m(E+r) \leq 3,$$

and this leads to a contradiction, since $m(E+r) = m(E)$ for all r. ■

Remark 1.6.2 *For any $A \subset \mathcal{M}$ with $m(A) > 0$ there exists $E \subset A$ such that E is not Lebesgue measurable.*

PROOF: Same argument as in the case $[0,1]$ works for the set $A \cap [0,1]$. ■

Example 1.6.3 *A Lebesgue measurable set which is not a Borel set.*

PROOF: Let us first construct the Cantor function in $[0,1]$. Define the following sequence of piecewise linear functions:

$f_0(x) = x$,

$f_1(x) = \frac{1}{2}$ if $x \in (\frac{1}{3}, \frac{2}{3})$ and then linearly with $f_1(0) = 0$ and $f_1(1) = 1$,

$f_2(x) = \frac{1}{4}$ if $x \in (\frac{1}{9}, \frac{2}{9})$, $f_2(x) = \frac{3}{4}$ if $x \in (\frac{7}{9}, \frac{8}{9})$ and then linearly with $f_1(0) = 0$ and $f_1(1) = 1$,

In this way, inductively we construct $f_n(x)$. This take the constant value $\frac{l}{2^k}$ in the set $I_{n,l}$ and it is piecewise linear in the complement of J_n.

Define $f(x) = \lim_n f_n(x)$.

We now prove that it is increasing, non constant, continuous in $[0,1]$ and derivable with $f'(x) = 0$ for all $x \in [0,1] \setminus C$, where C is the Cantor set.

Since f is pointwise limit of increasing functions is increasing. Let $x \notin C$. Then there exists n_0 for which $x \in I_{n_0,l}$ then $f_n(y) = f(y)$ for all $n \geq n_0$ and $y \in I_{n_0,l}$. So $f'(x) = 0$.

To see that f is continuous we shall show that f is uniform limit of continuous functions. Due to the construction we have

$$\begin{aligned}
\sup_{0 \leq x \leq 1} |f_n(x) - f_m(x)| &\leq \sum_{k=n}^{m-1} \sup_{0 \leq x \leq 1} |f_k(x) - f_{k+1}(x)| \\
&\leq \sum_{k=n}^{m-1} \sup_{0 \leq x \leq \frac{1}{3^k}} |f_k(x) - f_{k+1}(x)| \\
&\leq \sum_{k=n}^{m-1} \left|\frac{1}{2^{k+1}} - (\frac{3}{2})^k \frac{1}{3^{k+1}}\right| \\
&\leq \frac{1}{6} \sum_{k=n}^{\infty} \frac{1}{2^k}.
\end{aligned}$$

Now we construct $g : [0,1] \to [0,2]$ given by $g(x) = x + f(x)$. We have that g is a homeomorphism. Of course g is strictly increasing and continuous, and hence there exists g^{-1} continuous and strictly increasing.

Taking into account that $g(I_{k,l}) = c_{k,l} + I_{k,l}$ we get that

$$\begin{aligned} m(g(C)) &= m(g([0,1] \setminus \cup_k J_k)) \\ &= m([0,2] \setminus g(\cup_k J_k)) \\ &= 2 - \sum_k m(g(J_k)) \\ &= 2 - \sum_k \sum_{l=1}^{2^{k-1}} m(c_{k,l} + I_{k,l}) \\ &= 2 - \sum_k \sum_{l=1}^{2^{k-1}} m(I_{k,l}) \\ &= 2 - m([0,1] \setminus C) = 1. \end{aligned}$$

We then have that $g(C)$ is a Borel set (using the fact, to be proved in Exercise 1.7.5, that $g(\mathcal{B}([0,1])) \subset \mathcal{B}([0,2]))$ and $m(g(C)) > 0$. So we can find $E \notin \mathcal{M}([0,2])$ with $E \subset g(C)$. Now consider $A = g^{-1}(E))$. This is a Lebesgue measurable set (since $A \subset C$) but $A \notin \mathcal{B}([0,1])$ (since $g(A) = E \notin \mathcal{B}([0,2])$).

∎

1.7 Exercises

Exercise 1.7.1 *(i) Let $\mathcal{R}$ be a ring. Show that $(\mathcal{R}, \triangle, \cap)$ is a ring in the algebraic sense.*

(ii) Let $\mathcal{M}$ be a non-empty family of sets in X and $\mathcal{R}(\mathcal{M})$ the ring generated by $\mathcal{M}$. Show that any set in $\mathcal{R}(\mathcal{M})$ can be covered by a finite union of sets in $\mathcal{M}$.

Exercise 1.7.2 *Let $f : X \to Y$ be a function, $\mathcal{A}$ an algebra over Y. Let $\sigma(\mathcal{A})$ denote the σ-algebra generated by $\mathcal{A}$ and $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$. Then $\sigma(f^{-1}(\mathcal{A}))$ coincides with $f^{-1}(\sigma(\mathcal{A}))$.*

Exercise 1.7.3 *Let (X, Σ, μ) be a measure space and let (A_n) be a sequence in Σ such that A_j intersects at most one other set in the sequence. Show that*

$$\mu(\cup_{j=1}^{\infty} A_j) \leq \sum_{n=1}^{\infty} \mu(A_j) \leq 2\mu(\cup_{j=1}^{\infty} A_j).$$

Exercise 1.7.4 *Study whether or not the following set functions are outer measures.*

(i) Let $X = (a_{i,j})$ *be a matrix in* M_{10} *and define* $\lambda(A) = card\{j : a_{ij} \in A\}$.

(ii) Let $X = \mathbb{N}$ *and define* $\lambda(A) = limsup\frac{card(A\cap\{1,2,\ldots,k\})}{k}$.

(iii) Let $X = \mathbb{Z}$ *and define* $\lambda(A) = 0$ *for* $A = \varnothing$,

$$\lambda(A) = \frac{a}{a+1} \text{ if } A \text{ is finite, where } a = sup\{|n| : n \in A\} ,$$

$$\lambda(A) = 1 \text{ if } A \text{ is infinite.}$$

(iv) Let X *be a metric space with distance* d *and let* $\alpha > 0$. *Define*

$$\lambda_\alpha = sup_{\varepsilon>0} \inf\{\sum_{k=1}^{\infty}(\delta(A_k))^\alpha : A = \cup A_k, \delta(A_k) < \varepsilon\}\},$$

where $\delta(A_k) = diam(A_k) = sup\{d(x, y) : x, y \in A_k\}$.

Exercise 1.7.5 *Let* X, Y *be topological spaces and let* $g : X \to Y$ *be continuous. Show that the* $\mathcal{B}(Y) \subset g(\mathcal{B}(X))$.

Exercise 1.7.6 *Let* I *be an open interval in* $\mathbb{R}$ *and* $F : I \to \mathbb{R}$ *a continuous and strictly increasing function. Show that the Borel-Stieljes measure defined by* F *coincides with the measure image by* F^{-1} *of the Lebesgue measure on the Borel sets in* $F(I)$.

Exercise 1.7.7 *Describe the Lebesgue-Stieltjes measure* m_F *associated to the following functions:*

(i) $F(x) = [x]$,

(ii) $F(x) = \chi_{[0,\infty)}(x)$,

(iii) $F(x) = (x - 1)^+$.

Exercise 1.7.8 *Let* $F(x) = \frac{-1}{x}\chi_{(0,1)} + (log(x) - 1)\chi_{[1,\infty)}$.

(i) Find an unbounded Borel set A *with* $0 < m_F(A) < \infty$.

(ii) Find an open set G *such that* $0 \in G'$ *and* $m_F(G) < \infty$.

Exercise 1.7.9 *Let* m_F *be the Lebesgue-Stieltjes measure* $(0, \infty)$ *associated to* $F(x) = x^\alpha$ *for some* $\alpha > 0$ *and let* $A = \cup_{n=1}^{\infty}(n, \frac{n^2+1}{n})$.

Find the values of α *so that* $m_F(A) < \infty$.

Exercise 1.7.10 *Let* $\phi(t) = \sum_{n=1}^{\infty} n\chi_{(n,\frac{2n+1}{2})}(t)$.

Find F *for the Lebesgue-Stieltjes measure associated to* F *to coincide with the measure image by* ϕ *of the Lebesgue measure* m, *that is* $m_F = \phi(m)$.

Exercise 1.7.11 *Let* $F : (0,\infty) \to \mathbb{R}$ *be given by* $F(t) = log(t)$. *Show that*

(i) $m_F = exp(m)$ *where* m *is the Lebesgue measure and* $exp(x) = e^x$.

(ii) m_F *is invariant under dilations.*

(iii) If $\mu : \mathcal{B}((0,\infty)) \to [0,\infty]$ *is a measure finite over bounded intervals and invariant under dilations then* $\mu = Cm_F$ *for some constant* $C > 0$.

(iv) Find an unbounded open set $G \subset (0,\infty)$ *such that* $0 \in G'$ *and* $m_F(G) < \infty$.

Chapter 2

Measurable and Integrable functions

2.1 Measurable functions

Definition 2.1.1 *Let (X, Σ) be a measurable space. A function $f : X \to [0, \infty]$ is called measurable if the sets $E_\alpha = \{x \in X : f(x) > \alpha\} \in \Sigma$ for all $\alpha > 0$.*

Some equivalent formulations are given in the following proposition.

Proposition 2.1.2 *Let (X, Σ) be a measurable space and $f : X \to [0, \infty]$. The following are equivalent*

(i) f is measurable.
(ii) $\{x \in X : f(x) \leq \alpha\} \in \Sigma$ for all $\alpha > 0$.
(iii) $\{x \in X : f(x) < \alpha\} \in \Sigma$ for all $\alpha > 0$.
(iv) $\{x \in X : f(x) \geq \alpha\} \in \Sigma$ for all $\alpha > 0$.
(v) $\{x \in X : f(x) = \infty\} \in \Sigma$ and $f^{-1}(B) \in \Sigma$ for all Borel set $B \subset \mathbb{R}^+$.
(vi) $\{x \in X : f(x) = \infty\} \in \Sigma$ and $f^{-1}(G) \in \Sigma$ for all open set $G \subset \mathbb{R}^+$.
(vii) $\{x \in X : f(x) = \infty\} \in \Sigma$ and $f^{-1}([a, b)) \in \Sigma$ for all $0 \leq a < b < \infty$.

Proof: $(i) \iff (ii)$ It follows from $\{x \in X : f(x) \leq \alpha\} = X \setminus \{x \in X : f(x) > \alpha\}$.

$(ii) \iff (iii)$ It follows since $\{x \in X : f(x) < \alpha\} = \cup_{n \in \mathbb{N}} \{x \in X : f(x) \leq \alpha - \frac{1}{n}\}$ and $\{x \in X : f(x) \leq \alpha\} = \cap_{n \in \mathbb{N}} \{x \in X : f(x) < \alpha + \frac{1}{n}\}$

$(iii) \iff (iv)$ It follows from $\{x \in X : f(x) \geq \alpha\} = X \setminus \{x \in X : f(x) < \alpha\}$.

$(i) \Longrightarrow (v)$ Write $\{x \in X : f(x) = \infty\} = \cap_{n\in\mathbb{N}}\{x \in X : f(x) > n\} \in \Sigma$.
Define $\mathcal{R} = \{B \in \mathcal{B}([0,\infty)) : f^{-1}(B) \in \Sigma\}$.
This is σ-algebra and contains $(a, b]$ for all $0 \le a < b$ since

$$f^{-1}((a,b]) = \{x \in X : f(x) > a\} \cap \{x \in X : f(x) \le b\} \in \Sigma.$$

$(v) \Longrightarrow (vi)$ It is obvious.

$(vi) \Longrightarrow (vii)$ Note that $[0, b)$ is open in $[0,\infty)$ and $[a, b) = \cap_{n\ge n_0}(a - 1/n, b)$ for $0 < 1/n_0 < a$.

$(vii) \Longrightarrow (i)$ It follows since

$$\{x \in X : f(x) > \alpha\} = \{x \in X : f(x) = \infty\} \cup (\cup_n\{x \in X : f(x) \in [\alpha+1/n, n)\}).$$

■

Definition 2.1.3 *Let (X,Σ) be a measurable space. A function $f : X \to [0,\infty)$ is called a simple function if it takes a finite number of different values. In particular, $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $\mathcal{A}_i = f^{-1}(\{\alpha_i\})$ are disjoint sets and $\alpha_i > 0$ are different for $i = 1, 2, ..., n$.*

Note that f is measurable if and only if $A_i \in \Sigma$ for $i = 1, 2, ..., n$..

Remark 2.1.1 *Let us denote by $\mathcal{S}$ the set of simple measurable functions. This is a vector space and also we have that if $s_1, s_2 \in \mathcal{S}$ then $s_1.s_2 \in \mathcal{S}$, $\max(s_1, s_2) \in \mathcal{S}$ and $\min(s_1, s_2) \in \mathcal{S}$. In general $\lim_n s_n \notin \mathcal{S}$ for a sequence $(s_n) \subset \mathcal{S}$, for instance $f = \sum_{n=1}^\infty n\chi_{[n,n+1/2]}$ is not simple, but it is limit of simple functions.*

Theorem 2.1.4 *Let $f : X \to [0,\infty]$ be a measurable function. There exists a sequence s_n of simple functions, such that $0 \le s_1 \le s_2 \le ... \le f$ and $\lim_n s_n(x) = f(x)$ for all $x \in X$.*

Moreover, if f a bounded function then $\lim_n s_n(x) = f(x)$ uniformly in $x \in X$.

PROOF: For each $n \in \mathbb{N}$ we denote by $E_{n,k} = f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n}))$, for $k = 1, 2, ..., n2^n$ and $E_{n,n2^n+1} = f^{-1}([n,\infty])$ and define

$$s_n = \sum_{k=1}^{n2^n+1} \frac{k-1}{2^n}\chi_{E_{n,k}}.$$

Let us show first that $s_n(x)$ is increasing for all $x \in X$. Fix $n \in \mathbb{N}$ and $x \in X$.

If $f(x) \geq n+1$ then $s_n(x) = n \leq s_{n+1}(x) = n+1$.

If $n \leq f(x) < n+1$ then $s_n(x) = n$ and $s_{n+1}(x) = \frac{k-1}{2^{n+1}}$ for some $k \geq n2^{n+1}+1$. Hence $s_n(x) \leq s_{n+1}(x)$.

If $f(x) < n$ then there exist k, j so that $f(x) \in [\frac{k-1}{2^n}, \frac{k}{2^n}) \cap [\frac{j-1}{2^{n+1}}, \frac{j}{2^{n+1}})$. Hence either $\frac{j-1}{2} = k-1$ or $\frac{j-1}{2} = k$. In the first case $s_n(x) = \frac{k-1}{2^n} = \frac{j-1}{2^{n+1}} = s_{n+1}(x)$ and in the second case $s_n(x) = \frac{k-1}{2^n} = \frac{j-3}{2^{n+1}} \leq \frac{j-1}{2^{n+1}} = s_{n+1}(x)$.

Let us now show that $\lim_{n\to\infty} s_n(x) = f(x)$.

The case $f(x) = \infty$ we have $s_n(x) = n$ for all $n \in \mathbb{N}$.

Assume $f(x) < \infty$. Take $n_0 = n_0(x) \in \mathbb{N}$ so that $f(x) \leq n_0$. For all $n > n_0$ we have that there exists $k \in \{1, 2, .., n2^n\}$ such that $f(x) \in E_{n,k}$. Hence for all $n \geq n_0$ we have

$$f(x) - s_n(x) = f(x) - \frac{k-1}{2^n} < \frac{1}{2^n}.$$

Note that if f is bounded n_0 is the same for all x and the previous estimate holds uniformly in $x \in X$. This finishes the proof. ∎

Definition 2.1.5 *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. A function $f : X_1 \to X_2$ is called (Σ_1, Σ_2)-measurable if $f^{-1}(A) \in \Sigma_1$ for all $A \in \Sigma_2$.*

For $\Sigma_2 = \mathcal{B}(X_2)$ where (X_2, τ) is a topological space, a function $f : X_1 \to X_2$ is $(\Sigma_1, \mathcal{B}(X_2))$-measurable (usually called simply measurable) if and only if $f^{-1}(A) \in \Sigma_1$ for all $A \in \tau$.

If X and Y are topological spaces, $f : X \to Y$ is said to be Borel measurable if $f^{-1}(G) \in \mathcal{B}(X)$ for all open set $G \subset X$.

For $X = Y = \mathbb{R}$ a function $f : \mathbb{R} \to \mathbb{R}$ is called Lebesgue measurable if it is $(\mathcal{M}, \mathcal{B})$-measurable, or equivalently $f^{-1}((a, b]) \in \mathcal{M}$ for all $a < b$.

Remark 2.1.2 *Given a measurable space (X, Σ) and a function $f : X \to [0, \infty]$. Considering $[0, \infty]$ as the compactification of the topological space $\mathbb{R}^+$ then f is measurable (according to Definition 2.1.1) if and only if f is $(\Sigma, \mathcal{B}([0, \infty]))$-measurable (according to Definition 2.1.5).*

Equivalently f is measurable if and only if $X_\infty = \{x \in X : f(x) = \infty\} \in \Sigma$ and $f|_{X \setminus X_\infty}$ is $(\Sigma, \mathcal{B}(\mathbb{R}^+))$-measurable.

Let us establish some simple results about measurability on composition of functions.

Lemma 2.1.6 *(i) Let (X_1, Σ_1), (X_2, Σ_2) and (X_3, Σ_3) be measurable spaces and functions $f : X_1 \to X_2$ and $g : X_2 \to X_3$. If f is (Σ_1, Σ_2)-measurable and g is (Σ_2, Σ_3)-measurable then $g \circ f$ is (Σ_1, Σ_3)-measurable.*

(ii) Let (X_1, Σ_1) be a measurable space,let X_2 and X_3 be topological spaces. If $f : X_1 \to X_2$ is measurable and $g : X_2 \to X_3$ is continuous then $g \circ f$ is measurable.

(iii) Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces and let Y be a topological space. If $f_1 : X_1 \to \mathbb{R}^n$ and $f_2 : X_1 \to \mathbb{R}^m$ are measurable and $\phi : \mathbb{R}^n \times \mathbb{R}^m \to Y$ is continuous then $h : X \to Y$ given by $h(x) = \phi(f_1(x), f_2(x))$ is measurable.

Proof: (i) and (ii) are immediate.

(iii) Consider $\psi : X \to \mathbb{R}^n \times \mathbb{R}^m$ given by $\psi(x) = (f_1(x), f_2(x))$ and note that

$$\psi^{-1}((a_1, b_1] \times ... \times (a_n, b_n] \times (a'_1, b'_1] \times ... \times (a'_m, b'_m])$$
$$= (f_1)^{-1}((a_1, b_1] \times ... \times (a_n, b_n]) \cap (f_2)^{-1}((a'_1, b'_1] \times ... \times (a'_m, b'_m]).$$

Since any open set in $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ is numerable union of sets $(a_1, b_1] \times ... \times (a_n, b_n] \times (a'_1, b'_1] \times ... \times (a'_m, b'_m]$ we have that ψ is measurable and then $h = \phi \circ \psi$ too. ∎

Let us recollect several operations on functions which are stable under measurability.

Proposition 2.1.7 *Let $(X; \Sigma)$ be a measurable space and let $\lambda \geq 0$ and $f, g, \{f_n\}$ be measurable functions.*

Then (with the convections $0 \cdot \infty = 0$, $1/0 = \infty$ and $1/\infty = 0$) λf, $f + g$, $f.g$, $\max\{f, g\}$, $\min\{f, g\}$ $1/f$, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are measurable functions.

Proof: Write $X_\infty = f^{-1}(\{\infty\})$, $X'_\infty = g^{-1}(\{\infty\})$, $f_1 = f|_{X \setminus X_\infty}$ and $g_1 = g|_{X \setminus X'_\infty}$ Hence

$$\{f + g > \alpha\} = X_\infty \cup X'_\infty \cup \{f_1 + g_1 > \alpha\},$$

$$\{fg > \alpha\} = \{f = \infty, g > 0\} \cup \{g = \infty, f > 0\} \cup \{f_1 g_1 > \alpha\}$$

Now, using (iii) in Lemma 2.1.6 we have that $\phi(f_1(x), f_2(x))$ for $\phi(t, s) = t+s$ or $\phi(t, s) = ts$ are measurable. Hence we obtain the measurability of $f + g$ or fg. The other cases follow from

$$\{\lambda f > \alpha\} = \{f > \frac{\alpha}{\lambda}\}.$$

$$\{\frac{1}{f} > \alpha\} = \{f = 0\} \cup \{0 < f < \frac{1}{\alpha}\}.$$

$$\{\max(f, g) \leq \alpha\} = \{f \leq \alpha\} \cap \{g \leq \alpha\}.$$

$$\{\min(f, g) > \alpha\} = \{f > \alpha\} \cup \{g > \alpha\}.$$

$$\{\sup_n f_n \leq \alpha\} = \cap_n \{f_n \leq \alpha\}.$$

$$\{\inf_n f_n > \alpha\} = \cup_n \{f_n > \alpha\}.$$

The *limsup* and *liminf* follows from the previous ones. ■

Remark 2.1.3 *The supremum of measurable functions needs not be measurable.*

Let E be the Vitali set in $[0, 1]$ and write $\chi_E = \sup_{x \in E} \chi_{\{x\}}$.

Definition 2.1.8 *Let $f : X \to \mathbb{R}$ be a function, then $f^+ = max(f, 0)$ and $f^- = -min(f, 0)$.*

Clearly, $f^+ = f\chi_{\{f \geq 0\}}$, $f^- = -f\chi_{\{f \leq 0\}}$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Proposition 2.1.9 *Let (X, Σ) be a measurable space and $f : X \to \mathbb{R}$. The following are equivalent.*

(i) f is $(\Sigma, \mathcal{B}(\mathbb{R}))$-measurable.

(ii) $\{x : f(x) > \beta\} \in \Sigma$ for all $\beta \in \mathbb{R}$.

(iii) f^+ and f^- are measurable.

PROOF: $(i) \implies (ii)$ $\{x : f(x) > \beta\} = f^{-1}((\beta, \infty)) \in \Sigma$.

$(ii) \implies (iii)$ Let $\alpha > 0$ then $\{x : f^+(x) > \alpha\} = \{x : f(x) > \alpha\} \in \Sigma$ and $\{x : f^-(x) > \alpha\} = \{x : f(x) < -\alpha\} = X \setminus \cap_n \{x : f(x) > -\alpha - 1/n\} \in \Sigma$.

$(iii) \implies (i)$ Let G be an open set. Then $f^{-1}(G) = (f^+)^{-1}(G \cap (0, \infty)) \cup (f^-)^{-1}(-G \cap [0, \infty)) \in \Sigma$. ■

Proposition 2.1.10 *Let (X, Σ) be a measurable space and $f : X \to \mathbb{C}$. The following are equivalent.*

(i) f is $(\Sigma, \mathcal{B}(\mathbb{C}))$-measurable.

(ii) $\Re f$ and $\Im f$ are measurable.

(iii) $|f|$ is measurable and there exists a measurable function $\alpha : X \to \mathbb{T} = \{z : |z| = 1\}$ such that $f = |f|\alpha$.

PROOF: $(i) \Longrightarrow (ii)$ It follows from composition since $z \to \Re(z)$ and $z \to \Im(z)$ are continuous.

$(ii) \Longrightarrow (iii)$ If follows from Lemma 2.1.6 that $|f|$ is measurable, since $|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$. Take now $E = \{x : f(x) = 0\} \in \Sigma$. Consider $\pi : \mathbb{C} \setminus \{0\} \to \mathbb{T}$ given by $\pi(z) = \frac{z}{|z|}$ and define $\alpha(x) = \pi(f(x) + \chi_E(x))$. We can apply Lemma 2.1.6 to get that α is measurable and, of course, $f = |f|\alpha$.

$(iii) \Longrightarrow (i)$ Using $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by $(z_1, z_2) \to z_1 z_2$ we get that the product of measurable functions is measurable and gives the result. ∎

Let us list several operations which preserve measurability, whose proofs are left to the reader.

Proposition 2.1.11 *Let $(X; \Sigma)$ be a measurable space and let $\lambda \in \mathbb{K}$ and $f, g, \{f_n\}$ be measurable functions from X into $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

Then

(i) λf, $f + g$, $f.g$, $1/f$ are measurable.

(ii) For $\mathbb{K} = \mathbb{R}$, $\max\{f, g\}$, $\min\{f, g\}$ $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are measurable functions.

(iii) If $\lim_n f_n(x) = F(x)$ for all $x \in X$ then F is measurable.

Definition 2.1.12 *Let (X, Σ) be a measurable space. A measurable function $f : X \to \mathbb{C}$ is called a simple function if it takes a finite number of different values. In particular, $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $\mathcal{A}_i = f^{-1}(\{\alpha_i\}) \in \Sigma$ are disjoint sets and α_i are the non zero values for $i = 1, 2, ..., n$.*

Corollary 2.1.13 *Let $f : X \to \mathbb{C}$. Then f is measurable if and only if there exists a sequence (s_n) of simple functions such that $|s_n| \le |f|$ and $f(x) = \lim_{n\to\infty} s_n(x)$ for all $x \in X$.*

PROOF: Since the limit of measurable functions is measurable we simply need to prove the "only if" part. Assume $f = u + iv$ be measurable and write also $u = u^+ - u^-$ and $v = v^+ - v^-$. We can apply Theorem 2.1.4 and find increasing sequences $(r_n), (t_n), (p_n)$ and (q_n) of non-negative simple functions converging to u^+, u^-, v^+ and v^- respectively.

The functions $s_n = r_n - t_n + i(p_n - q_n)$ are simple complex-valued functions and converge to f. Also we have

$$\begin{aligned} |s_n| &= \sqrt{(r_n - t_n)^2 + (p_n - q_n)^2} \\ &\le \sqrt{(r_n + t_n)^2 + (p_n + q_n)^2} \end{aligned}$$

$$\begin{aligned} &\leq \sqrt{(u^+ + u^-)^2 + (v^+ + v^-)^2} \\ &= \sqrt{u^2 + v^2} = |f|. \end{aligned}$$

■

2.2 Some types of convergence.

Definition 2.2.1 *Let (X, Σ, μ) a measure space. We say that a property P holds almost everywhere with respect to (X, Σ, μ) (in short μ-a.e.) if there exists the set $A \in \Sigma$ $\mu(A) = 0$ such that the property holds in $X \setminus A$.*

Let us point out some facts to see the difference between complete and not complete measure spaces regarding the behaviour "a.e."

Remark 2.2.1 *(X, Σ, μ) a measure space. Let f be a measurable function and $f = g$ μ-a.e. Then g needs not be measurable.*

Indeed, the example can be produced for non complete measure space. Consider $(\mathbb{R}, \mathcal{B}, m)$ the Borel measure space on $\mathbb{R}$. There exists $A \in \mathcal{M} \setminus \mathcal{B}$ with $m(A) = 0$ such that there is $B \in \mathcal{B}$ satisfying $A \subset B$ and $m(B) = 0$. Now define $f = 1$ and $g = \chi_{\mathbb{R}\setminus B} + 2\chi_{B\setminus A}$. We have that $f = g$ m-a.e since $\{f(x) \neq g(x)\} = B$, but $g^{-1}(\{0\}) = A \notin \mathcal{B}$.

Remark 2.2.2 *(X, Σ, μ) a measure space. Let f_n be measurable functions and $\lim_n f_n = f$ μ-a.e. Then f needs not be measurable.*

Indeed, let $(\mathbb{R}, \mathcal{B}, m)$ be the Borel σ-algebra on $\mathbb{R}$. As above tak $A \in \mathcal{M}\setminus\mathcal{B}$ with $m(A) = 0$ and $B \in \mathcal{B}$ such that $A \subset B$ and $m(B) = 0$. Now define $f_n = 0$ and $f = \chi_A$. We have that $\lim_n f_n = f$ m-a.e. since $\{f(x) \neq 0\} \subset B$.

Proposition 2.2.2 *Let (X, Σ, μ) be a complete measure space.*

(i) If $f : X \to [0, \infty]($ or $\mathbb{C})$ is measurable and $f = g$ μ-a.e. then g is measurable.

(ii) If $f_n : X \to [0, \infty]($ or $\mathbb{C})$ is a sequence of measurable functions and $\lim f_n = f$ μ-a.e. then f is measurable.

PROOF: (i) Assume $f : X \to [0, \infty]$ and $f(x) = g(x)$ for $x \notin A$ and $A \in \Sigma$ with $\mu(A) = 0$. Given $\alpha > 0$ we have that

$$\{x : g(x) > \alpha\} = \{x \in A : g(x) > \alpha\} \cup \{x \notin A : f(x) > \alpha\}.$$

Since $\{x \in A : g(x) > \alpha\} \subset A$ then $\{x \in A : g(x) > \alpha\} \in \Sigma$ and also $\{x \notin A : f(x) > \alpha\} \in \Sigma$, which gives the result.

The case $f : X \to \mathbb{C}$ is done in a similar way.

(ii) Assume $f(x) = \lim f_n(x)$ for all $x \notin A$ and $A \in \Sigma$ with $\mu(A) = 0$. Denote $X_1 = X \setminus A$ and Σ_1 the induced σ-algebra. The restrictions of f_n to X_1 are measurable with respect to Σ_1 and then f restricted to X_1 is measurable with respect to Σ_1. Define $g(x) = f(x)$ for $x \in X_1$ and $g(x) = 0$ for $x \in A$. Obviously g is measurable and $f = g$ μ-a.e., hence using (i) we get that f is measurable. ∎

Theorem 2.2.3 *Let (X, Σ, μ) be a measure space and $(X, \bar{\Sigma}, \bar{\mu})$ its complection. Then $f : X \to [0, \infty]$ (respect. $f : X \to \mathbb{C}$) is measurable with respect to $\bar{\Sigma}$ if and only if there exists $g : X \to [0, \infty]$ (respect. $g : X \to \mathbb{C}$) measurable with respect to Σ such that $f = g$ μ-a.e.*

PROOF: Assume that $f : X \to [0, \infty]$ is measurable with respect to $\bar{\Sigma}$. Then using Theorem 2.1.4 we get (s_n) simple functions measurable with respect to Σ and we can write, with $s_0 = 0$,

$$f = \sum_{n=1}^{\infty} (s_n - s_{n-1}) = \sum_{i=1}^{\infty} c_i \chi_{E_i}$$

where $c_i > 0$ and $E_i \in \bar{\Sigma}$ for all $i \in \mathbb{N}$.

We now choose $A_i, B_i \in \Sigma$ so that $A_i \subset E_i \subset B_i$ and $\mu(B_i \setminus A_i) = 0$. Define $g = \sum_{i=1}^{\infty} c_i \chi_{A_i}$. It is measurable with respect to Σ. If $N = \cup_i (B_i \setminus A_i)$ we have that $f(x) = g(x)$ for $x \in X \setminus N$ (since $x \notin N$ implies that for each i one gets $x \notin B_i$ or $x \in A_i$) and $\mu(N) = 0$.

The case $f : X \to \mathbb{C}$ follows from the previous one in the usual way and it is left to the reader.

Assume now $f = g$ μ-a.e. for some $g : X \to [0, \infty]$ (respect. $g : X \to \mathbb{C}$) measurable with respect to Σ. Using that g is also measurable with respect $\bar{\Sigma}$ and Proposition 2.2.2 we get that f is measurable with respect $\bar{\Sigma}$. ∎

Corollary 2.2.4 *$f : \mathbb{R} \to \mathbb{C}$ is Lebesgue measurable if and only if there exists $g : \mathbb{R} \to \mathbb{C}$ Borel measurable such that $f = g$ m-a.e.*

Definition 2.2.5 *Let (X, Σ, μ) be a measure space and $B \in \Sigma$.*

A sequence (f_n) is said to converge uniformly in B to f if

$$\lim_{n\to\infty} \sup_{x\in B} |f(x) - f_n(x)| = 0.$$

A sequence (f_n) is said to converge almost uniformly to f if there exists $A \in \Sigma$ with $\mu(A) = 0$ such that f_n converges to f uniformly in $X \setminus A$, i.e.

$$\lim_{n\to\infty} \sup_{x\notin A} |f(x) - f_n(x)| = 0.$$

Remark 2.2.3 *The almost uniform convergence is weaker than the uniform convergence.*

It suffices to take a sequence uniformly convergent to a function and modify the limit function in a set of measure cero to get an almost uniform limit which is not uniform.

Remark 2.2.4 *The pointwise convergence is weaker that the almost uniform convergence.*

Take $f_n = \chi_{[n,n+1]}$ in $(\mathbb{R}, \mathcal{M}, m)$. Of course $f_n(x)$ converges pointwise to zero, but for any set $A \in \mathcal{M}$ with $m(A) = 0$ we have that for each $n \in \mathbb{N}$ there is $x_n \in [n, n+1] \notin A$ (otherwise there exists $k \in \mathbb{N}$ such that $[k, k+1] \subset A$).

Hence $\sup_{x\notin A} |f_n(x)| \geq \sup_n |f_n(x_n)| = 1$.

Theorem 2.2.6 *(Egorov's theorem) Let (X, Σ, μ) be a finite measure space. Let $f_n : X \to [0, \infty]$ be finite a.e. measurable functions (i.e. $\mu(\{f_n = \infty\}) = 0$) and let $f(x) = \lim_n f_n(x)$ a.e is measurable and finite a.e. (i.e.$\mu(\{f = \infty\}) = 0$ and there exists $C \in \Sigma, \mu(C) = 0$ such that $\lim_n f_n(x) = f(x), x \notin C$). Then for any $\delta > 0$ there exists $A \in \Sigma$ with $\mu(A) < \delta$ such that f_n converges to f uniformly in $X \setminus A$.*

PROOF: We can assume that f_n, f take values in $\mathbb{R}^+$ and that $f_n(x)$ converges to $f(x)$ for all $x \in X$. In other case, we put $A_n = \{f_n = \infty\}$, $B = \{f = \infty\}$ and C with $\mu(C) = 0$ such that $\lim f_n(x) = f(x)$ if $x \notin C$. We work in $X_1 = X \setminus N$ where $N = (\cup_n A_n) \cup B \cup C$.

Let us first prove that $\forall \delta, \varepsilon > 0$ there exists $A_{\varepsilon,\delta} \in \Sigma$ and $n_{\varepsilon,\delta} \in \mathbb{N}$ such that $\mu(A_{\varepsilon,\delta}) < \delta$ and $|f_m(x) - f(x)| < \varepsilon$, for all $x \notin A_{\varepsilon,\delta}$ and $m \geq n_{\varepsilon,\delta}$.

Indeed, write

$$A_n(\varepsilon) = \{x \in X : |f_m(x) - f(x)| \geq \varepsilon \text{ for some } m \geq n\}.$$

It is a decreasing sequence $A_{n+1}(\varepsilon) \subset A_n(\varepsilon)$ and $\cap_n A_n(\varepsilon) = \varnothing$.

Since $\mu(X) < \infty$ then $\lim_n \mu(A_n(\varepsilon)) = 0$. Therefore there exist $n_0 = n_{\varepsilon,\delta} \in \mathbb{N}$ and $A_{\varepsilon,\delta} = A_{n_0}(\varepsilon)$ such that $\mu(A_{n_0}(\varepsilon)) < \delta$. Note that $|f_m(x) - f(x)| < \varepsilon$ if $x \notin A_{\varepsilon,\delta}$ and $m \geq n_0$.

Now, given $k \in \mathbb{N}$ and $\delta > 0$, we apply the previous result for ε and δ being $\varepsilon = \frac{1}{k}$ and $\frac{\delta}{2^k}$ to get $B_k \in \Sigma$, $n_k \in \mathbb{N}$ for which $\mu(B_k) < \frac{\delta}{2^k}$ and $|f_m(x) - f(x)| < \frac{1}{k}$ for $x \notin B_k$ and $m \geq n_k$. Finally take $A = \cup_k B_k$. We have $\mu(A) < \delta$ and f_n converges uniformly in $X \setminus A$.

Indeed, given $\eta > 0$ we first get k_0 for which $1/k < \eta$ for $k \geq k_0$. Now for all $m \geq n_{k_0}$ and $x \notin A$ this implies $x \notin B_k$ and then $|f_m(x) - f(x)| < 1/k < \eta$.

■

Remark 2.2.5 *If $\mu(X) = \infty$ then Egorov's theorem does not hold.*

Take $(\mathbb{R}, \mathcal{B}, m)$ and $f_n = \chi_{[n,\infty)}$. Clearly $f_n \to 0$ pointwise, but if $A \in \mathcal{B}$ and $\mu(A) < \infty$ then $A \cap [n, \infty) \neq \varnothing$ for all $n \in \mathbb{N}$. This shows that $\sup_{x \notin A} |f_n(x)| \geq 1$ for all $n \in \mathbb{N}$ and (f_n) does not converge uniformly in A.

2.3 Integrable functions.

Definition 2.3.1 *Let (X, Σ, μ) a measure space and let $s : X \to [0, \infty]$ be a simple function. We define*

$$\int_X s d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i)$$

where α_i are the different non zero values of s.

We say that s is μ-integrable (or simply integrable) if $\int_X s d\mu < \infty$, which is equivalent to $\mu(E_i) < \infty$ for $\alpha_i > 0$.

We use the convection $0 \cdot \infty = 0$ and then we can write $\sum_{i=1}^n \alpha_i \mu(E_i)$ even when $\alpha_i = 0$.

Let us see that actually the definition is independent of the decomposition of s.

Proposition 2.3.2 *Let s be a simple function with non zero values α_i for $i = 1, ..., n$, say $s = \sum_{i=1}^n \alpha_i \chi_{E_i}$. Assume that also $s = \sum_{j=1}^m \beta_j \chi_{F_j}$ for $\beta_j \geq 0$ and $F_j \in \Sigma$ for $j \in 1, ..., m$. Then*

$$\int_X s d\mu = \sum_{j=1}^{m} \beta_j \mu(F_j).$$

PROOF: Assume first that F_j are pairwise disjoint. For each $i \in 1, ..., n$, consider $M_i = \{j \in \{1, ..., m\} : \beta_j = \alpha_i\}$ Hence M_i are pairwise disjoint, $\{1, ..., m\} = \cup_{i=1}^n M_i$ and $E_i = \cup_{j \in M_i} F_j$. Therefore

$$\begin{aligned} \sum_{i=1}^n \alpha_i \mu(E_i) &= \sum_{i=1}^n \alpha_i \sum_{j \in M_i} \mu(F_j) \\ &= \sum_{i=1}^n \sum_{j \in M_i} \beta_j \mu(F_j) \\ &= \sum_{j=1}^m \beta_j \mu(F_j) \end{aligned}$$

Now we show the general case.
For $s = \beta_1 \chi_{F_1} + \beta_2 \chi_{F_2}$ where $F_1, F_2 \in \Sigma$ not necessarily disjoint.
We can write

$$s = (\beta_1 + \beta_2)\chi_{F_1 \cap F_2} + \beta_1 \chi_{F_1 \setminus F_2} + \beta_2 \chi_{F_2 \setminus F_1}.$$

Applying the previous case

$$\int_X s d\mu = (\beta_1 + \beta_2)\mu(F_1 \cap F_2) + \beta_1 \mu(F_1 \setminus F_2) + \beta_2 \mu(F_2 \setminus F_1) = \beta_1 \mu(F_1) + \beta_2 \mu(F_2).$$

For $s = \beta_1 \chi_{F_1} + \beta_2 \chi_{F_2} + \beta_3 \chi_{F_3}$ where $F_1, F_2, F_3 \in \Sigma$ not necessarily pairwise disjoint.
We can write

$$\begin{aligned} s &= (\beta_1 + \beta_2 + \beta_3)\chi_{F_1 \cap F_2 \cap F_3} \\ &+ (\beta_1 + \beta_2)\chi_{(F_1 \cap F_2) \setminus F_3} + (\beta_1 + \beta_3)\chi_{(F_1 \cap F_3) \setminus F_2} + (\beta_2 + \beta_3)\chi_{(F_2 \cap F_3) \setminus F_1} \\ &+ \beta_1 \chi_{F_1 \setminus (F_2 \cup F_3)} + \beta_2 \chi_{F_2 \setminus (F_1 \cup F_3)} + \beta_3 \chi_{F_3 \setminus (F_1 \cup F_2)}. \end{aligned}$$

Applying the case of disjoint sets

$$\begin{aligned} \int_X s d\mu &= (\beta_1 + \beta_2 + \beta_3)\mu(F_1 \cap F_2 \cap F_3) \\ &+ (\beta_1 + \beta_2)\mu((F_1 \cap F_2) \setminus F_3) \\ &+ (\beta_1 + \beta_3)\mu((F_1 \cap F_3) \setminus F_2) \\ &+ (\beta_2 + \beta_3)\mu((F_2 \cap F_3) \setminus F_1) \\ &+ \beta_1 \mu(F_1 \setminus (F_2 \cup F_3)) + \beta_2 \mu(F_2 \setminus (F_1 \cup F_3)) + \beta_3 \mu(F_3 \setminus (F_1 \cup F_2)) \\ &= \beta_1 \mu(F_1) + \beta_2 \mu(F_2) + \beta_3 \mu(F_3). \end{aligned}$$

Now if $s = \sum_{i=1}^n \beta_i \chi_{F_i}$ where $F_i \in \Sigma$ not necessarily pairwise disjoint. This general case follows same argument with a rather more complicate notation. Set $F_{i,j} = F_i \cap F_j$ for $i \neq j$ and $F_{i_1,i_2,...,i_j} = F_{i_1} \cap F_{i_2} \cap ... \cap F_{i_j}$ for different $i_1, ..., i_j$.

We can write

$$\begin{aligned} s &= \sum_{i=1}^n \beta_i \chi_{F_i \setminus (\cup_{j \neq i} F_j)} \\ &+ \sum_{i \neq j} (\beta_i + \beta_j) \chi_{F_{i,j} \setminus \cup_{(i',j') \neq (i,j)} F_{i',j'}} \\ &+ \sum_{(i_1,i_2,i_3)} (\beta_{i_1} + \beta_{i_2} + \beta_{i_3}) \chi_{F_{i_1,i_2,i_3} \setminus \cup_{(i_1',i_2',i_3') \neq (i_1,i_2,i_3)} F_{i_1',i_2',i_3}} \\ &+ .... \\ &+ (\sum_{i=1}^n \beta_i) \chi_{F_1 \cap ... \cap F_n}. \end{aligned}$$

Now using the result for disjoint sets and adding up the measures the result is complete. ∎

Definition 2.3.3 *Let (X, Σ, μ) a measure space and let $f : X \to [0, \infty]$ be a measurable function. We define*

$$\int_X f d\mu = \sup\{ \int_X s d\mu : 0 \leq s \leq f, \ s \text{ simple } \}.$$

We say that f is μ-integrable if $\int_X f d\mu < \infty$.

For each $E \in \Sigma$ we define $\int_E f d\mu = \int_X f \chi_E d\mu$.

Proposition 2.3.4 *Let $f, g : X \to [0, \infty]$ be measurable functions, $\lambda > 0$ and $E, F \in \Sigma$.*

(i) If $f \leq g$ then $\int_X f d\mu \leq \int_X g d\mu$.

(ii) If $E \subset F$ then $\int_E f d\mu \leq \int_F g d\mu$.

(iii) If s is a simple function, $E \in \Sigma$ then

$$\int_E s d\mu = \sum_{i=1}^n \alpha_i \mu(E \cap E_i) = \sup\{ \int_E t d\mu : 0 \leq t \leq s : t \text{ simple } \}.$$

(iv) $\int_X \lambda f d\mu = \lambda \int_X f d\mu$.

Proof: (i) and (iv) are obvious.

(ii) and (iii) follow from (i). ∎

Theorem 2.3.5 *(The Lebesgue monotone convergence theorem) Let $f_n : X \to [0,\infty]$ be measurable functions such that $f_n(x) \leq f_{n+1}(x)$ and $\lim_n f_n(x) = f(x)$ for all $x \in X$. Then*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

PROOF: Using Proposition 2.3.4 we have

$$\int_X f_n d\mu \leq \int_X f_{n+1} d\mu \leq \int_X f d\mu.$$

Define $M = \sup_n \int_X f_n d\mu = \lim_n \int_X f_n d\mu$.

Of course $M \leq \int_X f d\mu$. Let us show that $\int_X f d\mu \leq M$.

We may assume $M < \infty$. For each simple function $s = \sum_{i=1}^n \alpha_i \chi_{E_i}$ such that $0 \leq s \leq f$ and each $0 < c < 1$ we consider the sequence

$$A_n = \{x \in X : f_n(x) \geq cs(x)\}$$

which form an increasing sequence of measurable sets in Σ.

Observe that $X = \cup_n A_n$. Indeed, there exists n_0 so that $f_n(x) \geq cs(x)$ for all $n \geq n_0$. If $s(x) = 0$ we can take $n_0 = 1$ and if $s(x) > 0$ then $f(x) = \lim_n f_n(x) > cs(x)$ and we can apply the definition of limit.

Since

$$\int_{A_n} cs d\mu \leq \int_{A_n} f_n d\mu \leq \int_X f_n d\mu \leq M,$$

and $\lim_n \mu(A_n \cap E_i) = \mu(E_i)$ and therefore $\lim_n \int_{A_n} cs d\mu = \int_X cs d\mu$ we get that $\int_X cs d\mu \leq M$. Since this holds for all $s \leq f$ and $0 < c < 1$ we have $\int_X f d\mu \leq M$. ■

Corollary 2.3.6 *(The Fatou Lemma) Let $f_n : X \to [0,\infty]$ be measurable functions. Then*

$$\int_X (\liminf_{n\to\infty} f_n) d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.$$

PROOF: Define $g_n = \inf\{f_k : k \geq n\}$. They are measurable functions, $g_n \leq g_{n+1}$ and $g_n \leq f_n$ for all $n \in \mathbb{N}$.

Applying Theorem 2.3.5 we have

$$\begin{aligned}\int_X (\liminf_{n\to\infty} f_n) d\mu &= \int_X (\lim_{n\to\infty} g_n) d\mu = \lim_{n\to\infty} \int_X g_n d\mu \\ &= \liminf_{n\to\infty} \int_X g_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu.\end{aligned}$$

■

Proposition 2.3.7 *Let $f, g : X \to [0, \infty]$ be measurable functions, $\alpha, \beta > 0$ and $E, F \in \Sigma$.*

(i) $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$.

(ii) If $E \cap F = \varnothing$ *then* $\int_{E\cup F} f d\mu = \int_E f d\mu + \int_F f d\mu$.

(iii) If $f \le g$ *then* $\int_X f d\mu \le \int_X g d\mu$.

(iv) $\mu(\{x \in X : f(x) > \alpha\}) \le \frac{1}{\alpha} \int_X f d\mu$ *for all* $\alpha > 0$.

(v) If $\mu(E) = 0$ *then* $\int_E f = 0$.

(vi) $\int_X f d\mu = 0$ *if and only if* $f = 0$ *μ-a.e.*

(vii) If f is μ-integrable then $\mu(\{x \in X : f(x) = \infty\}) = 0$.

Proof: (i) follows from Proposition 2.3.2 for simple functions. The general case then follows by combining Theorem 2.1.4 and the Lebesgue monotone convergence theorem.

(ii) Note that $f\chi_{E\cup F} = f\chi_E + f\chi_F$ and apply (i).

(iii) For each s simple function with $0 \le s \le f$ one has $0 \le s \le g$. Then the result follows from the definition.

(iv) Note that $\alpha\chi_{E_\alpha} \le f\chi_{E_\alpha}$ where $E_\alpha = \{x \in X : f(x) > \alpha\}$ and integrate.

(v) Let s simple with $0 \le s \le f$. Then $\int_E s d\mu = \sum_{i=1}^n \alpha_i \mu(E \cap E_i) = 0$. Then $\int_E f d\mu = 0$.

(vi) Assume $f = 0$ μ-a.e. Hence $\mu(\{x : f(x) > 0\} = 0$ and then, using (iv) we get

$$\int_X f d\mu = \int_{\{f>0\}} f d\mu + \int_{\{f=0\}} f d\mu = 0$$

Conversely, assume $\mu(\{x : f(x) > 0\} > 0$. Since $\{x : f(x) > 0\} = \cup_{n\in\mathbb{N}}\{x : f(x) > \frac{1}{n}\}$ we have $\mu(\{x : f(x) > \frac{1}{n_0}\}) > 0$ for some n_0. Hence

$$\int_X f d\mu \ge \int_{\{f>\frac{1}{n_0}\}} f d\mu > \frac{1}{n_0}\mu(\{x : f(x) > \frac{1}{n_0}\}) > 0.$$

This gives the direct implication.

Conversely, assume $f = 0$ μ-a.e. then if $E_0 = \{x : f(x) > 0\}$ we have $\mu(E_0) = 0$ and $f = f\chi_{E_0}$. Now using (iv) we get that $\int_X f d\mu = \int_{E_0} f d\mu = 0$.

(vii) The set $\{x \in X : f(x) = \infty\} = \cap_{n\in\mathbb{N}}\{x \in X : f(x) > n\}$. Hence (iii) gives that

$$\mu(\{x \in X : f(x) = \infty\}) \le \frac{1}{n}\int_X f d\mu$$

for all $n \in \mathbb{N}$. Passing to the limit we get $\mu(\{x \in X : f(x) = \infty\}) = 0$. ∎

Corollary 2.3.8 *Let $f_n : X \to [0,\infty]$ be measurable functions. Then*

$$\int_X (\sum_{n=1}^{\infty} f_n) d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

PROOF: Apply Theorem 2.3.5 for $g_n = \sum_{k=1}^{n} f_k$. ∎

Proposition 2.3.9 *Let $f : X \to [0,\infty]$ be a measurable funtion.*
Define

$$\nu(E) = \int_E f d\mu, \quad E \in \Sigma.$$

(i) ν is a measure on Σ.
(ii) ν is finite if and only if f is μ-integrable.
(iii) If $\mu(E) = 0$ then $\nu(E) = 0$.
(iv) If ν is finite then $\lim\limits_{\mu(E)\to 0} \nu(E) = 0$.

PROOF: (i) Let $\{E_n\}$ be a sequence of pairwise disjoint measurable sets. Since $f\chi_{\cup E_n} = \sum_{n=1}^{\infty} f\chi_{E_n}$ we have that

$$\nu(\cup_{n\in\mathbb{N}} E_n) = \int_X f\chi_{\cup E_n} d\mu = \sum_{n=1}^{\infty} \int_X f\chi_{E_n} d\mu = \sum_{n=1}^{\infty} \nu(E_n).$$

(ii) It is obvious.
(iii) follows from (iv) in Proposition 2.3.7.
(iv) Note that $\lim\limits_{\mu(E)\to 0} \nu(E) = 0$ means that for all $\varepsilon > 0$ there exists $\delta > 0$ so that $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$.

If $\nu(X) < \infty$ then for each $\varepsilon > 0$ there exist a simple function s such that

$$\int_X f d\mu \le \int_X s d\mu + \frac{\varepsilon}{2}$$

Now for $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ we have that

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i \cap E) \le (\sum_{i=1}^{n} \alpha_i)\mu(E).$$

Take $\delta < \frac{\varepsilon}{2\sum_{i=1}^{n} \alpha_i}$. Now if $\mu(E) < \delta$ then $\int_E f d\mu < \varepsilon$. ∎

Definition 2.3.10 *Let $f : X \to \mathbb{C}$ be a measurable function. f is said to be μ-integrable if $\int_X |f| d\mu < \infty$.*

Moreover if $f = u + iv$ is μ-integrable we define

$$\int_X f d\mu = \int_X u^+ d\mu - \int_X u^- d\mu + i \int_X v^+ d\mu - i \int_X v^- d\mu.$$

Remark 2.3.1 *In particular*

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

Lemma 2.3.11 *Let $f : X \to \mathbb{R}$ be a measurable function $f = f_1 - f_2$ for some $f_i; X \to \mathbb{R}^+$ which are μ-integrable. Then $f$is μ-integrable and*

$$\int_X f d\mu = \int_X f_1 d\mu - \int_X f_2 d\mu.$$

PROOF: Using that $|f| \leq |f_1| + |f_2|$ one gets the integrability of f.

Now notice that $f^+ - f^- = f_1 - f_2$. Hence $f^+ + f_2 = f^- + f_1$ and using the linearity of the integral for non-negative functions one obtains

$$\int_X f^+ d\mu + \int_X f_2 d\mu = \int_X f^- d\mu + \int_X f_1 d\mu.$$

Therefore

$$\int_X f d\mu = \int_X f_1 d\mu - \int_X f_2 d\mu.$$

■

Proposition 2.3.12 *Let f and g be μ-integrable functions and let α, β be complex numbers. Then $\alpha f + \beta g$ is μ-integrable and*

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

PROOF: Since $|\alpha f + \beta g| \leq |\alpha||f| + |\beta||g|$ and

$$\int_X |\alpha f + \beta g| d\mu \leq |\alpha| \int_X |f| d\mu + |\beta| \int_X |g| d\mu < \infty.$$

It suffices to show the result for real valued functions. The case of complex values follows immediately from the previous one using that if $f = u + iv$ then

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

Hence for $f = u + iv$, $g = u' + iv'$, $\alpha = a + ib$ and $\beta = a' + ib'$ one has $\alpha f + \beta g = (au - bv) + (a'u' - b'v') + i(av + bu) + i(a'v' + b'u')$.

Now use the result for real valued functions to get

$$\begin{aligned}
\int_X (\alpha f + \beta g) d\mu &= \int_X (au - bv) d\mu + \int_X (a'u' - b'v') d\mu \\
&+ i \int_X (av + bu) d\mu + i \int_X (a'v' + b'u') d\mu \\
&= a \int_X u d\mu - b \int_X v d\mu + a' \int_X u' d\mu - b' \int_X v' d\mu \\
&+ ia \int_X v d\mu + ib \int_x u d\mu + ia' \int_X v d\mu + ib' \int_x u d\mu \\
&= (a + ib) \int_X (u + iv) d\mu + (a' + ib') \int_X (u' + iv') d\mu.
\end{aligned}$$

Assume first that $\alpha \in \mathbb{R}$ and $f : X \to \mathbb{R}$ is μ-integrable. Since $g^+ = \frac{1}{2}(|g| + g)$ and $g^- = \frac{1}{2}(|g| - g)$ for any real-valued function, $\alpha f = (\alpha^+ - \alpha^-)(f^+ - f^-)$ and $|\alpha f| = (\alpha^+ + \alpha^-)(f^+ + f^-)$ then we conclude that

$$(\alpha f)^+ = \alpha^+ f^+ + \alpha^- f^-, \quad (\alpha f)^- = \alpha^+ f^- + \alpha^- f^+.$$

These give

$$\begin{aligned}
\int_X (\alpha f) d\mu &= \int_X \alpha^+ f^+ + \alpha^- f^- d\mu \\
&- \int_X \alpha^+ f^- + \alpha^- f^+ d\mu \\
&= \alpha^+ \int_X (f^+ - f^-) d\mu - \alpha^- \int_X (f^+ - f^-) d\mu \\
&= \alpha \int_X f d\mu.
\end{aligned}$$

Let $f, g : X \to \mathbb{R}$ be μ-integrable. Then $f + g = (f^+ + g^+) - (f^- + g^-)$ and we can apply Lemma 2.3.11 to conclude that

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

∎

Proposition 2.3.13 *If f is μ-integrable then $|\int_X f d\mu| \leq \int_X |f| d\mu$.*

Proof: Assume that $f : X \to \mathbb{R}$, and write $f = f^+ - f^-$. Then

$$|\int_X f d\mu| = |\int_X f^+ d\mu - \int_X f^- d\mu| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X |f| d\mu.$$

In the general case, assume $\int_X f d\mu = z \in \mathbb{C} \setminus \{0\}$. Take $\alpha = \frac{|z|}{z}$ and observe that

$$|\int_X f d\mu| = \alpha \int_X f d\mu = \int_X \alpha f d\mu = \int_X \Re(\alpha f) d\mu \leq \int_X |\alpha f| d\mu = \int_X |f| d\mu.$$

∎

Theorem 2.3.14 *(The Lebesgue dominated convergence theorem) Let $f_n : X \to \mathbb{C}$ a sequence of measurable functions and $f(x) = \lim_n f_n(x)$ for all $x \in X$. If there exists $g \geq 0$ μ-integrable such that $|f_n(x)| \leq g(x)$ for all $x \in X$ then f is μ-integrable and*

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0.$$

In particular $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

Proof: Since $|f| \leq g$ we have that f is μ-integrable. Observe now that $|f_n - f| \leq 2g$ and write $h_n = 2g - |f_n - f|$. Using Fatou's Lemma we have

$$\int_X 2g d\mu \leq \liminf \int_X h_n d\mu = \int_X 2g d\mu - \limsup \int |f_n - f| d\mu.$$

This shows that $\limsup \int |f_n - f| d\mu = 0$. Now using Proposition 2.3.13 we obtain that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$. ∎

Corollary 2.3.15 *(Bounded convergence theorem) Let (X, Σ, μ) be a finite measure space. Let f_n be a sequence of measurable functions such there exists $M > 0$ for which $|f_n(x)| \leq M$ for all $n \in \|$ and $x \in X$. If $\lim_n f_n = f$ then $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$.*

Proof: Note that constant functions are μ-integrable for finite measures μ. So the result follows from the dominated convergence theorem. ∎

Remark 2.3.2 *If $\mu(X) = \infty$ then $\sup_n |f_n| < \infty$ and $\lim_{n\to\infty} f_n = f$ do not imply $\int_X f d\mu = \lim_{n\to\infty} f_n d\mu$.*

Take $f_n = \frac{1}{n}\chi_{(0,n)}$. We have that $f_n \leq 1$, $f_n \to 0$ as $n \to \infty$ but $\int_{(0,\infty)} f_n dm = 1$ for all $n \in \mathbb{N}$.

Definition 2.3.16 *We define the equivalence relation $f \approx g$ if $f = g$ μ-a.e. and we denote by $L^1(\mu)$ the set of equivalence classes of μ-integrable functions.*

Let us denote $||f||_1 = \int_X |f| d\mu$.

Proposition 2.3.17 *$(L^1(\mu), ||.||_1)$ is a Banach space.*

PROOF: Let us first show that $L^1(\mu)$ is a vector space.

This actually shows that $||f + g||_1 \le ||f||_1 + ||g||_1$.

Obviously $||\alpha f||_1 = |\alpha| ||f||_1$.

The other property to get a norm is that $\int_X |f| d\mu = 0$ implies that $f = 0$ μ-a.e.

Let us show the completeness. It is equivalent to show that $\sum_{n=1}^{\infty} ||f_n||_1 < \infty$ implies that the series $\sum_{n=1}^{\infty} f_n$ converges in $L^1(\mu)$.

Now, using that $\sum_{n=1}^{\infty} ||f_n||_1 = \int_X \sum_{n=1}^{\infty} |f_n| d\mu < \infty$ we get that the series $\sum_{n=1}^{\infty} |f_n| < \infty$ is convergent μ-a.e, and hence there exists $A \in \Sigma$ such that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for all $x \notin A$.

Define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for $x \notin A$ and $f(x) = 0$ for $x \in A$. Observe first that f is μ-integrable, since $\int_X |f| d\mu \le \int_X \sum_{n=1}^{\infty} |f_n| d\mu < \infty$.

Now

$$\lim_n \left(\sum_{k=1}^{n} f_k(x) - f(x)\right) = 0, x \notin A \text{ and}$$

$$|\sum_{k=1}^{n} f_k(x) - f(x)| \le 2|f(x)|, x \notin A.$$

Therefore, from the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_X |\sum_{k=1}^{n} f_k - f| d\mu = \lim_{n\to\infty} \int_{X\setminus A} |\sum_{k=1}^{n} f_k - f| d\mu = 0.$$

∎

Corollary 2.3.18 *Simple integrable functions are dense in $L^1(\mu)$.*

PROOF: Using Proposition 2.1.13 we get that if f is μ-integrable there exists a sequence of simple functions such that $s_n \to f$ pointwise and $|s_n| \le |f|$. Now, the dominated convergence theorem gives $\lim_{n\to\infty} \int_X |f - s_n| = 0$. ∎

Proposition 2.3.19 *Let $f : X \to \mathbb{C}$ be μ-integrable. Then*

(i) $\lim_{\mu(E)\to 0} \int_E |f|d\mu = 0$, *i.e. for all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\mu(E) < \delta$ *implies* $\int_E |f|d\mu < \varepsilon$.

(ii) For all $\varepsilon > 0$ *there exists* $B \in \Sigma$ *with* $\mu(B) < \infty$ *and* $\int_{X\setminus B} |f|d\mu < \varepsilon$.

Proof:

(i) Note that, using the dominated Lebesgue theorem, for each $\varepsilon > 0$ there exist a simple function s such that

$$\int_X |f - s|d\mu \le \frac{\varepsilon}{2}.$$

Now for $s = \sum_{i=1}^n \alpha_i \chi_{E_i}$ we have that

$$\int_E |s|d\mu \le \sum_{i=1}^n |\alpha_i|\mu(E_i \cap E) \le (\sup_{i=1,\dots n} |\alpha_i|)\mu(E).$$

Hence if $\delta < \frac{\varepsilon}{2\sup_{i=1,\dots,n}|\alpha_i|}$ and $\mu(E) < \delta$ we get that $\int_E |f|d\mu < \varepsilon$.

(ii) Take s_n simple functions such that s_n converges to f and $|s_n| \le |f|$ and apply the dominated convergence theorem. Given $\varepsilon > 0$ there is n_0 so that $\int_X |f - s_{n_0}|d\mu < \varepsilon$. Take $B = \cup_{i=1}^m E_i$ where $s_{n_0} = \sum_{i=1}^n \alpha_i \chi_{E_i}$. One has that $\mu(B) < \infty$ since s_{n_0} is μ-integrable and

$$\int_{X\setminus B} |f|d\mu \le \int_{X\setminus B} |f - s_{n_0}|d\mu < \varepsilon.$$

■

Definition 2.3.20 *A sequence* $\{f_n\}$ *of μ-integrable functions is called equi-integrable if for every* $\varepsilon > 0$

(a) there exist $\delta > 0$ *such that implies that*

$$\mu(E) < \delta \Longrightarrow \sup_{n\in\mathbb{N}} \int_E |f_n|d\mu < \varepsilon$$

(b) and there exists $B \in \Sigma$ *with* $\mu(B) < \infty$ *and*

$$\sup_{n\in\mathbb{N}} \int_{X\setminus B} |f_n|d\mu < \varepsilon.$$

Remark 2.3.3 *In particular if* $|f_n| \le g$ *for some μ-integrable function g the sequence* $\{f_n\}$ *is equi-integrable from (iii) and (iv) in 2.3.19.*

Next theorem provides a converse to the Dominated convergence theorem.

Theorem 2.3.21 *(Vitali's theorem) Let f_n be a sequence of μ-integrable functions converging to f. The following are equivalent:*
(i) $\{f_n\}$ is equi-integrable
(ii) f is μ-integrable and $\lim_{n\to\infty}\int_X |f_n - f|d\mu = 0$.

PROOF: (i) $\Longrightarrow$)(ii) Given $0 < \varepsilon < 1$ there exist $\delta > 0$ and $B \in \Sigma$ with $\mu(B) < \infty$ such that

$$\sup_n \int_{X\setminus B} |f_n|d\mu < \varepsilon, \quad \sup_n \int_E |f_n|d\mu < \varepsilon \text{ whenever } \mu(E) < \delta.$$

Hence, using Fatou's lemma, $\int_{X\setminus B}|f|d\mu \le \liminf_n \int_{X\setminus B}|f_n|d\mu \le \varepsilon$.

Therefore $\int_X |f - f_n|d\mu \le 2\varepsilon + \int_B |f - f_n|d\mu$.

Now from Egorov's theorem (since $\mu(B) < \infty$) we can find $A \subset B$, $A \in \Sigma$ and $\mu(A) < \delta$ such that f_n converges to f uniformly in $B \setminus A$.

Putting everything together we obtain

$$\begin{aligned}\int_B |f - f_n|d\mu &\le \int_{B\setminus A}|f - f_n|d\mu + \int_A |f - f_n|d\mu \\ &\le \int_{B\setminus A}|f - f_n|d\mu + \sup_n \int_A |f - f_n|d\mu\end{aligned}$$

Applying Fatou's Lemma again and $\mu(A) < \delta$ we can say that

$$\int_B |f - f_n|d\mu \le \int_{B\setminus A}|f - f_n|d\mu + 2\varepsilon.$$

Using that $\lim_n \sup_{x\in B\setminus A}|f_n(x) - f(x)| = 0$ we have $|f_n - f| \le M\chi_{B\setminus A}$ for some constant $M > 0$. So we can use the bounded convergence theorem and get $\lim_n \int_{B\setminus A}|f - f_n|d\mu = 0$.

Finally this shows that $\lim_n \int_X |f - f_n|d\mu = 0$. This also gives that f is μ-integrable since $\int_X |f|d\mu \le \int_X |f - f_n|d\mu + \int_X |f_n|d\mu < \infty$ after choosing n such that $\int_X |f - f_n|d\mu < 1$.

(ii) $\Longrightarrow$(i) Note that

$$|\int_E f_n d\mu| \le \int_E |f_n - f|d\mu + |\int_E f d\mu|$$

for all $n \in \mathbb{N}$ and $E \in \Sigma$. Hence for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}, \delta_0 > 0$ for which $\int_X |f_n - f|d\mu < \varepsilon/2$ for all $n \ge n_0$ and $|\int_E f d\mu| < \varepsilon/2$ if $\mu(E) < \delta_0$.

For each for $k \leq n_0$, Proposition 2.3.19 provides $\delta_k > 0$ such that $|\int_E f_k d\mu| < \varepsilon/2$ if $\mu(E) < \delta_k$. Now take $\delta < \min\{\delta_k, \delta_0\}$ to conclude part (a) in the definition of equi-integrability.

On the other hand, from Proposition 2.3.19 again we choose $B_0 \in \Sigma$ with $\mu(B_0) < \infty$ and $\int_{X\setminus B_0} |f| d\mu < \varepsilon/2$ and $B_k \in \Sigma$ for $1 \leq k \leq n_0$ such that $\mu(B_k) < \infty$ and $\int_{X\setminus B_k} |f_k| d\mu < \varepsilon$. Denote $B = \cup_{k=0}^{n_0} B_k$. Therefore

$$\sup_{n\geq n_0} \int_{X\setminus B_0} |f_n| d\mu \leq \sup_{n\geq n_0} \int_{X\setminus B_0} |f - f_n| d\mu + \varepsilon/2 < \varepsilon$$

and

$$\sup_n \int_{X\setminus B} |f_n| d\mu < \varepsilon.$$

■

2.4 Exercises

Exercise 2.4.1 *Let (X, Σ) be a measurable space y (A_n) be a sequence of measurable sets such that $\cup_{n\in\mathbb{N}} A_n = X$.*

i) Given a functions f defined on X such that $f_n = f\chi_{A_n}$ is measurable with respect to the σ-algebra $\Sigma_n = \{E \cap A_n : E \in \Sigma\}$ for all $n \in \mathbb{N}$. Show that f is Σ- measurable.

ii) Assume A_n are pairwise disjoint and let f_n be Σ_n-measurable functions defined on A_n. If f is defined on X in such a way that $f(x) = f_n(x)$ for $x \in A_n$. Show that f is Σ- measurable.

iii) Assume A_n is increasing sequence of measurable sets and let f_n be Σ_n-measurable functions defined in A_n and such that $f_n(x) = f_{n+1}(x)$ for $x \in A_n$. If f is defined in X such that $f(x) = f_n(x)$ for $x \in A_n$. Show that f is Σ-measurable.

Exercise 2.4.2 *Let (X, Σ) be a measurable space, $S \in \Sigma$ and Σ_S the σ-algebra induced over S. Let f be a map from X into a topological space Y and let $y \in Y$. Show that f is Σ_S-measurable if and only if $\bar{f}$ given by $f = f\chi_S + y\chi_{X\setminus S}$ is Σ-measurable.*

Exercise 2.4.3 *Let (X, Σ) be a measurable space and let $f : X \to [0, \infty]$ be Σ-measurable. Show that $f = \sum_{n=1}^{\infty} c_n \chi_{A_n}$ for certain $c_n \geq 0$ and $A_n \in \Sigma$.*

Exercise 2.4.4 *Let (Y, τ) be a topological space and $(f_\alpha)_{\alpha \in J}$ a family of maps from X into Y. Show that there exists a minimum σ-algebra over X such that f_α are measurable and get its description. (It is called the σ-algebra generated by (f_α) and denoted $\sigma(f_\alpha : \alpha \in J)$.)*

Exercise 2.4.5 *Let $(\mathbb{R}, \Sigma)$ be a measurable space where $\Sigma = \{B \subseteq \mathbb{R} : B \text{ numerable or } X \backslash B \text{ numerable}\}$. Find a characterization of the Σ-measurable functions $f : \mathcal{R} \longrightarrow \mathcal{R}$.*

Exercise 2.4.6 *Let (X, Σ) be a measurable space and $f, g : X \to [0, \infty]$ be Σ-measurable. Show that the set $\{x \in X : f(x) = g(x)\}$ belongs to Σ.*

Exercise 2.4.7 *Show that the set of points where converge a sequence of complex measurable functions is measurable.*

Exercise 2.4.8 *Let I be an open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$.*

i) If f is piecewise monotone then f is measurable.

ii) Assume f is derivable. Show that f' is Borel-measurable.

Exercise 2.4.9 *Let (X, Σ) be a measurable space and $f : X \to \mathbb{R}^n$ given by $f = (f_1, f_2, ..., f_n)$.*

Show that f is Σ-measurable if and only if f_i are Σ-measurable for all $1 \leq i \leq n$.

Exercise 2.4.10 *(i) Show that $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable if and only if f^2 is Lebesgue-measurable and $\{x : f(x) > 0\}$ is a Lebesgue-measurable set.*

(ii) Give a non-measurable function f such that f^2 is measurable.

Exercise 2.4.11 *Let (X, Σ, μ) a measure space, $A \in \Sigma$, Σ_A the σ-algebra induced over A and μ_A measure concentrated on A.*

(i) Let f be Σ-measurable (from X into either $[0, \infty]$ or $\mathbb{C}$).

Show that f is μ_A-integrable if and only if $\int_A |f| d\mu < \infty$. Moreover if $E \in \Sigma$ then $\int_E f d\mu_A = \int_{E \cap A} f d\mu$.

(ii) Let f a Σ_A-measurable function and $f_0 = f\chi_A$ the Σ-measurable extension defined on X.

Show that f is μ_A-integrable if and only if f_0 is μ-integrable. Moreover if $E \in \Sigma_A$ then $\int_E f d\mu_A = \int_{E \cap A} f_0 d\mu$.

Exercise 2.4.12 *Let (X, Σ) be a measurable space and let (μ_n) be a sequence of measures on it. Define $\mu(E) = \sum_{n \in \mathbb{N}} \mu_n(E)$.*

Describe the μ-integrability in terms of the μ_n-integrability.

Exercise 2.4.13 *Let (X, Σ, μ) be a measure space and let $\phi : X \longrightarrow Y$ be a map. Show that $g : Y \longrightarrow \mathbb{R}$ is $\phi(\mu)$-integrable if and only if $g \circ \phi$ is μ-integrable. Moreover*

$$\int_\Omega g \circ \phi d\mu = \int_E g d\phi(\mu).$$

Exercise 2.4.14 *Let (X, Σ) a measurable space, $a \in X$ and δ_a the Dirac mass concentrated in a.*

(i) Describe the δ_a-integrability and find the integral with respect to δ_a.

(ii) Let μ be defined over $\mathcal{P}(\mathbb{N})$ given by $\mu = \sum_{n=1}^\infty 2^{-n}\delta_n$.

Describe the μ-integrability and the integral with respect μ.

Calculate $\int_\mathbb{N} f d\mu$ para $f(n) = n$.

Exercise 2.4.15 *Let (X, Σ, μ) be a measure space and let $f_n : X \to [0, \infty]$ be measurable functions. Show that*

i) $\int_X (sup_{k\in\mathbb{N}} f_k) d\mu \leq \sum_{k\in\mathbb{N}} \int_X f_k d\mu$.

ii) If $f_{j_1}.f_{j_2}...f_{j_{n+1}} = 0$ for any $(j_1, j_2, ...j_{n+1})$ then

$$\sum_{k\in\mathbb{N}} \int_X f_k d\mu \leq n \int_X (sup_{k\in\mathbb{N}} f_k) d\mu.$$

What does it mean for $f_k = \chi_{A_k}$.?

Exercise 2.4.16 *Let $f : \mathbb{R}^n \longrightarrow [0, \infty)$ be Lebesgue-measurable. let μ be the measure defined on $\mathcal{M}(\mathbb{R}^n)$ by $\mu(A) = \int_A f(x)dx$. Show that $g : \mathbb{R}^n \longrightarrow \mathbb{C}$ is μ-integrable if and only if $g \cdot f$ is Lebesgue integrable. In such a case, $\int g d\mu = \int g(x)f(x)dx$.*

Exercise 2.4.17 *Study the μ-integrability of f, calculating its integral whenever it does exist:*

a) $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \nu)$, ν the counting measure and $f(n) = e^{-|n|}$ for $n \in \mathbb{Z}$.

b) $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$,$\nu$ the counting measure and $f(n) = \frac{(-1)^n}{n+1}$ for $n \in \mathbb{N}$.

c) $((0, \infty), \mathcal{B}((0, \infty)), \mu)$, $d\mu(x) = e^{-x}dx$ and $f = \sum_{n=1}^\infty n\chi_{[n-1,n)}$.

d) $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, $d\mu(x) = |sen\ x|dx$ and $f(x) = \frac{1}{x}\chi_{\mathcal{R}\setminus\{0\}}$.

e) $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu)$, $\mu = \phi(m)$ where $\phi(t) = (e^{-|t|}cos\ t, e^{-|t|}sen\ t)$ and $f(x, y) = xy$.

f) $([0, \frac{\pi}{2}], \mathcal{B}([0, \frac{\pi}{2}]), m)$ and $f(x) = sen\ x\chi_{[0,\frac{\pi}{2}]\cap\mathbb{Q}} + cos\ x\chi_{[0,\frac{\pi}{2}]\cap\mathbb{R}\setminus\mathbb{Q}}$.

g) $([0, \frac{\pi}{2}], \mathcal{B}([0, \frac{\pi}{2}]), m)$ and $f(x) = sen\ x\chi_{[0,\frac{\pi}{2}]\cap\{x:cos\ x\in\mathbb{Q}\}} + sen^2\ x\chi_{[0,\frac{\pi}{2}]\cap\{x:cos\ x\in\mathbb{R}\setminus\mathbb{Q}\}}$.

Exercise 2.4.18 *Study the m_F-integrability de f for the Lebesgue-Stieltjes measure m_F, computing its integral when possible:*

a) $I = (0, \infty)$, $F(x) = (x-1)^+, f(x) = x^\alpha$ $(\alpha \in \mathbb{R})$.

b) $I = (0, 1)$, $F(x) = -[\frac{1}{x}], f(x) = x^\alpha$ $(\alpha \in \mathbb{R})$.

c) $I = (0, 1)$, $F(x) = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{[\frac{1}{n+1}, \frac{1}{n})}(x), f(x) = x$.

d) $I = \mathbb{R}$, $F(x) = \int_{(0,\infty)} |sen\ t| dt, f(x) = \frac{1}{x} \chi_{\mathbb{R}\setminus\{0\}}$.

e) $I = (0, \infty)$, $F(x) = -e^{-x}, f(x) = \sum_{n=1}^{\infty} n \chi_{(n-1,n)}(x)$.

f) $I = (0, \infty)$, $F(x) = log\ x, f(x) = x^{-\alpha} \chi_{(0,1)} + x^{-\beta} \chi_{(1,\infty)}$, $(\alpha, \beta > 0)$.

Exercise 2.4.19 *Find the following limits:*

a) $\lim_{n\to\infty} \int_0^\infty e^{-nx} cos\ nx sen\ \frac{x}{n} dx.$

b) $\lim_{n\to\infty} \int_1^\infty e^{\frac{x}{n}} x^{-2} dx.$

c) $\lim_{n\to\infty} \int_0^n e^{-\frac{x}{n}} x^{-\frac{1}{2}} log\ x dx.$

d) $\lim_{n\to\infty} \int_0^1 sen(\frac{\pi n}{2n+x})) dx.$

e) $\lim_{n\to\infty} \int_0^n e^{-ax} (1 + \frac{x}{n})^n dx,\ (a > 0).$

f) $\lim_{t\to 0^+} \frac{1}{t} \sum_{n=1}^{\infty} \frac{1}{n} arctag\ \frac{t}{n}.$

Exercise 2.4.20 *Let (X, Σ, μ) be a measure space and $f, g : X \to \mathbb{R}$ $\mu-$integrable functions. Discuss whether or not the following functions are also μ-integrable. (Give conditions to get afirmative answers and also counterexamples for the negative ones).*

f^2, $f^{\frac{1}{3}}$, $arctag\ f$, $\sqrt{|f|} + \sqrt{|g|}$, $f.g$, $\sqrt{fg}$, $sen(\frac{1}{1+|f|})$, $\sqrt{|f|^2 + |g|^2}$, $\frac{f}{1+|g|}$, $|f|^\alpha (\alpha \in \mathbb{R})$.

Exercise 2.4.21 *Let (X, Σ, μ) be a measure space and let $f : X \to [0, \infty]$ be $\mu-$integrable. Show that $\{x \in X : f(x) > 0\}$ is a countable union of sets of finite measure.*

Exercise 2.4.22 *Give an example where the inequality of Fatou's lemma is strict.*

Exercise 2.4.23 *Let $((-\pi, \pi], \mathcal{B}((-\pi, \pi]), m)$ be the Lebesgue measure space over $\mathcal{B}((-\pi, \pi])$. Consider $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathcal{B}(\mathbb{T})$ and the measure $\mu = \phi(m)$ where $\phi(t) = e^{it}$. Show that* *(i) μ is an invariant under rotations probalility measure, i.e.* $(\mu(\mathbb{T}) = 1)$ *and* $\mu(\lambda A) = \mu(A)$ *for* $|\lambda| = 1$.

(ii) A measurable function $f : \mathbb{T} \to \mathbb{C}$ *is μ-integrable if and only if* $g(t) = f(\phi(t))$ *is m-integrable. Moreover* $\int_{\mathbb{T}} f d\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi} g dm$.

Exercise 2.4.24 *Let Γ be a C^1-curve defined in $\mathbb{R}^n$ and let $\phi : [a,b] \to \mathbb{R}^n$ be a parametrization of Γ. On the borelians of Γ we define the measure m_γ given by $m_\Gamma = \phi(\mu)$ where μ corresponds to the measure over $[a,b]$ with density $||\phi'||$, i.e. $d\mu = ||\phi'||dm$.*

(i) Show that m_Γ is independent of the chossen parametrization.

(ii) Characterize the m_Γ-integrability and show that for integrable functions one has

$$\int_\Gamma f dm_\Gamma = \int_a^b f(\phi(t))||\phi'(t)||dt.$$

Exercise 2.4.25 *Let $s \in \mathbb{C}$ and $f : (0,\infty) \to \mathbb{C}$ given by $f(x) = x^{s-1}e^{-x}$.*

Show that f is integrable if and only if $\Re s > 0$.

(Recall that $\Gamma(s) = \int_{(0,\infty)} x^{s-1}e^{-x}dx$).

Exercise 2.4.26 *Let $a, s \in \mathbb{C}$ and $f : (0,\infty) \to \mathbb{C}$ given by $f(x) = x^s e^{-ax}$.*

(i) Find the values of a and s for f to be integrable.

Find the integral in the cases $a > 0$ and integrable f.

(ii) Show that if $\Re s > 1$ then

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_{(0,\infty)} \frac{x^{s-1}}{e^x - 1} dx.$$

Exercise 2.4.27 *Let $z \in \mathbb{C}$ and $f : (0,\infty) \to \mathbb{C}$ given by $f(t) = cos\ zte^{-t^2}$. Show that f is integrable for all $z \in \mathbb{C}$ and compute its integral.*

Exercise 2.4.28 *Prove, justifying the computations, that*

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}.$$

Exercise 2.4.29 *Show, using the image measure, the following well-known change of variable result. Let $X : \mathbb{R} \longrightarrow \mathbb{R}$ be of class C^1, strictly increasing and bijective and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable. Then*

$$\int_{\mathbb{R}} (g \circ X)(t) dt = \int_{\mathbb{R}} g(x)(X^{-1}(x))' dx.$$

Exercise 2.4.30 *Let* $f : X \longrightarrow [0, \infty]$ *be measurable and let* μ *be a finite measure. Show that* f *is* μ*-integrable if and only if* $\sum_n \mu(\{x \in X : f(x) \geq n\}) < \infty$.

Exercise 2.4.31 *Let* f, f_n *be non-negativeintegrable functions such that*

a) $\lim f_n = f$ *a.e.*

b) $\lim \int f_n d\mu = \int f d\mu$

Show that $\lim \int |f_n - f| d\mu = 0$.

Chapter 3

The product measure and Fubini's theorem

3.1 The product measure

Definition 3.1.1 *Let* (X, Σ_1) *and* (Y, Σ_2) *be measurable spaces. We define* $\mathcal{R} = \{A \times B : A \in \Sigma_1, B \in \Sigma_2\}$ *the family of measurable rectangles and*

$$\mathcal{A} = \{E = \cup_{k=1}^{n} A_k \times B_k : A_k \times B_k \in \mathcal{R}, (A_k \times B_k) \cap (A_l \times B_l) = \varnothing, k \neq l\}$$

the elementary sets in the product $X \times Y$*.*

We denote by $\Sigma_1 \otimes \Sigma_2 = \sigma(\mathcal{A}) = \sigma(\mathcal{R})$ *the product* σ*-algebra over* $X \times Y$*.*

Proposition 3.1.2 *The family* $\mathcal{A}$ *is an algebra over* $X \times Y$*.*

Proof: Of course $\varnothing \in \mathcal{A}$.

Let $E \in \mathcal{R}$, say $E = A \times B$ where $A \in \Sigma_1$ and $B \in \Sigma_2$. Note that

$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times B) \cup (X \setminus A) \times (Y \setminus B) \cup (A \times (Y \setminus B)) \in \mathcal{A}.$$

Observe now that the intersection of two rectangles is a rectangle, since $(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B')$, what gives that

$$(X \times Y) \setminus (\cup_{k=1}^{n} (A_k \times B_k)) = \cap_{k=1}^{n} (X \times Y) \setminus (A_k \times B_k) \in \mathcal{A}.$$

Finally any finite union of pairwise disjoint elementary sets is elementary. Using $\cup_{i=1}^{n} A_i = \cup_{i=1} (A_i \setminus \cup_{j=1}^{i-1} B_j)$ one concludes the result. ∎

Proposition 3.1.3 *Let $n = k + l$, $k, l \in \mathbb{N}$. Then $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^l)$.*

PROOF: Recall that $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{E})$ where

$$\mathcal{E} = \{(a_1, b_1] \times ... \times (a_n, b_n] : a_i < b_i, i = 1, 2, ..., n\}$$

and $\mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^l) = \sigma(\mathcal{R})$ where

$$\mathcal{R} = \{A \times B : A \in \mathcal{B}(\mathbb{R}^k), B \in \mathcal{B}(\mathbb{R}^l)\}.$$

Hence it suffices to see that $\mathcal{E} \subset \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^l)$ and $\mathcal{R} \subset \mathcal{B}(\mathbb{R}^n)$.

Obviously $\mathcal{E} \subset \mathcal{R}$.

Now, given $A \times B \in \mathcal{R}$ we have

$$A \times B = (A \times \mathbb{R}^l) \cap (\mathbb{R}^k \times B).$$

We shall only see that $A \times \mathbb{R}^l \in \mathcal{B}(\mathbb{R}^n)$ for all $A \in \mathcal{B}(\mathbb{R}^k)$. Define

$$\Sigma = \{A \in \mathcal{B}(\mathbb{R}^k) : A \times \mathbb{R}^l \in \mathcal{B}(\mathbb{R}^n)\}.$$

Clearly it is a σ-algebra and contains the open sets, so the proof is complete.

■

Definition 3.1.4 *Let E be a subset of $X \times Y$, $x \in X$ and $y \in Y$. We call x-section (respect. y-section) of E the sets $E_x = \{y \in Y : (x, y) \in E\}$ (repect. $E^y = \{x \in X : (x, y) \in E\}$.)*

Let $f : X \times Y \to Z$ be function, $x \in X$ and $y \in Y$. We call the x-section (respect. y-section) of f the functions $f_x : Y \to Z$ (respect. $f^y : X \to Z$) where $f_x(y) = f^y(x) = f(x, y)$.

Proposition 3.1.5 *Let Z be a topological space or $Z = [0, \infty]$, $E \subset X \times Y$, $x \in X$ and $y \in Y$. Then*

(i) if $E \in \Sigma_1 \otimes \Sigma_2$ then $E_x \in \Sigma_2$ and $E^y \in \Sigma_1$,

(ii) if $f : X \times Y \to Z$ is $\Sigma_1 \otimes \Sigma_2$-measurable then f_x is Σ_2-measurable and f^y is Σ_1-measurable.

PROOF: (i) Let $\Sigma = \{E \in \Sigma_1 \otimes \Sigma_2 : E_x \in \Sigma_2\}$. It is clear that $(\cup_n E_n)_x = \cup(E_n)_x$ and $((X \times Y) \setminus E)_x = Y \setminus E_x$. Hence Σ is σ-algebra.

On the other hand, if $E = A \times B \in \mathcal{R}$ then $E_x = B$ for $x \in A$ and $E_x = \varnothing$ for $x \notin A$. Hence $\mathcal{R} \subset \Sigma$. This shows that $\Sigma = \Sigma_1 \otimes \Sigma_2$.

Similarly the case of y-sections.

To see (ii) note that, for each open set $G \subset Z$,

$$\{y \in Y : f_x(y) \in G\} = (\{(x, y) \in X \times Y : f(x, y) \in G\})_x$$

and apply part (i). ∎

Theorem 3.1.6 *Let (X, Σ_1, μ) and (Y, Σ_2, ν) be finite measure spaces and let $E \in \Sigma_1 \otimes \Sigma_2$. Then*

(i) $x \to \nu(E_x)$ is Σ_1-measurable and $y \to \mu(E^y)$ is Σ_2-measurable, and

(ii) $\int_X \nu(E_x)d\mu = \int_Y \mu(E^y)d\nu$.

Proof: Case 1.- Assume $E = A \times B \in \mathcal{R}$. Obviously

$$\nu(E_x) = \nu(B)\chi_A(x), \quad \mu(E^y) = \mu(A)\chi_B(y),$$

and

$$\mu(A)\nu(B) = \int_X \nu(E_x)d\mu = \int_Y \mu(E^y)d\nu.$$

Case 2.- $E = \cup_{k=1}^n A_k \times B_k$ where $A_k \times B_k$ are pairwise disjoint. Then

$$\nu(E_x) = \sum_{k=1}^n \nu(B_k)\chi_{A_k}(x), \quad \mu(E^y) = \sum_{k=1}^n \mu(A_k)\chi_{B_k}(y),$$

and

$$\sum_{k=1}^n \mu(A_k)\nu(B_k) = \int_X \nu(E_x)d\mu = \int_Y \mu(E^y)d\nu.$$

In general, define

$$\Sigma = \{E \in \Sigma_1 \otimes \Sigma_2 : \text{ satisfies (i) and (ii)}\}.$$

We shall prove that Σ is a monotone class. Since it contains $\mathcal{A}$ we will have $\mathcal{M}(\mathcal{A}) = \Sigma = \Sigma_1 \otimes \Sigma_2$ and the proof will be finished.

Let $\{E_n\}$ be an increasing sequence in Σ. Taking into account that $\nu((E_n)_x)$ (respect. $\mu((E_n)^y)$) are increasing sequences of Σ_2-measurable (respect. Σ_1-measurable) functions and converges to $\nu((\cup_n E_n)_x)$ (respect. to $\nu((\cup_n E_n)^y)$) we get that $\cup_n E_n$ verifies (i).

Using also the monotone convergence theorem we have that

$$\lim_n \int_X \nu(E_x)d\mu = \int_X \nu((\cup_n E_n)_x)d\mu,$$

$$\lim_n \int_Y \mu(E^y)d\nu = \int_Y \mu((\cup_n E_n)^y)d\nu.$$

Hence $\int_Y \mu((\cup_n E_n)^y)d\nu = \int_X \nu((\cup_n E_n)_x)d\mu$ and $\cup_n E_n \in \Sigma$.

Let $\{E_n\}$ be an decreasing sequence in Σ. Taking into account that $\nu((E_n)_x)$ (respect. $\mu((E_n)^y)$) are decreasing sequences of Σ_2-measurable (respect. Σ_1-measurable) functions and, using that μ and ν are finite, converges to $\nu((\cap_n E_n)_x)$ (respect. to $\nu((\cap_n E_n)^y)$) we get that $\cap_n E_n$ verifies (i).

Since $\nu((E_n)_x) \leq \nu(Y)$ for all $x \in X$ and all $n \in \mathbb{N}$ (respect. $\mu((E_n)^y) \leq \mu(X)$ for all $y \in Y$ and all $n \in \mathbb{N}$), then, using the bounded convergence theorem we have that

$$\lim_n \int_X \nu(E_x)d\mu = \int_X \nu((\cap_n E_n)_x)d\mu,$$

$$\lim_n \int_Y \mu(E^y)d\nu = \int_Y \mu((\cap_n E_n)^y)d\nu.$$

Hence $\int_Y \mu((\cap_n E_n)^y)d\nu = \int_X \nu((\cap_n E_n)_x)d\mu$ and $\cap_n E_n \in \Sigma$. ∎

Corollary 3.1.7 *Let (X, Σ_1, μ) and (Y, Σ_2, ν) be σ-finite measure spaces and let $E \in \Sigma_1 \otimes \Sigma_2$. Then*

(i) $x \to \nu(E_x)$ is Σ_1-measurable and $y \to \mu(E^y)$ is Σ_2-measurable.

(ii) $\int_X \nu(E_x)d\mu = \int_X \mu(E^y)d\nu$.

PROOF: Write $X = \cup_{n=1}^\infty X_n$ with $X_n \in \Sigma_1$ for all $n \in \mathbb{N}$, $\mu(X_n) < \infty$ and $X_n \cap X_{n'} = \varnothing$ if $n \neq n'$, and $Y = \cup_{m=1}^\infty Y_m$ with $Y_m \in \Sigma_2$ for all $m \in \mathbb{N}$, $\nu(Y_m) < \infty$ and $Y_m \cap Y_{m'} = \varnothing$ if $m \neq m'$.

Let $n, m \in \mathbb{N}$ and define $\mu_n(A) = \mu(A \cap X_n)$ and $\nu_m(B) = \nu(B \cap Y_m)$. Since (X, Σ_1, μ_n) and (Y, Σ_2, ν_m) are finite measure spaces, then $x \to \nu_m(E_x)$ is Σ_1-measurable and $y \to \mu_n(E^y)$ is Σ_2-measurable and also

$$\int_{X_n} \nu(E_x \cap Y_m)d\mu = \int_{Y_m} \mu(E^y \cap X_n)d\nu.$$

Now $\nu(E_x) = \sum_{m=1}^\infty \nu_m(E_x)$ and $\mu(E^y) = \sum_{n=1}^\infty \mu_n(E^y)$ and hence $x \to \nu(E_x)$ is Σ_1-measurable and $y \to \mu(E^y)$ is Σ_2-measurable.

Moreover

$$\begin{aligned}\int_X \nu(E_x)d\mu &= \sum_{m=1}^\infty \int_X \nu_m(E_x)d\mu \\ &= \sum_{m=1}^\infty \sum_{n=1}^\infty \int_{X_n} \nu(E_x \cap Y_m)d\mu\end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{Y_m}\mu(E^y\cap X_n)d\nu$$
$$= \sum_{n=1}^{\infty}\int_{Y}\mu(E^y\cap X_n)d\nu$$
$$= \int_X \mu_n(E^y)d\nu.$$

■

Definition 3.1.8 *Let (X,Σ_1,μ) and (Y,Σ_2,ν) be σ-finite measure spaces. We define the measure*

$$\mu\otimes\nu(E)=\int_X \nu(E_x)d\mu=\int_X \mu(E^y)d\nu$$

for all $E\in\Sigma_1\otimes\Sigma_2$.

Then $(X\times Y,\Sigma_1\otimes\Sigma_2,\mu\otimes\nu)$ is a σ-finite measure space.

Remark 3.1.1 *$\mu\otimes\nu(A\times B)=\mu(A)\nu(B)$ for all $A\in\Sigma_1$ and $B\in\Sigma_2$.*

Remark 3.1.2 *Let $E\in\Sigma_1\otimes\Sigma_2$. Then*

$$\int_{X\times Y}\chi_E(x,y)d\mu\otimes\nu=\int_X(\int_Y\chi_{E_x}d\nu)d\mu(x)=\int_Y(\int_X\chi_{E^y}d\mu)d\nu(y).$$

Remark 3.1.3 *The σ-finiteness is necessary for the iterated integrals to coincide.*

Indeed, let $X=Y=[0,1]$, $\mu=m$ be the Lebesgue measure and let ν be the counting measure. Take $E=\{(x,x):0\le x\le 1\}$. Note that $E_x=\{x\}$ and $E^y=\{y\}$, and hence $\nu(E_x)=1$ and $\mu(E^y)=0$, for all $x,y\in[0,1]$. Therefore $1=\int_X\nu(E_x)d\mu\neq\int_Y\mu(E^y)d\nu=0$.

Definition 3.1.9 *In the case $X=Y=\mathbb{R}$, $\Sigma_1=\Sigma_2=\mathcal{B}(\mathbb{R})$ and $\mu=\nu=m$ the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, we define the Lebesgue measure on $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})=\mathcal{B}(\mathbb{R}^2)$ as the product measure $m\otimes m$ and it is denoted m_2.*

Inductively $m_n=m\otimes m_{n-1}$ is defined on $\mathcal{B}(\mathbb{R}^n)$ for all $n\in\mathbb{N}$. It follows from the unicity of the Hahn theorem that $m_n=m_k\otimes m_l$ for any $n=k+l$.

Remark 3.1.4 *The space $(\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2),m_2)$ is a σ-finite not complete measure space.*

Indeed, let E and C be the Vitali and the Cantor sets in $[0,1]$ respectively. Then $E\times C\notin\mathcal{B}(\mathbb{R}^2)$ (due to the fact $(E\times C)^y=E$ for any $y\in C$) but $E\times C\subset[0,1]\times C$ and $m_2([0,1]\times C)=0$.

Definition 3.1.10 *We denote by $(\mathbb{R}^2, \mathcal{M}(\mathbb{R}^2), m_2)$ the complection of the space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), m_2)$ and it is called the Lebesgue measure space in $\mathbb{R}^2$.*

Similarly $(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n), m_n)$ for all $n \in \mathbb{N}$.

Remark 3.1.5 *Due to the fact that $m \otimes m$ is not complete in $\mathcal{M}(\mathbb{R}) \otimes \mathcal{M}(\mathbb{R})$ (see Remark 3.1.4) then $\mathcal{M}(\mathbb{R}) \otimes \mathcal{M}(\mathbb{R}) \neq \mathcal{M}(\mathbb{R}^2)$.*

Theorem 3.1.11 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $A \in \mathcal{M}(\mathbb{R}^n)$. Then $T(A) \in \mathcal{M}(\mathbb{R}^n)$ and $m_n(T(A)) = |det(T)| m_n(A)$.*

PROOF: Case 1: T is a bijection and that $A \in \mathcal{B}(\mathbb{R}^n)$.

Note that, since $\|T(x) - T(y)\| \leq \|T\| \|x - y\|$, one get the continuity. Define now $\Sigma = \{A \in \mathcal{B}(\mathbb{R}^n) : T(A) \in \mathcal{B}(\mathbb{R}^n)\}$. Clearly it is a σ-algebra which contains the open sets. Hence $\Sigma = \mathcal{B}(\mathbb{R}^n)$.

On the other hand $\mu(A) = m_n(T(A))$ defines a measure on $\mathcal{B}(\mathbb{R}^n)$ such that $\mu((a_1, b_1] \times ... \times (a_n, b_n]) = m_n(T((a_1, b_1] \times ... \times (a_n, b_n]))$

Note that $(a_1, b_1] \times ... \times (a_n, b_n] = (a_1, ..., , a_n) + (0, b_1 - a_1] \times ... \times (0, b_n - a_n]$. Hence its measure equals $m_n(T((0, b_1 - a_1] \times ... \times (0, b_n - a_n]))$.

Since T is continuous we may assume that $b_i - a_i \in \mathbb{Q}$ for $i = 1, ..., n$. Standard arguments show that $m_n(T((0, b_1 - a_1] \times ... \times (0, b_n - a_n])) = (b_1 - a_1)...(b_n - a_n) m_n(T((0, 1] \times ... \times (0, 1]))$.

Hence we simply need to see that if $Q_n = (0, 1] \times ... \times (0, 1]$ then

$$m_n(T(Q_n)) = |det(T)|. \tag{3.1}$$

Any automorphism in $\mathbb{R}^n$ can be decomposed into products of automorphisms of the following types:

(i) $(T(e_1), ..., T(e_n))$ is a permutation of $(e_1, ..., e_n)$,

(ii)$T(e_1) = \alpha e_1$ and $T(e_i) = e_i$ for $i = 2, ..., n$, or

(iii) $T(e_1) = e_1 + e_2$ and $T(e_i) = e_i$ for $i = 2, ..., n$.

Hence it suffices to show (3.1) for these cases.

In the first one $T(Q_n) = Q_n$ and $det(T) = \pm 1$, in the second case $T(Q_n) = [0, \alpha) \times Q_{n-1}$ and $det(T) = \alpha$ and in the third case $T(Q_n) = \{(x_1, x_1 + x_2, x_3, ..., x_n) : 0 \leq x_i < 1\}$ and $det(T) = 1$. Since $T(Q_n) = A_2 \times Q_{n-2}$ where clearly $m_2(A_2) = 1$ we get the result.

Case 2: T is not bijection and $A \in \mathcal{B}(\mathbb{R}^n)$.

The image $T(A)$ would lie in a proper subspace. Hence the result follows by showing that any subspace has measure zero. Actually, due to the previous case, by composing with automorphims we may assume that $T(A) \subset \{x \in \mathbb{R}^n : x_1 = 0\} = \{0\} \times \mathbb{R}^{n-1}$ which is clearly of measure zero.

Case 3: $A \in \mathcal{M}(\mathbb{R}^n)$. Now the measure μ and $|det(T)|m_n$ are σ-finite measures such that coincide on the generating family and then coincide on $\mathcal{B}(\mathbb{R}^n)$. Now their complections must coincide and the result extends then to Lebesgue measurable sets. ∎

3.2 Fubini theorem

Theorem 3.2.1 *Let (X, Σ_1, μ) and (Y, Σ_2, ν) be σ-finite measure spaces and let $f : X \times Y \to [0, \infty]$ be $\Sigma_1 \otimes \Sigma_2$-measurable. Then*
(i) $x \to \int_Y f_x d\nu$ is Σ_1-measurable and $y \to \int_X f^y d\mu$ is Σ_2-measurable.
(ii) $\int_{X\times Y} f d\mu \otimes \nu = \int_X (\int_Y f_x d\nu) d\mu = \int_Y (\int_X f^y d\mu) d\nu$.

PROOF: For $f = \chi_E$ this coincides with Corollary 3.1.7.

For simple functions $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$ we have $f_x = \sum_{i=1}^n \alpha_i \nu((E_i)_x)$ and $f^y = \sum_{i=1}^n \alpha_i \mu((E_i)^y)$. Hence f^y and f_x are Σ_1 and Σ_2-measurable respectively. Moreover

$$\int_{X\times Y} f d\mu \otimes \nu = \sum_{i=1}^n \alpha_i \mu \otimes \nu(E_i)$$

coincides with

$$\sum_{i=1}^n \alpha_i \int_X \nu((E_i)_x) d\mu = \int_X (\int_Y f_x d\nu) d\mu$$

and

$$\sum_{i=1}^n \alpha_i \int_Y \mu((E_i)^y) d\nu = \int_Y (\int_X f^y d\mu) d\nu.$$

For a general $\Sigma_1 \otimes \Sigma_2$-measurable function $f : X \times Y \to [0, \infty]$, take a sequence of simple functions s_n which increases to f. For each $x \in X$ and $y \in Y$ we have that $(s_n)_x$ and $(s_n)^y$ are Σ_2 and Σ_1-measurable simple functions which increase to f_x and f^y respectively. To obtain (i) and (ii) we simply need to apply the monotone convergence theorem several times.

Note that $\int_Y (s_n)_x d\nu$ and $\int_X (s_n)^y d\mu$ increase to $\int_Y f_x d\nu$ and $\int_X f^y d\mu$ respectively. Hence $x \to \int_Y f_x d\nu$ and $y \to \int_X f^y d\mu$ are Σ_2 and Σ_1-measurable respectively. Moreover

$$\lim_n \int_X (\int_Y (s_n)_x d\nu) d\mu = \int_X (\int_Y f_x d\nu) d\mu,$$

$$\lim_n \int_Y (\int_X (s_n)^y d\mu) d\nu = \int_Y (\int_X f^y d\mu) d\nu,$$

which both coincide with

$$\lim_n \int_{X\times Y} s_n d\mu\otimes\nu = \int_{X\times Y} f d\mu\otimes\nu.$$

■

Theorem 3.2.2 *(Fubini theorem) Let (X,Σ_1,μ) and (Y,Σ_2,ν) be σ-finite measure spaces and let $f : X\times Y \to \mathbb{C}$ be $\mu\otimes\nu$-integrable. Then*

(i) f_x is ν-integrable μ-a.e. and f^y is μ-integrable ν-a.e.

(ii) $x \to \int_Y f_x d\nu$ is μ-integrable and $y \to \int_X f^y d\mu$ is ν-integrable.

(iii) $\int_{X\times Y} f d\mu\otimes\nu = \int_X(\int_Y f_x d\nu)d\mu = \int_Y(\int_X f^y d\mu)d\nu$.

Proof: Assume first that $f : X\times Y \to [0,\infty)$ is $\mu\otimes\nu$-integrable. We can apply Theorem 3.2.1 and get

$$\int_{X\times Y} f d\mu\otimes\nu = \int_X(\int_Y f_x d\nu)d\mu = \int_Y(\int_X f^y d\mu)d\nu < \infty.$$

Hence $\int_Y f_x d\nu < \infty$ μ-a.e. and $\int_X f^y d\mu < \infty$ ν-a.e. and $x \to \int_Y f_x d\nu$ and $y \to \int_X f^y d\mu$ are μ and ν-integrable respectively.

The general case follows from the previous one applied to the decompositon $f = u^+ - u^- + iv^+ - iv^-$. ■

Corollary 3.2.3 *Let (X,Σ_1,μ) and (Y,Σ_2,ν) be σ-finite measure spaces and let $f : X\times Y \to \mathbb{C}$ be $\Sigma_1\otimes\Sigma_2$-measurable.*

If $\int_X(\int_Y |f(x,y)|d\nu(y))d\mu(x) < \infty$ then f is $\mu\otimes\nu$-integrable and

$$\int_{X\times Y} f d\mu\otimes\nu = \int_X(\int_Y f_x d\nu)d\mu = \int_Y(\int_X f^y d\mu)d\nu.$$

Proof: Apply Theorem 3.2.1 to the function $|f|$ to get that it is $\mu\otimes\nu$-integrable and then apply Theorem 3.2.2. ■

3.3 Applications

Definition 3.3.1 *Let $n\in\mathbb{N}$, $n\geq 2$, and denote $||x|| = \sqrt{x_1^2+...x_n^2}$, the open unit ball by $B_n = \{u\in\mathbb{R}^n : ||u|| < 1\}$ and the unit sphere by $S_{n-1} = \{u\in\mathbb{R}^n : ||u|| = 1\}$.*

Clearly any $x \neq 0, x \in \mathbb{R}^n$ *can be decomposed as* $x = ru$ *where* $r = ||x|| > 0$ *and* $u = x/||x|| \in S_{n-1}$*, what allows us to say that* $\mathbb{R}^n \setminus \{0\} = (0,\infty) \times S_{n-1}$.

Let us denote by $\pi : B_n \setminus \{0\} \to S_{n-1}$ *the continuous projection defined by* $\pi(x) = \frac{x}{||x||}$.

We denote by $(S_{n-1}, \mathcal{B}(S_{n-1}), \sigma_{n-1})$ *the measure space defined by the image measure of n-times the Lebesgue measure on the* $\mathcal{B}(B_n \setminus \{0\})$*, that is* $\sigma_{n-1}(A) = nm_n(\hat{A})$ *where* $\hat{A} = \{ru : 0 < r < 1, u \in A\}$.

Theorem 3.3.2 *(Integration in polar coordinates) Let* $n \in \mathbb{N}$*,* $n \geq 2$ *and let* $f : \mathbb{R}^n \to [0, \infty]$ *be Borel measurable.*

Then the Lebesgue measure space $(\mathbb{R}^n \setminus \{0\}, \mathcal{B}(\mathbb{R}^n \setminus \{0\}), m_n)$ *coincides with* $((0,\infty) \times S_{n-1}, \mathcal{B}((0,\infty)) \otimes \mathcal{B}(S_{n-1}), r^{n-1}dr \otimes \sigma_{n-1})$ *and*

$$\int_{\mathbb{R}^n} f(x)dm_n(x) = \int_{(0,\infty)} (\int_{S_{n-1}} f(ru)d\sigma_{n-1}(u))r^{n-1}dr.$$

Proof: To see the coincidence between $\mathcal{B}((0,\infty)) \otimes \mathcal{B}(S_{n-1})$ and $\mathcal{B}(\mathbb{R}^n \setminus \{0\})$ it suffices to show that rectangles $A \times S_{n-1}$ and $(0,\infty) \times B$ belong to $\mathcal{B}(\mathbb{R}^n \setminus \{0\})$ for $A \in \mathcal{B}((0,\infty))$ and $B \in \mathcal{B}(S_{n-1})$, and that for any open set $G \subset \mathbb{R}^n$ one has that $G \setminus \{0\} \in \mathcal{B}((0,\infty)) \otimes \mathcal{B}(S_{n-1})$.

Since $\mathbb{R}^n \setminus \{0\}$ is homoeomorphic to $(0,\infty) \times S_{n-1}$ then any $G \subset \mathbb{R}^n$ open is a product of to open sets and hence belongs to $\mathcal{B}((0,\infty)) \otimes \mathcal{B}(S_{n-1})$. On the other hand, defining

$$\Sigma_1 = \{A \in \mathcal{B}((0,\infty) : A \times S_{n-1} \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})\},$$

$$\Sigma_1 = \{B \in \mathcal{B}(S_{n-1}) : (0,\infty) \times B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})\},$$

we have σ-algebras containing the open sets, so $\Sigma_1 = \mathcal{B}((0,\infty)$ and $\Sigma_2 = \mathcal{B}(S_{n-1})$.

To see that $m_n = r^{n-1}dr \otimes d\sigma_{n-1}$ it suffices to see that both measures coincide on rectangles $[r_1, r_2) \times B$ where B is open in S_{n-1} (due to the fact that both are σ-finite and they would coincide on $\mathcal{A}$) and the extension to the $\sigma(\mathcal{A})$ is unique.

Given $[r_1, r_2) \times B \in (0,\infty) \times S_{n-1}$ one gets $(r_1, r_2) \times B = r_2\hat{B} \setminus r_1\hat{B}$ and then

$$\begin{aligned} m_n([r_1, r_2) \times B) &= (r_2^n - r_1^n)m_n(\hat{B}) \\ &= (\int_{r_1}^{r_2} r^{n-1}dr)\sigma_{n-1}(B) \\ &= r^{n-1}dr \otimes d\sigma_{n-1}([r_1, r_2) \times B). \end{aligned}$$

Since both measures coincide then for simple functions $f = \sum_{i=1}^{m} \alpha_i \chi_{E_i}$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} f dm_n &= \int_{\mathbb{R}^n \setminus \{0\}} f dm_n \\
&= \sum_{i=1}^{m} \alpha_i \int_{\mathbb{R}^n \setminus \{0\}} \chi_{E_i} dm_n \\
&= \sum_{i=1}^{m} \alpha_i \int_0^\infty r^{n-1} (\int_{S_{n-1}} \chi_{E_i}(ru) d\sigma_{n-1}(u)) dr \\
&= \int_0^\infty r^{n-1} (\int_{S_{n-1}} f(ru) d\sigma_{n-1}(u)) dr
\end{aligned}$$

The standard argument using the monotone convergence theorem gives the general case. ∎

Corollary 3.3.3 *Let $\phi : [0, \infty) \to [0, \infty]$ be a measurable function and $f : \mathbb{R}^n \to [0, \infty]$ be the radial function $f(x) = \phi(||x||)$. Then*

$$\int_{\mathbb{R}^n} f dm_n = n v_n \int_0^\infty \phi(r) r^{n-1} dr$$

where $v_n = m_n(B_n)$.

PROOF: Applying the previous theorem we get

$$\int_{\mathbb{R}^n} f dm_n = \int_0^\infty r^{n-1} \int_{S_{n-1}} f(ru) d\sigma_{n_1} = nm(B_n) \int_0^\infty \phi(r) r^{n-1} dr$$

∎

Proposition 3.3.4 $\Gamma(\frac{1}{2}) = \int_{\mathbb{R}} e^{-t^2} dm_1(t) = \sqrt{\pi}$

PROOF: Note that $\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt = 2 \int_0^\infty e^{-t^2} dt$. Using Fubini we have

$$\begin{aligned}
(\int_{\mathbb{R}} e^{-t^2} dm_1(t))^2 &= \int_{\mathbb{R}^2} e^{-||x||^2} dm_2(x) \\
&= 2 v_2 \int_0^\infty r e^{-r^2} dr \\
&= v_2 \int_0^\infty e^{-r} dr = v_2 = \pi
\end{aligned}$$

∎

Proposition 3.3.5 *Let $n \in \mathbb{N}$. Then $v_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$.*

Proof: Write

$$B_n = \{x \in \mathbb{R}^n : ||x|| < 1\} = \{(t, x') \in \mathbb{R}\times\mathbb{R}^{n-1} : ||x'|| < 1, |t| < \sqrt{1 - ||x'||^2}\}.$$

Hence

$$\begin{aligned} v_n &= \int_{||x'||<1} m_1(\{t \in \mathbb{R} : |t| < \sqrt{1 - ||x'||^2}\}) dm_{n-1}(x') \\ &= \int_{||x'||<1} 2\sqrt{1 - ||x'||^2} dm_{n-1}(x') \\ &= (n-1)v_{n-1} \int_0^1 r^{n-2} 2\sqrt{1 - r^2} dr \\ &= (n-1)v_{n-1} \int_0^1 t^{\frac{n-3}{2}} \sqrt{1 - t} dt \\ &= (n-1)v_{n-1} B(\frac{n-1}{2}, \frac{3}{2}) \\ &= (n-1)v_{n-1} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+1)} \end{aligned}$$

Applying this formula again one gets

$$v_n = (n-1)v_{n-1}\frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+1)} = (n-1)(n-2)v_{n-2}\frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{n}{2}+1)}\frac{\Gamma(\frac{n-2}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{n-1}{2}+1)}.$$

Since $\Gamma(p+1) = p\Gamma(p)$ we obtain

$$v_n = 2(n-2)v_{n-2}\frac{\Gamma(\frac{n-2}{2})\Gamma^2(\frac{3}{2})}{\Gamma(\frac{n}{2}+1)}.$$

Repeating the process we finally have for $1 \le k \le n-1$

$$v_n = 2^{k-1}(n-k)v_{n-k}\frac{\Gamma(\frac{n-k}{2})\Gamma^k(\frac{3}{2})}{\Gamma(\frac{n}{2}+1)},$$

which, for $k = n-1$ and using Proposition 3.3.4 gives $v_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$.

∎

Definition 3.3.6 *Let (X, Σ, μ) be a σ-finite measure space. Given a measurable function $f : X \to [0, \infty]$ we define the distribution function of f by $\lambda_f(t) = \mu(\{x \in X : f(x) > t\})$.*

Theorem 3.3.7 *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a derivable function with $\phi(0) = 0$ and $\phi'(t) \geq 0$. If $f : X \to [0, \infty]$ is a measurable function then*

$$\int_X \phi(f(x))d\mu(x) = \int_0^\infty \phi'(t)\lambda_f(t)dt.$$

PROOF: Consider the function $F(x,t) = f(x) - t$ defined on $X \times (0,\infty)$. It is measurable with respect to the product σ-algebra. Consider the density measure $\nu(E) = \int_E \phi'(t)dm(t)$ on $\mathcal{B}((0,\infty))$. Then

$$\begin{aligned} \mu \otimes \nu(\{(x,t) : F(x,t) > 0\}) &= \int_X \nu(\{t : f(x) > t > 0\})d\mu \\ &= \int_{(0,\infty)} \mu(\{f(x) > t\})d\nu. \end{aligned}$$

Hence

$$\int_X \phi(f(x))d\mu(x) = \int_X (\int_{(0,f(x))} \phi'(t)dm(t))d\mu = \int_0^\infty \lambda_f(t)\phi'(t)dt.$$

■

Corollary 3.3.8 *Let (X, Σ, μ) be a σ-finite measure space, $1 \leq p < \infty$ and f measurable. Then $\int_X |f|^p d\mu = \int_0^\infty pt^{p-1}\lambda_f(t)dt$.*

3.4 Exercises

Exercise 3.4.1 *Let X, Y be non empty sets, and let $\mathcal{M} \subset \mathcal{P}(X)$ y $\mathcal{R} \subset \mathcal{P}(Y)$ such that $X \in \mathcal{M}$ y $Y \in \mathcal{R}$. Show that $\sigma(\mathcal{M} \times \mathcal{R}) = \sigma(\mathcal{M}) \otimes \sigma(\mathcal{R})$.*

Exercise 3.4.2 *Let (X, Σ_1, μ) and (Y, Σ_2, ν) be σ-finite measure spaces. Denote by $\hat{X}$ e $\hat{Y}$ their complections and by $X\hat{\otimes}Y$ the completion of $X \otimes Y$ with respect the product measure. Does it hold that $\hat{X} \otimes \hat{Y} = X\hat{\otimes}Y$.?*

Exercise 3.4.3 *Let (X, Σ_1, μ) and (Y, Σ_2, ν) be σ-finite complet meausure spaces and let $(X \otimes Y, \Sigma_1\hat{\otimes}\Sigma_2, \mu\hat{\otimes}\nu)$ be the complection of $(X \times Y, \Sigma_1 \otimes \Sigma_2, \mu \otimes \nu)$.*

Show that if $A \in \Sigma_1\hat{\otimes}\Sigma_2$ and $\mu\hat{\otimes}\nu(A) = 0$,then $\nu(A_x) = 0$ μ-a. e. and $\mu(A_y) = 0$ $\nu-$a.e.

Deduce then that Fubini's theorem holds true for non negative $\hat{\Sigma}_1 \otimes \hat{\Sigma}_2$-measurable functions or for $\hat{\Sigma}_1 \otimes \hat{\Sigma}_2$-integrable functions.

Exercise 3.4.4 *(Integration by parts) Let μ be a σ-finite Borel measure on $[a,b]$ for $-\infty \leq a < b \leq \infty$. Given μ-integrable functions f, g, we define, for $x \geq a$,*

$$F(x) = \int_{[a,x]} f d\mu, \qquad G(x) = \int_{[a,x]} g d\mu.$$

Show that , if we write $F(a^-) = 0$, then

$$\int_{[a,b]} f(x)G(x)d\mu(x) = F(b)G(b) - \int_{[a,b]} F(x^-)g(x)d\mu(x).$$

Exercise 3.4.5 *Let (X, Σ, μ) be σ-finite measure space and $f : X \to [0, \infty]$ be a measurable function. For $E \subset \Sigma$ we define*

$$R(f, E) = \{(x, y) \in E \times \mathcal{R} : \qquad 0 \leq y < f(x)\}$$

and $F(y) = \mu(\{x \in E : f(x) > y\})$, $y > 0$ (called the distribution function f over E).

Show that, for the Lebesgue measure m over $\mathbb{R}$, we have

$$\int_E f d\mu = (\mu \otimes m)(R(f, E)) = \int_0^\infty F(y)dm(y)$$

and for $0 < p < \infty$ one has

$$\int_X f^p d\mu = \int_0^\infty pt^{p-1}F(t)dm(t).$$

Exercise 3.4.6 *Let (X, Σ, μ) be σ-finite measure space, $I = (a, \infty)$ where $-\infty \leq a < \infty$ and $f : X \to I$ measurable with distribution function F.*

(i) Let $\phi : I \longrightarrow \mathbb{R}$ be C^1 non decreasing function with $\phi(a^+) = 0$. Then

$$\int_X \phi(f)d\mu = \int_0^\infty \phi'(t)F(t)dt.$$

(ii) Let $\phi : I \to \mathbb{R}$ be a non decreasing continuous function such that $\phi(a^+) = 0$. Then

$$\int_X \phi(f)d\mu = \int_0^\infty F(t)dm_\phi(t).$$

Exercise 3.4.7 *Let $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ be the measure space for the counting measure ν. Let (X, Σ, μ) be a measure space. Define, for $E \in \Sigma \otimes \mathcal{P}(\mathbb{N})$, the measure*

$$\mu \otimes \nu(E) = \sum_{n=1}^\infty \nu(E_n).$$

Show that f defined from $\mathbb{N}\times X$ into $[0,\infty]$ (or into $\mathbb{C}$) is measurable if and only if the sections f_n are Σ-measurables for all $n\in\mathbb{N}$.

Show that f is $\mu\otimes\nu$-integrable if and only if the series $\sum_{n=1}^{\infty}\int_X |f_n|d\mu$ is convergent. In such a case

$$\int_{\mathbb{N}\times X} fd(\mu\otimes\nu)=\sum_{n=1}^{\infty}\int_X f_n d\mu=\int_X\sum_{n=1}^{\infty} f_n d\mu$$

Exercise 3.4.8 *Let $f,g:[0,\pi/2]\to\mathbb{R}$ be given by $f(x)=\frac{1}{2}$ and $g(x)=sin^2(x)$.*

(i) Describe $\mu=f(m)$ and $\nu=g(m)$.

(ii) Compute $\mu\otimes\nu(\{(x,y)\in\mathbb{R}^2 : y<4x^2\})$.

Exercise 3.4.9 *Let $f:\mathbb{R}^k\times\mathbb{N}\to\mathbb{R}$ be given by*

$$f(x,y)=nx_1\chi_{\{(x,n):||x||\le\frac{1}{n}\}}.$$

Let μ be a measure over $\mathbb{N}$ such that $\mu(\{n\})=\frac{1}{n^\beta}$. Find the values of β for f to be $m_k\otimes\mu$-integrable, where m_k is the Lebesgue measure over $\mathbb{R}^k$. For such values calculate the integral $\int_{\mathbb{R}^k\times\mathbb{N}} fdm_k\otimes\mu$.

Exercise 3.4.10 *Study the integrability on $\mathbb{R}^2$ the following functions:*

(i) $f(x,y)=\frac{sin(x)cos(xy)}{x}\chi_{[0,\infty)\times[0,a]}(x,y)$ for $a>0$.

(ii) $f(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$.

(iii) $f(x,y)=\frac{y^2\sin^2(x)}{x^2(x^2+y^2)(x^2+y^2+m^2)}\chi_{\mathbb{R}^2-\{(0,0)\}}(x,y)$.

(iv)$f(x,y)=\frac{y^2sen^2x}{x^2(x^2+y^2)(x^2+y^2+m^2)}\chi_{\mathbb{R}^2\setminus\{(0,0)\}}(x,y)$ for $m>0$.

Exercise 3.4.11 *Let $X=Y=\mathbb{N}$ and μ the counting measure. Study the $\mu\otimes\mu$-integrability of $f=\sum_n(2-2^{-n})\chi_{\{(n,n)\}}-\sum_n(-2+2^{-n})\chi_{\{(n+1,n)\}}$.*

Exercise 3.4.12 *Let $f:\mathbb{R}^k\setminus\{0\}\to\mathbb{R}$ be given by $f(x)=\frac{sen(\frac{1}{||x||})-1}{||x||^k(1-||x||)}$.*

Show that it does not exist $\int_{\{||x||<1\}} f(x)dx$ but it does exist the principal value $\lim_{\epsilon\to 0}\int_{\{\epsilon<||x||<1\}} f(x)dx$. Compute such a value.

Exercise 3.4.13 *Find the values of α for the following functions to ve integrable and compute the value of their integrals.*

(i)$\int_{\mathbb{R}^k}\frac{dx}{(1+||x||^2)^\alpha}$.

(ii)$\int_{\{||x||<r\}}||x||^\alpha dx$.

(iii) $\int_{\{||x||<1\}}\frac{x_1^2-x_2^2+x_3^2+...+(-1)^{k+1}x_k^2}{||x||^\alpha}dx$.

(iv)$\int_{\{||x||<1\}}\frac{|x_1|+...+|x_k|}{||x||}dx$.

Exercise 3.4.14 *Find out the integrals* $I_k = \int_{B_k} |x_1 ... x_k| dm_k(x)$ *and* $J_k = \int_{S_{k-1}} |u_1 ... u_k| d\sigma_{k-1}(u)$, *where* B_k *is the closed unit ball of* $\mathbb{R}^k$ *and* S_{k-1} *the sphere* $||x|| = 1$.

Exercise 3.4.15 *Compute the integral* $\int_A (\beta + \alpha_1 x_1 + ... + \alpha_k x_k) dm_k$ *where* $A = \{x \in \mathbb{R}^k : ||x - a|| < r\}$, $a \in \mathbb{R}^k$, $\alpha_i, \beta \in \mathbb{R}$ *y* $r > 0$.

Exercise 3.4.16 *Find out in terms of the* Γ *function the value of the integral* $\int_{\mathbb{R}^k} x_1^n e^{-(\sum_{i=1}^k a_i x_i^2)} dm_k(x)$ *for* $a_i > 0$ *and* $n \in \mathbb{N}$.

Exercise 3.4.17 *Compute the Lebesgue measure of the set* $A_n = \{x \in \mathbb{R}^n : x_j > 0, \sum_{j=1}^n x_j < 1\}$.

Use it to show that $\int_{\{x_i > 0, i=1,..,n\}} e^{-(x_1 + ... + x_n)^2} dx = \frac{\Gamma(n/2+1)}{n!}$.

Exercise 3.4.18 *Compute the Lebesgue measure of the following sets.*

(i) $A_n = \{x \in \mathbb{R}^n : \sum_{j=1}^n |x_j| \leq 1\}$.

(ii)$B_n = \{x \in \mathbb{R}^n : \sum_{j=1}^n |x_j|^2 \leq 1\}$.

(iii)$C_n = \{x \in \mathbb{R}^n : max|x_j| \leq 1\}$.

(iv) $D_n = \{x \in \mathbb{R}^n : |x_j| + |x_n| \leq a, j = 1, 2, .., n-1\}$ *for* $a > 0$.

Exercise 3.4.19 *Compute the measure of the following sets:*

(i) $A = \{(x, y, z, u) : (x+y)^2 + (z+u)^2 < 1, |x-y| + |z-u| < 1\}$.

(ii) $B = \{x = (x', x") \in \mathbb{R}^{k+j} : ||x'|| \leq 1, ||x'|| ||x"|| \leq 1\}$.

(iii) $A = \{\lambda_1 v_1 + ... + \lambda_n v_n : 0 \leq \lambda_j \leq 1, j = 1, 2, ..., n\}$, *where* $v_1, ..., v_n$ *are linearly independent vectors in* $\mathbb{R}^n$.

(iv) $B = \{x \in \mathbb{R}^n : \alpha_j < x.v_j < \beta_j, j = 1, 2, ..., n\}$ *where* $v_1, ..., v_n$ *are linearly independent vectors in* $\mathbb{R}^n$ *and* $x.v$ *denotes the scalar product and* $\alpha_j < \beta_j$ *for all* j.

Chapter 4

The Radon-Nikodym Theorem

4.1 Complex and real measures.

Let us start with the following result to motivate the next definitions.

Proposition 4.1.1 *Let* (X, Σ, μ) *be a measure space and* $f : X \to \mathbb{C}$ *be* μ*-integrable.*

Then $\nu(E) = \int_E f d\mu$ *for* $E \in \Sigma$ *defines a complex-valued set function on* Σ *with the following properties:*

(i) $\nu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \nu(E_n)$ *for any sequence* $\{E_n\}$ *of pairwise disjoint measurable sets.*

(ii) If $\mu(E) = 0$ *then* $\nu(E) = 0$.

(iii) $\lim_{\mu(E)\to 0} \nu(E) = 0$, *i.e. for all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\mu(E) < \delta$ *implies* $|\nu(E)| < \varepsilon$.

PROOF: (i) Let $\{E_n\}$ be a sequence of pairwise disjoint measurable sets. Let us write $f\chi_{\cup E_n} = \sum_{n=1}^{\infty} f\chi_{E_n} = \lim_n \sum_{k=1}^{n} f\chi_{E_k}$. We have that $g_n = \sum_{k=1}^{n} f\chi_{E_k} = f\chi_{\cup_{k=1}^{n} E_k}$ and $|g_n| = \sum_{k=1}^{n} |f|\chi_{E_k} = |f|\chi_{\cup_{k=1}^{n} E_k}$. Since $|g_n| \leq |f|$, the dominated convergence theorem implies that

$$\nu(\cup_{n\in\mathbb{N}} E_n) = \int_X f\chi_{\cup E_n} d\mu = \sum_{n=1}^{\infty} \int_X f\chi_{E_n} d\mu = \sum_{n=1}^{\infty} \nu(E_n).$$

(ii) It is obvious.

(iii) Given $\varepsilon > 0$ take a simple function s such that $\int_X |f - s| d\mu \leq \frac{\varepsilon}{2}$.

Now for $s = \sum_{i=1}^n \alpha_i \chi_{E_i}$ we have that

$$|\int_E s d\mu| \leq \sum_{i=1}^n |\alpha_i| \mu(E_i \cap E) \leq (\sup_{i=1,\dots n} |\alpha_i|)\mu(E).$$

Hence if $\delta < \frac{\varepsilon}{2\sup_{i=1,\dots n}|\alpha_i|}$ and $\mu(E) < \delta$ we get that $|\int_E f d\mu| < \varepsilon$.

∎

Definition 4.1.2 *Let (X, Σ) be a measurable space. A complex measure is a set function $\mu : \Sigma \to \mathbb{C}$ such that for any sequence $\{E_n\}$ of pairwise disjoint sets in Σ*

$$\mu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n).$$

Remark 4.1.1 *(i) Observe that the convergence of the series $\sum_{n=1}^\infty \mu(E_n)$ for any complex measure μ and any sequence $\{E_n\}$ of pairwise disjoint sets in Σ is unconditional, since any permutation converges to the same sum.*

(ii) The condition $\mu(\varnothing) = 0$ follows from the definition.

(iii) If (X, Σ, μ) is a measure space and $f : X \to \mathbb{C}$ is μ-integrable then Proposition 4.1.1 shows that $\nu(E) = \int_E f d\mu$ is a complex measure.

Actually there exists μ_1 a non negative measure such that $|\nu(E)| \leq \mu_1(E)$ for all $E \in \Sigma$. It suffices to take $\mu_1(E) = \int_E |f| d\mu$.

Let us see first that this last remark holds true for any complex measure, that is there exists always a non negative measure λ such that $|\mu(E)| \leq \lambda(E)$ for all $E \in \Sigma$.

Definition 4.1.3 *Let $\mu : \Sigma \to \mathbb{C}$ be a complex measure and $E \in \Sigma$, we define the variation of μ on E, as*

$$|\mu|(E) = \sup\{\sum_{n=1}^n |\mu(E_n)| : E = \cup_{n=1}^\infty E_n, E_n \in \Sigma, E_n \cap E_m = \varnothing \text{ for } n \neq m\}.$$

Remark 4.1.2 *If μ is a complex measure, $A \subset B$ and $A, B \in \Sigma$ then $|\mu|(A) \leq |\mu|(B)$.*

Theorem 4.1.4 *Let $\mu : \Sigma \to \mathbb{C}$ be a complex measure. Then*

(i) $(X, \Sigma, |\mu|)$ is a measure space.

(ii) If λ is a non-negative measure such that $|\mu(E)| \leq \lambda(E)$ for all $E \in \Sigma$ then $|\mu| \leq \lambda$.

PROOF: (i) Let us see first that $|\mu|$ is a non-negative measure. Of course $|\mu|(\varnothing) = 0$ since $\mu(\varnothing) = 0$.

Let $\{A_m\}$ be a sequence of pairwise disjoint sets in Σ and $A = \cup_{m=1}^{\infty} A_m$. Assume now that $A = \cup_{n=1}^{\infty} E_n$ where $\{E_n\}$ are pairwise disjoint sets in Σ. Note that $E_n = \cup_{m=1}^{\infty} E_n \cap A_m$ and $A_m = \cup_{n=1}^{\infty} E_n \cap A_m$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} |\mu(E_n)| &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\mu(E_n \cap A_m)| \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\mu(E_n \cap A_m)| \\ &\leq \sum_{m=1}^{\infty} |\mu|(A_m)| \end{aligned}$$

This shows that $|\mu|(A) \leq \sum_{m=1}^{\infty} |\mu|(A_m)$.

Let us now see the converse. We may assume $|\mu|(A) < \infty$ and then, from Remark 4.1.2, $|\mu|(A_k) < \infty$ for all k.

Given $k \in \mathbb{N}$ and $\varepsilon > 0$ there exist $E_{n,k} \in \Sigma$ pairwise disjoint such that $A_k = \cup_{n=1}^{\infty} E_{n,k}$ and $|\mu|(A_k) < \sum_{n=1}^{\infty} |\mu(E_{n,k})| + \varepsilon/2^k$.

Therefore, since $\cup_{n,k} E_{n,k} = A$, we have

$$\sum_{k=1}^{\infty} |\mu|(A_k) < \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\mu(E_{n,k})| + \sum_{k=1}^{\infty} \varepsilon/2^k < |\mu|(A) + \varepsilon.$$

(ii) Observe that for any partition $E = \cup_n E_n$ we have

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq \sum_{n=1}^{\infty} \lambda(E_n) = \lambda(E).$$

Hence $|\mu|(E) \leq \lambda(E)$. ∎

Lemma 4.1.5 *Let ν_1, ν_2 be complex measures and $\alpha_1, \alpha_2 \in \mathbb{C}$ and $E \in \Sigma$. Then $\alpha_1\nu_1 + \alpha_2\nu_1$ is a complex measure. Moreover*

$$|\alpha_1\nu_1 + \alpha_2\nu_2|(E) \leq |\alpha_1||\nu_1|(E) + |\alpha_2||\nu_2|(E).$$

Proposition 4.1.6 *Let (X, Σ, μ) be a measure space and $f : X \to \mathbb{C}$ be μ-integrable and $\nu(E) = \int_E f d\mu$. Then $|\nu|(E) = \int_E |f| d\mu$ for all $E \in \Sigma$.*

PROOF: From Theorem 4.1.4 we have $|\nu|(E) \leq \int_E |f| d\mu$ for all $E \in \Sigma$. Assume now that s is a simple μ-integrable function, say $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$ and denote $\nu_s(E) = \int_E s d\mu$.

Therefore for $E \in \Sigma$, choosing the partition, $E = \cup_{i=1}^n (E \cap A_i) \cup (E \cap (X \setminus \cup_{i=1}^n A_i)$, we get

$$\sum_{i=1}^n |\nu_s(A_i \cap E)| = \sum_{i=1}^n |\alpha_i| \mu(A_i \cap E) = \int_E |s| d\mu \leq |\nu_s|(E).$$

In general, given $\varepsilon > 0$ take a μ-integrable simple function s so that $\int_X |f - s| d\mu < \varepsilon/2$. Now for any $E \in \Sigma$,

$$\begin{aligned} |\nu|(E) &\leq |\nu - \nu_s|(E) + |\nu_s|(E) \\ &\leq \int_E |f - s| d\mu + \int_E |s| d\mu \\ &\leq \int_X |f - s| d\mu + \int_E |s - f| d\mu + \int_E |f| d\mu \\ &\leq \varepsilon + \int_E |f| d\mu \end{aligned}$$

This concludes the result. ■

Lemma 4.1.7 *Let $z_1, z_2, ..., z_n \in \mathbb{C}$. There exists $S \subset \{1, 2, ..., n\}$ such that*

$$\frac{1}{\pi} \sum_{k=1}^n |z_k| \leq |\sum_{k \in S} z_k| \leq \sum_{k=1}^n |z_k|.$$

PROOF: Let $z_k = |z_k| e^{i\theta_k}$. For each $\theta \in [-\pi, \pi)$ define $S(\theta) = \{k \in \{1, 2, ..., n\} : cos(\theta_k - \theta) > 0\}$ and $f(\theta) = \sum_{k=1}^n |z_k| cos^+(\theta_k - \theta)$.

Clearly $f(\theta) = \sum_{k \in S(\theta)} |z_k| cos(\theta_k - \theta) = \Re(\sum_{k \in S(\theta)} e^{-i\theta} z_k)$.

Hence $f(\theta) \leq |\sum_{k \in S(\theta)} e^{-i\theta} z_k| = |\sum_{k \in S(\theta)} z_k|$.

Integrating over $[-\pi, \pi)$ we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(\theta) d\theta &= \sum_{k=1}^n |z_k| \int_{-\pi}^{\pi} cos^+(\theta_k - \theta) d\theta \\ &= \sum_{k=1}^n |z_k| \int_{-\pi}^{\pi} cos^+(\theta) d\theta \\ &= 2 \sum_{k=1}^n |z_k|. \end{aligned}$$

On the other hand, since f is continuous, say that α is a point where it attains the maximum, we have

$$\int_{-\pi}^{\pi} f(\theta)d\theta \leq 2\pi f(\alpha) = 2\pi|\sum_{k\in S(\alpha)} z_k|,$$

which concludes the proof. ∎

Lemma 4.1.8 *Let μ be a complex measure. If $|\mu|(E) = \infty$ there exist $A, B \in \Sigma$ such that $A \cup B = E$ with $|\mu|(A) = \infty$ and $|\mu(B)| > 1$.*

Proof: There exists $\{E_n\}$ such that $E = \cup_n E_n$ with $\sum_n |\mu(E_n)| = \infty$. Let $N \in \mathbb{N}$ be such that

$$\sum_{n=1}^{N} |\mu(E_n)| > \pi(1 + |\mu(E)|).$$

Applying Lemma 4.1.7 to $z_k = \mu(E_k)$ one can take $A = \cup_{k\in S} E_k$ and $B = E \setminus A$.

Hence

$$|\mu(A)| = |\sum_{k\in S} \mu(E_k)| > \frac{1}{\pi}\sum_{k=1}^{N} |\mu(E_k)| > 1 + |\mu(E)|.$$

Also $|\mu(B)| \geq |\mu(E) - \mu(A)| > 1$.

Now, since $|\mu|$ is a measure, then $|\mu|(A) = \infty$ or $|\mu|(B) = \infty$. ∎

Theorem 4.1.9 *If μ is a complex measure then $|\mu|(X) < \infty$.*

Proof: If $|\mu|(X) = \infty$ then, applying consecutively Lemma 4.1.8, we can find a sequence of pairwise disjoint sets $E_n \in \Sigma$ with $|\mu(E_n)| > 1$. Hence $E = \cup_n E_n \in \Sigma$ and $\mu(E) = \sum_n \mu(E_n)$ but the series can not converge since $\mu(E_n)$ does not converges to zero. ∎

Given a μ-integrable function $f : X \to \mathbb{R}$ we have that $\nu(E) = \int_E f d\mu$ can be decomposed as $\nu = \nu^+ - \nu^-$ where $\nu^+(E) = \int_E f^+ d\mu$ and $\nu^-(E) = \int_E f^- d\mu$.

We shall see that this decomposition is actually true for any real measure.

Definition 4.1.10 *A complex measure μ such that $\mu(E) \subset \mathbb{R}$ for all $E \in \Sigma$ is called a real measure. For such a measure μ we define $\mu^+ = \frac{|\mu|+\mu}{2}$ and $\mu^+ = \frac{|\mu|-\mu}{2}$. Both are non-negative finite measures and*

$$\mu = \mu^+ - \mu^- \text{ and } |\mu| = \mu^+ - \mu^-.$$

Proposition 4.1.11 *Let (X, Σ, μ) be a measure space and let $f : X \to \mathbb{R}$ be a μ-integrable function. If $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$ then*

$$\nu^+(E) = \int_E f^+ d\mu, \quad \nu^-(E) = \int_E f^- d\mu.$$

Proof: This follows from Proposition 4.1.6. ∎

Theorem 4.1.12 *(Jordan decomposition theorem) Let $\mu : \Sigma \to \mathbb{C}$ be a function set. Then μ is a complex measure if and only if there exist $\{\mu_i\}_{i=1}^4$ finite non-negative measures such that $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$.*

Theorem 4.1.13 *Let μ be a real measure and $E \in \Sigma$. Then*

$$\mu^+(E) = \sup\{\mu(F) : F \subset E, F \in \Sigma\},$$

$$\mu^-(E) = -\inf\{\mu(F) : F \subset E, F \in \Sigma\}.$$

Proof: Since $\nu = -\mu$ verifies that $\nu^+ = \mu^-$ and

$$\sup\{\nu(F) : F \subset E, F \in \Sigma\} = -\inf\{\mu(F) : F \subset E, F \in \Sigma\},$$

it suffices to see the part corresponding to μ^+.

Let $F \in \Sigma$ and $F \subset E$, $\mu(F) \leq \mu^+(F) \leq \mu^+(E)$. Hence

$$\mu^+(E) \geq \sup\{\mu(F) : F \subset E, F \in \Sigma\}.$$

Given $\varepsilon > 0$ there exists $\{A_n\}$ pairwise disjoint such that $E = \cup_n A_n$ and $|\mu|(E) < \sum_{n=1}^{\infty} |\mu(A_n)| + \varepsilon$.

Write $S = \{n : \mu(A_n) \geq 0\}$, $F = \cup_{n\in S} A_n$ and $G = \cup_{n\notin S} A_n$. Of course $E = F \cup G$ and $F \cap G = \varnothing$. Observe that

$$|\mu|(E) < \sum_{n=1}^{\infty} |\mu(A_n)| + \varepsilon = \sum_{n\in S} \mu(A_n) - \sum_{n\notin S} \mu(A_n) + \varepsilon = \mu(F) - \mu(G) + \varepsilon.$$

Since $\mu(E) = \mu(F) + \mu(G)$ we get that $\mu^+(E) < \mu(F) + \varepsilon$, what finishes the proof. ∎

Lemma 4.1.14 *Let μ be a real measure and $E \in \Sigma$. Then there exists $F \in \Sigma$ and $F \subset E$ such that $\mu^+(E) = \mu(F)$.*

PROOF: Take $F_n \in \Sigma$ with $F_n \subset E$ such that $\mu^+(E) \geq \mu(F_n) > \mu^+(E) - \frac{1}{2^n}$. Define $F = \limsup F_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} F_k \subset E$.

Of course $\mu(F) \leq \mu^+(E)$, $\mu(F) = \lim_{n\to\infty} \mu(\cup_{k=n}^{\infty} F_k)$ and

$$\mu(\cup_{k=n}^{\infty} F_k) = \mu(F_n) + \sum_{k=1}^{\infty} \mu(F_{n+k} \setminus \cup_{j=0}^{k-1} F_{n+j}).$$

Now observe that

$$\begin{aligned} -\frac{1}{2^{n+k}} + \mu^+(E) &\leq \mu(F_{n+k}) \\ &= \mu(F_{n+k} \setminus \cup_{j=0}^{k-1} F_{n+j}) + \mu(\cup_{j=0}^{k-1} F_{n+j}) \\ &\leq \mu(F_{n+k} \setminus \cup_{j=0}^{k-1} F_{n+j}) + \mu^+(E) \end{aligned}$$

Therefore $-\frac{1}{2^{n+k}} \leq \mu(F_{n+k} \setminus \cup_{j=0}^{k-1} F_{n+j})$ and then

$$\mu(\cup_{k=n}^{\infty} F_k) \geq \mu(E^+) - \frac{1}{2^n} - \sum_{k=1}^{\infty} \frac{1}{2^{n+k}} = \mu(E^+) - \frac{1}{2^{n-1}}.$$

The proof is completed by taking limits. ∎

Theorem 4.1.15 *(Hahn decomposition theorem) Let μ be a real measure. There exist $A, B \in \Sigma$ with $X = A \cup B$, $A \cap B = \varnothing$ such that $\mu(E) \geq 0$ for all $E \subset A$ and $\mu(E) \leq 0$ for all $E \subset B$.*

PROOF: Let us apply Lemma 4.1.14 for $E = X$ and find $A \in \Sigma$ such that $\mu^+(X) = \mu(A)$. Write $B = X \setminus A$.

If $E \subset A$ then

$$\mu(A) = \mu(E) + \mu(A \setminus E) \leq \mu(E) + \mu^+(X) = \mu(E) + \mu(A).$$

This shows that $\mu(E) \geq 0$ for all $E \subset A$.

On the other hand, if $E \subset B$,

$$\mu(A) = \mu^+(X) \geq \mu(A \cup E) = \mu(A) + \mu(E).$$

This shows that $\mu(E) \leq 0$ for all $E \subset B$. ∎

4.2 The theorem and its proof.

In this chapter we want to find out conditions on two measures μ and ν on a σ-algebra Σ to get that ν has a density with respect to μ, i.e. $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$ and some non-negative measurable function f.

Definition 4.2.1 *Given two measures μ and ν on a measurable space (X, Σ) we say that ν is absolutely continuous with respect to μ (or μ-continuous), to be denoted $\nu << \mu$ if $\mu(E) = 0$ implies $\nu(E) = 0$.*

If μ is a measure and ν is a complex measure we also say that ν is μ-continuous if $\nu(E) = 0$ for all $E \in \Sigma$ such that $\mu(E) = 0$.

Theorem 4.2.2 *Let μ and ν be measures with $\nu(X) < \infty$. Then $\nu << \mu$ if and only if $\lim_{\mu(E)\to 0} \nu(E) = 0$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) < \delta$ then $\nu(E) < \varepsilon$.*

Proof: Assume that there exists $\varepsilon > 0$ and $E_n \in \Sigma$ so that $\mu(E_n) < 1/2^n$ but $\nu(E_n) \geq \varepsilon$.

Take $E = \limsup E_n$. Let us see that $\mu(E) = 0$ and $\nu(E) \geq \varepsilon$.

Since $\nu(X) < \infty$ and $\cup_{k=n}^{\infty} E_k$ is decreasing then

$$\nu(E) = \lim_n \nu(\cup_{k=n}^{\infty} E_k) \geq \varepsilon.$$

On the other hand $\mu(\cup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k) < 1/2^n$. Hence $\mu(E) \leq 1/2^n$ for all n, and then $\mu(E) = 0$. ∎

Remark 4.2.1 *The finiteness of ν is necessary.*

Consider μ the Lebesgue measure on $[0, 1]$ and $\nu(E) = \int_E \frac{1}{t} dm(t)$. Clearly $E_n = [0, 1/n]$ verifies that $\mu(E_n)$ converges to 0, but $\nu(E_n) = \infty$ for all n.

Theorem 4.2.3 *Let (X, Σ, μ) and (X, Σ, ν) be finite spaces. Then $\nu << \mu$ if and only if there exists a μ-integrable function f (and μ-a.e. unique) such that $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$.*

Proof: Let us see first the uniqueness. Assume that there are f, g measurables and non-negative such that $\int_E f d\mu = \int_E g d\mu$ for all $E \in \Sigma$. Then $\int_E (f - g) d\mu = 0$ for all $E \in \Sigma$ and $f - g$ is μ-integrable. Hence $f = g$ μ-a.e.

To see the existence let us consider

$$\mathcal{G} = \{g : X \to [0, \infty] \text{ measurable} : \int_E g d\mu \leq \nu(E), E \in \Sigma\}.$$

Define $A = \{\int_X g d\mu : g \in \mathcal{G}\}$. Clearly is a non-empty set bounded by $\nu(X)$. Put $M = \ supA$ and select $g_n \in \mathcal{G}$ such that $\int_X g_n d\mu \geq M - 1/n$.

Note first that $f_n = \max\{g_1, g_2, ..., g_n\} \in \mathcal{G}$. By induction, assume that $f_{n-1} \in \mathcal{G}$ and since $f_n = \max\{g_n, f_{n-1}\}$ we have

$$\begin{aligned} \int_E f_n d\mu &= \int_{E \cap \{f_{n-1} \geq g_n\}} g_n d\mu + \int_{E \cap \{f_{n-1} < g_n\}} f_{n-1} d\mu \\ &\leq \nu(E \cap \{f_{n-1} \geq g_n\}) + \nu(E \cap \{f_{n-1} \geq g_n\}) = \nu(E). \end{aligned}$$

Define now $f = \sup_n f_n$. Using the monotone convergent theorem we get that $\int_E f d\mu = \lim_n \int_E f_n d\mu \leq \nu(E)$. This proves that $f \in \mathcal{G}$.

Since $M - 1/n \leq \int_X g_n d\mu \leq \int_X f_n d\mu \leq \int_X f d\mu \leq M$ for all $n \in \mathbb{N}$, we obtain $M = \int_X f d\mu$.

Our aim is to show that $\int_E f d\mu = \nu(E)$ for all $E \in \Sigma$. Since $\nu(E) - \int_E f d\mu$ is a non-negative measure, it suffices to see that $\nu(X) = M$.

Assume that $\nu(X) > M$. Using that $\mu(X) < \infty$ we get $\varepsilon > 0$ so that $\nu(X) > \int_X (f + \varepsilon) d\mu$. Define now $\alpha(E) = \nu(E) - \int_E (f + \varepsilon) d\mu$. We have that α is a real measure. Consider A, B the Hahn decomposition of the measure α. Therefore $\nu(E) \geq \int_E (f + \varepsilon) d\mu$, for $E \subset A$ and $\nu(E) \leq \int_E (f + \varepsilon) d\mu$ for $E \subset B$.

Define $g = f\chi_B + (f + \varepsilon)\chi_A$. We have that $g \in \mathcal{G}$. Indeed,

$$\int_E g d\mu = \int_{E \cap B} f d\mu + \int_{E \cap A} (f + \varepsilon) d\mu \leq \nu(E \cap B) + \nu(E \cap A) = \nu(E).$$

This gives that $\int_X g d\mu = \int_X f d\mu + \varepsilon \mu(A) \leq M$. Hence $\mu(A) = 0$ and, using that $\nu << \mu$, we have that $\nu(A) = 0$ which implies $\alpha \leq 0$. This leads to a contradiction because $\nu(X) \leq \int_X (f + \varepsilon) d\mu$. ∎

Remark 4.2.2 *The Radon-Nikodym theorem does not hold without assumptions on the measures μ and ν.*

Take μ the counting measure on $[0, 1]$ and ν the Lebesgue measure on $[0, 1]$. Since there are no μ-null sets besides $\varnothing$ we have $\nu << \mu$, but $m(E) = \int_E f d\mu$ for all $E \in \mathcal{B}([0, 1])$ leads to $f(x) = \int_{\{x\}} f d\mu = m(\{x\}) = 0$ for all $x \in [0, 1]$.

Theorem 4.2.4 *Let (X, Σ, μ) and (X, Σ, ν) be σ-finite spaces. Then $\nu << \mu$ if and only if there exists a measurable function $f : X \to [0, \infty]$ (and μ-a.e. unique) such that $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$.*

PROOF: Let us write $X = \cup_m X_m = \cup_n Y_n$ where $X_m \in \Sigma$ are pairwise disjoint and $Y_n \in \Sigma$ are pairwise disjoint, $\mu(X_m) < \infty$ and $\mu(Y_n) < \infty$ for all $n, m \in \mathbb{N}$. Hence if $X_{n,m} = X_n \cap Y_m$ we have $X = \cup_{(n,m)\in\mathbb{N}^2} X_{n,m}$ where $\mu(X_{n,m}) < \infty$ and $\nu(X_{n,m}) < \infty$.

For fixed $n, m \in \mathbb{N}$ we can consider $(X, \Sigma, \mu_{m,n})$ and $(X, \Sigma, \nu_{m,n})$ where $\mu_{m,n}(E) = \mu(E \cap X_{m,n})$ and $\nu_{m,n}(E) = \nu(E \cap X_{m,n})$ for all $E \in \Sigma$. Note that $\nu_{n,m} << \mu_{n,m}$ because $\mu(E \cap X_{m,n}) = 0$ implies $\nu(E \cap X_{m,n}) = 0$. We can then apply Theorem 4.2.3 to get $f_{n,m}$ such that, for all $E \in \Sigma$,

$$\nu(E \cap X_{n,m}) = \int_E f_{n,m} d\mu_{n,m}.$$

Define $f = \sum_{(n,m)\in\mathbb{N}^2} f_{n,m}\chi_{X_{n,m}}$. It is measurable and

$$\begin{aligned}
\nu(E) &= \sum_{(n,m)\in\mathbb{N}^2} \nu(E \cap X_{n,m}) \\
&= \sum_{(n,m)\in\mathbb{N}^2} \int_E f_{n,m} d\mu_{n,m} \\
&= \int_E \sum_{(n,m)\in\mathbb{N}^2} f_{n,m}\chi_{X_{n,m}} d\mu \\
&= \int_E f d\mu.
\end{aligned}$$

The uniqueness follows from the fact that $f = g$ μ-a.e. in $X_{n,m}$ for all $(n, m) \in \mathbb{N}^2$ and hence $f = g$ μ-a.e. ∎

Theorem 4.2.5 *Let (X, Σ, μ) be a σ-finite space and (X, Σ, ν) any measure space. Then $\nu << \mu$ if and only if there exists a measurable function $f : X \to [0, \infty]$ (and μ-a.e. unique) such that $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$.*

PROOF: We may assume that there is $E_0 \in \Sigma$ such that $\nu(E_0) < \infty$, otherwise $f = \infty$. We first deal with $\mu(X) < \infty$. Define

$$\mathcal{C} = \{E \in \Sigma : \nu : \Sigma_E \to [0, \infty] \ \sigma\text{-finite}\},$$

where $\Sigma_E = \{E \cap A : A \in \Sigma\}$. Observe that $\mathcal{C} \neq \varnothing$ since $E_0 \in \mathcal{C}$.

Let $S = \sup\{\mu(E) : E \in \mathcal{C}\}$. Note that $S \leq \mu(X)$. Take $E_n \in \mathcal{C}$ with $\lim_n \mu(E_n) = S$. Consider $X_1 = \cup_n E_n$. Clearly $X_1 \in \mathcal{C}$, hence $\mu(X_1) = S$

since $\mu(E_n) \leq \mu(X_1) \leq S$ for all $n \in \mathbb{N}$. Using Theorem 4.2.4 there exists $f_1 : X_1 \to [0, \infty]$ measurable with respect Σ_{X_1} such that

$$\nu(A \cap X_1) = \int_{A \cap X_1} f_1 d\mu$$

for all $A \in \Sigma$.

Let us define $f(x) = f_1(x)$ for $x \in X_1$ and $f(x) = \infty$ for $x \in X \setminus X_1$. We have that f is Σ-measurable and

$$\nu(E) = \int_{E \cap X_1} f_1 d\mu + \nu(E \cap (X \setminus X_1)).$$

Now if $\mu(E \cap (X \setminus X_1)) > 0$ then $\nu(E \cap (X \setminus X_1)) = \infty = \int_E f d\mu$, since $\nu(E \cap (X \setminus X_1)) < \infty$ implies that $X_2 = X_1 \cup E \cap (X \setminus X_1) \in \mathcal{C}$ and $\mu(X_2) > S$.

On the other hand if $\mu(E \cap (X \setminus X_1)) = 0$ then $\nu(E \cap (X \setminus X_1)) = 0$ and then $\nu(E) = \int_{E \cap X_1} f_1 d\mu = \int_E f d\mu$.

Therefore $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$.

To see the uniqueness, assume f, g are measurable functions satisfying $\nu(E) = \int_E f d\mu = \int_E g d\mu$ for all $E \in \Sigma$. Since ν is σ-finite on Σ_{X_1}, using the uniqueness of Theorem 4.2.4 then there exists $A \in \Sigma_{X_1}$ with $\mu(A) = 0$ such that $f(x) = g(x)$ for all $x \notin A$.

Note that for all $E \in \Sigma$ we have

$$\nu(E \cap (X \setminus X_1)) = \int_{E \cap (X \setminus X_1)} f d\mu = \int_{E \cap (X \setminus X_1)} g d\mu.$$

Hence if $\mu(X \setminus X_1) = 0$ then $f = g$ μ-a.e. and in the case $\mu(X \setminus X_1) > 0$ we have as above that

$$\nu(X \setminus X_1) = \infty = \int_{X \setminus X_1} f d\mu = \int_{X \setminus X_1} g d\mu.$$

This implies that $f = g = \infty$ μ-a.e. in $X \setminus X_1$.

Indeed, if $\mu(\{x \in X \setminus X_1 : f(x) < \infty\} > 0$ gives $n \in \mathbb{N}$ so that $\mu(\{x \in X \setminus X_1 : f(x) \leq n\} > 0$. Hence

$$\nu(\{x \in X \setminus X_1 : f(x) \leq n\} \leq n\mu(\{x \in X \setminus X_1 : f(x) \leq n\} < \infty$$

and then $X_2 = X_1 \cup \{x \in X \setminus X_1 : f(x) \leq n\} \in \mathcal{C}$ and $\mu(X_2) > S$.

The case μ is σ-finite follows from the previous case in the usual way.

■

Theorem 4.2.6 *Let (X,Σ,μ) be a σ-finite space and ν a complex measure on Σ. Then $\nu << \mu$ if and only if there exists a μ-integrable function $f : X \to \mathbb{C}$ (and μ-a.e. unique) such that $\nu(E) = \int_E f d\mu$ for all $E \in \Sigma$.*

PROOF: Using the Jordan decomposition theorem we can write $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$ where ν_i are non-negative finite measures. Note that $\nu << \mu$ implies $\nu_i << \mu$ for $i = 1, 2, 3, 4$. Hence using Theorem 4.2.4 we find f_i μ-integrable such that $\nu_i(E) = \int_E f_i d\mu$ for all $E \in \Sigma$. Therefore $f = f_1 - f_2 + if_3 - if_4$ verifies the result.

The uniqueness follows from the same argument as in the previous cases.

■

4.3 Applications

Definition 4.3.1 *Let (X,Σ) be a measurable space and μ a measure (or a complex measure) over X. Given $A \in \Sigma$, we say that μ is concentrated in A if $\mu(E) = \mu(E \cap A)$ for all $E \in \Sigma$.*

In other words, for non-negative measures, μ is concentrated in A if and only if $\mu = \mu_A$, or $\mu(X \setminus A) = 0$.

Example 4.3.1 *(i) A density measure $\nu(E) = \int_E f d\mu$ is concentrated in $supp(f) = \{x \in X : f(x) \neq 0\}$ or in $X \setminus N$ for any $N \in \Sigma$ with $\mu(N) = 0$.*

(ii) Let μ be a measure, $A \in \Sigma$ and let $\phi : X \to Y$ be a function with $\phi(A) \in \phi(\Sigma)$. If μ is concentrated in A then the image measure $\phi(\mu)$ is concentrated in $\phi(A)$.

(iii) δ_x is concentrated in $\{x\}$.

Definition 4.3.2 *Let μ and ν be measures (or complex measures) on a measurable space (X,Σ). We say that they are mutually singular, denoted $\mu \perp \nu$, if there exist disjoint sets $A, B \in \Sigma$ such that μ is concentrated in A and ν is concentrated in B.*

Example 4.3.2 *(i) If f_1 and f_2 are measurable functions and $\mu(supp(f_1) \cap supp(f_2)) = \varnothing$ then the density measures $d\nu_1 = f_1 d\mu$ and $d\nu_2 = f_2 d\mu$ verify that $\nu_1 \perp \nu_2$.*

(ii) Let μ be a real measure, then $\mu^+ \perp \mu_-$.

(iii) $\delta_x \perp m$ for any $x \in [0,1]$ where m stands for the Lebesgue measure in $[0,1]$.

Theorem 4.3.3 *(The Lebesgue decomposition theorem) Let (X, Σ) be a measurable space. If μ and ν are σ-finite measures then there exist two unique measures ν_a and ν_s such that $\nu = \nu_a + \nu_s$ where $\nu_a << \mu$ and $\nu_s \perp \mu$.*

PROOF: Let us begin assuming $\nu(X) < \infty$. Using that $\nu << \nu + \mu$ we can apply the Radon -Nikodym theorem and find a μ-integrable function f such that, for all $E \in \Sigma$,

$$\nu(E) = \int_E f d\mu + \int_E f d\nu.$$

Define $\nu_s(E) = \nu(E \cap A)$ where $A = \{x \in X : f(x) \geq 1\}$ and $\nu_a(E) = \nu(E \cap B)$ where $B = \{x \in X : f(x) < 1\}$.

Obviuosly $\nu = \nu_a + \nu_s$.

If $\mu(E) = 0$ then

$$\nu_a(E) = \int_{E\cap B} f d\mu + \int_{E\cap B} f d\nu = \int_{E\cap B} f d\nu.$$

Hence $\int_{E\cap B}(1 - f)d\nu = 0$ since $\nu(E \cap B) = \int_{E\cap B} f d\nu$, which shows that $\nu(E \cap B) = \nu_a(E) = 0$ since $1 - f > 0$ in $E \cap B$. Therefore $\nu_a << \mu$.

On the other hand,

$$\nu(A) = \int_A f d\mu + \int_A f d\nu \geq \mu(A) + \nu(A).$$

Hence ν_s is concentrated in A and $\mu(A) = 0$, which shows that $\nu_s \perp \mu$.

Let us show now the uniqueness. Assume $\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$ where $\nu_a << \mu$, $\nu_s \perp \mu$, $\nu'_a << \mu$ and $\nu'_s \perp \mu$.

Consider the real measure $\alpha = \nu_a - \nu'_a = \nu_s - \nu'_s$. It is clear that $\alpha << \mu$. Let us see that $\alpha \perp \mu$.

If ν_s is concentrated in A and μ in B for some $A, B \in \Sigma$ and $A \cap B = \varnothing$ and also ν'_s is concentrated in A' and μ in B' for some $A', B' \in \Sigma$ and $A' \cap B' = \varnothing$ then we obtain that α is concentrated in $A \cup A'$ and μ in $B \cap B'$. Hence, since $\mu(E \cap (A \cup A')) = 0$, we get that $\alpha(E) = \alpha(E \cap (A \cup A')) = 0$ for all $E \in \Sigma$, which gives that $\nu_a = \nu'_a$ and $\nu_s = \nu'_s$.

Let us now show the σ-finite case. Write $X = \cup_n X_n$ where X_n are pairwise disjoint measurable sets with $\nu(X_n) < \infty$. Let us write $\nu = \sum_{n=1}^{\infty} \nu_n$ where $\nu_n(E) = \nu(E \cap X_n)$. Using the previous case we can find $(\nu_n)_a$ and $(\nu_n)_s$ such that $\nu_n = (\nu_n)_a + (\nu_n)_s$ where $(\nu_n)_a << \mu$ and $(\nu_n)_s \perp \mu$.

Define now $\nu_a = \sum_{n=1}^{\infty} (\nu_n)_a$ and $\nu_s = \sum_{n=1}^{\infty} (\nu_n)_s$.

Clearly $\nu_a << \mu$ and $\nu_s \perp \mu$ since ν_s is concentrated in $\cup_n A_n$ and μ is concentrated in $\cap_n B_n$ where $A_n \cap B_n = \varnothing$ for all $n \in \mathbb{N}$.

To show the uniqueness observe that if $\nu = \nu_a + \nu_s = \nu'_a + \nu'_s$ where $\nu_a << \mu$, $\nu_s \perp \mu$, $\nu'_a << \mu$ and $\nu'_s \perp \mu$ then we also would have that $(\nu_a)_n(E) = \nu_a(E \cap X_n)$ and $(\nu'_a)_n(E) = \nu'_a(E \cap X_n)$ verify that $(\nu_a)_n << \mu$, $(\nu_s)_n \perp \mu$, $(\nu'_a)_n << \mu$ and $(\nu'_s)_n \perp \mu$. Therefore $(\nu_a)_n = (\nu'_a)_n$ and $(\nu_s)_n = (\nu'_s)_n$. This gives that $\nu_a = \nu'_a$ and $\nu_s = \nu'_s$. ■

Using the Jordan decomposition theorem and the Lebesgue decomposition theorem for non-negative measures we easily get the following corollary.

Corollary 4.3.4 *Let (X, Σ) be a measurable space. If μ is a σ-finite measure and ν is a complex measure on Σ then there exist two unique complex measures ν_a and ν_s such that $\nu = \nu_a + \nu_s$ where $\nu_a << \mu$ and $\nu_s \perp \mu$.*

Definition 4.3.5 *Let (X, Σ, μ) be a measure space and let $f : X \to [0, \infty]$ be measurable. A non-negative number M is called an essential bound for f if $f \leq M$ μ-a.e., that is $\mu(\{x \in X : f(x) > M\}) = 0$.*

It is said to be essentially bounded if it has some essential bound.

A measurable function $f : X \to \mathbb{C}$ is said to be essentially bounded if $|f|$ is.

Recall thaf $f \approx g$ if $f = g$ μ-a.e. Hence if f is essentially bounded and $g \approx f$ then g is essentially bounded as well. We denote by $L^\infty(\mu)$ the space of equivalent classes of complex-valued essentially bounded functions.

We write $||f||_\infty = \inf\{M \geq 0 : f \leq M \quad \mu - a.e.\}$.

Proposition 4.3.6 *Let (X, Σ, μ) be a measure space. Then $(L^\infty(\mu), ||.||_\infty)$ is a Banach space.*

PROOF: Clearly $||f||_\infty = 0$ gives $f = 0$ μ-a.e. and $||\lambda f||_\infty = |\lambda| ||f||_\infty$.

Let us see the triangular inequality. Observe that

$$\{|f + g| > ||f||_\infty + ||g||_\infty\} \subset \{|f| > ||f||_\infty\} \cup \{|g| > ||g||_\infty\},$$

hence $\mu(\{|f+g| > ||f||_\infty + ||g||_\infty\}) = 0$ and then $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$.

To see the completeness. Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\mu)$.

Given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and $N_{n,k} \in \Sigma$ with $\mu(N_{n,k}) = 0$ such that $|f_n(x) - f_k(x)| < \varepsilon/2$ for $x \notin N_{n,k}$ and $n, k \geq n_0$.

Hence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$ for all $x \notin \cup_{n,k \geq n_0} N_{n,k}$. Define $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \notin \cup_{n,k\geq n_0} N_{n,k}$ and $f(x) = 0$ for $x \in \cup_{n,k\geq n_0} N_{n,k}$ where $\mu(\cup_{n,k\geq n_0} N_{n,k}) = 0$.

Therefore, taking limits as $k \to \infty$, one gets $|f_n(x) - f(x)| \le \varepsilon/2 < \varepsilon$ for $x \notin \cup_{n,k\ge n_0} N_{n,k}$ and $n \ge n_0$.

This implies that $||f_n - f||_\infty < \varepsilon$ for $n \ge n_0$ and the proof is complete.

∎

Theorem 4.3.7 *Let (X, Σ, μ) be a finite measure space. Then the dual of $L^1(\mu)$ is isometrically isomorphic to $L^\infty(\mu)$.*

Proof: Let $\Phi : L^\infty(\mu) \to (L^1(\mu))^*$ be defined by $\Phi(f) = \phi_f$ where $\phi_f(g) = \int_X fgd\mu$.

Observe first that if $f \in L^\infty(\mu)$ and $g \in L^1(\mu)$ then $fg \in L^1(\mu)$ since $|fg| \le ||f||_\infty |g|$, which says that ϕ_f is well defined. From the properties of the integral, it is linear and verifies $|\phi_f(g)| \le ||f||_\infty ||g||_1$. Therefore $\phi_f \in (L^1(\mu))^*$.

Also it follows from the properties of the integral that Φ is linear and from the previous estimate $||\Phi(f)|| \le ||f||_\infty$.

Therefore Φ is a continuous injective linear map with $||\Phi|| \le 1$.

Let us show that it is surjective and $||\Phi|| = 1$.

Assume first that $\mu(X) < \infty$. Given $\phi \in (L^1(\mu))^*$ we can define $\nu_\phi(E) = \phi(\chi_E)$ for any $E \in \Sigma$ (since $\chi_E \in L^1(\mu)$).

We first see that ν_ϕ is a complex measure. Indeed, if $\{E_k\}$ are pairwise disjoint sets in Σ then $\sum_k \chi_{E_k}$ converges in $L^1(\mu)$. Hence

$$\nu_\phi(\cup_k E_k) = \phi(\sum_k \chi_{E_k}) = \sum_k \phi(\chi_{E_k}) = \sum_k \nu_\phi(E_k).$$

On the other hand $|\nu_\phi(E)| \le ||\phi|| \mu(E)$, which implies that $\nu << \mu$. Using now the Radon-Nikodym theorem we obtain $f \in L^1(\mu)$ such that $\nu_\phi(E) = \int_X \chi_E f d\mu = \phi(\chi_E)$.

Note that $|\nu_\phi(E)| = |\int_E fd\mu| \le ||\phi|| \mu(E)$ for all $E \in \Sigma$. This gives $|\nu_\phi|(E) \le ||\phi|| \mu(E)$ for all $E \in \Sigma$.

Let us define $E_0 = \{x \in X : |f(x)| > ||\phi||\}$. Observe that $\mu(E_0) = 0$, since otherwise $|\nu_\phi|(E_0) = \int_{E_0} |f| d\mu > ||\phi|| \mu(E_0)$. This shows that $f \in L^\infty(\mu)$ and $||f||_\infty \le ||\phi||$.

Since ϕ is linear we get $\phi(s) = \int_X sfd\mu$ for all simple functions s and by approximation we actually get $\phi(g) = \int_X fgd\mu$ for all $g \in L^1(\mu)$, due to the facts that ϕ and $g \to \int_X fgd\mu$ are both continuous maps. This shows that $\Phi(f) = \phi_f = \phi$ and that $||f||_\infty = ||\phi_f|| = ||\phi||$.

To deal with the σ-finite case, we split $X = \cup_n X_n$ where X_n are pairwise disjoint and $\mu(X_n) < \infty$.

Given $\phi \in (L^1(\mu))^*$ consider $\phi_n(g) = \phi(g\chi_{X_n})$. Applying the previous argument to ϕ_n one gets $f_n \in L^\infty(\mu)$ such that $||f_n||_\infty \leq ||\phi_n|| \leq ||\phi||$ and $\phi_n = \Phi(f_n) = \phi_{f_n}$ for all $n \in \mathbb{N}$.

Defining $f = \sum_{n\in\mathbb{N}} f_n\chi_{X_n}$ we obtain a function such that $\Phi_f = \phi$ and the proof is finished. ∎

4.4 Exercises

Exercise 4.4.1 *Let μ be the measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by $\mu(A) = \int_A |x|dx$. Show that $\mu << m$ but $\lim_{|(E)\to 0} \mu(E) \neq 0$.*

Exercise 4.4.2 *Let identify $\mathbb{Q}$ with $(r_n)_{n\in\mathbb{N}}$ and define, for $n \in \mathbb{N}$, $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ the non-negative Borel function such that $\int f_n dx = 1$ which vanishes in the exterior of the closed interval of length $\frac{1}{2^n}$ centered on r_n. Let $\mu(A) = \int_A \sum f_n dx$ for a Borel set A.*

i) Show that $\sum f_n(x) < \infty$ m-a.e. $x \in \mathbb{R}$.

ii) Show that μ is σ-finite, $\mu << m$ and that every non empty open set A verifies that $\mu(A) = \infty$.

Exercise 4.4.3 *Let μ and η be σ-finite measures over $(X, \mathcal{A})$, such that $\eta << \mu$ and let g be the Radon-Nikodym derivative of η with respect to μ and let f be $\mathcal{A}$-measurable.*

Show that f is η-integrable if and only if fg es μ-integrable and $\int f d\eta = \int fg d\mu$.

Exercise 4.4.4 *Let X be a non numerable set, $\mathcal{M}$ the class of all numerable or co-numerable sets in X and let μ be the counting measure. Let $\eta(E) = 0$ for numerable sets E and $\eta(E) = \infty$ otherwise.*

Show that, although $\eta << \mu$, one cannot define the Radon-Nikodym derivative in this case.

Exercise 4.4.5 *Let λ, μ, η be σ-finite measures and let $f = \frac{d\lambda}{d(\lambda+\mu)}$, $g = \frac{d\lambda}{d(\lambda+\eta)}$, $F = \frac{d\lambda}{d(\lambda+\mu+\eta)}$.*

Justify their existence and get an expression of F in terms of f and g.

Exercise 4.4.6 *Let μ be $d\mu = e^{-\sqrt{ax^2+by^2}}dxdy$ defined on the Borel sets of $\mathbb{R}^2$. Let $v : \mathbb{R}^2 - \{0\} \longrightarrow S_1$ the projection on the unit sphere and denote by $\lambda = v(\mu)$ the image measure.*

Show that λ is absolutely continuous with respect to the Lebesgue measure σ on S_1 and compute $\frac{d\lambda}{d\sigma}$.

Exercise 4.4.7 *Let $(X, \mathcal{M})$ be a measurable space and let (P_n) be a sequence of probabilities over $\mathcal{M}$. Find a probability P such that $P_n << P$ for all $n \in \mathbb{N}$.*

Exercise 4.4.8 *Let μ be the counting measure over $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Show that a measure ν over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, is absolutely continuous with respect to μ if and only if there exists a sequence $\{a_n\}$ of non-negative real numbers such that $\nu = \sum_{n=1}^{\infty} a_n \delta_n$. Compute $\frac{d\nu}{d\mu}$ in this case.*

Exercise 4.4.9 *Let μ be the restriction of the Lebesgue measure m to the σ-algebra $\mathcal{F}$ generated by the vertical strips in the plane. If $\nu(A \times \mathbb{R}) = m(A \times (0,1))$. Show that $\nu << \mu$ but it does not have integral representation.*

Exercise 4.4.10 *Let μ be a probability measure and let ν be a σ-finite measure over $\mathbb{R}$ such that $\nu << \mu$. Show that the Radon-Nikodym derivative of f verifies*

$$\lim_{h \to 0} \frac{\nu(x-h, x+h]}{\mu(x-h, x+h]} = f(x)$$

on a set of μ-measure 1.

Exercise 4.4.11 *Let (X, Σ) be a measurable space. Denote by $L^0(X)$ the space of complex measurable functions and by $\mathcal{M}(X)$ the space of complex measures over Σ.*

(i) Let $\mu \in \mathcal{M}(X)$. Show that there exists a function $h \in L^1(|\mu|)$, essentially unique, such that $d\mu = hd|\mu|$. Moreover $|h(x)| = 1 \quad \mu - a.e.$.

We say that $f \in L^0(X)$ is $\mu-$integrable (denoted $f \in L^1(\mu)$) if $f.h \in L^1(|\mu|)$ and, in this case, we define $\int_E f d\mu = \int_E f.hd|\mu|$ for all $E \in \Sigma$.

Show that

(ii) If $\mu \in \mathcal{M}(X)$, $f \in L^1(\mu)$ and $E \in \Sigma$ then $\int_E f d\mu = \int_X \chi_E f d\mu$.

(iii) If $\mu \in \mathcal{M}(X)$, $f, g \in L^0(X)$, $f \in L^1(\mu)$ y $f = g \quad |\mu| - a.e.$ then $\int_X f d\mu = \int_X g d\mu$.

(iv) If $\mu \in \mathcal{M}(X)$ then $T : L^1(\mu) \to \mathbb{C}$ given by $T(f) = \int_X f d\mu$ is linear.

Exercise 4.4.12 *Let λ, μ be complex measures which are absolutely continuous with respect to a σ-finite measure ν. Show that for all $a, b \in \mathbb{C}$ one has*

$$\frac{d(a\lambda + b\mu)}{d\nu} = a\frac{d\lambda}{d\nu} + b\frac{d\mu}{d\nu}.$$

Exercise 4.4.13 *Let λ, μ, ν be σ-finite measures over (X, Σ) such that $\lambda << \mu$ and $\mu << \nu$. Show the following chain rule*

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}.\frac{d\mu}{d\nu}.$$

Exercise 4.4.14 *Let μ, ν be σ-finite measures over (X, Σ) such that $\nu << \mu$ y $\mu << \nu$. Show that*

$$\frac{d\nu}{d\mu} \neq 0 \quad \mu - a.e. \quad , \frac{d\mu}{d\nu} = \frac{1}{d\nu/d\mu} \quad \nu - a.e.$$

Exercise 4.4.15 *Let μ_1, ν_1 be σ-finite measures over (X_1, Σ_1) and let μ_2, ν_2 be σ-finite measures over (X_2, Σ_2).*

(i) If $\nu_i << \mu_i (i = 1, 2)$ then $\nu_1 \otimes \nu_2 << \mu_1 \otimes \mu_2$.

(ii) Compute $\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}$.

(iii) Describe, in the general case, the Lebesgue decomposition of $\nu_1 \otimes \nu_2$ with respect to $\mu_1 \otimes \mu_2$.

(iv) Show that $\nu_1 \otimes \nu_2 << \mu_1 \otimes \mu_2$ if and only if $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$.

(v) Show that $\nu_1 \otimes \nu_2$ is mutually singular to $\mu_1 \otimes \mu_2$ if and only if ν_1 is mutually singular to μ_1 or ν_2 is mutually singular with μ_2.

Exercise 4.4.16 *Let α, β be real measures defined over (X, Σ) and let μ be a σ-finite measure. Show that*

(i) $|\alpha + \beta| \leq |\alpha| + |\beta|$, $(\alpha + \beta)^+ \leq \alpha^+ + \beta^+$ and $(\alpha + \beta)^- \leq \alpha^- + \beta^-$.

(ii) $|\alpha + \beta| = |\alpha| + |\beta|$ if and only if α^+, α^- are mutually singular with respect to β^+, β^- respectively.

(iii) If α is absolutely continuous with respect to μ and β is mutually singular with respect to μ then α is mutually singular with respect to β.

(iv) If α is absolutely continuous with respect to μ and α is also mutually singular with respect to μ then $\alpha = 0$.

Exercise 4.4.17 *Let α, β be real measures defined over (X, Σ) and $(Y, \mathfrak{R})$ respectively.*

(i) Show that there exists a real measure $\alpha \otimes \beta$ *over* $\Sigma \otimes \mathcal{R}$ *such that* $\alpha \otimes \beta(A \times B) = \alpha(A)\beta(B)$ *for* $A \in \Sigma$ *and* $B \in \mathcal{R}$.

(ii) Find the Hahn decomposition of $\alpha \otimes \beta$ *from the Hanh descompositions of each factor.*

(iii) Compute $(\alpha \otimes \beta)^+$, $(\alpha \otimes \beta)^-$ *and* $|\alpha \otimes \beta|$ *in terms of those of* α *y* β.

Exercise 4.4.18 *Let* $\Sigma = \mathcal{B}([0,1])$ *and* $\mu(E) = m(E) + im(E \cap [0, \frac{1}{2}])$.

(i) Describe $|\mu|$ *in terms of* m.

(ii) Show that

$$\mu(E) \leq (Re\mu)^+(E) + (Re\mu)^-(E) + (Im\mu)^+(E) + (Im\mu)^-(E)$$

and that the inequality can be strict.

(iii) Find a Borel function h *such that* $|h| = 1$ *and* $\mu(E) = \int_E hd|\mu|$ *for all* $E \in \Sigma$.

Exercise 4.4.19 *For each Borel set in* $\mathbb{R}$ *define*

$$\mu(E) = \int_{E\cap(0,\infty)} \frac{sen^3 \pi t}{t^3} dt - \int_{E\cap(-\infty,0)} \frac{sen^3 \pi t}{t^3} dt.$$

(i) Show that μ *is a real measure and compute* $\mu(\mathbb{R})$.

(ii) Find the Hahn decomposition of $\mathbb{R}$ *relative to* μ.

(iii) Study the Radon-Nikodym derivative of $|\mu|$ *with respect to* m *and compute it if possible.*

Exercise 4.4.20 *Show that, although* $\mu(E) = 0$ *implies* $\nu(E) = 0$, *in general we do not have the* ϵ-δ *condition of absolute continuity :*

(i) $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, *the counting measure* ν *and* $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_n$.

(ii) $([0,1], \mathcal{B})$, $d\nu(t) = \frac{1}{t}dt$ *and the Lebesgue measure* μ.

(iii) $(\mathbb{R}, \mathcal{B})$, $\nu(E) = \sum_{n\in\mathbb{Z}} |n| m([n, n+1) \cap E)$ *and the Lebesgue measure* μ.

Exercise 4.4.21 *Let* $f(x) = \sqrt{1-x}$ *for* $x \leq 1$ *and* $f(x) = 0$ *for* $x > 1$ *and define* $\eta(E) = \int_E f(x)dx$. *Let* $g(x) = x^2$ *for* $x \geq 0$ *and* $g(x) = 0$ *for* $x < 0$ *and define* $\mu(E) = \int_E g(x)dx$. *Get the Lebesgue decomposition of* η *with respect to* μ.

Exercise 4.4.22 *Find the Lebesgue decomposition of the Lebesgue-Stieltjes measure given by the distribution function* $F(x) = (E[x])^2 - (x - E[x])^2$ *with respect to the Lebesgue measure.*