Abstract Let (Ω, Σ, μ) be a finite measure space, $1 \leq p < \infty$, X be a Banach space X and $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. We say that an X-valued function f is p-integrable with respect to $\mathcal{B}$ whenever $\sup_{\|y\|=1} \int_\Omega \|\mathcal{B}(f(w), y)\|^p d\mu < \infty$. We get an analogue to Hölder's inequality in this setting.

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HÖLDER INEQUALITY FOR FUNCTIONS INTEGRABLE WITH RESPECT TO BILINEAR MAPS

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1. Introduction

Throughout the paper $1 \leq p < \infty$, (Ω, Σ, μ) will be a finite complete measure space, X, Y and Z will stand for Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $\mathcal{B} : X \times Y \to Z$ will denote a bounded bilinear map. We denote by $L^0(X)$, $L^0_{\text{weak}}(X)$ and $L^0_{\text{weak}^*}(X^*)$ the spaces of strongly, weakly measurable and weak*-measurable functions and write $L^p(X)$, $L^p_{\text{weak}}(X)$ and $L^p_{\text{weak}^*}(X^*)$ for the space of functions in $L^0(X)$, $L^0_{\text{weak}}(X)$ and $L^0_{\text{weak}^*}(X^*)$ such that $\|f\| \in L^p(\mu)$, $\langle f, x^* \rangle \in L^p(\mu)$ for $x^* \in X^*$ and $\langle x, f \rangle \in L^p(\mu)$ for $x \in X$ respectively. Finally we use the notation $P^p(X)$ for the space of Pettis p-integrable functions $P^p(X) = L^p_{\text{weak}}(X) \cap L^0(X)$.

In this paper we shall consider spaces of X-valued functions which are p-integrable with respect to a bounded bilinear map $\mathcal{B} : X \times Y \to Z$, that is to say functions f satisfying the condition $\mathcal{B}(f, y) \in L^p(Z)$ for all $y \in Y$.

Although theses classes have been around for a long time in particular cases such us

$$\mathcal{B}_X = \mathcal{B} : X \times \mathbb{K} \to X, \qquad \mathcal{B}(x, \lambda) = \lambda x, \tag{1.1}$$

$$\mathcal{D}_X = \mathcal{D} : X \times X^* \to \mathbb{K}, \qquad \mathcal{D}(x, x^*) = \langle x, x^* \rangle, \tag{1.2}$$

$$\mathcal{D}_{1,X} = \mathcal{D}_1 : X^* \times X \to \mathbb{K}, \qquad \mathcal{D}_1(x^*, x) = \langle x, x^* \rangle, \tag{1.3}$$

or

$$\pi_Y : X \times Y \to X \hat{\otimes} Y, \qquad \pi_Y(x, y) = x \otimes y, \tag{1.4}$$

$$\tilde{\mathcal{O}}_Y : X \times \mathcal{L}(X, Y) \to Y, \qquad \tilde{\mathcal{O}}_Y(x, T) = T(x), \tag{1.5}$$

$$\mathcal{O}_{Y,Z} : \mathcal{L}(Y, Z) \times Y \to Z, \qquad \mathcal{O}_{Y,Z}(T, y) = T(y) \tag{1.6}$$

a systematic study for general bilinear maps has been iniciated in [6]. This approach has been used to extend the results on boundedness from $L^p(Y)$ to $L^p(Z)$ of operator-valued kernels by M. Girardi and L. Weiss [10] to the case where $K : \Omega \times \Omega' \to X$ is measurable and the integral operators are defined by

$$T_K(f)(w) = \int_{\Omega'} \mathcal{B}(K(w, w'), f(w')) d\mu'(w').$$

Also the reader is referred to [7] for the introduction of Fourier Analysis in the bilinear context. This allows to extend the results in [2, 4, 5] regarding convolution by means of bilinear maps and Fourier coefficients for functions in these wider classes.

Let us mention some notions that were relevant for developping the general theory (see [6]). Given $x \in X$ and $y \in Y$ we shall be denoting by $\mathcal{B}_x \in \mathcal{L}(Y, Z)$ and $\mathcal{B}^y \in \mathcal{L}(X, Z)$ the corresponding linear operators

$$\mathcal{B}_x(y) = \mathcal{B}(x, y) \text{ and } \mathcal{B}^y(x) = \mathcal{B}(x, y).$$

The triple $(Y, Z, \mathcal{B})$ is admissible for X if the map $x \to \mathcal{B}_x$ is injective from $X \to \mathcal{L}(Y, Z)$ and X is said to be $(Y, Z, \mathcal{B})$-normed (or normed by $\mathcal{B}$) if there exists $C > 0$ such that for all $x \in X$

$$\|x\| \leq C\|\mathcal{B}_x\|.$$

Given a bounded bilinear map $\mathcal{B} : X \times Y \to Z$, we can define the "adjoint" $\mathcal{B}^* : X \times Z^* \to Y^*$ by the formula

$$\langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x, y), z^*\rangle.$$

Note that

$$\mathcal{B}^* = \mathcal{D},\ (\pi_Y)^* = \widetilde{\mathcal{O}}_{Y^*} \text{ and } (\mathcal{O}_{Y,Z})^*(T, z^*) = \mathcal{O}_{Z^*,Y^*}(T^*, z^*).$$

Let us start with the following definitions:

Definition 1.1. (see [6]) We say that $f : \Omega \to X$ belongs to $L^0_{\mathcal{B}}(X)$ if $\mathcal{B}(f, y) \in L^0(Z)$ for any $y \in Y$. We write $\mathcal{L}^p_{\mathcal{B}}(X)$ for the space of functions f in $L^0_{\mathcal{B}}(X)$ such that

$$\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{L^p(Z)} : \|y\| = 1\} < \infty.$$

A function $f \in \mathcal{L}^p_{\mathcal{B}}(X)$ is said to belong to $L^p_{\mathcal{B}}(X)$ if there exists a sequence of simple functions $(s_n)_n \in \mathcal{S}(X)$ such that

$$s_n \to f \text{ a.e.} \quad \text{and} \quad \|s_n - f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} \to 0.$$

For $f \in L^p_{\mathcal{B}}(X)$ we write $\|f\|_{L^p_{\mathcal{B}}(X)}$ instead of $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$. Clearly one has that

$$\|f\|_{L^p_{\mathcal{B}}(X)} = \lim_{n\to\infty} \|s_n\|_{L^p_{\mathcal{B}}(X)}.$$

In particular

$$L^0_{\mathcal{B}}(X) = L^0(X), L^0_{\mathcal{D}}(X) = L^0_{\text{weak}}(X) \text{ and } L^0_{\mathcal{D}_1}(X^*) = L^0_{\text{weak}*}(X).$$

$$\mathcal{L}^p_{\mathcal{B}}(X) = L^p(X), \mathcal{L}^p_{\mathcal{D}}(X) = L^p_{\text{weak}}(X) \text{ and } \mathcal{L}^p_{\mathcal{D}_1}(X^*) = L^p_{\text{weak}*}(X^*).$$

$$L^p_{\mathcal{B}}(X) = L^p(X) \text{ and } L^p_{\mathcal{D}}(X) = P^p(X) (\text{see [11], page 54 for the case } p = 1).$$

Observe that $L^p(X) \subseteq L^p_{\mathcal{B}}(X)$ for any $\mathcal{B}$ and, that in general, $L^p_{\mathcal{B}}(X) \subsetneq \mathcal{L}^p_{\mathcal{B}}(X)$ (see [8] page 53, for the case $\mathcal{B} = \mathcal{D}$). It was shown in [6] that $\mathcal{L}^p_{\mathcal{B}}(X) \subset L^p_{\text{weak}}(X)$ if and only if X is $\mathcal{B}$-normed.

Clearly $f \in L^0_{\mathcal{B}}(X)$ and $g \in L^0(Y)$ implies that $\mathcal{B}(f, g) \in L^0(Z)$. Hence a natural question that arises is the following: If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, does $\mathcal{B}(f, g)$ belong to $L^{p_3}(Z)$ for any $f \in \mathcal{L}^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Y)$?

The answer is negative for any infinite dimensional Banach space X. Indeed, take $p_1 = p_2 = 2$ and $p_3 = 1$, let X be an infinite dimensional Banach space, $Y = X^*$ and $Z = \mathbb{K}$ and $\mathcal{B} = \mathcal{D}$. Take $(x_n) \in \ell^2_{\text{weak}}(X) \setminus \ell_2(X)$. This allows to find $(x^*_n) \in \ell_2(X^*)$ such that $\sum_n |\langle x_n, x^*_n\rangle| = \infty$. Consider now $\Omega = [0, 1]$ with the Lebesgue measure, $I_k = (2^{-k}, 2^{-k+1}]$ and define the functions $f = \sum_{k=1}^\infty 2^{\frac{k}{2}} x_k \mathbf{1}_{I_k}$ and $g = \sum_{k=1}^\infty 2^{\frac{k}{2}} x^*_k \mathbf{1}_{I_k}$. It is clear that $f \in \mathcal{L}^2_{\mathcal{D}}(X)$ with $\|f\|^2_{\mathcal{L}^2_{\mathcal{D}}(X)} = \sup\{\sum_{n=1}^\infty |\langle x_n, x^*\rangle|^2 : \|x^*\| = 1\}$ and $g \in L^2(X^*)$ with $\|g\|^2_{L^2(X^*)} = \sum_{n=1}^\infty \|x^*_n\|^2$ but $\mathcal{B}(f, g) = \sum_{k=1}^\infty 2^k \langle x_k, x^*_k\rangle \mathbf{1}_{I_k} \notin L^1$.

One might think that the difficulty comes from allowing the functions to belong to $\mathcal{L}^{p_1}_{\mathcal{B}}(X)$ instead of $L^{p_1}_{\mathcal{B}}(X)$. Let us then modify the question: Does $\mathcal{B}(f, g)$ belong to $L^{p_3}(Z)$ for any $f \in L^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Y)$?

The answer is again negative. If the result hold true we would have that there exists $M > 0$ such that $\|\mathcal{B}(s, t)\|_{L^1(Z)} \leq M\|s\|_{L^2_{\mathcal{B}}(X)}\|t\|_{L^2(Y)}$ for any $s \in \mathcal{S}(X)$ and $t \in \mathcal{S}(Y)$.

Select $X = Y = \ell_2$, $Z = \ell_1$ and $\mathcal{B} : \ell_2 \times \ell_2 \to \ell_1$ given by $\mathcal{B}((\lambda_n)_{n\in\mathbb{N}}, , (\beta_n)_{n\in\mathbb{N}}) = (\lambda_n\beta_n)_{n\in\mathbb{N}}$. Let us now consider $s_N = t_N = \sum_{k=1}^N 2^{\frac{k}{2}} e_k \mathbf{1}_{I_k}$ where e_k is the canonical basis and I_k are chosen as above. Hence $\mathcal{B}(s_N, y) = \sum_{k=1}^N 2^{\frac{k}{2}} \beta_k e_k \mathbf{1}_{I_k}$ for $y = (\beta_n)_{n\in\mathbb{N}} \in \ell_2$. Therefore $\|s_N\|_{L^2_{\mathcal{B}}(\ell_2)} \leq 1$. On the other hand $\|s_N\|_{L^2(\ell_2)} = \sqrt{N}$. Finally observe that $\mathcal{B}(s_N, s_N) = \sum_{k=1}^N 2^k e_k \mathbf{1}_{I_k}$ and $\|\mathcal{B}(s_N, s_N)\|_{L^1(\ell_1)} = N$. This contradicts (1).

Modifying the previous argument with $Z = \mathbb{K}$ and $\mathcal{B} = \mathcal{D}$ one can even show that there exist $f \in L^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Y)$ such that $\mathcal{B}(f, g) \notin L^{p_3}_{\text{weak}}(Z)$.

The objective of this paper is to present an analogue to Hölder inequality in the setting of vector-valued functions integrables with respect to bilinear maps. We shall then study the following general problem:

Problem: Let $1 \leq p_1, p_2, p_3 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. If $\mathcal{B}_1 : X \times X_1 \to X_2$ and $\mathcal{B}_2 : Y \times Y_1 \to Y_2$ are bounded bilinear maps, find $\mathcal{B}_3 : Z \times Z_1 \to Z_2$ such that for any $f \in \mathcal{L}^{p_1}_{\mathcal{B}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)$ one has $\mathcal{B}(f, g) \in \mathcal{L}^{p_3}_{\mathcal{B}_3}(Z)$.

2. A bilinear version of Hölder's Inequality.

It is well known and easy to see the following analogues of Hölder's inequality in the vector-valued setting: Let $1 \leq p_1, p_2, p_3 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$.

(1) If $f \in L^{p_1}_{\text{weak}}(X)$ and $g \in L^{p_2}$ then $fg \in L^{p_3}_{\text{weak}}(X)$.

(2) If $f \in P^{p_1}(X)$ and $g \in L^{p_2}$ then $fg \in P^{p_3}(X)$.

(3) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}$ then $fg \in L^{p_3}(X)$.

(4) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X^*)$ then $\langle f, g\rangle \in L^{p_3}$.

(5) If $f \in L^{p_1}(\mathcal{L}(X, Y))$ and $g \in L^{p_2}(X)$ then $f(w)(g(w)) \in L^{p_3}(Y)$.

Definition 2.1. We say that $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is a compatible triple if $\mathcal{B} : X \times Y \to Z$, $\mathcal{B}_1 : X \times X_1 \to X_2$ and $\mathcal{B}_2 : Y \times Y_1 \to Y_2$ are bounded bilinear maps and there exist a Banach space F and two bounded bilinear maps $\mathcal{P} : X_2 \times Y_2 \to F$ and $\widetilde{\mathcal{P}} : Z \times (X_1 \hat{\otimes} Y_1) \to F$ such that

$$\widetilde{\mathcal{P}}(\mathcal{B}(x, y), x_1 \otimes y_1) = \mathcal{P}(\mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1))$$

for all $x \in X, y \in Y$, $x_1 \in X_1$ and $y_1 \in Y_1$.

A general procedure of construction of such compatible triples of bilinear maps can be obtained as follows:

Proposition 2.2. *Let U be a Banach space, $\mathcal{B}_1 : X \times X_1 \to U$ and $\mathcal{B}_2 : Y \times Y_1 \to U^*$ be bounded bilinear maps . Define the bilinear map $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{B} : X \times Y \to \mathcal{L}(X_1, Y_1^*)$ by the formula*

$$\langle \mathcal{B}(x, y)(x_1), y_1\rangle = \langle \mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1)\rangle$$

for $x \in X$, $y \in Y$, $x_1 \in X_1$ and $y_1 \in Y_1$.

Proof. Using that $\mathcal{L}(X_1, Y_1^*) = (X_1 \hat{\otimes} Y_1)^*$ we also can write

$$\langle \mathcal{B}(x, y), x_1 \otimes y_1\rangle = \langle \mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1)\rangle.$$

This shows that $(\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2), \mathcal{B}_1, \mathcal{B}_2)$ is compatible by selecting $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D} : U \times U^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_1 : \mathcal{L}(X_1, Y_1^*) \times (X_1 \hat{\otimes} Y_1) \to \mathbb{K}$. □

Let us now give some more concrete examples of admissible triples:

Example 2.3. $(\mathcal{B}, \mathcal{B}_X, \mathcal{B}_Y)$ is a compatible triple for any $\mathcal{B} : X \times Y \to Z$.
In particular, $(\mathcal{D}_X, \mathcal{B}_X, \mathcal{B}_{X^*})$ or $(\mathcal{O}_{X,Y}, \mathcal{B}_X, \mathcal{B}_Y)$ are compatible triples.

Indeed, if $\mathcal{B} : X \times Y \to Z$, $\mathcal{B}_1 = \mathcal{B}_X : X \times \mathbb{K} \to X$ and $\mathcal{B}_2 = \mathcal{B}_Y : Y \times \mathbb{K} \to Y$ then select $F = Z$, $\mathcal{P} = \mathcal{B} : X \times Y \to Z$ and $\widetilde{\mathcal{P}} = \mathcal{B}_Z : Z \times \mathbb{K} \to Z$. Observe that $\widetilde{\mathcal{P}}(\mathcal{B}(x, y), \lambda\beta) = \mathcal{P}(\mathcal{B}(x, \lambda), \mathcal{B}(y, \beta))$. □

Example 2.4. $(\mathcal{B}, \mathcal{B}^*, \mathcal{B}_Y)$ is a compatible triple.

Indeed, if $\mathcal{B} : X \times Y \to Z$, $\mathcal{B}_1 = \mathcal{B}^* : X \times Z^* \to Y^*$ given by

$$\langle y, \mathcal{B}_1(x, z^*)\rangle = \langle \mathcal{B}(x, y), z^*\rangle$$

and $\mathcal{B}_2 = \mathcal{B}_Y : Y \times \mathbb{K} \to Y$ then we can select $F = \mathbb{K}$, $\mathcal{P} = (\mathcal{D}_1)_Y : Y^* \times Y \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_Z : Z \times Z^* \to \mathbb{K}$. □

Example 2.5. $(\pi_Y, \mathcal{B}_X, \mathcal{O}_{X^*})$ is a compatible triple.

Indeed, if $\mathcal{B} = \pi_Y : X \times Y \to X\hat{\otimes}Y$, $\mathcal{B}_1 = \mathcal{B}_X : X \times \mathbb{K} \to X$ and $\mathcal{B}_2 = \widetilde{\mathcal{O}}_{X^*} : Y \times \mathcal{L}(Y, X^*) \to X^*$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_X : X \times X^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_{X\hat{\otimes}Y} : X\hat{\otimes}Y \times \mathcal{L}(Y, X^*) \to \mathbb{K}$. The compatibility now follows from

$$\widetilde{\mathcal{P}}(\mathcal{B}(x, y), \lambda T) = \langle x \otimes y, \lambda T\rangle = \langle \lambda x, Ty\rangle = \mathcal{P}(\mathcal{B}_1(x, \lambda), \mathcal{B}_2(y, T)).$$

□

Example 2.6. Let $\mathcal{B} : \mathcal{L}(X, Z)\times\mathcal{L}(Y, Z^*) \to \mathcal{L}(Y, X^*)$ be given by $(T, S) \to T^*S$. Then $(\mathcal{B}, \mathcal{O}_{X,Z}, \mathcal{O}_{Y,Z^*})$ is a compatible triple.

Indeed, if $\mathcal{B}_1 = \mathcal{O}_{X,Z} : \mathcal{L}(X, Z) \times X \to Z$ and $\mathcal{B}_2 = \mathcal{O}_{Y,Z^*} : \mathcal{L}(Y, Z^*) \times Y \to Z^*$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_Z : Z\times Z^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = (\mathcal{D}_1)_{X\hat{\otimes}Y} : \mathcal{L}(Y, X^*)\times X\hat{\otimes}Y \to \mathbb{K}$ given by $\widetilde{\mathcal{P}}(T, x\otimes y) = \langle x, Ty\rangle$. Observe that the compatibility follows from the formula

$$\widetilde{\mathcal{P}}(\mathcal{B}(T, S), x \otimes y) = \langle x, T^*Sy\rangle = \langle Tx, Sy\rangle = \mathcal{P}(\mathcal{B}_1(T, x), \mathcal{B}_2(S, y)).$$

□

Theorem 2.7. (Hölder's inequality I) *Let $1 \le p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$. Assume that $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is a compatible triple for some F, $\mathcal{P}$ and $\widetilde{\mathcal{P}}$.*

(1) If $f \in \mathcal{L}^{p_1}_{\mathcal{B}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)$ then $\mathcal{B}(f, g) \in \mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

(2) If $f \in L^{p_1}_{\mathcal{B}_1}(X)$ and $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ then $\mathcal{B}(f, g) \in L^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

Moreover $\|\mathcal{B}(f, g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \le \|\mathcal{P}\|\|f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)}$.

Proof. *(1)* Let us first show that if $f \in L^0_{\mathcal{B}_1}(X)$ and $g \in L^0_{\mathcal{B}_2}(Y)$ then $h = \mathcal{B}(f, g) \in L^0_{\widetilde{\mathcal{P}}}(Z)$.

Indeed, if $x_1 \in X_1$ and $y_1 \in Y_1$ then $\widetilde{\mathcal{P}}(h, x_1 \otimes y_1) = \mathcal{P}(\mathcal{B}_1(f, x_1), \mathcal{B}_2(g, y_1))$. Now since $\mathcal{B}_1(f, x_1) \in L^0(X_2)$, $\mathcal{B}_2(g, y_1) \in L^0(Y_2)$ and $\mathcal{P}$ is continuous then $\widetilde{\mathcal{P}}(h, x_1 \otimes y_1) \in L^0(F)$. For general $\varphi \in X_1\hat{\otimes}Y_1$, assume $\varphi = \sum_n x^n_1 \otimes y^n_1$ with $\sum_n \|x^n_1\|\|y^n_1\| < \infty$. Then, using the continuity of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, one has

$$\widetilde{\mathcal{P}}(h, \varphi) = \lim_{N\to\infty} \sum_{k=1}^{N} \widetilde{\mathcal{P}}(\mathcal{B}_1(f, x^k_1), \mathcal{B}_2(g, y^k_1)) \in L^0(F).$$

Assume $f \in \mathcal{L}^{p_1}_{\mathcal{B}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)$. Let us show that $h \in \mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

If $x_1 \in X_1$ and $y_1 \in Y_1$ then

$$\begin{aligned}
(\int_\Omega \|\widetilde{\mathcal{P}}(h, x_1 \otimes y_1)\|^{p_3} d\mu)^{\frac{1}{p_3}} &= (\int_\Omega \|\mathcal{P}(\mathcal{B}_1(f, x_1), \mathcal{B}_2(g, y_1))\|^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\le \|\mathcal{P}\|(\int_\Omega (\|\mathcal{B}_1(f, x_1)\|\|\mathcal{B}_2(g, y_1))\|)^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\le \|\mathcal{P}\|(\int_\Omega \|\mathcal{B}_1(f, x_1)\|^{p_1} d\mu)^{\frac{1}{p_1}} (\int_\Omega \|\mathcal{B}_2(g, y_1)\|^{p_2} d\mu)^{\frac{1}{p_2}} \\
&\le \|\mathcal{P}\|\|f\|_{L^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}\|x_1\|\|y_1\|.
\end{aligned}$$

In general, for each $\varphi = \sum_n x^n_1 \otimes y^n_1 \in X_1\hat{\otimes}Y_1$, one has $\widetilde{\mathcal{P}}(h, \sum_n x^n_1 \otimes y^n_1) = \sum_n \widetilde{\mathcal{P}}(h, x^n_1 \otimes y^n_1)$. Therefore

$$\begin{aligned}
(\int_\Omega \|\widetilde{\mathcal{P}}(h, \sum_n x^n_1 \otimes y^n_1)\|^{p_3} d\mu)^{\frac{1}{p_3}} &\le \sum_n (\int_\Omega \|\mathcal{P}(\mathcal{B}_1(f, x^n_1), \mathcal{B}_2(g, y^n_1))\|^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\le \|\mathcal{P}\|(\sum_n \|x^n_1\|\|y^n_1\|)\|f\|_{L^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}
\end{aligned}$$

This gives $\|\mathcal{B}(f, g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \le \|\mathcal{P}\|\|f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)}$.

(2) Assume that f and g are simple functions. If $f = \sum_k x_k \mathbf{1}_{E_k} \in \mathcal{S}(X)$ and $g = \sum_p y_p \mathbf{1}_{F_p} \in \mathcal{S}(Y)$ then

$$h = \mathcal{B}(f,g) = \sum_{k,p} \mathcal{B}(x_k, y_p)\mathbf{1}_{E_k \cap F_p} \in \mathcal{S}(Z).$$

Now, if we take $f \in L^{p_1}_{\mathcal{B}_1}(X)$ and $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ then there exists $(f_n)_n \subseteq \mathcal{S}(X)$ and $(g_n)_n \subseteq \mathcal{S}(Y)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $\|f_n - f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \to 0$ and $\|g_n - g\|_{L^{p_2}_{\mathcal{B}_2}(Y)} \to 0$. Clearly $\mathcal{B}(f_n, g_n)$ are simple functions and converge to $\mathcal{B}(f,g)$ a.e.

Due to the previous result

$$\begin{aligned} \|\mathcal{B}(f_n,g_n) - \mathcal{B}(f,g)\|_{\mathcal{L}^{p_3}_{\tilde{\mathcal{P}}}(Z)} &\leq \|\mathcal{B}(f_n - f, g_n)\|_{\mathcal{L}^{p_3}_{\tilde{\mathcal{P}}}(Z)} + \|\mathcal{B}(f, g_n - g)\|_{\mathcal{L}^{p_3}_{\tilde{\mathcal{P}}}(Z)} \\ &\leq \|\mathcal{P}\|\|f_n - f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g_n\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)} \\ &+ \|\mathcal{P}\|\|f\|_{\mathcal{L}^{p_1}_{\mathcal{B}_1}(X)}\|g_n - g\|_{\mathcal{L}^{p_2}_{\mathcal{B}_2}(Y)} \end{aligned}$$

Taking limits the result is completed.

□

Let us point out a little improvement that can be achieved for the compatible triples in Proposition 2.2. Let us recall the following fact that will be used in the proof.

Lemma 2.8. *Let X be a Banach space, $1 \leq p < \infty$ and $(x_n^*)_n \subseteq X^*$. Then*

$$\sup\{(\sum_n |\langle x_n^*, x^{**}\rangle|^p)^{\frac{1}{p}} : \|x^{**}\| = 1\} = \sup\{(\sum_n |\langle x, x_n^*\rangle|^p)^{\frac{1}{p}} : \|x\| = 1\}$$

Corollary 2.9. (Hölder's inequality II) *Let X, X_1, Y, Y_1 and U be a Banach spaces and $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$. Let $\mathcal{B}_1 : X \times X_1 \to U$, $\mathcal{B}_2 : Y \times Y_1 \to U^*$ be bounded bilinear maps and let $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2) = \mathcal{B} : X \times Y \to \mathcal{L}(X_1, Y_1^*)$ be defined by the formula*

$$\langle \mathcal{B}(x,y)(x_1), y_1 \rangle = \langle \mathcal{B}_1(x, x_1), \mathcal{B}_2(y, y_1)\rangle.$$

If $f \in L^{p_1}_{\mathcal{B}_1}(X)$ and $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ then $\mathcal{B}(f,g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*))$.
Moreover $\|\mathcal{B}(f,g)\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1,Y_1^*))} \leq \|f\|_{L^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}$.

Proof. Assume first that f and g are simple functions. If $f = \sum_k x_k \mathbf{1}_{E_k} \in \mathcal{S}(X)$ and $g = \sum_p y_p \mathbf{1}_{F_p} \in \mathcal{S}(Y)$ then $h = \mathcal{B}(f,g) = \sum_{k,p} \mathcal{B}(x_k, y_p)\mathbf{1}_{E_k \cap F_p} \in \mathcal{S}(\mathcal{L}(X_1, Y_1^*))$. Note that $\mathcal{L}(X_1, Y_1^*) = (X_1 \hat{\otimes} Y_1)^*$. Hence from Lemma 2.8

$$\begin{aligned} \|h\|_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^*)} &= \sup\{(\sum_{k,p} |\langle \mathcal{B}(x_k, y_p), \psi\rangle|^{p_3} \mu(E_k \cap F_p))^{\frac{1}{p_3}} : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\ &= \sup\{(\sum_{k,p} |\langle \varphi, \mathcal{B}(x_k, y_p)\rangle|^{p_3} \mu(E_k \cap F_p))^{\frac{1}{p_3}} : \|\varphi\|_{X_1\hat{\otimes}Y_1} = 1\} \\ &= \|h\|_{L^{p_3}_{\text{weak}*}((X_1\hat{\otimes}Y_1)^*)}. \end{aligned}$$

We conclude, using Theorem 2.7, that

$$\|h\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1,Y_1^*))} \leq \|f\|_{L^{p_1}_{\mathcal{B}_1}(X)}\|g\|_{L^{p_2}_{\mathcal{B}_2}(Y)}.$$

Now, if we take $f \in L^{p_1}_{\mathcal{B}_1}(X)$ and $g \in L^{p_2}_{\mathcal{B}_2}(Y)$ then there exists $(f_n)_n \subseteq \mathcal{S}(X)$ and $(g_n)_n \subseteq \mathcal{S}(Y)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $\|f_n - f\|_{L^{p_1}_{\mathcal{B}_1}(X)} \to 0$ and $\|g_n - g\|_{L^{p_2}_{\mathcal{B}_2}(Y)} \to 0$. Clearly $\mathcal{B}(f_n, g_n) \to \mathcal{B}(f,g)$ a.e. and therefore $\mathcal{B}(f,g)$ is strongly measurable and

$$|\langle \mathcal{B}(f_n, g_n), \psi\rangle|^{p_3} \to |\langle \mathcal{B}(f,g), \psi\rangle|^{p_3} \text{ a.e.}$$

for all $\psi \in (X_1\hat{\otimes}Y_1)^{**}$.

To see that $\mathcal{B}(f,g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*))$ it suffices to show that $\mathcal{B}(f,g) \in L^{p_3}_{\text{weak}}(\mathcal{L}(X_1, Y_1^*))$.

Then using Fatou's Lemma and the inequality for simple functions we have that

$$\begin{aligned}
\|\mathcal{B}(f,g)\|^{p_3}_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^*)} &= \sup\{\int_\Omega |\langle \mathcal{B}(f,g),\psi\rangle|^{p_3}\,d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&= \sup\{\int_\Omega \lim_n |\langle \mathcal{B}(f_n,g_n),\psi\rangle|^{p_3}\,d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&\leq \sup\{\liminf_n \int_\Omega |\langle \mathcal{B}(f_n,g_n),\psi\rangle|^{p_3}\,d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&\leq \liminf_n \|\mathcal{B}(f_n,g_n)\|^{p_3}_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^{**})} \\
&\leq \liminf_n \|f_n\|^{p_3}_{L^{p_1}_{\mathcal{B}_1}(X)} \|g_n\|^{p_3}_{L^{p_2}_{\mathcal{B}_2}(Y)} \\
&= \|f\|^{p_3}_{L^{p_1}_{\mathcal{B}_1}(X)} \|g\|^{p_3}_{L^{p_2}_{\mathcal{B}_2}(Y)}.
\end{aligned}$$

□

Applying Theorem 2.7 to the examples given above one obtains the following applications.

Corollary 2.10. *Let $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map.*

(1) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X^*)$ then $\langle f, g\rangle \in L^{p_3}$.

(2) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}_{\widetilde{\mathcal{B}}_*}(Y)$ then $\mathcal{B}(f,g) \in L^{p_3}_{\text{weak}}(Z)$, where

$\widetilde{\mathcal{B}}_* : Y \times Z^* \to X^*$ is given by $\langle x, \widetilde{\mathcal{B}}_*(y, z^*)\rangle = \langle \mathcal{B}(x,y), z^*\rangle$.

(3) If $f \in L^{p_1}_{\mathcal{B}}(X)$ and $g \in L^{p_2}(Z^*)$ then $\mathcal{B}^*(f,g) \in L^{p_3}_{\text{weak}^*}(Y^*)$, where

$\mathcal{B}^* : X \times Z^* \to Y^*$ is given by $\langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x,y), z^*\rangle$.

(4) If $f \in L^{p_1}_{\tilde{\mathcal{O}}_{Y^*}}(X)$ and $g \in L^{p_2}(Y)$ then $f \otimes g \in L^{p_3}_{\text{weak}}(X \hat{\otimes} Y)$.

(5) If $f \in L^{p_1}_{\mathcal{O}_{X,Z}}(\mathcal{L}(X,Z))$ and $g \in L^{p_2}_{\mathcal{O}_{Y,Z^*}}(\mathcal{L}(Y,Z^*))$ and if we put $f^*(t) = f(t)^* \in \mathcal{L}(Z^*, X^*)$ then $f^*g \in L^{p_3}_{\text{weak}^*}(\mathcal{L}(Y, X^*))$.

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