Research & Exposition in Mathematics
24 (2000) 157–167

Bilinear maps and convolutions

Oscar Blasco*

Abstract. Let X, Y, Z be Banach spaces and let $u : X \times Y \to Z$ be a bounded bilinear map. Given a locally compact abelian group G, and two functions $f \in L^1(G, X)$ and $g \in L^1(G, Y)$, we define the u-convolution of f and g as the Z-valued function $f *_u g(t) = \int_G u(f(t-s), g(s)) d\mu_G(s)$ where $d\mu_G$ stands for the Haar measure on G.

We define the concepts of vector-valued approximate identity and summability kernel associated to a bounded bilinear map, showing the corresponding approximation result in this setting. A Haussdorf-Young type result associated to a bounded bilinear map is also presented under certain assumptions on the Banach space X.

1. Introduction

In this paper we shall be dealing with a notion which allows to understand the convolution of functions taking values in different Banach spaces. This is done by means of bilinear maps and it has been already considered in [1, 5] for the particular groups $\mathbb{T}$ and $\mathbb{R}$.

In this paper our main objetives will be to get some formulations of vector-valued approximate identities and summability kernels in this context and the study of Haussdorff-Young's theorem on the Fourier transform for this vector-valued and bilinear setting.

In what follows G will be an abelian and locally compact group, with addition as group operation, usually abreviated by LCA. We denote the Haar measure of the group, by μ_G, that is the unique (in abelian compact groups) or unique up to a positive constant factor (in LCA groups) probability Borel measure, which is invariant under traslations, and we use the notation of $L^p(G, X)$ ($1 \le p < \infty$), for the p-integrable Bochner functions, that is X-valued measurable functions for which $\|f\|_{L^p(G,X)} = (\int_G \|f(t)\|_X^p d\mu_G(t))^{1/p} < \infty$. The reader is referred to [10] and to [23] or [14] for an introduction to vector-valued Bochner integral and to harmonic analysis on groups respectively.

If G is a LCA group G, X, Y, Z are Banach spaces and $u : X \times Y \to Z$ is a bounded bilinear map we shall define the u-convolution by

$$f *_u g(t) = \int_G u(f(t-s), g(s)) d\mu_G(s) \in L^1(\mathbb{T}, Z)$$

for any pair of functions $f \in L^1(G, X)$ and $g \in L^1(G, Y)$.

*Supported by grant DGESIC PB98-1426

Let us mention some cases where this notion has been already used in the literature.

(1) Let $X = Y$ be a Banach algebra A and u the product on A $u(a, b) = ab$. For $f, g \in L^1(G, A)$ the convolution is then defined by

$$f * g(t) = \int_G f(t-s))g(s)d\mu_G(s) \in L^1(G, A).$$

(2) Let X be a Banach space (say over $\mathbb{R}$) and $u : X \times \mathbb{R} \to X$ given by $u(x, \lambda) = \lambda x$. Then if $f \in L^1(G, X)$ and $g \in L^1(G)$

$$f * g(t) = \int_G g(s)f(t-s)d\mu_G(s) \in L^1(G, X).$$

(3) Let X be a Banach space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $u : X \times X^* \to \mathbb{K}$ given by the duality pair $u(x, x^*) =< x, x^* >$. For $f \in L^1(G, X)$ and $g \in L^1(G, X^*)$ the convolution gives

$$f * g(t) = \int_G < f(t-s), g(s) > d\mu_G(s) \in L^1(G).$$

(4) Let X_1, X_2 be a couple of Banach spaces, put $X = X_1$, the space of linear continuous operators $L(X_1, X_2) = Y$ and the bilinear map $u : X_1 \times L(X_1, X_2) \to X_2$ given by $u(x, T) = T(x)$. Then for $f \in L^1(\mathbb{T}, X_1)$ and $g \in L^1(\mathbb{T}, L(X_1, X_2))$ the convolution means

$$f * g(t) = \int_G g(s)(f(t-s))d\mu_G(s) \in L^1(\mathbb{T}, X_2).$$

The reader is referred to [1] for the use of such a convolution in partial differential equations, to [17] for its connection with absolutely summing operators, and to [5, 2, 3] for different applications, mainly in the cases $G = \mathbb{T}$ and $G = \mathbb{R}$, when using the following particular bilinear maps :

(1) For $X = L^{p_1}(\mu), Y = L^{p_2}(\mu)$ and $Z = L^p(\mu)$ where μ is any measure and $1 \le p_1, p_2 \le \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, Hölder's inequality provides the bilinear map $u(f, g) = f.g$.

(2) For $X = L^{p_1}(\mathbb{R}^n), Y = L^{p_2}(\mathbb{R}^n)$ and $Z = L^p(\mathbb{R}^n)$ where $1 \le p_1, p_2 \le \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} \ge 1$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1$, Young's theorem provides the bilinear map via convolution $u(f, g) = f * g$

(3) Given three Banach spaces X_1, X_2, X_3, the composition of operators $u(T, S) = ST$ gives the bilinear map where $X = L(X_1, X_2), Y = L(X_2, X_3)$ and $Z = L(X_1, X_3)$. This example provides also nice aplications when considering operator ideals, such as p-absolutely summing operators or Schatten classes, instead that just the spaces of bounded operators.

(4) The bilinear map $u(x, y) = x \otimes y$ from $u : X \times Y \to X\hat{\otimes}Y$, where $X\hat{\otimes}Y$ the projective tensor product, also provides interesting applications.

The paper is divided into three sections. In the first section we define the convolution and prove some elementary properties of it. In particular we analyze the behaviour with respect to Fourier transform. In section 2 we introduce the notion of bounded approximate identity and summability kernel with respect a bilinear map $u : X \times Y \to X$ and prove the corresponding approximation result. Finally in Section 3 we consider the notion of Fourier type with respect to a group and prove a Hausdorff-Young type result about Fourier transform under the assumptions of Fourier type on X.

2. Preliminaries

Let us start off by mentioning that if $f \in L^1(G,X)$ and $g \in L^1(G,Y)$ then $\int_G \|f(t-s)\|_X \|g(s)\|_Y d\mu_G(s) < \infty$ for almost all $t \in G$. This allows us to give that following definition.

Definition 2.1. Let G be LCA group , X, Y, Z be Banach spaces and $u : X \times Y \to Z$ be a bounded bilinear map. For each $f \in L^1(G,X)$ and $g \in L^1(G,Y)$ we define for almost all $s \in G$

$$f *_u g(t) = \int_G u(f(t-s), g(s)) d\mu_G(s).$$

Remark 2.2. From Fubini's theorem we actually get $f *_u g \in L^1(G,Z)$ and

$$\|f *_u g\|_{L^1(G,Z)} \leq \|u\| \|f\|_{L^1(G,X)} \|g\|_{L^1(G,Y)}.$$

It is rather easy to extend several properties on the classical convolution to this general setting due to the following observations.

Remark 2.3. Let G be LCA group, X be a Banach space and $1 \leq p < \infty$. If f_s denotes the function $f_s(t) = f(t-s)$ then the same proof as in the scalar-valued case (see [23]) shows that the map $s \to f_s$ is uniformly continuous from G to $L^p(G,X)$.

Remark 2.4. Let G be LCA group , X, Y, Z be Banach spaces and $u : X \times Y \to Z$ be a bounded bilinear map. Then

$$\|f *_u g(t)\|_Z \leq \|u\| \int_G \|f(t-s)\|_X \|g(s)\|_Y d\mu_G(s)$$

for all $f \in L^1(G,X)$ and $g \in L^1(G,Y)$.

In particular we can state the following result, whose proof follows from the previous remarks and the analogous scalar-valued formulation.

Theorem 2.5. *(see [23] or [14]) Let G be LCA group , X, Y, Z be Banach spaces and $u : X \times Y \to Z$ be a bounded bilinear map.*

(1) If $1 < p < \infty$, $1/p + 1/q = 1$, $f \in L^p(G,X)$ and $g \in L^q(G,Y)$ then $f *_u g \in C_0(G,Z)$. Moreover

$$\|f *_u g\|_{L^\infty(G,Z)} \leq \|u\| \|f\|_{L^p(G,X)} \|g\|_{L^q(G,Y)}.$$

(2) If $1 \leq p < \infty$, $f \in L^1(G,X)$ and $g \in L^p(G,Y)$ then $f *_u g \in L^p(G,Z)$. Moreover

$$\|f *_u g\|_{L^p(G,Z)} \leq \|u\| \|f\|_{L^1(G,X)} \|g\|_{L^p(G,Y)}.$$

(3) If $1 \leq p, q \leq \infty$, $1/p + 1/q \geq 1$, $f \in L^p(G, X)$ *and* $g \in L^q(G, Y)$ *then* $f *_u g \in L^r(G, Z)$ *where* $1/r = 1/p + 1/q - 1$. *Moreover*

$$\|f *_u g\|_{L^r(G,Z)} \leq \|u\| \|f\|_{L^p(G,X)} \|g\|_{L^q(G,Y)}.$$

Let us denote by $\Gamma = \hat{G}$ the dual group and write γ for the characters in Γ. The Fourier transform of functions in $f \in L^1(G)$ is defined by the formula

$$\hat{f}(\gamma) = \int_G f(t)\overline{\gamma(t)} d\mu_G(t), \qquad \gamma \in \Gamma.$$

In the vector valued situation we can still define the Fourier transform for $f \in L^1(G, X)$ as in the previous formula, but now $\hat{f}(\gamma)$ means the Bochner integral and belongs to X.

Next result establishes that the vector-valued formulation behaves perfectly with respect to bilinear maps.

Theorem 2.6. *Let* G *be LCA group ,* X, Y, Z *be Banach spaces and* $u : X \times Y \to Z$ *be a bounded bilinear map.*

If $f \in L^1(G, X)$ *and* $g \in L^1(G, Y)$ *then*

$$\widehat{f *_u g}(\gamma) = u(\hat{f}(\gamma), \hat{g}(\gamma)).$$

Proof.

$$\begin{aligned}
\widehat{f *_u g}(\gamma) &= \int_G f *_u g(t)\overline{\gamma(t)} d\mu_G(t) \\
&= \int_G \Big(\int_G u(f(t-s), g(s)) d\mu_G(s) \Big) \overline{\gamma(t)} d\mu_G(t) \\
&= \int_G \Big(\int_G u(f(t-s)\overline{\gamma(t-s)}, g(s)\overline{\gamma(s)}) d\mu_G(s) \Big) d\mu_G(t) \\
&= \int_G \Big(\int_G u(f(t-s)\overline{\gamma(t-s)}, g(s)\overline{\gamma(s)}) d\mu_G(t) \Big) d\mu_G(s) \\
&= \int_G u(\int_G f(t-s)\overline{\gamma(t-s)} d\mu_G(t), g(s)\overline{\gamma(s)}) d\mu_G(s) \\
&= \int_G u(\hat{f}(\gamma)), g(s)\overline{\gamma(s)}) d\mu_G(s) \\
&= u(\hat{f}(\gamma), \int_G g(s)\overline{\gamma(s)} d\mu_G(s)) \\
&= u(\hat{f}(\gamma), \hat{g}(\gamma)).
\end{aligned}$$

■

To get an interesting applicaton, let us first recall the so-called Marcinkiewizc-Zygmund theorem (see [12]):

Let $1 \leq p_i < \infty$ for $i = 1, 2$, and let $(\Omega_i, \Sigma_i, \mu_i)$ measure spaces for $i = 1, 2$. If $T : L^{p_1}(\mu_1) \to L^{p_2}(\mu_2)$ is a bounded linear map then there exists $C > 0$ such that

$$\|(\sum_{j=1}^n |T(\phi_j)|^2)^{1/2}\|_{L^{p_2}(\mu_2)} \leq C\|u\| \|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{L^{p_1}(\mu_1)}$$

for any $n \in \mathbb{N}$ and any $\phi_1, ..., \phi_n \in L^{p_1}(\mu_1)$.

We now have the following analogue.

Corollary 2.7. *(The Bilinear Marcinkiewicz-Zygmund Theorem) Let* $1 \leq p_i < \infty$ *for* $i = 1,2,3$*, and let* $(\Omega_i, \Sigma_i, \mu_i)$ *measure spaces for* $i = 1,2,3$*. If* $u : L^{p_1}(\mu_1) \times L^{p_2}(\mu_2) \to L^{p_3}(\mu_3)$ *is a bounded bilinear map then there exists* $C > 0$ *such that*

$$\|(\sum_{j=1}^n |u(\phi_j, \psi_j)|^2)^{1/2}\|_{L^{p_3}} \leq C\|u\| \cdot \|(\sum_{j=1}^n |\phi_j|^2)^{1/2}\|_{L^{p_1}} \|(\sum_{j=1}^n |\psi_j|^2)^{1/2}\|_{L^{p_2}}$$

for any $n \in \mathbb{N}$*,* $\phi_1, ..., \phi_n \in L^{p_1}(\mu_1)$ *and* $\psi_1, ..., \psi_n \in L^{p_2}(\mu_2)$*.*

Proof. Let us define

$$f_1(t) = \sum_{j=1}^n \phi_j e^{i2^j t} \in L^1(\mathbb{T}, L^{p_1}(\mu_1)),$$

$$f_2(t) = \sum_{j=1}^n \psi_j e^{i2^j t} \in L^1(\mathbb{T}, L^{p_2}(\mu_2)).$$

Using Theorem 2.6 for $G = \mathbb{T}$ we get

$$f_1 *_u f_2(t) = \sum_{j=1}^n (\phi_j * \psi_j) e^{i2^j t}.$$

To complete the proof, let us recall that Kintchine's inequalites (see [24]) easily imply that for any $0 < p < \infty$, and any measure space (Ω, Σ, μ) one has

$$\|(\sum_{j=1}^n |h_j|^2)^{1/2}\|_{L^p(\mu)} \approx \|\sum_{j=1}^n h_j e^{i2^j t}\|_{L^1(\mathbb{T}, L^p(\mu))}$$

for any $n \in \mathbb{N}$ and any $h_1, ..., h_n \in L^p(\mu)$. Using this and (2) in Theorem 2.5 the result is achieved. ■

3. u-bounded approximate identities

Let us recall that a right bounded approximate identity (r.b.a.i) in a Banach algebra A is a directed net a_α satisfying that there exists $C > 0$ such that $\sup_\alpha \|a_\alpha\| \leq C$ and $aa_\alpha \to a$ for all $a \in A$. This is a replacement for different purposes of a right identity.

Inspired by these two notions we define the following concepts.

Definition 3.1. Let X, Y be Banach spaces and $u : X \times Y \to X$ be a bounded bilinear map . We say that $y_0 \in Y$ is a (u, X)-identity if $u(x, y_0) = x$ for all $x \in X$.

Example 3.2. (1) If A is a commutative Banach algebra with identity, then the identity is (u, J)-identity for any ideal J and $u : J \times A \to J$ given by $u(a, b) = a.b$. This applies to the cases of $A = M(G)$ and $J = L^1(G)$ or $A = C(K)$ for a compact space K and $J = \{f \in C(K) : f(t_0) = 0\}$.

(2) If X is a Banach space, $Y = L(X, X)$ and $u : X \times L(X, X) \to X$ is given by $u(x, T) = T(x)$ then id_X is a (u, X)-identity.

Definition 3.3. Let X, Y be Banach spaces and $u : X \times Y \to X$ be a bounded bilinear map. We say that a directed family $\{y_\alpha\} \in Y$ is a (u, X)-bounded approximate identity if there exists $C > 0$ such that $\sup_\alpha \|y_\alpha\|_Y \leq C$ and $u(x, y_\alpha) \to x$ for all $x \in X$.

Example 3.4. (1) Let A be Banach algebra and let X be a right A-module , i.e X is a Banach space for which there exists a bilinear bounded map $(x, a) \to x.a$ from $X \times A$ to X such that $(ab).x = a.(b.x)$ for all $x \in X$ and $a, b \in A$ (where ab denotes the product in the algebra).

The essential part of X, usually denoted by X_e, corresponds to the A-module generated by $XA = \{xa : x \in X, a \in A\}$. It is easy to see that if (a_j) is a right bounded approximate identity of A then (a_j) is also a $(\cdot, X_e)$-bounded approximate identity.

The reader is referred to [11] for connections with factorization theory.

(2) Let X be Banach space with a Schauder basis, say (x_j). Let $Y = L(X, X)$ and $u : X \times L(X, X) \to X$ given by $u(x, T) = T(x)$. Then the canonical projections $P_n(x) = \sum_{j=1}^n < x_j^*, x > x_j$ define an (u, X)-bounded approximate identity.

(3) Let $X = L^p([0, 1])$, $Y = L(X, X)$ and $u : X \times L(X, X) \to X$ given by $u(x, T) = T(x)$. If we consider the filtration of σ-algebras $\mathcal{F}_n$ generated by the dyadic intervals of length 2^{-n} we have that the conditional expectation operators $E_n(f) = E(f|\mathcal{F}_n)$ define an (u, X)-bounded approximate identity.

The reader is referred to [18] for more r.i. spaces X where same result holds.

A general procedure to get bounded approximate identities in the Banach algebra $L^1(G)$ is the use of the so-called summability kernels (see [15]). We shall define the corresponding notion adecuated to the bilinear formulation and the vector valued setting.

Definition 3.5. Let G be a LCA group, X and Y be Banach spaces and $u : X \times Y \to X$ be a bounded bilinear map .

We say that a directed family $\{K_\alpha\}$ of functions in $L^1(G, Y)$ is a (u, X)-summability kernel if

(1) $sup_\alpha \int_G \|K_\alpha(t)\|_Y d\mu_G(t) < \infty$,

(2) $\int_G K_\alpha(t) d\mu_G(t) = y_\alpha$ is a (u, X)-bounded approximate identity, and

(3) for any U neighborhood of 0

$$\lim_\alpha \int_{G \setminus U} \|K_\alpha(t)\|_Y d\mu_G(t) = 0.$$

This notion allows us to prove our next approximation result, which follows closely the classical approach.

Theorem 3.6. *Let G be a LCA group, X and Y be Banach spaces and $u : X \times Y \to X$ be a bounded bilinear map . If $\{K_\alpha\}$ is a (u, X)-summability kernel then*

(1) If $1 \leq p < \infty$ $\lim_\alpha \|f *_u K_\alpha - f\|_{L^p(G,X)} = 0$ for any $f \in L^p(G, X)$.

(2) $\lim_\alpha f *_u K_\alpha(t) = f(t)$ uniformly in G for any $f \in C_0(G, X)$.

Proof. Using (2) in Definition 3.5 one has

$$\begin{aligned} f *_u K_\alpha(t) - f(t) &= \int_G u(f(t-s), K_\alpha(s)) d\mu_G(s) - u(f(t), y_\alpha) \\ &+ u(f(t), y_\alpha) - f(t) \\ &= \int_G u(f(t-s) - f(t), K_\alpha(s)) d\mu_G(s) \\ &+ u(f(t), y_\alpha) - f(t). \end{aligned}$$

Therefore

$$\begin{aligned} \|f *_u K_\alpha(t) - f(t)\|_X &\leq \|u\| \int_G \|f(t-s) - f(t)\|_X \|K_\alpha(s)\|_Y d\mu_G(s) \\ &+ \|u(f(t), y_\alpha) - f(t)\|_X. \end{aligned}$$

To show (1) assume that $f \in L^p(G, X)$. Using the vector valued Minkowsky inequality we can write

$$\begin{aligned} \|f *_u K_\alpha - f\|_{L^p(G,X)} &\leq \|u\| \int_G \|f_s - f\|_{L^p(G,X)} \|K_\alpha(s)\|_Y d\mu_G(s) \\ &+ \|u(f, y_\alpha) - f\|_{L^p(G,X)}. \end{aligned}$$

Let $\varepsilon > 0$, Remark 2.3 gives that there exists a neighborhood of 0, say U, such that $\|f_s - f\|_{L^p(G,X)} < \varepsilon$ for $s \in U$.

Hence

$$\begin{aligned} \|f *_u K_\alpha - f\|_{L^p(G,X)} &\leq \varepsilon \|u\| \int_U \|K_\alpha(s)\|_Y d\mu_G(s) \\ &+ 2\|f\|_{L^p(G,X)} \int_{G \setminus U} \|K_\alpha(s)\|_Y d\mu_G(s) \\ &+ \|u(f, y_\alpha) - f\|_{L^p(G,X)}. \end{aligned}$$

Now an application of the property (3) in Definition 3.5 gives that

$$\lim_\alpha \int_{G \setminus U} \|K_\alpha(s)\|_Y d\mu_G(s) = 0$$

and the Lebesgue dominated convergence theorem, since $\lim_\alpha \|u(f(t), y_\alpha) - f(t)\|_X = 0$ and $\|u(f(t), y_\alpha) - f(t)\|_X \leq C\|f(t)\| \in L^p(G)$, implies

$$\lim_\alpha \|u(f, y_\alpha) - f\|_{L^p(G,X)} = 0.$$

Hence taking limits one gets

$$\lim_\alpha \|f *_u K_\alpha - f\|_{L^p(G,X)} \leq C\varepsilon$$

what gives (1).

To see (2) assume first that f has compact support, using then that f is uniformly continuous we can repeat the previous argument to get, for each $\varepsilon > 0$, a neigborhood of 0 so that

$$\begin{aligned} \|f *_u K_\alpha(t) - f(t)\|_X &\leq \varepsilon \|u\| \int_U \|K_\alpha(s)\|_Y d\mu_G(s) \\ &+ 2\|f\|_{L^\infty(G,X)} \int_{G\setminus U} \|K_\alpha(s)\|_Y d\mu_G(s) \\ &+ \|u(f(t), y_\alpha) - f(t)\|_X. \end{aligned}$$

Now observe that an ε-net argument easily gives that $\lim_\alpha \|u(f(t), y_\alpha) - f(t)\|_X = 0$ uniformly in $t \in supp(f)$, what together with property (3) in Definition 3.5 completes the proof for functions with compact support.

An elementary density argument takes care of the general case. ■

4. A Hausdorff-Young type Theorem

Let us recall that the Fourier transform can be defined for functions in $L^p(G)$ for $1 \leq p \leq 2$ and LCA groups G. In fact the Hausdorff-Young theorem still holds, that is the Fourier transform is bounded from $L^p(G)$ into $L^{p'}(\Gamma)$ where p' stands for the conjugate exponent for $1 < p \leq 2$ (see [14]).

Note that for linear combinations of the form $f = \sum_{j=1}^n \phi_j x_j$, $\phi_j \in L^p(G), x_j \in X, j = 1, ..., n,$ the Fourier transform is well defined by the formula

$$\mathcal{F}_G(f)(\gamma) = \hat{f}(\gamma) = \sum_{j=1}^n \hat{\phi}_j(\gamma) x_j.$$

Since $L^p(G) \otimes X$ is dense in $L^p(G, X)$ the following definition make sense.

Definition 4.1. Let $1 < p \leq 2$. A Banach space X is said have Fourier type p with respect to the group G if the operator $\mathcal{F}_G$ extends as a bounded operator from $L^p(G, X)$ to $L^{p'}(\Gamma, X)$.

The reader is referred to the survey paper by Garcia-Cuerva, Kazarian, Kolyada and Torrea ([13]) for a nice presentation of this property and its use connected with geometry of Banach spaces.

The notion was first introduced by J. Peetre ([20]) for the group $G = \mathbb{R}$ and was very useful in interpolation theory (see also [19] for groups). The use of other groups was of special interest and have been considered by many authors.

It is shown in [13] that Fourier type p with respect to $\mathbb{T}$, $\mathbb{R}$ and $\mathbb{Z}$ are all equivalent. It is not hard to see that X has Fourier type p (with respect to G) if and only if X^* has Fourier type p (with respect to Γ). Typical examples of spaces with Fourier type p are $L^r(\mu)$ for $p \leq r \leq p'$ or those obtained by interpolation between a Banach and a Hilbert space.

One of the most relevant results on this notion is the celebrated theorem by J. Bourgain which establishes that the B-convexity can be characterized by having Fourier type bigger than 1 (see [8, 9]) or the book [22] . The reader is also referred to [7, 6, 13] for a equivalent formulation in terms of the validity of certain inequality due to Hardy for functions in the space $H^1(\mathbb{T}, X)$, defined by means of X-valued atoms.

Another basic theorem on Fourier type corresponds to the case $p = 2$. It was shown by S. Kwapien (see [16]) that Fourier type 2 is equivalent to being isomorphic to a Hilbert space.

Here we shall present the following result.

Theorem 4.2. *Let X, Y be Banach spaces, let $u : X \times X \to Y$ be a bounded bilinear map. If $1 < p \leq \frac{4}{3}$ and Y has Fourier type $s = \frac{p}{2-p}$ with respect to G then there exists a constant $C > 0$ such that*

$$\Big(\int_\Gamma \|u(\hat{f}(\gamma), \hat{f}(\gamma))\|_Y^{\frac{p'}{2}} d\mu_\Gamma(\gamma)\Big)^{\frac{1}{2}} \leq C\|u\|\|f\|_{L^p(G,X)}$$

for all $f \in L^p(G, X)$.

Proof. Take $p_1 = p_2 = p$ and r given by $\frac{1}{r} = \frac{2}{p} - 1$. Then $r = s$ and $s' = \frac{p'}{2}$. Our assumption on p gives that $1 < s \leq 2$, what allows to apply the Fourier type condition. This together with Theorem 2.6, and Young's theorem (see (3) in Theorem 2.5) give

$$\Big(\int_\Gamma \|u(\hat{f}(\gamma), \hat{f}(\gamma))\|_Y^{\frac{p'}{2}} d\mu_\Gamma(\gamma)\Big)^{\frac{p'}{2}} \leq C\|f *_u f\|_{L^s(G,Y)} \leq C\|u\|\|f\|^2_{L^p(G,X)}. \qquad \blacksquare$$

Theorem 4.3. *Let $n \in \mathbb{N}$ and let A be a Banach algebra. If $\frac{n}{2} < p \leq \frac{2n}{3}$ and A has Fourier type $s = \frac{p}{n-p}$ with respect to G then there exists a constant $C > 0$ such that*

$$\Big(\int_\Gamma \|\hat{f}(\gamma).^{(n}.\hat{f}(\gamma))\|_A^{\frac{p}{2p-n}} d\mu_\Gamma(\gamma)\Big)^{\frac{1}{n}} \leq C\|f\|_{L^p(G,A)}$$

for all $f \in L^p(G, A)$.

Proof. Iterating part (3) in Theorem 2.5 one gets that if $\frac{1}{p_1} + .. + \frac{1}{p_n} \geq 1$ and $\frac{1}{r} = \frac{1}{p_1} + .. + \frac{1}{p_n} - 1$ then

$$\|f_1 * ... * f_n\|_{L^r(G,A)} \leq \|f_1\|_{L^{p_1}(G,A)} \cdots \|f_n\|_{L^{p_n}(G,A)}$$

for $f_i \in L^{p_i}(G, A)$ for $i = 1, ..., n$.

Applying this to $p_i = p$ and $f_i = f$ for $i = 1, ..., n$ one gets $\frac{1}{r} = \frac{n-p}{p}$, $s = r$ and $r' = \frac{p}{2p-n}$. The assumption gives again $1 < s \leq 2$ and therefore

$$\Big(\int_\Gamma \|\hat{f}(\gamma).^{(n}.\hat{f}(\gamma))\|_A^{\frac{p}{2p-n}} d\mu_\Gamma(\gamma)\Big)^{\frac{2p-n}{p}} \leq C\|f * .^n. * f\|_{L^s(G,A)} \leq C\|f\|^n_{L^p(G,A)}$$

for all $f \in L^p(G, A)$. $\blacksquare$

References

[1] H. Amann, *Operator-valued Fourier multipliers, vector valued Besov spaces and applications*, Math. Nachr., Vol. **186**, 1997, pp. 15-56.

[2] J.L. Arregui, O. Blasco, *On the Bloch space and convolution of functions in the L^p-valued case*, Collectanea Math., Vol. **48**, 1997, pp. 363-373.

[3] J.L. Arregui, O. Blasco, *Convolution of three functions by means of bilinear maps and applications*, Illinois J., Vol. **43**, 1999, pp. 264-280.

[4] J. Berg and J. Löfstrom, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin and New York, 1973.

[5] O. Blasco, *Convolution by means of bilinear maps*, Contemporary Math., Vol. **38**, 2000, pp. 21-36.

[6] O. Blasco, *Vector-valued Hardy inequalities and B-convexity*, Ark. Mat. Vol. **232**, 1999, pp. 85-103.

[7] O. Blasco and A. Pelczynski, *Theorems of Hardy and Paley for vector valued analytic functions and related classes of Banach spaces*, Trans. Amer. Math. Soc., Vol. **323**, 1991, pp. 335-367.

[8] J. Bourgain, *A Hausdorff-Young inequality for B-convex spaces*, Pacific J. Math., Vol. **101**, 1982, pp. 255-262.

[9] J. Bourgain, *Vector-valued Hausdorff-Young inequality and applications*, Geometric Aspects in Functional Analysis, Israel Seminar (GAFA) 1986-87. Lecture Notes in Math., Vol. **1317**, pp. 239-249.

[10] J. Diestel-J.J. Uhl, *Vector measures*, American Mathematical Society, Mathematical Surveys and Monographs, Volume **15**, 1977.

[11] R. Doran and J. Wichman, *Approximate identities and factorization in Banach modules*, Lecture Notes in Math., Vol. **768**, Springer Verlag, Berlin, 1979.

[12] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weigthed norm inequalities and related topics*, North-Holland, Amsterdam, 1985.

[13] J. Garcia-Cuerva, K.S. Kazarian, V.I. Kolyada, and J.L. Torrea , *Vector-valued Hausdorff-Young inequality and applications*, Russian Math. Surveys, Vol. **53**, 1998, pp. 435-513

[14] E. Hewitt, K.A: Ross, *Abstract Harmonic Analysis, vol I, II*, Springer Verlag, New York, 1970.

[15] Y. Katznelson, *Introduction to Harmonic Analysis*, John Wiley and Sons., New York, 1968.

[16] S. Kwapien, *Isomorphic characteriztions of inner product spaces by orthogonal series with vector-valued coefficients*, Studia Math., Vol. **44**, 1972, pp. 583-595.

[17] S. Kwapien, A. Pelczynski, *Absolutely summing operators and translation-invariant spaces on compact abelian groups*, Math. Nachr., Vol. **94**, 1980, pp. 303-340.

[18] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces II*, Springer Verlag, New York, 1979.

[19] M. Milman, *Complex interpolation and geometry of Banach spaces*, Ann. Mat. Pura Appl. Vol. **136**, 1984, pp. 317-328.

[20] J. Peetre, *Sur la transformation de Fourier des fonctions a valeurs vectorielles*, Rend. Sem. Mat. Univ. Padova, Vol. **42**, 1969 2, pp. 15-46.

[21] A. Pelczynski, *On commesurate sequences of characters*, Proc. Amer. Math. Soc., Vol. **104**, 1988 , pp. 525-531.

[22] A. Pietsch, J. Wenzel, *Orthonormal systems and Banach space geometry*, Cambridge Univ. Press, Cambrigde, 1998.

[23] W. Rudin, *Fourier Analysis on groups*, Interscience Publishers, New York, 1962.

[24] A. Zygmund, *Trigonometric series*, Cambrigde Univ. Press. New York, 1959.

Oscar Blasco
Departamento de Análisis Matemático
Universidad de Valencia
46100 Burjassot
Valencia
Spain
Oscar.Blasco@uv.es