Vector-valued functions integrable with respect to bilinear maps

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Abstract

Let (Ω, Σ, μ) be a finite measure space, $1 \leq p < \infty$, X be a Banach space X and $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. We say that an X-valued function f is p-integrable with respect to $\mathcal{B}$ whenever $\sup_{\|y\|=1} \int_\Omega \|\mathcal{B}(f(w), y)\|^p d\mu < \infty$. We identify the spaces of functions integrable with respect to the bilinear maps arising from Hölder's and Young's inequalities, and also present an analogue to Hölder's inequality in this setting. We apply the theory to give conditions on X-valued kernels for the boundedness of integral operators $T_\mathcal{B}(f)(w) = \int_{\Omega'} \mathcal{B}(k(w, w'), f(w'))d\mu'(w')$ from $L^p(Y)$ into $L^p(Z)$, extending the results known in the operator-valued case, corresponding to $\mathcal{B} : \mathcal{L}(X, Y) \times X \to Y$ given by $\mathcal{B}(T, x) = Tx$.

Key words: Vector-valued functions; Pettis and Bochner integrals; bilinear maps.

1 Introduction

In this paper we shall consider spaces of X-valued functions which are integrable with respect to bilinear maps, that is to say functions f satisfying the condition $\mathcal{B}(f, y) \in L^1(Z)$ for all $y \in Y$ for some bounded bilinear map $\mathcal{B} : X \times Y \to Z$. The motivation for our study comes from two different sources: On the one hand, the recent paper by M. Girardi and L. Weiss [9], where

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conditions on operator-valued kernels $K : \Omega \times \Omega' \to \mathcal{L}(X, Y)$ for the integral operator

$$T_K(f)(w) = \int_{\Omega'} K(w, w')(f(w'))d\mu'(w')$$

to be bounded from $L^p(X)$ to $L^p(Y)$ were given, and, on the other hand, the papers [3–5] where the notion of convolution by means of bilinear maps was introduced and applied in different contexts.

Operator-valued multipliers and operator-valued singular integrals has been considered by different authors. An introduction to the general theory and its applications can be found in [1,8]. We shall deal here with more general bilinear maps in our study and present a basic introduction to the spaces which can be defined with this notion of integrability. These will allow, among other things, to get that the conditions appearing on the kernels for the boundedness of integral operators can be understood as certain integrability conditions with respect to the corresponding bilinear maps. This approach also shows that between the class of Pettis integrable functions and the Bochner integrable ones, there are many others, corresponding to integrable with respect to other bilinear maps. These classes are the natural ones where the results on convolution by means of bilinear maps obtained in [3–5] still hold true.

The paper is organized as follows: First we introduce the spaces, consider basic properties on the triples $(Y, Z, \mathcal{B})$ formed by two Banach spaces Y and Z and a bounded bilinear map $\mathcal{B} : X \times Y \to Z$ which play some important role in the development of the theory and present the examples of natural triples that naturally appear for any Banach space X. The second section is devoted to present some version of Hölder's inequality in this setting. In section 3 we identify the spaces of p integrable functions with respect to concrete examples of bilinear mapas based on Hölder's and Young's inequalities and also use some inequalities borrowed from the theory of Hardy spaces to understand the Poisson kernel $r \to P_r$ as a function in our spaces for certain bilinear maps . The last section concludes with the analogues of the results in [9] in our more general situation.

Throughout the paper $1 \leq p < \infty$, (Ω, Σ, μ) stands for a finite complete measure space and X denotes a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Recall that an X-valued function $f : \Omega \to X$ is said to be strongly measurable if there exists a sequence of simple functions, $(s_n)_n \subseteq \mathcal{S}(X)$, which converges to f a.e. and to be weakly measurable if $\langle f, x^* \rangle$ is measurable for any $x^* \in X^*$. In the case of dual spaces X^* a function is called weak*-measurable if $\langle x, f \rangle$ is measurable for any $x \in X$. We denote by $L^0(X)$, $L^0_{\text{weak}}(X)$ and $L^0_{\text{weak}^*}(X^*)$ the spaces of strongly, weakly measurable and weak*-measurable functions. We write $L^p(X)$, $L^p_{\text{weak}}(X)$ and $L^p_{\text{weak}^*}(X^*)$ for the space of functions in $L^0(X)$, $L^0_{\text{weak}}(X)$ and $L^0_{\text{weak}^*}(X^*)$ such that $\|f\| \in L^p(\mu)$, $\langle f, x^* \rangle \in L^p(\mu)$ for $x^* \in X^*$ and $\langle x, f \rangle \in L^p(\mu)$ for $x \in X$ respectively. Finally we use the notation $P^p(X)$

for the space of Pettis p-integrable functions $P^p(X) = L^p_{\text{weak}}(X) \cap L^0(X)$.

2 Integrablility with respect to bilinear maps.

Definition 1 *Let Y and Z be Banach spaces and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. We say that $f : \Omega \to X$ is $(Y, Z, \mathcal{B})$-measurable if $\mathcal{B}(f, y) \in L^0(Z)$ for any $y \in Y$. We shall denote the class of such functions by $L^0_{\mathcal{B}}(X)$.*

Given a Banach space X there are many standard ways to find triples Y, Z and $\mathcal{B}$ where $\mathcal{B} : X \times Y \to Z$ becomes a bounded bilinear map.

The basic ones are:

$$\mathcal{B}_X = \mathcal{B} : X \times \mathbb{K} \to X, \qquad \mathcal{B}(x, \lambda) = \lambda x. \tag{1}$$

$$\mathcal{D}_X = \mathcal{D} : X \times X^* \to \mathbb{K}, \qquad \mathcal{D}(x, x^*) = \langle x, x^* \rangle. \tag{2}$$

Note that $L^0_{\mathcal{B}}(X) = L^0(X)$ and $L^0_{\mathcal{D}}(X) = L^0_{\text{weak}}(X)$.

Natural generalizations of (1) and (2) are the following: For any other Banach space Y one has

$$\pi_Y : X \times Y \to X \hat{\otimes} Y, \qquad \pi_Y(x, y) = x \otimes y. \tag{3}$$

$$\widetilde{\mathcal{O}}_Y : X \times \mathcal{L}(X, Y) \to Y, \qquad \widetilde{\mathcal{O}}_Y(x, T) = T(x). \tag{4}$$

In the case of dual spaces X^* we have also

$$\mathcal{D}_{1,X} = \mathcal{D}_1 : X^* \times X \to \mathbb{K}, \qquad \mathcal{D}_1(x^*, x) = \langle x, x^* \rangle. \tag{5}$$

Note that $L^0_{\mathcal{D}_1}(X^*) = L^0_{\text{weak}*}(X)$.

A generalization of (5) correspond to the case $X = \mathcal{L}(Y, Z)$ which plays an important role in what follows: Denote consider

$$\mathcal{O}_{Y,Z} : \mathcal{L}(Y, Z) \times Y \to Z, \qquad \mathcal{O}_{Y,Z}(T, y) = T(y). \tag{6}$$

In the particular case $Y = Z$ one can also consider,

$$\mathcal{C}_E : \mathcal{L}(E, E) \times \mathcal{L}(E, E) \to \mathcal{L}(E, E), \qquad \mathcal{C}_E(T, S) = TS. \tag{7}$$

Actually (7) is just the product on a Banach algebra A:

$$\mathcal{P}r : A \times A \to A, \qquad \mathcal{P}r(a,b) = ab. \tag{8}$$

Given a bounded bilinear map $\mathcal{B} : X \times Y \to Z$, we can define the "adjoint" $\mathcal{B}^* : X \times Z^* \to Y^*$ by the formula

$$\langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x,y), z^*\rangle.$$

Note that

$$\mathcal{B}^* = \mathcal{D},\ (\pi_Y)^* = \widetilde{\mathcal{O}}_{Y^*} \text{ and } (\mathcal{O}_{Y,Z})^*(T, z^*) = \mathcal{O}_{Z^*,Y^*}(T^*, z^*).$$

Definition 2 *We write $\mathcal{L}^p_{\mathcal{B}}(X)$ for the space of functions f in $L^0_{\mathcal{B}}(X)$ such that*

$$\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f,y)\|_{L^p(Z)} : \|y\| = 1\} < \infty.$$

Clearly $\|f+g\|_{\mathcal{L}^p_{\mathcal{B}}(X)} \leq \|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} + \|g\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$ and $\|\lambda f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = |\lambda| \|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$ for f, g in $\mathcal{L}^p(\mathcal{B})$ and $\lambda \in \mathbb{K}$, but in general the $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = 0$ does not imply $f = 0$ a.e. (It suffices to take $\mathcal{B}$ such that there exists $x \neq 0$ for which $\mathcal{B}(x,y) = 0$ for all $y \in Y$, and select $f = x\mathbf{1}_\Omega$).

Observe that $L^p(X) \subset \mathcal{L}^p_{\mathcal{B}}(X)$ for any bounded bilinear map $\mathcal{B}$. Also one has $\mathcal{L}^p_{\mathcal{B}}(X) = L^p(X)$, $\mathcal{L}^p_{\mathcal{D}}(X) = L^p_{\text{weak}}(X)$ and $\mathcal{L}^p_{\mathcal{D}_1}(X^*) = L^p_{\text{weak}*}(X^*)$.

Remark 3 *Observe that simple functions, say $s = \sum_{k=1}^n x_k \mathbf{1}_{A_k}$, $x_k \in X$, and pairwise disjoint sets A_k, belong to $\mathcal{L}^p_{\mathcal{B}}(X)$. Actually*

$$\|s\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{(\sum_{k=1}^n \|\mathcal{B}(x_k, y)\|^p \mu(A_k))^{\frac{1}{p}} : \|y\| = 1\}$$

A simple duality argument gives

$$\|s\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{\|\sum_{k=1}^n \mathcal{B}^*(x_k, z_k^*)\mu(A_k)^{\frac{1}{p}}\| : (\sum_{k=1}^n \|z_k^*\|^{p'})^{\frac{1}{p'}} = 1\}.$$

Definition 4 *A function $f \in \mathcal{L}^p_{\mathcal{B}}(X)$ is said to belong to $L^p_{\mathcal{B}}(X)$ if there exists a sequence of simple functions $(s_n)_n \in \mathcal{S}(X)$ such that*

$$s_n \to f \text{ a.e.} \qquad \text{and} \qquad \|s_n - f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} \to 0.$$

For $f \in L^p_{\mathcal{B}}(X)$ we write $\|f\|_{L^p_{\mathcal{B}}(X)}$ instead of $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$. Clearly one has that

$$\|f\|_{L^p_{\mathcal{B}}(X)} = \lim_{n\to\infty} \|s_n\|_{L^p_{\mathcal{B}}(X)}.$$

Remark 5 *Let $\Omega = [0,1]$ with the Lebesgue measure. Let $f = \sum_{k=1}^{\infty} 2^k x_k \mathbf{1}_{I_k}$ where $x_k \in X$ and $I_k = (2^{-k}, 2^{-k+1}]$ for $k \in \mathbb{N}$.*

It is elementary to see that $f \in \mathcal{L}^p_{\mathcal{B}}(X)$ if and only if $\sup_{\|y\|=1} \sum_{k=1}^{\infty} \|\mathcal{B}(x_k, y)\|^p < \infty$. From this it follows that if $\lim_{N\to\infty} \sup_{\|y\|=1} \sum_{k=N}^{\infty} \|\mathcal{B}(x_k, y)\|^p = 0$ then $f \in L^p_{\mathcal{B}}(X)$.

Remark 6 *(i) $L^p(X) \subseteq L^p_{\mathcal{B}}(X)$ for any $\mathcal{B}$ and $\mathcal{L}^p_{\mathcal{B}}(X) = L^p_{\mathcal{B}}(X) = L^p(X)$.*

(ii) $L^p_{\mathcal{D}}(X) = P^p(X)$ (see [10], page 54 for the case $p = 1$).

(iii) $L^p_{\mathcal{B}}(X) \subsetneq \mathcal{L}^p_{\mathcal{B}}(X)$ (see [6] page 53, for the case $\mathcal{B} = \mathcal{D}$).

(iv) Let $f : \Omega \to \mathcal{L}(X, Y)$ belong to $\mathcal{L}^1_{\mathcal{O}_{X,Y}}(\mathcal{L}(X, Y))$ and denote $f(w) = T_w$. Then, for any $A \in \Sigma$, there exists $T_A \in \mathcal{L}(X, Y)$ such that $T_A x = \int_\Omega T_t x d\mu$ for $x \in X$.

As expected the bilinear map $\mathcal{B}$ defines the smallest space in the scale $\{L^p_{\mathcal{B}}(X) :$ $\mathcal{B}$ bilinear and bounded $\}$. One might expect the space of Pettis p-integrable functions, $L^p_{\mathcal{D}}(X)$, to be the biggest in the scale. We shall now see that the inclusion $L^p_{\mathcal{B}}(X) \subset P^p(X)$ holds true only among certain class of bilinear maps.

Given $x \in X$ and $y \in Y$ we shall be denoting by $\mathcal{B}_x \in \mathcal{L}(Y, Z)$ and $\mathcal{B}^y \in \mathcal{L}(X, Z)$ the corresponding linear operators

$$\mathcal{B}_x(y) = \mathcal{B}(x, y) \text{ and } \mathcal{B}^y(x) = \mathcal{B}(x, y).$$

Definition 7 *Let Y and Z be Banach spaces and $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. We shall say that the triple $(Y, Z, \mathcal{B})$ is admissible for X if the map $x \to \mathcal{B}_x$ is injective from $X \to \mathcal{L}(Y, Z)$, i.e. $\mathcal{B}(x, y) = 0$ for all $y \in Y$ implies $x = 0$.*

Notice that if $(Y, Z, \mathcal{B})$ is admissible for X if and only if $(Z^*, Y^*, \mathcal{B}^*)$ is.

It is elementary to see that examples in (1)-(7) are admissible triples. In the example (8) the admissibility condition becomes "no zero divisors" and holds true for Banach algebras with identity or with bounded approximation of the identity.

Definition 8 *Let Y and Z be Banach spaces and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. X is said to be $(Y, Z, \mathcal{B})$-normed (or normed by $\mathcal{B}$) if there exists $C > 0$ such that for all $x \in X$*

$$\|x\| \leq C\|\mathcal{B}_x\|.$$

This simply means X can be understood as a subspace of $\mathcal{L}(Y, Z)$ and that

$|||x||| = \|\mathcal{B}_x\|$ *defines an equivalent norm on* X.

Remark 9 *(i) If* X *is* $(Y, Z, \mathcal{B})$*-normed then* $(Y, Z, \mathcal{B})$ *is an admissible triple.*

(ii) X *is* $(Y, Z, \mathcal{B})$*-normed if and only if it is* $(Z^*, Y^*, \mathcal{B}^*)$*-normed.*

Remark 10 *Let* X *be* $(Y, Z, \mathcal{B})$ *normed and* $f \in L^p_{\mathcal{B}}(X)$*. Then the function* $\widetilde{f} : \Omega \to \mathcal{L}(Y, Z)$ *given by* $\widetilde{f}(w) = \mathcal{B}_{f(w)}$ *belongs to* $L^p_{\mathcal{O}_{Y,Z}}(\mathcal{L}(Y, Z))$*. Moreover*

$$\|\widetilde{f}\|_{L^p_{\mathcal{O}_{Y,Z}}(\mathcal{L}(Y,Z))} = \|f\|_{L^p_{\mathcal{B}}(X)}.$$

Proposition 11 *Let* X, Y *and* Z *be Banach spaces and let* $\mathcal{B} : X \times Y \to Z$ *be a bounded bilinear map. The following are equivalent:*

(1) X *is* (Y, Z, B)*-normed.*
(2) For each $x^* \in X^*$ *there exists a functional* $\varphi_{x^*} \in \mathcal{L}(Y, Z)^*$ *such that*

$$\langle x, x^* \rangle = \varphi_{x^*}(\mathcal{B}_x) \text{ for all } x \in X.$$

PROOF. Assume that X is $(Y, Z, \mathcal{B})$-normed and denote by $\hat{X} = \{\mathcal{B}_x : x \in X\} \subseteq \mathcal{L}(Y, Z)$. By assumption $\hat{X}$ is a closed subspace of $\mathcal{L}(Y, Z)$. Given $x^* \in X^*$ the map $B_x \to \langle x^*, x\rangle$ defines bounded functional in $(\hat{X})^*$. Now, by the Hahn-Banach theorem there is an extension φ_{x^*} to $(\mathcal{L}(Y, Z))^*$.
The converse is immediate. □

Of course, given a Banach space X there are many triples $(Y, Z, \mathcal{B})$ for which X is $(Y, Z, \mathcal{B})$-normed. In particular the ones considered in the examples (1)-(7).

However it is also easy to produce examples of admissible triples which are not $(Y, Z, \mathcal{B})$-normed:

Example 12 *Let* $X = \ell_p$ *for* $1 \le p < 2$*,* $Y = \ell_2$*,* $Z = \ell_1$ *and* $B : \ell_p \times \ell_2 \to \ell_1$ *given by*

$$B((\alpha_n)_n, (\beta_n)_n) = (\alpha_n\beta_n)_n.$$

Then ℓ_p *is not* (Y, Z, B)*-normed.*

Theorem 13 *Let* X, Y *and* Z *be Banach spaces and let* $\mathcal{B} : X \times Y \to Z$ *be a bounded bilinear map. The following are equivalent:*

(1) X *is* $(Y, Z, \mathcal{B})$*-normed.*
(2) $L^p_{\mathcal{B}}(X) \subset P^p(X)$ *for all* $1 \le p < \infty$*.*
(3) $L^p_{\mathcal{B}}(X) \subset P^p(X)$ *for some* $1 \le p < \infty$*.*

PROOF.
(1)⇒(2) Let $1 \leq p < \infty$ and let $s = \sum_{k=1}^{n} x_k \mathbf{1}_{A_k} \in \mathcal{S}(X)$. Let us write

$$\begin{aligned} \|s\|_{P^p(X)} &= \sup\{(\sum_{k=1}^{n} |\langle x_k, x^*\rangle|^p \mu(A_k))^{\frac{1}{p}} : \|x^*\| = 1\} \\ &= \sup\{(|\langle \sum_{k=1}^{n} x_k \mu(A_k)^{\frac{1}{p}} \alpha_k, x^*\rangle| : \|x^*\| = 1, \|\alpha\|_{\ell_{p'}} = 1\} \end{aligned}$$

For each $x^* \in X^*$ and $\|\alpha\|_{\ell_{p'}} = 1$, using Proposition 11 one gets

$$\langle \sum_{k=1}^{n} x_k \mu(A_k)^{\frac{1}{p}} \alpha_k, x^*\rangle = \varphi_{x^*}(\mathcal{B}_{\sum_{k=1}^{n} x_k \mu(A_k)^{\frac{1}{p}} \alpha_k}).$$

Hence

$$\begin{aligned} \|s\|_{P^p(X)} &\leq \sup\{\|\varphi_{x^*}\| \|\mathcal{B}(\sum_{k=1}^{n} \alpha_k x_k \mu(A_k)^{\frac{1}{p}}, y)\| : \|x^*\| = 1, \|\alpha\|_{\ell_{p'}} = 1, \|y\| = 1\} \\ &\leq \sup\{\|\varphi_{x^*}\| \sum_{k=1}^{n} \|\mathcal{B}(x_k \mu(A_k)^{\frac{1}{p}}, y)\| \ |\alpha_k| : \|x^*\| = 1, \|\alpha\|_{\ell_{p'}} = 1, \|y\| = 1\} \\ &= M \|s\|_{L^p_{\mathcal{B}}(X)} \end{aligned}$$

Now if we take a function $f \in L^p_{\mathcal{B}}(X)$ then there exists $(s_n)_n \in \mathcal{S}(X)$ convergent to f a.e and in the norm $\|\cdot\|_{L^p_{\mathcal{B}}(X)}$. Since $(|\langle s_n, x^*\rangle|^p)_n$ converges to $(|\langle f, x^*\rangle|^p)$ a.e. , Fatou's Lemma implies that

$$\begin{aligned} \|f\|^p_{P^p(X)} &= \sup\{\int_{\Omega} \lim_n |\langle s_n(w), x^*\rangle|^p d\mu : \|x^*\| = 1\} \\ &\leq \sup\{\liminf_n \int_{\Omega} |\langle s_n(w), x^*\rangle|^p d\mu : \|x^*\| = 1\} \\ &\leq \liminf_n \|s_n\|^p_{P^p(X)} \\ &\leq M^p \liminf_n \|s_n\|^p_{L^p_{\mathcal{B}}(X)} \\ &\leq M^p \|f\|^p_{L^p_{\mathcal{B}}(X).} \end{aligned}$$

(2)⇒(3) Obvious.

(3)⇒(1) Assume (3), fix $x \in X$ and consider the simple function

$$\begin{aligned} f_x \colon \quad \Omega &\to X \\ w &\mapsto x\mu(\Omega)^{-\frac{1}{p}} \mathbf{1}_{\Omega}(w) \end{aligned}$$

Since $\|f_x\|_{P^p(X)} = \|x\|$ and $\|f_x\|_{L^p_{\mathcal{B}}(X)} = \|\mathcal{B}_x\|$ one gets (1). □

Proposition 14 *Let X be a $(Y, Z, \mathcal{B})$-normed space and $f \in L^1_{\mathcal{B}}(X)$. For each $E \in \Sigma$ there exists a unique $x_E \in X$ such that for any $y \in Y$*

$$\mathcal{B}(x_E, y) = \int_E \mathcal{B}(f(w), y) d\mu.$$

The value $x_E = (\mathcal{B}) \int_E f d\mu$ is called the $\mathcal{B}$-integral of f over E.

PROOF. Note that the uniqueness follows from the bilinearity of $\mathcal{B}$ and the admissibility of the triple.

To show the existence, observe that if $f \in L^1(X)$ then x_E can be taken the Bochner integral of f over E, $\int_E f d\mu$, using that $\mathcal{B}^y \in \mathcal{L}(X, Z)$ and $B^y(x_E) = \int_E B^y(f) d\mu$ for any $y \in Y$.

Now, if $f \in L^1_{\mathcal{B}}(X)$ and $(s_n)_n$ is the sequence of simple functions of the definition then we have

$$\int_E \mathcal{B}(f(w), y) d\mu = \lim_n \mathcal{B}(x_{n,E}, y),$$

for $E \in \Sigma$ and $y \in Y$ where $x_{n,E} = \int_E s_n d\mu$.

The fact that X is $(Y, Z, \mathcal{B})$-normed implies that there exists $\lim_n x_{n,E} \in X$, say x_E. Indeed,

$$\begin{aligned} \|x_{n,E} - x_{m,E}\| &\leq C \sup\{\|\mathcal{B}_{x_{n,E} - x_{m,E}}(y)\| : \|y\| = 1\} \\ &\leq C \sup\{\|\mathcal{B}(s_n - s_m, y)\|_{L^1(Z)} : \|y\| = 1\} \\ &\leq C \|s_n - s_m\|_{L^1_{\mathcal{B}}(X)}. \end{aligned}$$

Finally we have $\int_E \mathcal{B}(f(w), y) d\mu = \lim_n \mathcal{B}(x_{n,E}, y) = \mathcal{B}(\lim_n x_{n,E}, y) = \mathcal{B}(x_E, y)$.

□

Remark 15 *If X be (Y, Z, B)-normed space and $f \in L^1_{\mathcal{B}}(X)$ then*

$$x_E = (\mathcal{B}) \int_E f d\mu = (P) \int_E f d\mu$$

for any $E \in \Sigma$ where $(P) \int_E f d\mu(w)$ stands for the Pettis integral over E.

3 A bilinear version of Hölder's Inequality.

It is well known and easy to see the following analogues of Hölder's inequality in the vector-valued setting: Let $1 \le p_1, p_2, p_3 \le \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$.

(1) If $f \in L^{p_1}_{\text{weak}}(X)$ and $g \in L^{p_2}$ then $fg \in L^{p_3}_{\text{weak}}(X)$.
(2) If $f \in P^{p_1}(X)$ and $g \in L^{p_2}$ then $fg \in P^{p_3}(X)$.
(3) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}$ then $fg \in L^{p_3}(X)$.
(4) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X^*)$ then $\langle f, g\rangle \in L^{p_3}$.
(5) If $f \in L^{p_1}(\mathcal{L}(X,Y))$ and $g \in L^{p_2}(X)$ then $f(w)(g(w)) \in L^{p_3}(Y)$.

Clearly $f \in L^0_{\mathcal{B}}(X)$ and $g \in L^0(Y)$ implies that $\mathcal{B}(f,g) \in L^0(Z)$. Hence a natural question that arises is the following: Does $\mathcal{B}(f,g)$ belong to $L^{p_3}(Z)$ for any $f \in \mathcal{L}^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Y)$?

The answer is negative for any infinite dimensional Banach space X.

Indeed, take $p_1 = p_2 = 2$ and $p_3 = 1$, let X be an infinite dimensional Banach space, $Y = X^*$ and $Z = \mathbb{K}$ and $\mathcal{B} = \mathcal{D}$. Take $(x_n) \in \ell^2_{\text{weak}}(X) \setminus \ell_2(X)$. This allows to find $(x_n^*) \in \ell_2(X^*)$ such that $\sum_n |\langle x_n, x_n^*\rangle| = \infty$. Consider now $\Omega = [0,1]$ with the Lebesgue measure, $I_k = (2^{-k}, 2^{-k+1}]$ and define the functions $f = \sum_{k=1}^\infty 2^{\frac{k}{2}} x_k \mathbf{1}_{I_k}$ and $g = \sum_{k=1}^\infty 2^{\frac{k}{2}} x_k^* \mathbf{1}_{I_k}$. It is clear that $f \in \mathcal{L}^2_{\mathcal{D}}(X)$ with $\|f\|^2_{\mathcal{L}^2_{\mathcal{D}}(X)} = \sup\{\sum_{n=1}^\infty |\langle x_n, x^*\rangle|^2 : \|x^*\| = 1\}$ and $g \in L^2(X^*)$ with $\|g\|^2_{L^2(X^*)} = \sum_{n=1}^\infty \|x_n^*\|^2$ but $\mathcal{B}(f,g) = \sum_{k=1}^\infty 2^k \langle x_k, x_k^*\rangle \mathbf{1}_{I_k} \notin L^1$.

One might think that the difficulty comes from allowing functions to belong to $\mathcal{L}^{p_1}_{\mathcal{B}}(X)$ instead of $L^{p_1}_{\mathcal{B}}(X)$. Let us then modify the question: Does $\mathcal{B}(f,g)$ belong to $L^{p_3}(Z)$ for any $f \in L^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Y)$?

The answer is again negative. If the result hold true we would have that there exists $M > 0$ such that $\|\mathcal{B}(s,t)\|_{L^1(Z)} \le M \|s\|_{L^2_{\mathcal{B}}(X)} \|t\|_{L^2(Y)}$ for any $s \in \mathcal{S}(X)$ and $t \in \mathcal{S}(Y)$.

Select $X = Y = \ell_2$, $Z = \ell_1$ and $\mathcal{B} : \ell_2 \times \ell_2 \to \ell_1$ given by $\mathcal{B}((\lambda_n)_{n\in\mathbb{N}},, (\beta_n)_{n\in\mathbb{N}}) = (\lambda_n \beta_n)_{n\in\mathbb{N}}$. Let us now consider $s_N = \sum_{k=1}^N 2^{\frac{k}{2}} e_k \mathbf{1}_{I_k}$ where e_k is the canonical basis and I_k are chosen as above. Hence $\mathcal{B}(s_N, y) = \sum_{k=1}^N 2^{\frac{k}{2}} \beta_k e_k \mathbf{1}_{I_k}$ for $y = (\beta_n)_{n\in\mathbb{N}} \in \ell_2$. Therefore $\|s_N\|_{L^2_{\mathcal{B}}(\ell_2)} \le 1$. Let us also take $t_N = \sum_{k=1}^N 2^{\frac{k}{2}} e_k \mathbf{1}_{I_k}$ which gives $\|t_N\|_{L^2(\ell_2)} = \sqrt{N}$. Finally observe that $\mathcal{B}(s_N, t_N) = \sum_{k=1}^N 2^k e_k \mathbf{1}_{I_k}$ and $\|\mathcal{B}(s_N, t_N)\|_{L^1(\ell_1)} = N$. This contradicts (3).

Modifying the previous argument with $Z = \mathbb{K}$ and $\mathcal{B} = \mathcal{D}$ one can even show that there exist $f \in L^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Y)$ such that $\mathcal{B}(f,g) \notin L^{p_3}_{\text{weak}}(Z)$.

To establish some bilinear version of Hölder's inequality we need to put together different bilinear maps. We shall then study the following general problem:

Problem: Let $1 \leq p_1, p_2, p_3 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. If $\mathcal{B}_1 : X \times X_1 \to X_2$ and $\mathcal{B}_2 : Y \times Y_1 \to Y_2$ are bounded bilinear maps, find $\mathcal{B}_3 : Z \times Z_1 \to Z_2$ such that for any $f \in \mathcal{L}^{p_1}_{\mathcal{B}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)$ one has $\mathcal{B}(f, g) \in \mathcal{L}^{p_3}_{\mathcal{B}_3}(Z)$.

Definition 16 *We say that $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is a compatible triple if $\mathcal{B} : X \times Y \to Z$, $\mathcal{B}_1 : X \times X_1 \to X_2$ and $\mathcal{B}_2 : Y \times Y_1 \to Y_2$ are bounded bilinear maps and there exist a Banach space F and two bounded bilinear maps $\mathcal{P} : X_2 \times Y_2 \to F$ and $\widetilde{\mathcal{P}} : Z \times (X_1 \hat{\otimes} Y_1) \to F$ such that*

$$\widetilde{\mathcal{P}}(\mathcal{B}(x, y), x_1 \otimes y_1) = \mathcal{P}(\mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1))$$

for all $x \in X, y \in Y$, $x_1 \in X_1$ and $y_1 \in Y_1$.

A general procedure of construction of such compatible triples of bilinear maps can be obtained as follows:

Example 17 *Let U be a Banach space, $\mathcal{B}_1 : X \times X_1 \to U$ and $\mathcal{B}_2 : Y \times Y_1 \to U^*$ be bounded bilinear maps . Define the bilinear map $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{B} : X \times Y \to \mathcal{L}(X_1, Y_1^*)$ defined by the formula*

$$\langle \mathcal{B}(x, y)(x_1), y_1 \rangle = \langle \mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1) \rangle$$

for $x \in X$, $y \in Y$, $x_1 \in X_1$ and $y_1 \in Y_1$.

Using that $\mathcal{L}(X_1, Y_1^*) = (X_1 \hat{\otimes} Y_1)^*$ we also can write

$$\langle \mathcal{B}(x, y), x_1 \otimes y_1 \rangle = \langle \mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1) \rangle.$$

Note that $(\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2), \mathcal{B}_1, \mathcal{B}_2)$ is compatible, selecting $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D} : U \times U^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_1 : \mathcal{L}(X_1, Y_1^*) \times (X_1 \hat{\otimes} Y_1) \to \mathbb{K}$.

Let us now give some more concrete examples of admissible triples:

Example 18 *$(\mathcal{B}, \mathcal{B}_X, \mathcal{B}_Y)$ is a compatible triple for any $\mathcal{B} : X \times Y \to Z$.*

In particular, $(\mathcal{D}_X, \mathcal{B}_X, \mathcal{B}_{X^*})$ or $(\mathcal{O}_{X,Y}, \mathcal{B}_X, \mathcal{B}_Y)$ are compatible triples.

Indeed, if $\mathcal{B} : X \times Y \to Z$, $\mathcal{B}_1 = \mathcal{B}_X : X \times \mathbb{K} \to X$ and $\mathcal{B}_2 = \mathcal{B}_Y : Y \times \mathbb{K} \to Y$ then select $F = Z$, $\mathcal{P} = \mathcal{B} : X \times Y \to Z$ and $\widetilde{\mathcal{P}} = \mathcal{B}_Z : Z \times \mathbb{K} \to Z$. Observe that $\widetilde{\mathcal{P}}(\mathcal{B}(x, y), \lambda\beta) = \mathcal{P}(\mathcal{B}(x, \lambda), \mathcal{B}(y, \beta))$. □

Example 19 *$(\mathcal{B}, \mathcal{B}^*, \mathcal{B}_Y)$ is a compatible triple.*

Indeed, if $\mathcal{B} : X \times Y \to Z$, $\mathcal{B}_1 = \mathcal{B}^* : X \times Z^* \to Y^*$ given by

$$\langle \mathcal{B}_1(x, z^*), y\rangle = \langle \mathcal{B}(x, y), z^*\rangle$$

and $\mathcal{B}_2 = \mathcal{B}_Y : Y \times \mathbb{K} \to Y$ then we can select $F = \mathbb{K}$, $\mathcal{P} = (\mathcal{D}_1)_Y : Y^* \times Y \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_Z : Z \times Z^* \to \mathbb{K}$. □

Example 20 $(\pi_Y, \mathcal{B}_X, \mathcal{O}_{X^*})$ *is a compatible triple.*

Indeed, if $\mathcal{B} = \pi_Y : X \times Y \to X\hat{\otimes}Y$, $\mathcal{B}_1 = \mathcal{B}_X : X \times \mathbb{K} \to X$ and $\mathcal{B}_2 = \widetilde{\mathcal{O}}_{X^*} : Y \times \mathcal{L}(Y, X^*) \to X^*$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_X : X \times X^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_{X\hat{\otimes}Y} : X\hat{\otimes}Y \times \mathcal{L}(Y, X^*) \to \mathbb{K}$. The compatibility now follows from

$$\widetilde{\mathcal{P}}(\mathcal{B}(x, y), \lambda T) = \langle x \otimes y, \lambda T\rangle = \langle \lambda x, Ty\rangle = \mathcal{P}(\mathcal{B}_1(x, \lambda), \mathcal{B}_2(y, T)).$$

□

Example 21 *Let* $\mathcal{B} : \mathcal{L}(X, Z) \times \mathcal{L}(Y, Z^*) \to \mathcal{L}(Y, X^*)$ *be given by* $(T, S) \to T^*S$*. Then* $(\mathcal{B}, \mathcal{O}_{X,Z}, \mathcal{O}_{Y,Z^*})$ *is a compatible triple.*

Indeed, if $\mathcal{B}_1 = \mathcal{O}_{X,Z} : \mathcal{L}(X, Z) \times X \to Z$ and $\mathcal{B}_2 = \mathcal{O}_{Y,Z^*} : \mathcal{L}(Y, Z^*) \times Y \to Z^*$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_Z : Z \times Z^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = (\mathcal{D}_1)_{X\hat{\otimes}Y} : \mathcal{L}(Y, X^*) \times X\hat{\otimes}Y \to \mathbb{K}$ given by $\widetilde{\mathcal{P}}(T, x \otimes y) = \langle x, Ty\rangle$.

Observe that the compatibility follows from the formula

$$\widetilde{\mathcal{P}}(\mathcal{B}(T, S), x \otimes y) = \langle x, T^*Sy\rangle = \langle Tx, Sy\rangle = \mathcal{P}(\mathcal{B}_1(T, x), \mathcal{B}_2(S, y)).$$

□

Theorem 22 (Hölder's inequality I) *Let* $1 \leq p_1, p_2, p_3 < \infty$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$*. Assume that* $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ *is a compatible triple for some* F, $\mathcal{P}$ *and* $\widetilde{\mathcal{P}}$.

(1) If $f \in \mathcal{L}^{p_1}_{\mathcal{B}_1}(X)$ *and* $g \in \mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)$ *then* $\mathcal{B}(f, g) \in \mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.
(2) If $f \in L^{p_1}_{\mathcal{B}_1}(X)$ *and* $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ *then* $\mathcal{B}(f, g) \in L^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

Moreover $\|\mathcal{B}(f, g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \leq \|\mathcal{P}\|\|f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)}$.

PROOF. *(1)* Let us first show that if $f \in L^0_{\mathcal{B}_1}(X)$ and $g \in L^0_{\mathcal{B}_2}(Y)$ then $h = \mathcal{B}(f, g) \in L^0_{\widetilde{\mathcal{P}}}(Z)$.

Indeed, if $x_1 \in X_1$ and $y_1 \in Y_1$ then $\widetilde{\mathcal{P}}(h, x_1 \otimes y_1) = \mathcal{P}(\mathcal{B}_1(f, x_1), \mathcal{B}_2(g, y_1))$. Now since $\mathcal{B}_1(f, x_1) \in L^0(X_2)$, $\mathcal{B}_2(g, y_1) \in L^0(Y_2)$ and $\mathcal{P}$ is continuous then

$\widetilde{\mathcal{P}}(h, x_1 \otimes y_1) \in L^0(F)$. For general $\varphi \in X_1 \hat{\otimes} Y_1$, assume $\varphi = \sum_n x_1^n \otimes y_1^n$ with $\sum_n \|x_1^n\| \|y_1^n\| < \infty$. Then, using the continuity of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, one has

$$\widetilde{\mathcal{P}}(h, \varphi) = \lim_{N \to \infty} \sum_{k=1}^{N} \widetilde{\mathcal{P}}(\mathcal{B}_1(f, x_1^k), \mathcal{B}_2(g, y_1^k)) \in L^0(F).$$

Assume $f \in \mathcal{L}^{p_1}_{\mathcal{B}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)$. Let us show that $h \in \mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

If $x_1 \in X_1$ and $y_1 \in Y_1$ then

$$\begin{aligned}
(\int_\Omega \|\widetilde{\mathcal{P}}(h, x_1 \otimes y_1)\|^{p_3} d\mu)^{\frac{1}{p_3}} &= (\int_\Omega \|\mathcal{P}(\mathcal{B}_1(f, x_1), \mathcal{B}_2(g, y_1))\|^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\leq \|\mathcal{P}\| (\int_\Omega (\|\mathcal{B}_1(f, x_1)\| \|\mathcal{B}_2(g, y_1))\|)^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\leq \|\mathcal{P}\| (\int_\Omega \|\mathcal{B}_1(f, x_1)\|^{p_2} d\mu)^{\frac{1}{p_2}} (\int_\Omega \|\mathcal{B}_2(g, y_1)\|^{p_1} d\mu)^{\frac{1}{p_1}} \\
&\leq \|\mathcal{P}\| \|f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)} \|x_1\| \|y_1\|.
\end{aligned}$$

In general, for each $\varphi = \sum_n x_1^n \otimes y_1^n \in X_1 \hat{\otimes} Y_1$, one has $\widetilde{\mathcal{P}}(h, \sum_n x_1^n \otimes y_1^n) = \sum_n \widetilde{\mathcal{P}}(h, x_1^n \otimes y_1^n)$. Therefore

$$\begin{aligned}
(\int_\Omega \|\widetilde{\mathcal{P}}(h, \sum_n x_1^n \otimes y_1^n)\|^{p_3} d\mu)^{\frac{1}{p_3}} &\leq \sum_n (\int_\Omega \|\mathcal{P}(\mathcal{B}_1(f, x_1^n), \mathcal{B}_2(g, y_1^n))\|^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\leq \|\mathcal{P}\| (\sum_n \|x_1^n\| \|y_1^n\|) \|f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}
\end{aligned}$$

This gives $\|\mathcal{B}(f,g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \leq \|\mathcal{P}\| \|f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)} \|g\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)}$.

(2) Assume that f and g are simple functions. If $f = \sum_k x_k \mathbf{1}_{E_k} \in \mathcal{S}(X)$ and $g = \sum_p y_p \mathbf{1}_{F_p} \in \mathcal{S}(Y)$ then

$$h = \mathcal{B}(f, g) = \sum_{k,p} \mathcal{B}(x_k, y_p) \mathbf{1}_{E_k \cap F_p} \in \mathcal{S}(Z).$$

Now, if we take $f \in L^{p_1}_{\mathcal{B}_1}(X)$ and $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ then there exists $(f_n)_n \subseteq \mathcal{S}(X)$ and $(g_n)_n \subseteq \mathcal{S}(Y)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $\|f_n - f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \to 0$ and $\|g_n - g\|_{L^{p_2}_{\mathcal{B}_2}(Y)} \to 0$. Clearly $\mathcal{B}(f_n, g_n)$ are simple functions and converge to $\mathcal{B}(f, g)$ a.e.

Due to the previous result

$$\begin{aligned}\|\mathcal{B}(f_n,g_n)-\mathcal{B}(f,g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} &\leq \|\mathcal{B}(f_n-f,g_n)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)}+\|\mathcal{B}(f,g_n-g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)}\\ &\leq \|\mathcal{P}\|\|f_n-f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g_n\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)}\\ &+\|\mathcal{P}\|\|f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g_n-g\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)}\end{aligned}$$

Taking limits the result is completed. □

Let us point out a little improvement that can be achieved for the compatible triples of in Example 17. Let us recall the following fact that will be used in the proof.

Lemma 23 *Let X be a Banach space, $1\leq p<\infty$ and $(x_n^*)_n\subseteq X^*$. Then*

$$\sup\{(\sum_n|\langle x_n^*,x^{**}\rangle|^p)^{\frac{1}{p}}:\|x^{**}\|=1\}=\sup\{(\sum_n|\langle x,x_n^*\rangle|^p)^{\frac{1}{p}}:\|x\|=1\}$$

Theorem 24 (Hölder's inequality II) *Let X,X_1,Y,Y_1 and U be a Banach spaces and $1\leq p_1,p_2,p_3<\infty$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$. Let $\mathcal{B}_1:X\times X_1\to U$, $\mathcal{B}_2:Y\times Y_1\to U^*$ be bounded bilinear maps and let $\mathcal{B}(\mathcal{B}_1,\mathcal{B}_2)=\mathcal{B}:X\times Y\to\mathcal{L}(X_1,Y_1^*)$ be defined by the formula*

$$\langle\mathcal{B}(x,y)(x_1),y_1)\rangle=\langle\mathcal{B}_1(x,x_1),\mathcal{B}_2(y,y_1)\rangle.$$

If $f\in L^{p_1}_{\mathcal{B}_1}(X)$ and $g\in L^{p_2}_{\mathcal{B}_2}(Y)$ then $\mathcal{B}(f,g)\in P^{p_3}(\mathcal{L}(X_1,Y_1^*))$.

Moreover $\|\mathcal{B}(f,g)\|_{L^{p_3}_{\mathrm{weak}}(\mathcal{L}(X_1,Y_1^*))}\leq\|f\|_{L^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}$.

PROOF. Assume first that f and g are simple functions. If $f=\sum_k x_k\mathbf{1}_{E_k}\in\mathcal{S}(X)$ and $g=\sum_p y_p\mathbf{1}_{F_p}\in\mathcal{S}(Y)$ then $h=\mathcal{B}(f,g)=\sum_{k,p}\mathcal{B}(x_k,y_p)\mathbf{1}_{E_k\cap F_p}\in\mathcal{S}(\mathcal{L}(X_1,Y_1^*))$. Note that $\mathcal{L}(X_1,Y_1^*)=(X_1\hat{\otimes}Y_1)^*$. Hence from Lemma 23

$$\begin{aligned}\|h\|_{L^{p_3}_{\mathrm{weak}}((X_1\hat{\otimes}Y_1)^*)} &=\sup\{(\sum_{k,p}|\langle\mathcal{B}(x_k,y_p),\psi\rangle|^{p_3}\mu(E_k\cap F_p))^{\frac{1}{p_3}}:\|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}}=1\}\\ &=\sup\{(\sum_{k,p}|\langle\varphi,\mathcal{B}(x_k,y_p)\rangle|^{p_3}\mu(E_k\cap F_p))^{\frac{1}{p_3}}:\|\varphi\|_{X_1\hat{\otimes}Y_1}=1\}\\ &=\|h\|_{L^{p_3}_{\mathrm{weak}*}((X_1\hat{\otimes}Y_1)^*)}.\end{aligned}$$

We conclude, using Theorem 22, that

$$\|h\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1,Y_1^*))} \leq \|f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}.$$

Now, if we take $f \in L^{p_1}_{\mathcal{B}_1}(X)$ and $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ then there exists $(f_n)_n \subseteq \mathcal{S}(X)$ and $(g_n)_n \subseteq \mathcal{S}(Y)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $\|f_n - f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \to 0$ and $\|g_n - g\|_{L^{p_2}_{\mathcal{B}_2}(Y)} \to 0$. Clearly $\mathcal{B}(f_n, g_n) \to \mathcal{B}(f,g)$ a.e. and therefore $\mathcal{B}(f,g)$ is strongly measurable and

$$|\langle \mathcal{B}(f_n, g_n), \psi)\rangle|^{p_3} \to |\langle \mathcal{B}(f,g), \psi)\rangle|^{p_3} \text{ a.e.}$$

for all $\psi \in (X_1 \hat{\otimes} Y_1)^{**}$.

To see that $\mathcal{B}(f,g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*))$ it suffices to show that $\mathcal{B}(f,g) \in L^{p_3}_{weak}(\mathcal{L}(X_1, Y_1^*))$.

Then using Fatou's Lemma and the inequality for simple functions we have that

$$\begin{aligned}
\|\mathcal{B}(f,g)\|^{p_3}_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^*)} &= \sup\{\int_\Omega |\langle \mathcal{B}(f,g), \psi\rangle|^{p_3} d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&= \sup\{\int_\Omega \lim_n |\langle \mathcal{B}(f_n,g_n), \psi\rangle|^{p_3} d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&\leq \sup\{\liminf_n \int_\Omega |\langle \mathcal{B}(f_n,g_n), \psi\rangle|^{p_3} d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&\leq \liminf_n \|\mathcal{B}(f_n,g_n)\|^{p_3}_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^{**})} \\
&\leq \liminf_n \|f_n\|^{p_3}_{L^{p_1}_{\mathcal{B}_1}(X)} \|g_n\|^{p_3}_{L^{p_2}_{\mathcal{B}_2}(Y)} \\
&= \|f\|^{p_3}_{L^{p_1}_{\mathcal{B}_1}(X)} \|g\|^{p_3}_{L^{p_2}_{\mathcal{B}_2}(Y)}.
\end{aligned}$$

□

Corollary 25 *Let $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map.*

(1) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X^*)$ then $\langle f, g\rangle \in L^{p_3}$.
(2) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}_{\widetilde{\mathcal{B}}_*}(Y)$ then $\mathcal{B}(f,g) \in L^{p_3}_{\text{weak}}(Z)$, where
$\widetilde{\mathcal{B}}_* : Y \times Z^* \to X^*$ is given by $\langle x, \widetilde{\mathcal{B}}_*(y, z^*)\rangle = \langle \mathcal{B}(x,y), z^*\rangle$.
(3) If $f \in L^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Z^*)$ then $\mathcal{B}^*(f,g) \in L^{p_3}_{\text{weak}^*}(Y^*)$, where
$\mathcal{B}^* : X \times Z^* \to Y^*$ is given by $\langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x,y), z^*\rangle$.
(4) If $f \in L^{p_1}_{\widetilde{\mathcal{O}}_{Y^*}}(X)$ and $g \in L^{p_2}(Y)$ then $f \otimes g \in L^{p_3}_{\text{weak}}(X \hat{\otimes} Y)$.

(5) *If* $f \in L^{p_1}_{\mathcal{O}_{X,Z}}(\mathcal{L}(X,Z))$ *and* $g \in L^{p_2}_{\mathcal{O}_{Y,Z^*}}(\mathcal{L}(Y,Z^*))$ *and if we put* $f^*(t) = f(t)^* \in \mathcal{L}(Z^*,X^*)$ *then* $f^*g \in L^{p_3}_{\text{weak}^*}(\mathcal{L}(Y,X^*))$.

4 Some concrete examples.

We now will see more concrete examples of spaces and bilinear maps where the theory can give nice applications.

Example 26 (Hölder's bilinear map) *Let* (Ω_1,η) *be a* σ*-finite measure space, let* $1 \le p_1,p_2,p_3 \le \infty$ *and* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and consider*

$$\mathcal{H}_{p_1,p_2} : L^{p_1}(\eta) \times L^{p_2}(\eta) \to L^{p_3}(\eta), \qquad (f,g) \to fg.$$

It is clear that $L^{p_1}(\eta)$ is $(L^{p_2}(\eta), L^{p_3}(\eta), \mathcal{H}_{p_1,p_2})$-normed.

In particular for $\Omega_1 = \mathbb{N}$ with the counting measure, one has for $p = p_3$:

Proposition 27 *Let* $1 \le p_1 < \infty$, $1 \le p_2 \le \infty$, $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\mathcal{H}_{p_1,p_2} : \ell_{p_1} \times \ell_{p_2} \to \ell_{p_3}$. *If* $f = (f_n) \in \mathcal{L}^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})$ *then*

$$\|f\|_{\mathcal{L}^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})} = \|(f_n)\|_{\ell_{p_1}(L^{p_3})}.$$

PROOF. Note that

$$\begin{aligned}
\|f\|_{\mathcal{L}^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})} &= \sup\{(\int_\Omega \|(f_n(w)\beta_n)_n\|^{p_3}_{\ell_{p_3}} d\mu)^{\frac{1}{p_3}} : \|(\beta_n)_n\|_{\ell_{p_2}} = 1\} \\
&= \sup\{(\sum_{n=1}^{\infty} (\|f_n\|_{L^{p_3}(\mu)}|\beta_n|)^{p_3})^{\frac{1}{p_3}} : \|(\beta_n)_n\|_{\ell_{p_2}} = 1\} \\
&= \|(\|f_n\|_{L^{p_3}})_n\|_{\ell_{p_1}} = \|(f_n)_n\|_{\ell_{p_1}(L^{p_3}(\mu))}
\end{aligned}$$

□

Example 28 (Young's bilinear map) *Let* G *be locally compact abelian group,* $1 \le p_1,p_2 \le \infty$ *and* $1/p_1 + 1/p_2 \ge 1$. *Let* $1 \le p_3 \le \infty$ *with* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ *and consider*

$$\mathcal{Y}_{p_1,p_2} : L^{p_1}(G) \times L^{p_2}(G) \to L^{p_3}(G), \qquad (f,g) \to f * g.$$

Proposition 29

(1) $L^p(\mathbb{R})$ *is* $(L^1(\mathbb{R}), L^p(\mathbb{R}), \mathcal{Y}_{p,1})$*-normed for any* $1 \le p < \infty$.

(2) $(L^2(\mathbb{R}), L^2(\mathbb{R}), \mathcal{Y}_{1,2})$ *is an admissible triple for* $L^1(\mathbb{R})$ *but* $L^1(\mathbb{R})$ *is not* $(L^2(\mathbb{R}), L^2(\mathbb{R}), \mathcal{Y}_{1,2})$*-normed.*

PROOF. *(1)* Since $L^1(\mathbb{R})$ has a bounded approximation of the identity then

$$\|f\|_p = \sup\{\|f * g\|_p : \|g\|_1 = 1\} = \sup\{\|\mathcal{Y}_{p,1}(f,g)\|_p : \|g\|_1 = 1\}.$$

(2) Note that

$$\sup\{\|f * g\|_2 : \|g\|_2 = 1\} = \sup\{\|\mathcal{Y}_{1,2}(f,g)\|_p : \|g\|_2 = 1\} = \|\hat{f}\|_\infty$$

which is not equivalent to $\|f\|_1$. □

In particular for $G = \mathbb{R}$ with the Lebesgue measure, the norm in the spaces $\mathcal{L}^p_{\mathcal{Y}_{p_1,p_2}}(L^{p_1})$ can be easily described in some cases.

Proposition 30 *Let* $1 \le p_1 < \infty$*,* $1 \le p_2 \le \infty$ *with* $\frac{1}{p_1} + \frac{1}{p_2} \ge 1$*. Let* $1 \le p_3 \le \infty$ *with* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - 1$*.*

(1) $\mathcal{L}^p_{\mathcal{Y}_{p_1,1}}(L^{p_1}(\mathbb{R})) = L^p(L^{p_1}(\mathbb{R}))$ *for any* $1 \le p < \infty$*.*
Moreover $\|f\|_{\mathcal{L}^p_{\mathcal{Y}_{p_1,1}}(L^{p_1}(\mathbb{R}))} = \|f\|_{L^p(L^{p_1}(\mathbb{R}))}$*.*

(2) If $f \in L^0(L^1(\mathbb{R}))$ *then*

$$\|f\|_{\mathcal{L}^2_{\mathcal{Y}_{1,2}}(L^1(\mathbb{R}))} = \sup_{x\in\mathbb{R}} (\int_\Omega |\hat{f}_w(x)|^2 d\mu)^{\frac{1}{2}}.$$

PROOF. *(1)* Assume $f \in L^0_{\mathcal{Y}_{p_1,1}}(L^{p_1}(\mathbb{R}))$ then, Proposition 29 and Theorem 13 give that f is weakly measurable and, due to the separability of $L^{p_1}(\mathbb{R})$, we conclude that $f \in L^0(L^{p_1}(\mathbb{R}))$. Assuming that $f : \Omega \to L^{p_1}(\mathbb{R})$ is given by $w \mapsto f_w$ and taking a bounded approximation of the identity in $L^1(\mathbb{R})$, say g_n, one has

$$\begin{aligned}\|f\|_{L^p(L^{p_1}(\mathbb{R}))} &= (\int_\Omega \|f_w\|^p_{L^{p_1}(\mathbb{R})} d\mu)^{\frac{1}{p}} \\ &= (\int_\Omega \lim_{n\to\infty} \|f_w * g_n\|^p_{L^{p_1}(\mathbb{R})} d\mu)^{\frac{1}{p}} \\ &\le \sup\{(\int_\Omega \|f_w * g\|^p_{L^{p_1}(\mathbb{R})} d\mu)^{\frac{1}{p}} : \|g\|_{L^1(\mathbb{R})} = 1\} \\ &= \|f\|_{\mathcal{L}^p_{\mathcal{Y}_{p_1,1}}(L^{p_1}(\mathbb{R}))}\end{aligned}$$

The other inclusion and inequality of norms are always true.

(2) Now if $f \in L^0(L^1(\mathbb{R}))$ then $f : \Omega \to L^1(\mathbb{R})$ given by $w \mapsto f_w$ and we have (using Plancherel's identity and Fubini's theorem) that

$$\begin{aligned}
\|f\|_{L^2_{\mathcal{Y}_{1,2}}(L^1(\mathbb{R}))} &= \sup\{(\int_\Omega \|f_w * g\|^2_{L^2(\mathbb{R})} d\mu)^{\frac{1}{2}} \ : \ \|g\|_{L^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\Omega \|\widehat{f_w * g}\|^2_{L^2(\mathbb{R})} d\mu)^{\frac{1}{2}} \ : \ \|g\|_{L^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\Omega \int_\mathbb{R} |\widehat{f_w}(x)\widehat{g}(x)|^2 dx d\mu)^{\frac{1}{2}} \ : \ \|\widehat{g}\|_{L^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\mathbb{R} (\int_\Omega |\widehat{f_w}(x)|^2 d\mu)|\widehat{g}(x)|^2 dx)^{\frac{1}{2}} \ : \ \|\widehat{g}\|_{L^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\mathbb{R} (\int_\Omega |\widehat{f_w}(x)|^2 d\mu) \ |h(x)| dx)^{\frac{1}{2}} \ : \ \|h\|_{L^1(\mathbb{R})} = 1\} \\
&= \sup_{x \in \mathbb{R}} (\int_\Omega |\hat{f}_w(x)|^2 d\mu)^{\frac{1}{2}}
\end{aligned}$$

□

We now show two interesting examples given in terms of the Poisson kernel.

Let $P_r(\theta) = \frac{1-r^2}{|1-r\,e^{i\theta}|^2}$ for $0 \le r < 1$ denote the Poisson kernel in $\mathbb{D}$. Due to the facts $\|P_r\|_1 = 1$ for all $0 < r < 1$ and $\|P_r\|_\infty = \frac{1+r}{1-r}$ one gets, for $1 < p < \infty$, the estimate $\|P_r\|_p \le (\frac{1+r}{1-r})^{-\frac{1}{p'}}$.

Definition 31 *Let $1 \le p \le \infty$ and write $\mathcal{P}_p : [0,1) \to L^p(\mathbb{T})$ for the function $\mathcal{P}_p(r) = P_r$*

Clearly $\mathcal{P}_p$ is continuous on $[0,1)$ but unbounded for $1 < p \le \infty$. Actually it is well known that

$$C_1(1-r)^{-\frac{1}{p'}} \le \|\mathcal{P}_p(r)\|_p \le C_2(1-r)^{-\frac{1}{p'}}.$$

This shows that $\mathcal{P}_p \notin L^{p'}([0,1), L^p(\mathbb{T}))$. Nevertheless we can define some bilinear maps $\mathcal{B}$ such that $\mathcal{P}_p$ belongs to $L^p_\mathcal{B}([0,1), L^p(\mathbb{T}))$.

For such a purpose we will apply two important inequalities from the theory of Hardy spaces. We refer the reader to [7] for the non explained notation.

Proposition 32 *Let $1 \le p < \infty$ and let $\mathcal{H} : L^p(\mathbb{T}) \times H^{p'}(\mathbb{D}) \to \mathbb{C}$ be defined* *by*

$$\mathcal{H}(\phi, f) = \int_{-\pi}^{\pi} \phi(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

Then $\mathcal{P}_p \in \mathcal{L}^{p'}_{\mathcal{H}}([0,1), L^p(\mathbb{T}))$.

PROOF. Observe first that $\mathcal{H}$ is a bounded bilinear map but $(H^{p'}(\mathbb{D}), \mathbb{C}, \mathcal{H})$ is not an admissible triple.

Since $P_r * f(\theta) = f(r\,e^{i\theta})$ for any $f \in H^p(\mathbb{D})$ then $f(r) = P_r * f(0)$ and hence $\mathcal{H}(\mathcal{P}_p(r), f) = f(r)$ for $f \in H^{p'}(\mathbb{D})$. Then, applying Fèjer-Riesz's inequality (see [7] page 46)

$$\Big(\int_0^1 |\mathcal{H}(\mathcal{P}_p(r), f)|^{p'} dr\Big)^{\frac{1}{p'}} = \Big(\int_0^1 |f(r)|^{p'} dr\Big)^{\frac{1}{p'}} \leq C\|f\|_{p'}.$$

Hence $\|\mathcal{P}_p\|_{\mathcal{L}^{p'}_{\mathcal{H}}([0,1), L^p(\mathbb{T}))} \leq C$. □

Proposition 33 *Let* $1 \leq p_1 < p_2 < \infty$ *and take* p *such that* $\frac{1}{p'} = \frac{1}{p_1} - \frac{1}{p_2}$. *Let us define* $\mathcal{C} : L^p(\mathbb{T}) \times H^{p_1}(\mathbb{D}) \to H^{p_2}(\mathbb{D})$ *by*

$$\mathcal{C}(\phi, f) = \phi * f.$$

Then $\mathcal{P}_p \in \mathcal{L}^{p'}_{\mathcal{C}}(d\mu_{p_1,p_2,p'}, L^p(\mathbb{T}))$ *with the measure* $d\mu_{p_1,p_2,p'}(r) = (1-r)^{p'(\frac{1}{p_1}-\frac{1}{p_2})-1} dr$.

PROOF. Observe that Young's inequality implies that $\mathcal{C}$ is a bounded bilinear map because $\frac{1}{p_2} = \frac{1}{p} + \frac{1}{p_1} - 1$, but $(H^{p_1}(\mathbb{D}), H^{p_2}(\mathbb{D}), \mathcal{C})$ is not an admissible triple.

If $f \in H^{p_1}(\mathbb{D})$ then we have $\mathcal{C}(\mathcal{P}_p(r), f) = f_r$ where $f_r(e^{i\theta}) = f(r\,e^{i\theta})$.

Recall that Hardy-Littlewood's inequality (see [7] page 87) establishes that for $1 \leq p < q < \infty$ and $\lambda \geq p$, there exists a constant $C > 0$ such that

$$\Big(\int_0^1 (1-r)^{\lambda(\frac{1}{p}-\frac{1}{q})-1} M_q(f,r)^\lambda dr\Big)^{\frac{1}{\lambda}} \leq C\|f\|_{H^p(\mathbb{D})} \text{ for all } f \in H^p(\mathbb{D})$$

where $M_p(f,r) = \|f_r\|_p$.

Therefore, applying the previous inequality for $\lambda = p'$, $p = p_1$ and $q = p_2$, one gets

$$\Big(\int_0^1 \|\mathcal{C}(\mathcal{P}_p(r),f)\|^{p'}_{H^{p_2}(\mathbb{D})} d\mu_{p_1,p_2,p'}(r)\Big)^{\frac{1}{p'}} = \Big(\int_0^1 \|f_r\|^{p'}_{H^{p_2}(\mathbb{D})} d\mu_{p_1,p_2,p'}(r)\Big)^{\frac{1}{p'}}$$
$$= \Big(\int_0^1 M^{p'}_{p_2}(f,r)(1-r)^{p'(\frac{1}{p_1}-\frac{1}{p_2})-1} dr\Big)^{\frac{1}{p'}}$$
$$\leq C\|f\|_{H^{p_1}(\mathbb{D})}.$$

Therefore $\|\mathcal{P}_p\|_{\mathcal{L}^{p'}_{\mathcal{C}}(d\mu_{p_1,p_2,p'},L^p(\mathbb{T}))} \leq C$. □

5 Integral operators by means of bilinear maps.

Throughout this section $(\Omega,\Sigma,d\mu(w))$ and $(\Omega',\Sigma',d\mu'(w'))$ are finite complete measure spaces, X is a Banach space and $k:\Omega\times\Omega'\to X$ belong to $L^0_{\mathcal{B}}(\Omega\times\Omega',X)$ for some $(Y,Z,\mathcal{B})$ is an admissible triple for X. Our objective is to study the boundedness of the integral operator associated to $\mathcal{B}$ given by

$$\begin{array}{rccl} T^{\mathcal{B}}_k\colon & L^p(\Omega',Y) & \to & L^p(\Omega,Z) \\ & g & \mapsto & T^{\mathcal{B}}_k(g)(\omega) = \int_{\Omega'} \mathcal{B}(k(w,w'),g(w'))d\mu'(w') \end{array}$$

As usual, denote by

$$\begin{array}{rccl} k_w = k(w,\cdot)\colon & \Omega' & \to & X \\ & w' & \mapsto & k(w,w') \end{array} \qquad \begin{array}{rccl} k^{w'} = k(\cdot,w')\colon & \Omega & \to & X \\ & w & \mapsto & k(w,w') \end{array}.$$

We also write $\mathcal{K}(w) = k_w$ and $\mathcal{K}'(w') = k^{w'}$.

We now introduce similar conditions to the ones appearing in [9] in our more general setting.

Definition 34 *We say that $k:\Omega\times\Omega'\to X$ satisfies the condition $(\mathrm{C}^{\mathcal{B}}_0)$ if*

(1) $k_\omega \in L^1_{\mathcal{B}}(\Omega',X)$ a.e. in Ω, and
(2) for each $y\in Y$ and $E\in\Sigma'$ the function

$$\begin{array}{rccl} T^{\mathcal{B}}_k(y,E)\colon & \Omega & \to & Z \\ & \omega & \mapsto & \int_E \mathcal{B}(k(\omega,\omega'),y)d\mu'(w') \end{array}$$

belongs to $L^0(\Omega,Z)$.

Remark 35 *If the kernel k satisfies $(\mathrm{C}_0^{\mathcal{B}})$ then the operator*

$$\begin{array}{rcl} T_k^{\mathcal{B}}\colon\ \mathcal{S}(\Omega', Y) & \to & L^0(\Omega, Z) \\ g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int\limits_{\Omega'} \mathcal{B}(k(\omega,\omega'), g(\omega'))d\mu'(w') \end{array}$$

is well defined.

Remark 36 *If $\mathcal{K} \in L^0(\Omega, L^1_{\mathcal{B}}(\Omega', X))$ then k satisfies $(\mathrm{C}_0^{\mathcal{B}})$.*

Definition 37 *We say that $k : \Omega \times \Omega' \to X$ satisfies the condition $(\mathrm{C}_1^{\mathcal{B}})$ if*

(1) $k^{\omega'} \in L^1_{\mathcal{B}}(\Omega, X)$ a.e. in Ω',
(2) there exists a constant $\mathrm{C}_1^{\mathcal{B}} > 0$ such that

$$\mu'(\{\omega' \in \Omega' : \|k^{\omega'}\|_{L^1_{\mathcal{B}}(\Omega,X)} > \mathrm{C}_1^{\mathcal{B}}\} = 0.$$

Remark 38 *If $\mathcal{K}' \in L^\infty(\Omega', L^1_{\mathcal{B}}(\Omega, X))$ then k satisfies $(\mathrm{C}_1^{\mathcal{B}})$ with*

$$\mathrm{C}_1^{\mathcal{B}} \leq \|\mathcal{K}'\|_{L^\infty(\Omega', L^1_{\mathcal{B}}(\Omega,X))}.$$

Proposition 39 *Let $\mathcal{B} : X \times Y \to Z$ bounded bilinear map and let $k : \Omega \times \Omega' \to X$ a kernel satisfying $(\mathrm{C}_0^{\mathcal{B}})$. If k satisties $(\mathrm{C}_1^{\mathcal{B}})$ then the integral operator*

$$\begin{array}{rcl} T_k^{\mathcal{B}}\colon\ \mathcal{S}(\Omega', Y) & \to & L^1(\Omega, Z) \\ g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int\limits_{\Omega'} \mathcal{B}(k(\omega,\omega'), g(\omega'))d\mu'(w') \end{array}$$

can be continuously extended to $L^1(\Omega', Y)$ and with norm bounded by $\mathrm{C}_1^{\mathcal{B}}$.

PROOF. Let $g = \sum_{k=1}^n y_k \mathbf{1}_{E_k}$. Then

$$T_k^{\mathcal{B}}(g)(w) = \sum_{k=1}^n \int\limits_{E_k} \mathcal{B}(k(\omega,\omega'), y_k)d\mu'(w')$$

Therefore

$$\begin{aligned} \int\limits_{\Omega} \|T_k^{\mathcal{B}}(g)(w)\|d\mu(w) &\leq \int\limits_{\Omega} \sum_{k=1}^n \int\limits_{E_k} \|\mathcal{B}(k(\omega,\omega'), y_k)\|d\mu'(w')d\mu(w) \\ &= \sum_{k=1}^n \|y_k\| \int\limits_{E_k} (\int\limits_{\Omega} \|\mathcal{B}(k(\omega,\omega'), \frac{y_k}{\|y_k\|})\|d\mu(w))d\mu'(w') \end{aligned}$$

$$\leq \sum_{k=1}^{n} \|y_k\| \int_{E_k} \|k^{w'}\|_{L^1_{\mathcal{B}}(\Omega,X)} d\mu'(w')$$

$$\leq \mathrm{C}_1^{\mathcal{B}} \sum_{k=1}^{n} \|y_k\| \mu'(E_k)$$

Now extend by the density of the simple functions on $L^1(\Omega', Y)$. □

We can get similar sufficient conditions for the boundedness on vector-valued L^p-spaces for $p > 1$.

Definition 40 *Let $1 < p < \infty$. We say that $k : \Omega \times \Omega' \to X$ satisfies $(\mathrm{C}_\mathrm{p}^{\mathcal{B}})$ if*

(1) $k^{\omega'} \in L^p_{\mathcal{B}}(\Omega, X)$ a.e. in Ω',
(2) $w' \to \|k^{w'}\|_{L^p_{\mathcal{B}}(\Omega,X)}$ belongs to $L^{p'}(\Omega')$.

Remark 41 *If $\mathcal{K}' \in L^{p'}(\Omega', L^p_{\mathcal{B}}(\Omega, X))$ then k satisfies $(\mathrm{C}_\mathrm{p}^{\mathcal{B}})$.*

Proposition 42 *Let $1 < p < \infty$ and $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. If $k : \Omega \times \Omega' \to X$ is a kernel satisfying $(\mathrm{C}_0^{\mathcal{B}})$ and $(\mathrm{C}_\mathrm{p}^{\mathcal{B}})$ then the integral operator*

$$\begin{array}{rccl} T_k^{\mathcal{B}}: & \mathcal{S}(\Omega', Y) & \to & L^p(\Omega, Z) \\ & g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int_{\Omega'} \mathcal{B}(k(\omega, \omega'), g(\omega'))d\mu'(w') \end{array}$$

can be continuously extended to $L^p(\Omega', Y)$.

PROOF. Let $g = \sum_{k=1}^n y_k \mathbf{1}_{E_k}$ and $T_k^{\mathcal{B}}(g)(w) = \int_{\Omega'} \mathcal{B}(k(\omega, \omega'), g(w'))d\mu'(w')$. Using Minkowski's inequality one gets Therefore

$$(\int_{\Omega} \|T_k^{\mathcal{B}}(g)(w)\|^p d\mu(w))^{\frac{1}{p}} \leq \int_{\Omega'} (\int_{\Omega} \|\mathcal{B}(k(\omega, \omega'), g(w'))\|^p d\mu(w))^{\frac{1}{p}} d\mu'(w')$$

$$\leq \int_{\Omega'} \|k^{\omega'}\|_{L^p_{\mathcal{B}}(\Omega,X)} \|g(w')\| d\mu'(w')$$

$$\leq (\int_{\Omega'} \|k_{w'}\|^{p'}_{L^p_{\mathcal{B}}(\Omega,X)} d\mu(w'))^{\frac{1}{p'}} \|g\|_{L^p(\Omega',Y)}$$

Now extend by the density of the simple functions on $L^p(\Omega', Y)$. □

Recall that $\mathcal{B}^*$ denotes the adjoint $\mathcal{B}^* : X \times Z^* \to Y^*$ given by $\langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x, y), z^*\rangle$. We write $\widetilde{k} : \Omega' \times \Omega \to X$ for the map $\widetilde{k}(w', w) = k(w, w')$.

Proposition 43 *Let* $\mathcal{B} : X \times Y \to Z$ *bounded bilinear map. If* $k : \Omega \times \Omega' \to X$ *satisfies* $(\mathrm{C}_0^{\mathcal{B}})$ *and* $\widetilde{k}$ *satisfies* $(\mathrm{C}_1^{\mathcal{B}^*})$ *then the integral operator*

$$\begin{array}{rccl} T_k^{\mathcal{B}}: & \mathcal{S}(\Omega', Y) & \to & L^\infty(\Omega, Z) \\ & g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int\limits_{\Omega'} \mathcal{B}(k(\omega, \omega'), g(\omega'))d\mu'(w') \end{array}$$

can be continuously extended to $\overline{\mathcal{S}(\Omega', Y)}^{L^\infty(\Omega', Y)}$ *with norm bounded by* $\mathrm{C}_1^{\mathcal{B}^*}$.

PROOF. Take $g \in \mathcal{S}(\Omega', Y)$. The condition $(\mathrm{C}^0_{\mathcal{B}})$ provides the measurability of the function $T_k^{\mathcal{B}}(g) : \Omega \to Z$. Then, for those $w \in \Omega$ for which $k_w \in L^1_{\mathcal{B}}(\Omega', X)$, we have that

$$\begin{aligned} \|T_k^{\mathcal{B}}(g)(w)\| &= \sup\{|\int\limits_{\Omega'} \langle \mathcal{B}(k(w, w'), g(w')), z^*\rangle d\mu'(w')| : \|z^*\| = 1\} \\ &= \sup\{|\int\limits_{\Omega'} \langle g(w'), \mathcal{B}^*(k(w, w'), z^*)\rangle d\mu'(w')| : \|z^*\| = 1\} \\ &\leq \|g\|_{L^\infty(\Omega', Y)} \|k_w\|_{L^1_{\mathcal{B}^*}(\Omega', X)}. \end{aligned}$$

Hence $\|T_k^{\mathcal{B}}(g)\|_{L^\infty(\Omega', Y)} \leq \mathrm{C}_1^{\mathcal{B}^*} \|g\|_{L^\infty(\Omega', Y)}$. □

The boundedness of the operator in the case $1 < p < \infty$ can also be deduced now of the previous propositions by means of interpolation.

Lemma 44 *(see [9], page 198).Let* $1 < p < \infty$ *and let* $T : \mathcal{S}(\Omega', Y) \to L^1(\Omega, Z) + L^\infty(\Omega, Z)$ *be a linear map and there exist* $c_1, c_2 > 0$ *such that*

$$\|T(g)\|_{L^1(\Omega', Y)} \leq c_1 \|g\|_{L^1(\Omega', Y)} \qquad \text{and} \qquad \|T(g)\|_{L^\infty(\Omega', Y)} \leq c_\infty \|g\|_{L^\infty(\Omega', Y)}$$

for all $g \in \mathcal{S}(\Omega', Y)$. *Then there exists a linear extension* $T : L^p(\Omega', Y) \to L^p(\Omega, Z)$ *with norm bounded by* $c_1^{\frac{1}{p}} c_\infty^{\frac{1}{p'}}$.

Theorem 45 *Let* $1 < p < \infty$, *let* $\mathcal{B} : X \times Y \to Z$ *be a bounded bilinear map. If* $k : \Omega \times \Omega' \to X$ *is a kernel satisfying* $(\mathrm{C}_0^{\mathcal{B}})$, k *satisfies* $(\mathrm{C}_1^{\mathcal{B}})$ *and* $\widetilde{k}$ *satisfies* $(\mathrm{C}_1^{\mathcal{B}^*})$ *then the integral operator*

$$\begin{array}{rccl} T_k^{\mathcal{B}}: & \mathcal{S}(\Omega', Y) & \to & L^\infty(\Omega, Z) \\ & g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int\limits_{\Omega'} \mathcal{B}(k(\omega, \omega'), g(\omega'))d\mu'(w') \end{array}$$

can be continuously extended to $T_k^{\mathcal{B}} : L^p(\Omega', Y) \to L^p(\Omega, Z)$ *with norm bounded by* $(\mathrm{C}_1^{\mathcal{B}})^{\frac{1}{\mathrm{p}}} (\mathrm{C}_1^{\mathcal{B}^*})^{\frac{1}{\mathrm{p}'}}$.

We finish this section mentioning some results about the extension of the operator to $L^\infty(Y))$ whose proofs can be obtained from the obvious modifications in the operator-valued case (see [9]).

Theorem 46 *Let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. If $k : \Omega\times\Omega' \to X$ satisfies $(\mathrm{C}_0^{\mathcal{B}})$ and $\widetilde{k}$ satisfies $(\mathrm{C}_1^{\mathcal{B}^*})$ then the integral operator*

$$\begin{array}{llll} T_k^{\mathcal{B}}: & \mathcal{S}(\Omega', Y) & \to & L^\infty(\Omega, Z) \\ & g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int\limits_{\Omega'} \mathcal{B}(k(\omega,\omega'), g(\omega'))d\mu'(w') \end{array}$$

can be continuously extended to $S_k^{\mathcal{B}} : L^\infty(\Omega', Y) \to L^\infty_{\mathrm{weak}*}(\Omega, Z^{**})$ given by

$$\langle z^*, S_k^{\mathcal{B}}(g)(w)\rangle = \int\limits_{\Omega'} \langle \mathcal{B}(k(w,w'), g(w')), z^*\rangle d\mu'(w')$$

for each $z^* \in Z^*$, $w \in \Omega$ and $g \in L^\infty(\Omega', Y)$ with norm bounded by $\mathrm{C}_1^{\mathcal{B}^*}$.

Theorem 47 *Let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. Assume that $k : \Omega \times \Omega' \to X$ satisfies $(\mathrm{C}_0^{\mathcal{B}})$ and $\widetilde{k}$ satisfies $(\mathrm{C}_1^{\mathcal{B}^*})$ and that Z does not contain a copy of c_0. Then $T_k^{\mathcal{B}}$ has a continuous extension to $T_k^{\mathcal{B}} : L^\infty(\Omega', Y) \to L^\infty(\Omega, Z)$.*

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