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Type and Cotype in Vector-Valued Nakano Sequence Spaces

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Given a sequence of Banach spaces $\{X_n\}_n$ and a sequence of real numbers $\{p_n\}_n$ in $[1,\infty)$, the vector-valued Nakano sequence spaces $\ell(\{p_n\},\{X_n\})$ consist of elements $\{x_n\}_n$ in $\prod_n X_n$ for which there is a constant $\lambda > 0$ such that $\sum_n(\|x_n\|/\lambda)^{p_n} < \infty$. In this paper we find the conditions on the Banach spaces X_n and on the sequence $\{p_n\}_n$ for the spaces $\ell(\{p_n\},\{X_n\})$ to have cotype q or type p. © 2001 Elsevier Science

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1. INTRODUCTION

Let $\phi = (\phi_n)_n$ be a sequence of Young functions; that is, $\phi_n\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and convex function such that $\phi_n(x) > 0$ for $x > 0$ and $\phi_n(0) = 0$ for all $n \in \mathbb{N}$. Recall that ℓ^ϕ denotes the Musielak–Orlicz sequence space, consisting of the sequences $(\alpha_n)_n \in \mathbb{R}^{\mathbb{N}}$ which satisfy $\sum_n \phi_n(\lambda|\alpha_n|) < \infty$ for some $\lambda > 0$. This becomes a Banach space under the (Luxemburg) norm

$$\|(\alpha_n)_n\|_\phi = \inf\left\{k > 0 \Big/ \sum_n \phi_n\left(\frac{|\alpha_n|}{k}\right) \le 1\right\}.$$

The reader is referred to [6, 11, 15] for a general study of these classes. We shall be dealing with the Nakano sequence spaces corresponding to $\phi_n(x) = x^{p_n}$ for some sequence $(p_n)_n \subseteq [1,\infty)$ and denoted by $\ell(\{p_n\})$ (see [2, 12, 14]).

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Actually we shall deal with the vector-valued situation. Namely, if $(X_n)_n$ is a sequence of Banach spaces, $1 \le p \le \infty$ and $(p_n)_n$ a sequence of real numbers with $1 \le p_n < \infty$, then we use the notations $\ell^p(\{X_n\})$ for the space of elements $(x_n)_n$ in $\prod_n X_n$ with the norm $\|(x_n)_n\|_{\ell^p(\{X_n\})} = (\sum_n \|x_n\|_{X_n}^p)^{1/p}$ and $\ell(\{p_n\},\{X_n\})$ for the space of elements $(x_n)_n$ in $\prod_n X_n$ (to be also written $\sum_n x_n \otimes e_n$) such that the sequence $(\|x_n\|_{X_n})_n$ belongs to $\ell(\{p_n\})$.

Our aim is to study the Rademacher type and cotype of $\ell(\{p_n\},\{X_n\})$. Let us recall (see [10, 8]) that a Banach space X is said to have type p, with $1 \le p \le 2$ (resp. cotype q, with $2 \le q < \infty$), if there exists a constant $C > 0$ such that

$$\int_0^1 \left\| \sum_{k=1}^{n} x_k r_k(t) \right\| dt \le C \left(\sum_{k=1}^{N} \|x_k\|^p \right)^{1/p}$$

$$\left(\text{resp.} \left(\sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \le C \int_0^1 \left\| \sum_{k=1}^{N} x_k r_k(t) \right\| dt \right)$$

for any $x_1, x_2, \ldots, x_n \in X$ and where r_k stand for the Rademacher functions in $[0,1]$. The least constant for which the inequality is valid (independently of the chosen vectors) is denoted by $T_p(X)$ (resp. $C_q(X)$).

In the paper [4], Kamińska and Turett defined in the frame of the spaces L^Φ (over a nonatomic measure space) the conditions Δ_q and Δ^{*p} for the Musielak–Orlicz function Φ, which turned out to be equivalent to the Banach space L^Φ to have Rademacher cotype q and type p, respectively. Later, Katirtzoglou (see [7]) considered the discrete case, adapted the forementioned conditions to the sequence space ℓ^ϕ, and got similar conclusions.

As a consequence, it was shown that if $q_0 = \limsup_n p_n$ and $p_0 = \liminf_n p_n$, then the Nakano space $\ell(\{p_n\})$ has cotype q for every $q > \max\{p_0, 2\}$ and it does not have cotype q for any $q < \max\{q_0, 2\}$, while it has type p for every $1 \le p < \min\{p_0, 2\}$ and does not have type p when $p > \min\{p_0, 2\}$. From our results, it will becomes clear for which sequences $\{p_n\}$ with $q_0 = \limsup_n p_n$ and $p_0 = \liminf_n p_n$ the space $\ell(\{p_n\})$ has cotype q_0 and type p_0.

Our main theorems (see Theorems 3.1 and 3.2 below) show that the cotype q_0 (resp. type p_0) of the space $\ell(\{p_n\},\{X_n\})$ really depends upon the uniform bound of the constants $C_{q_0}(X_n)$ (resp. $T_{p_0}(X_n)$) together with the existence of a constant $0 < C < 1$ such that $\sum_{p_n > q} C^{1/(p_n - q)} < \infty$ (resp. $\sum_{p_n < p} C^{1/(p - p_n)} < \infty$).

In particular, if $1 < p \le 2 \le q < \infty$, one obtains that $\ell(\{p_n\})$ has cotype q for $p_n = q + 1/\log(n+1)$ but not for $p_n = q + 1/\sqrt{\log(n+1)}$ and also, that $\ell(\{p_n\})$ has type p for $p_n = p - 1/\log(n+1)$ but not for $p_n = p - 1/\sqrt{\log(n+1)}$. In particular, we have that $\ell(\{p_n\})$ is isomorphic to ℓ^2 for $p_n = 2 + (-1)^n/\log(n+1)$.

The paper is divided into three sections. In Section 2, we present a characterization of the embedding $\ell(\{p_n\},\{X_n\}) \subseteq \ell(\{q_n\},\{Y_n\})$ that will be the key point in our considerations. In Section 3, we give the proof of the main results. Our proof will be direct and it will not be based upon the characterizations of type and cotype achieved in [7] for general Musielak–Orlicz spaces. In Section 4, we get some equivalent formulations of the conditions δ_q and δ^{*p} in the setting of Nakano sequence spaces, and also we get several equivalent formulations of the cotype q and type p conditions for these spaces.

2. PRELIMINARIES ON MUSIELAK–ORLICZ SPACES

We shall be using different properties and results from the general theory on Musielak–Orlicz spaces.

A condition on $\phi = (\phi_n)$ which is rather important in the study of these spaces is the so-called condition δ_2. Let us recall that $\phi \in \delta_2$ if there exist K, δ positive constants and $(c_n)_n \in \ell^1$ of nonnegative numbers such that

$$\phi_n(x) \leq \delta \Rightarrow \phi_n(2x) \leq K\phi_n(x) + c_n, \tag{1}$$

for every $n = 1, 2, \ldots$ and $x > 0$.

A related condition appears when looking at embeddings between different Musielak–Orlicz spaces.

THEOREM 2.1 ([11, Theorem 8.11]). *Let φ, ψ be two Musielak–Orlicz functions. The inclusion $\ell^\varphi \subseteq \ell^\psi$ holds if and only if there exist numbers $\delta > 0$, $K_1 > 0$, $K_2 > 0$, and a sequence $(a_n)_n$ of nonnegative numbers with $\sum_n a_n < \infty$ such that*

$$\varphi_n(u) < \delta \Rightarrow \psi_n(u) \leq K_1 \varphi_n(K_2 u) + a_n,$$

for $u \geq 0$ and $n = 1, 2, \ldots$.

Moreover, the norm convergence in ℓ^φ is stronger than the norm convergence in ℓ^ψ.

Some useful facts whose obvious proof is left to the reader are included in the following lemma.

LEMMA 2.1. *Let $(p_n)_n \subseteq [1, \infty)$, let $(X_n)_n$ be a sequence of Banach spaces, and let $x_n \in X_n$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$.*

(a) *The map* j_n *given by* $j_n(x_n) = x_n \otimes e_n$ *is an isometric embedding from* X_n *into the space* $\ell(\{p_n\},\{X_n\})$ *for each* $n = 1, 2, \dots$.

(b) *The map* J *given by* $J((\alpha_n)_n) = \sum_n \alpha_n x_n \otimes e_n$ *is an isometric embedding from* $\ell(\{p_n\})$ *into* $\ell(\{p_n\},\{X_n\})$.

The next theorem is essentially known, but we include here a proof for the sake of completeness.

THEOREM 2.2. *Let* $(p_n)_n, (q_n)_n \subseteq [1, \infty)$ *and let* $(X_n)_n, (Y_n)_n$ *be two families of Banach spaces. Then*

$$\ell(\{p_n\},\{X_n\}) \subseteq \ell(\{q_n\},\{Y_n\})$$

if and only if there exist $0 < C < 1$ *such that*

$$\sum_{p_n > q_n} C^{p_n q_n/(p_n - q_n)} < \infty$$

and inclusions $i_n\colon X_n \to Y_n$ *such that* $\sup_n \|i_n\| < \infty$.

Proof. Assume first that $\ell(\{p_n\},\{X_n\}) \subseteq \ell(\{q_n\},\{Y_n\})$. Let I be the inclusion map between those spaces and let us assume it has norm A. Using Lemma 2.1, we construct $i_n\colon X_n \to Y_n$ as the composition $\pi_n I j_n$, where π_n is the canonical projection from $\ell(\{q_n\},\{Y_n\})$ onto Y_n. Clearly, $\|i_n\| \le A$ for $n = 1, 2, \dots$.

Part (b) of Lemma 2.1 allows us to get that $\ell(\{p_n\}) \subseteq \ell(\{q_n\})$. Now, by Theorem 2.1, there exist $K_1, K_2, \delta > 0$ and $(a_n)_n \ge 0$ such that $(a_n)_n \in \ell^1$ verifying

$$0 \le x^{p_n} \le \delta \Rightarrow x^{q_n} \le K_1 (K_2 x)^{p_n} + a_n,$$

for every $n = 1, 2, \dots$.

We may assume without loss of generality $K_1 = K_2 = K > 1$ and $\delta = \frac{1}{K}$. Then

$$a_n \ge \max\{x^{q_n} - K^{p_n + 1} x^{p_n} : 0 \le x \le K^{-(1/p_n)}\}.$$

We put $f_n(x) := x^{q_n} - K^{p_n+1} x^{p_n}$.

For $n \in \mathbb{N}$ with $p_n \le q_n$, the inequality is redundant, because $f_n(x)$ is negative in $]0, 1[$.

For $n \in \mathbb{N}$ with $p_n > q_n$, it is clear that $f_n(x)$ is nonnegative in the interval $]0, (1/K^{p_n+1})^{1/(p_n - q_n)}$ [and it reaches its maximum at the point $x_{\max, n} = (q_n / p_n K^{p_n+1})^{1/(p_n - q_n)}$, which holds $0 < x_{\max, n} < K^{-(1/p_n)}$.

Then

$$\begin{aligned}
a_n &\geq \max\{f_n(x) : 0 \leq x \leq K^{-(1/p_n)}\} \\
&= f_n\left(\left(\frac{q_n}{p_n K^{p_n+1}}\right)^{1/(p_n-q_n)}\right) \\
&= \left(\frac{q_n}{p_n K^{p_n+1}}\right)^{q_n/(p_n-q_n)}\left(1 - \frac{q_n}{p_n}\right) \\
&= \left(\frac{q_n}{p_n}\right)^{q_n/(p_n-q_n)}\left(\frac{q_n^{1/q_n-1/p_n}}{K^{1+1/p_n}}\right)^{p_n q_n/(p_n-q_n)}\left(\frac{1}{q_n} - \frac{1}{p_n}\right).
\end{aligned}$$

Using the behavior of the functions $x^{x/(1-x)}$ in $[0,1]$ and $x^{1/x}$ in $[1,\infty)$, we easily get

$$e^{-1} \leq \left(\frac{q_n}{p_n}\right)^{q_n/(p_n-q_n)} \leq 1$$

and

$$\frac{1}{K^2} \leq \frac{q_n^{1/q_n-1/p_n}}{K^{1+1/p_n}} \leq \frac{q_n^{1/q_n}}{K} \leq \frac{e^{1/e}}{K}.$$

From these facts, we can find constants C_1 and C_2 such that

$$C_1 C_2^{p_n q_n/(p_n-q_n)}\left(\frac{1}{q_n} - \frac{1}{p_n}\right) \leq f_n(x_{\max,n}) \leq a_n.$$

Therefore, we have

$$\sum_{p_n>q_n} C_2^{p_n q_n/(p_n-q_n)}\left(\frac{1}{q_n} - \frac{1}{p_n}\right) < \infty.$$

If the set $\{n \in \mathbb{N}: p_n > q_n\}$ is finite, then there is nothing to prove. Otherwise, taking $0 < C < C_2$, we can say that the series

$$\sum_{p_n>q_n} C^{1/(1/q_n-1/p_n)}$$

converges.

For the converse, let us assume that series $\sum_{p_n>q_n} C^{p_n q_n/(p_n-q_n)}$ converges for some $0 < C < 1$ and take $(a_n)_n \in \ell(\{p_n\})$. Therefore, there exists $K_0 > 0$ such that $\sum_{n\in\mathbb{N}}(|a_n|/K)^{p_n} \leq 1$ for any $K \geq K_0$.

In order to see that $(a_n)_n \in \ell(\{q_n\})$, we shall show that $\sum_{n\in\mathbb{N}}(|a_n|/T)^{q_n} < \infty$ for any $T \geq K_0/C$.

For such a purpose, let us consider

$$A = \left\{n \in \mathbb{N}\colon p_n > q_n, |a_n| \le TC^{p_n/(p_n - q_n)}\right\}$$

and

$$B = \left\{n \in \mathbb{N}\colon p_n > q_n, |a_n| > TC^{p_n/(p_n - q_n)}\right\},$$

and now let us split the sum as follows:

$$\sum_{n \in \mathbb{N}} \left(\frac{|a_n|}{T}\right)^{q_n} = \sum_{p_n \le q_n} \left(\frac{|a_n|}{T}\right)^{q_n} + \sum_{n \in A} \left(\frac{|a_n|}{T}\right)^{q_n} + \sum_{n \in B} \left(\frac{|a_n|}{T}\right)^{q_n}.$$

Now, observe that

$$\sum_{p_n \le q_n} \left(\frac{|a_n|}{T}\right)^{q_n} \le \sum_{n \in \mathbb{N}} \left(\frac{|a_n|}{T}\right)^{p_n} \le 1,$$

$$\sum_{n \in A} \left(\frac{|a_n|}{T}\right)^{q_n} \le \sum_{p_n > q_n} C^{1/(1/q_n - 1/p_n)} < \infty$$

and

$$\begin{aligned}
\sum_{n \in B} \left(\frac{|a_n|}{T}\right)^{q_n} &= \sum_{n \in B} \left(\frac{|a_n|}{T}\right)^{p_n} \left(\frac{|a_n|}{T}\right)^{q_n - p_n} \\
&\le \sum_{n \in \mathbb{N}} \left(\frac{|a_n|}{T}\right)^{p_n} \left(\frac{1}{C}\right)^{p_n} \\
&\le \sum_{n \in \mathbb{N}} \left(\frac{|a_n|}{CT}\right)^{p_n} \le 1.
\end{aligned}$$

Therefore, are have $\ell(\{p_n\}) \subseteq \ell(\{q_n\})$.

Finally, if we have the inclusions $i_n\colon X_n \to Y_n$ uniformly bounded, then the map $\sum_n x_n \otimes e_n \mapsto \sum_n i_n(x_n) \otimes e_n$ is a bounded inclusion,

$$\begin{aligned}
\left\| \sum_n i_n(x_n) \otimes e_n \right\|_{\ell(\{q_n\}, \{Y_n\})} &= \left\| \sum_n \| i_n(x_n) \|_{Y_n} e_n \right\|_{\ell(\{q_n\})} \\
&\le C' \left\| \sum_n \| i_n(x_n) \|_{Y_n} e_n \right\|_{\ell(\{p_n\})} \\
&\le C'K \left\| \sum_n \|x_n\|_{X_n} e_n \right\|_{\ell(\{p_n\})} \\
&= C'K \left\| \sum_n x_n \otimes e_n \right\|_{\ell(\{p_n\}, \{X_n\})}
\end{aligned}$$

for some constants C' and K. ∎

COROLLARY 2.1 (see [12] for the scalar case). *Let $(p_n)_n, (q_n)_n \subseteq [1, \infty)$ and let $(X_n)_n, (Y_n)_n$ be two families of Banach spaces. Then we have $\ell(\{p_n\}, \{X_n\}) = \ell(\{q_n\}, \{Y_n\})$ if and only if $X_n = Y_n$ and there exist $0 < K, K' < \infty$ such that*

$$K \|x_n\|_{X_n} \leq \|x_n\|_{Y_n} \leq K' \|x_n\|_{X_n},$$

for all $n \in \mathbb{N}$ and $(x_n)_n \in \ell(\{p_n\}, \{X_n\})$ and there exists $0 < C < 1$ such that

$$\sum_{p_n \neq q_n} C^{1/|1/q_n - 1/p_n|} < \infty.$$

COROLLARY 2.2. *Let $1 \leq p < \infty$, $(p_n)_n \subseteq [1, \infty)$. Then*:

(i) $\ell(\{p_n\}) \subseteq \ell^p \Leftrightarrow \exists 0 < C < 1/\sum_{p_n > p} C^{1/(p_n - p)} < \infty$.

(ii) $\ell^p \subseteq \ell(\{p_n\}) \Leftrightarrow \exists 0 < C < 1/\sum_{p_n < p} C^{1/(p - p_n)} < \infty$.

3. PROOF OF THE MAIN THEOREM

Let us start by getting a useful necessary condition on the Nakano sequence spaces having cotype q.

LEMMA 3.1. *Let $2 \leq q < \infty$. If $\ell(\{p_n\}, \{X_n\})$ has cotype q, then*

(i) *X_n has cotype q for all $n = 1, 2, \ldots$ with* $\sup_n C_q(X_n) < \infty$, *and*

(ii) *there exists a constant $A > 0$ such that*

$$\inf\left\{\lambda > 0 \colon \sum_k \left(\frac{|a_k|}{\lambda}\right)^{p_{n_k}/q} \leq 1\right\} \geq A,$$

for all (a_n) such that $\sum_k |a_k| = 1$ and all $(n_k)_k \subseteq \mathbb{N}$.

Proof. From Lemma 2.1, it is obvious that $\ell(\{p_n\})$ and X_n have cotype q for $n = 1, 2, \ldots$. Moreover, $C_q(X_n)$ and $C_q(\ell(\{p_n\}))$ are bounded by $C_q(\ell(\{p_n\}, \{X_n\}))$ for $n = 1, 2, \ldots$.

Let $(n_k)_k \subseteq \mathbb{N}$ and let $(\alpha_k)_k$ be any real sequence with $\sum_k |\alpha_k|^q = 1$. Applying the condition of cotype for the vectors $x_k = \alpha_k \otimes e_{n_k}$ in $\ell(\{p_n\})$, we have

$$1 = \left(\sum_k |\alpha_k|^q\right)^{1/q} \leq C \int_0^1 \left\| \sum_k r_k(t) \alpha_k e_{n_k} \right\| dt \leq C \left\| \sum_k \alpha_k e_{n_k} \right\|.$$

Hence

$$\frac{1}{C} \leq \left\| \sum_k \alpha_k e_{n_k} \right\| = \inf\left\{\lambda > 0 \colon \sum_k \left(\frac{|\alpha_k|}{\lambda}\right)^{p_{n_k}} \leq 1\right\}.$$

Using now the notation $a_k = |\alpha_k|^q$, $k = 1, 2, \dots,$ the proof is finished taking $A = 1/C^{1/q}$. ∎

LEMMA 3.2. *Let $(p_n)_n \subseteq [1, \infty)$ and let $q \geq 2$. Then there exists $0 < \delta < 1$ such that, whenever $p_n > q$, the function*

$$\varphi_n(x) = \begin{cases} 2x^q - x^{p_n}, & 0 \leq x \leq \delta, \\ \varphi_n(\delta) + \varphi_n'(\delta^-)(x - \delta), & x > \delta, \end{cases}$$

is a Young function and $\varphi_n(x^{1/q})$ is a concave function.

Moreover, if $\ell(\{p_n\}) \subseteq \ell^q$ and

$$\phi_n(x) = \begin{cases} x^{p_n}, & p_n \leq q, \\ \varphi_n(x), & p_n > q, \end{cases}$$

that $\ell^{\phi} = \ell(\{p_n\})$ (with equivalent norms).

Proof. Let $n \in \mathbb{N}$ such that $p_n > q$. It is easy to check that the function $2x^q - x^{p_n}$ is convex in the interval $(0, x_n)$, where $x_n = (2q(q-1)/p_n(p_n - 1))^{1/(p_n - q)}$, and that $0 < \delta = \inf_{q < t < \infty}(2q(q-1)/t(t-1))^{1/(t-q)} < 1$. Hence, if we define

$$\varphi_n(x) = \begin{cases} 2x^q - x^{p_n}, & 0 \leq x \leq \delta, \\ \varphi_n(\delta) + \varphi_n'(\delta^-)(x - \delta), & x > \delta, \end{cases}$$

then φ_n is a Young function. Clearly, we also have that $\varphi_n(x^{1/q})$ is a concave function.

Let us now define the Musielak–Orlicz function $\phi = (\phi_n)$ given by

$$\phi_n(x) = \begin{cases} x^{p_n}, & p_n \leq q, \\ \varphi_n(x), & p_n > q. \end{cases}$$

First we shall prove, using Theorem 2.1, that $\ell^{\phi} \subseteq \ell(\{p_n\})$. For $n \in \mathbb{N}$ such that $p_n \leq q$, the inequality is obvious. For $n \in \mathbb{N}$ such that $p_n > q$, $\phi_n(x) \leq \delta^q$ yields $0 \leq x \leq \delta$ and then we get $\phi_n(x) = \varphi_n(x) = 2x^q - x^{p_n} \geq x^{p_n}$.

Finally, if we assume $\ell(\{p_n\}) \subseteq \ell^q$, we have that there exist $\delta_0 < 1$, $K_1, K_2 > 0$, and a sequence $(a_n)_n$ of nonnegative numbers with $(a_n)_n \in \ell^1$ such that, if $x^{p_n} \leq \delta_0$, then

$$x^q \leq K_1(K_2 x)^{p_n} + a_n.$$

A look at Corollary 2.2 shows that $(p_n)_n$ has to be bounded. Let δ' be defined as $\delta' = \min\{\delta_0, \delta^{\max_n p_n}\}$, and observe now that if $p_n > q$ and

$x^{p_n} \le \delta'$, then

$$\phi_n(x) = 2x^q - x^{p_n} \le 2x^q \le (2K_1)(K_2 x)^{p_n} + 2a_n.$$

Therefore, $\ell(\{p_n\}) \subseteq \ell^{\phi}$ and the proof is over. ∎

THEOREM 3.1. *Let $2 \le q < \infty$, let $(p_n)_n \subseteq [1,\infty)$, and let $(X_n)_n$ be a family of Banach spaces. Then the following assertions are equivalent*:

(1) *$\ell(\{p_n\},\{X_n\})$ has cotype q.*

(2) *X_n has cotype q for $n = 1,2,\dots$ with* $\sup_n C_q(X_n) < \infty$, *and there exists* $0 < C < 1$ *such that* $\sum_{p_n > q} C^{1/(p_n - q)} < \infty$.

Proof. Let us assume $\ell(\{p_n\},\{X_n\})$ has cotype q. Lemma 2.1 gives again that $\ell(\{p_n\})$ and X_n have cotype q and $\sup_n C_q(X_n) < \infty$. Let us assume now that, for any $0 < C < 1$, the series $\sum_n C^{m_n} = +\infty$, where $m_n = 1/(p_n/q - 1)$ if $p_n > q$, and $m_n = +\infty$ (or $C^{m_n} = 0$) if $p_n \le q$.

We shall see that, for any $0 < \varepsilon < 1$, we can find a sequence $(a_n)_n$ such that

$$\sum_n |a_n| = 1 \quad \text{and} \quad \sum_n \left(\frac{|a_n|}{\varepsilon}\right)^{p_n/q} \le 1.$$

This shows that

$$\inf_{\sum_k |a_k| = 1} \inf\left\{\lambda > 0 \colon \sum_k \left(\frac{|a_k|}{\lambda}\right)^{p_k/q} \le 1\right\} = 0,$$

which, according to Lemma 3.1, leads to a contradiction.

Given $\varepsilon > 0$, let $(a_n)_n$ be defined as follows: Since $\sum_n \varepsilon^{m_n} = \infty$, we can find $k \in \mathbb{N}$ so that

$$\sum_{n=1}^{k} \varepsilon^{m_n} \le \frac{1}{\varepsilon} \quad \text{and} \quad \sum_{n=1}^{k+1} \varepsilon^{m_n} > \frac{1}{\varepsilon}.$$

Let $a_n = \varepsilon\varepsilon^{m_n}$ for $n = 1,2,\dots,k$, $a_{k+1} = 1 - \varepsilon\sum_{n=1}^{k} \varepsilon^{m_n} \le \varepsilon\varepsilon^{m_{k+1}}$, and $a_n = 0$ for $n \ge k+2$.

Now $\sum_n |a_n| = 1$ and $(|a_n|/\varepsilon)^{1/m_n} \le \varepsilon$ hold trivially and

$$\sum_n \left(\frac{|a_n|}{\varepsilon}\right)^{p_n/q} = \sum_n \left(\frac{|a_n|}{\varepsilon}\right)\left(\frac{|a_n|}{\varepsilon}\right)^{1/m_n} \le \frac{1}{\varepsilon} \sum_n |a_n| \varepsilon = 1.$$

For the converse, invoke first part (i) in Corollary 2.2, to deduce that, in our situation, $\ell(\{p_n\}) \subseteq \ell^q$. Now applying Lemma 3.2, we can consider a

Musielak–Orlicz function $\phi = (\phi_n)$ such that $\ell^{\phi} = \ell(\{p_n\})$ with equivalent norms (denoted $\|.\|_{\phi}$ and $\|.\|$, respectively) and where the $\phi_n(x^{1/q})$ are concave.

Since the ϕ_n are convex functions, we have that $\sum_{n=1}^{\infty}\phi_n(|x_n|/1 + \sum\phi_n(|x_n|)) \leq 1$ and, therefore, $\|x\|_{\phi} \leq 1 + \sum_{n=1}^{\infty}\phi_n(|x(n)|)$ for any $x \in \ell^{\phi}$.

Let $M \in \mathbb{N}$ and let $y_1, y_2, \dots, y_M \in \ell(\{p_n\}, \{X_n\})$ and $(a_k)_{k=1}^{M}$ be positive scalars with $\sum_{k=1}^{M} a_k^{q'} = 1$, where $1/q + 1/q' = 1$. Then

$$\begin{aligned}
\sum_{k=1}^{M} a_k \|y_k\| &= \sum_{k=1}^{M} a_k^{q'} \left\| \frac{1}{a_k^{q'-1}} y_k \right\|_{\phi} \\
&\leq C_1 \sum_{k=1}^{M} a_k^{q'} \left\| \frac{1}{a_k^{q'-1}} y_k \right\|_{\phi} \\
&\leq C_1 \sum_{k=1}^{M} a_k^{q'} \left(1 + \sum_{n=1}^{\infty} \phi_n \left(\frac{\|y_k(n)\|_{X_n}}{a_k^{q'-1}} \right) \right) \\
&= C_1 \left(\sum_{k=1}^{M} a_k^{q'} + \sum_{k=1}^{M} \sum_{n=1}^{\infty} a_k^{q'} \phi_n \left(\frac{\|y_k(n)\|_{X_n}}{a_k^{q'-1}} \right) \right) \\
&= C_1 \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{M} a_k^{q'} \phi_n \left(\left(\frac{\|y_k(n)\|_{X_n}^q}{a_k^{(q'-1)q}} \right)^{1/q} \right) \right);
\end{aligned}$$

now using that $\phi_n(t^{1/q})$ are concave functions,

$$\begin{aligned}
&\leq C_1 \left(1 + \sum_{n=1}^{\infty} \phi_n \left(\left(\sum_{k=1}^{M} a_k^{q'} \frac{\|y_k(n)\|_{X_n}^q}{a_k^{(q'-1)q}} \right)^{1/q} \right) \right) \\
&\leq C_1 \left(1 + \sum_{n=1}^{\infty} \phi_n \left(\left(\sum_{k=1}^{M} \|y_k(n)\|_{X_n}^q \right)^{1/q} \right) \right).
\end{aligned}$$

Using, for every $n \in \mathbb{N}$, the formula of cotype with the elements $y_1(n), y_2(n), \dots, y_M(n)$ of X_n,

$$\left(\sum_{k=1}^{M} \|y_k(n)\|_{X_n}^q \right)^{1/q} \leq K \int_0^1 \left\| \sum_{k=1}^{M} r_k(t) y_k(n) \right\|_{X_n} dt,$$

and with the Minkowski integral inequality,

$$\begin{aligned}\left\| \left\{ \left(\sum_{k=1}^{M} \| y_k(n) \|_{X_n}^q \right)^{1/q} \right\}_{n=1}^{\infty} \right\|_{\phi} &\le K \left\| \left\{ \int_0^1 \left\| \sum_{k=1}^{M} r_k(t) y_k(n) \right\|_{X_n} dt \right\}_{n=1}^{\infty} \right\|_{\phi} \\ &\le K \int_0^1 \left\| \left\{ \sum_{k=1}^{M} r_k(t) y_k(n) \right\}_{n=1}^{\infty} \right\|_{\phi} dt.\end{aligned}$$

This allows us to conclude that there exists a positive constant C_2 such that, if $\int_0^1 \|\{\sum_{k=1}^M r_k(t) y_k(n)\}_{n=1}^\infty\|_\phi \, dt \le C_2$, then $\{(\sum_{k=1}^M \|y_k(n)\|_{X_n}^q)^{1/q}\}_{n=1}^\infty$ belongs to the unit ball of ℓ^ϕ.

Therefore, the previous estimates show that whenever we have the relation $\int_0^1 \|\sum_{k=1}^M r_k(t) y_k\| \, dt \le C_3$, it holds that $(\sum_{k=1}^M \|y_k\|^q)^{1/q} \le 2C_1$, where C_3 appears by the isomorphism between the norms $\|\cdot\|$ and $\|\cdot\|_\phi$. Now we can conclude that $\ell(\{p_n\}, \{X_n\})$ has cotype q. ∎

To deal with the notion of type on these spaces, we shall use some general duality arguments. We first point out the following simple fact.

LEMMA 3.3. *Let $\phi = (\phi_n)_n$ given by $\phi_n(x) = x^{p_n}$. Then ϕ satisfies the condition δ_2 if and only if $(p_n)_n$ is a bounded sequence.*

Proof. Assume the condition δ_2. Using (1), one has a constant $K > 0$ such that

$$2^{p_n} x^{p_n} \le K x^{p_n} + c_n$$

whenever $0 < x \le \delta^{1/p_n}$. This inequality applied to $x = \delta^{1/p_n}$ implies that $(p_n)_n$ is bounded.

For the converse implication, take $\delta = 1$, $c_n = 0$ for all $n \in \mathbb{N}$ and $K = 2^{\sup_n \{p_n\}}$. ∎

Remark 3.1. We recall that a Banach space is B-convex if and only if it has type p, for some $p > 1$ (see, for instance, [1]) and that, in the case of B-convex spaces, X having type p is equivalent to X^* having cotype p' (where $1/p + 1/p' = 1$) (see [1, 13]).

It was shown in [3] that ℓ^ϕ is B-convex if and only if ϕ and ϕ^* satisfy condition δ_2. Using Lemma 3.3, this result can be read as follows: $\ell(\{p_n\})$ is B-convex if and only if $1 < \liminf_n p_n \le \limsup_n p_n < \infty$.

THEOREM 3.2. *Let* $1 < p \leq 2$, *let* $(p_n)_n \subseteq [1, \infty)$, *and let* $(X_n)_n$ *be a family of Banach spaces. Then the following assertions are equivalent*:

(1) $\ell(\{p_n\}, \{X_n\})$ *has type* p.

(2) X_n *has type* p *for* $n = 1, 2, \ldots$ *with* $\sup_n T_p(X_n) < \infty$, $(p_n)_n$ *is bounded, and there exists* $0 < C < 1$ *such that* $\sum_{p_n < p} C^{1/(p - p_n)} < \infty$.

Proof. Assume $\ell(\{p_n\}, \{X_n\})$ has type p. Then every X_n and also $\ell(\{p_n\})$ has type p (see Lemma 2.1). Therefore, $\ell(\{p_n\})$ is B-convex, and Remark 3.1 implies that $(p_n)_n$ is bounded. Therefore, $\ell(\{p'_n\})$, which coincides with $\ell(\{p_n\})^*$ (see [11]), has cotype $p' < \infty$ (see [13]). From Theorem 3.1, we have a constant $C > 0$ such that $\sum_{p'_n > p'} C^{1/(p'_n - p')} < \infty$. Since

$$\sum_{p'_n > p'} C^{1/(p'_n - p')} = \sum_{p_n < p} \left(C^{(p_n - 1)(p - 1)}\right)^{1/(p - p_n)},$$

then we get (2).

Conversely, it is easy to prove, arguing as above, that $\ell(\{p_n\})$ has type p. Also we have that $\ell(\{p_n\})$ is p-convex and r-concave for some $r < \infty$ (see [7, Thms. 9a and 9b and Prop. 14]). Now a slight modification in the proof of [5, Thm. 4], that we shall present here, yields that $\ell(\{p_n\}, \{X_n\})$ has type p. We shall prove that there exists a constant $A > 0$ so that, for every finite family $y_1, y_2, \ldots, y_M \in \ell(\{p_n\}, \{X_n\})$, the inequality

$$\left(\int_0^1 \left\| \sum_{k=1}^M r_k(t) y_k \right\|^r dt\right)^{1/r} \leq A \left(\sum_{k=1}^M \|y_k\|^p\right)^{1/p}$$

holds. Therefore, denoting by $\|\cdot\|$ the norm in $\ell(\{p_n\}, \{X_n\})$, we can use (1) the r-concavity of $\ell(\{p_n\})$, (2) the uniform boundedness of type constants and the lattice structure of $\ell(\{p_n\})$, and (3) the p-convexity of $\ell(\{p_n\})$, to produce the chain of inequalities

$$\begin{aligned}
&\left(\int_0^1 \left\| \sum_{k=1}^M r_k(t) y_k \right\|^r dt\right)^{1/r} \\
&= \frac{1}{2^{M/r}} \left(\sum_{\varepsilon_i \in \{1, -1\}^M} \left\| \sum_k \varepsilon_i(k) y_k \right\|^r\right)^{1/r} \\
&= \frac{1}{2^{M/r}} \left(\sum_{\varepsilon_i \in \{1, -1\}^M} \left\| \left(\left\| \sum_k \varepsilon_i(k) y_k(j) \right\|_{X_j} \right)_j \right\|_{\ell(\{p_n\})}^r\right)^{1/r}
\end{aligned}$$

$$\text{(1)} \leq B\frac{1}{2^{M/r}}\left\|\left(\left(\sum_{\varepsilon_i\in\{1,-1\}^M}\left\|\sum_k \varepsilon_i(k)y_k(j)\right\|_{X_j}^r\right)^{1/r}\right)_j\right\|_{\ell(\{p_n\})}$$

$$= B\left\|\left(\left(\int_0^1\left\|\sum_{k=1}^M r_k(t)y_k(j)\right\|_{X_j}^r dt\right)^{1/r}\right)_j\right\|_{\ell(\{p_n\})}$$

$$\text{(2)} \leq BC\left\|\left(\left(\sum_k \|y_k(j)\|_{X_j}^p\right)^{1/p}\right)_j\right\|_{\ell(\{p_n\})}$$

$$\text{(3)} \leq BCD\left(\sum_k \left\|\left(\|y_k(j)\|_{X_j}\right)_j\right\|_{\ell(\{p_n\})}^p\right)^{1/p}$$

$$= BCD\left(\sum_k \|y_k\|^p\right)^{1/p}$$

for some constants $B, C, D > 0$. ∎

4. FINAL REMARKS

Let us mention that, in a much more general setting, it was found in [7] that the cotype q and the type p for the Musielak–Orlicz sequence spaces ℓ^ϕ can be described by the conditions called δ^q and δ^{*p} (together with δ_2), respectively.

DEFINITION 4.1 [7]. A Musielak–Orlicz function $\phi = (\phi_n)$ satisfies the condition δ^q ($q \geq 1$) if there are positive constants K, δ and a nonnegative sequence $(c_n)_n$ in ℓ^1 such that, for every $n \in \mathbb{N}$, $x > 0$ and $\lambda > 1$,

$$\phi_n(\lambda x) \leq K\lambda^q[\phi_n(x) + c_n] \tag{2}$$

whenever $\phi_n(\lambda x) \leq \delta$.

A Musielak–Orlicz function $\phi = (\phi_n)$ satisfies the condition δ^{*p} ($p \geq 1$) if there are positive constants K, δ and a nonnegative sequence $(c_n)_n$ in ℓ^1 such that, for every $n \in \mathbb{N}$, $x > 0$ and $\lambda > 1$,

$$\phi_n(\lambda x) \geq K\lambda^p[\phi_n(x) - c_n]$$

whenever $\phi_n(\lambda x) \leq \delta$.

THEOREM 4.1 ([7, Thm. 9a]). *Let $\phi = (\phi_n)_n$ be a Musielak–Orlicz function and let $2 \leq q < \infty$. The Musielak–Orlicz sequence space ℓ^ϕ is a space of cotype q if and only if ϕ satisfies the condition δ^q.*

THEOREM 4.2 ([7, Thm. 9b]). *Let* $1 < p \leq 2$ *and* $\phi = (\phi_n)_n$ *be a Musielak–Orlicz function satisfying*

$$\lim_{u \to 0} \phi_n(u)/u = 0 \quad \textit{and} \quad \lim_{u \to \infty} \phi_n(u)/u = \infty$$

for every $n \in \mathbb{N}$. *Then the Musielak–Orlicz sequence space* ℓ^ϕ *is a space of type p if and only if* ϕ *satisfies the conditions* δ_2 *and* δ^{*p}.

LEMMA 4.1. *Let* $\phi = (\phi_n)_n$ *given by* $\phi_n(x) = x^{p_n}$.

(i) $\phi = (\phi_n)_n$ *satisfies condition* δ^q *if and only if there exist a positive constant K and a sequence* $(c_n)_n \in \ell^1$ *of nonnegative numbers such that*

$$x^q \leq K(x^{p_n} + c_n)$$

for every $n \in \mathbb{N}$ *such that* $p_n > q$ *and* $0 < x \leq 1$.

(ii) $\phi = (\phi_n)_n$ *satisfies condition* δ^{*p} *if and only if there exist a positive constant K and a sequence* $(c_n)_n \in \ell^1$ *of nonnegative numbers such that*

$$x^p \geq K(x^{p_n} - c_n)$$

for every $n \in \mathbb{N}$ *such that* $p_n < p$ *and* $0 < x \leq 1$.

Proof. (i) Let us assume condition δ^q. Then (2) gives the existence of $K, \delta > 0$ and a nonnegative sequence $(c_n) \in \ell^1$ such that

$$\lambda^{p_n - q} x^{p_n} \leq K(x^{p_n} + c_n)$$

for all $x > 0$, $\lambda \geq 1$ such that $\lambda x \leq \delta^{1/p_n}$, $n \in \mathbb{N}$.

Taking $A = \max\{K, (1/\delta), 1\}$, we can say that

$$\lambda^{p_n - q} x^{p_n} \leq A(x^{p_n} + c_n)$$

whenever $\lambda x \leq A^{-(1/p_n)}$. Now fix $0 < x \leq A^{-(1/p_n)}$ and n such that $p_n > q$. Taking the supremum over $1 \leq \lambda \leq A^{-(1/p_n)}/x$, we get

$$(A^{1/p_n} x)^q \leq A^2(x^{p_n} + c_n).$$

In other words,

$$y^q \leq A y^{p_n} + A^2 c_n$$

for $0 < y \leq 1$ and n such that $p_n > q$.

Conversely, let us assume that there exist a positive constant K and a sequence $(c_n) \in \ell^1$ of nonnegative numbers such that

$$x^q \leq K(x^{p_n} + c_n)$$

for every n such that $p_n > q$ and $0 < x \leq 1$. Define $c'_n = 0$ for n such that $p_n \leq q$, $c'_n = c_n/K$ for n such that $p_n > q$, $\delta = 1$, and $K' = \max\{K^2, 1\}$. Let $\lambda \geq 1$, $x > 0$, with $\lambda x \leq 1$. Then, obviously, for $p_n \leq q$, one has that

$$(\lambda x)^{p_n} \leq \lambda^q x^{p_n} \leq K'\lambda^q(x^{p_n} + c'_n).$$

On the other hand, for $p_n > q$, one has that

$$(\lambda x)^{p_n} \leq (\lambda x)^q \leq \lambda^q K'(x^{p_n} + c'_n).$$

(ii) It is similar to (i) and is left to the interested reader. ∎

Now we can state the following equivalent formulations:

THEOREM 4.3. *Let $(p_n)_n \subseteq [1, \infty)$ and let $2 \leq q < \infty$. Then the following statements are equivalent*:

(i) *$\ell(\{p_n\})$ has cotype q.*

(ii) *$\ell(\{p_n\}) \subseteq \ell^q$.*

(iii) *There exist a positive constant K and a sequence $(c_n)_n \in \ell^1$ of nonnegative numbers such that*

$$x^q \leq K(x^{p_n} + c_n)$$

for every $n \in \mathbb{N}$ such that $p_n > q$ and $0 < x \leq 1$.

(iv) *There exists $0 < C < 1$ such that $\sum_{p_n > q} C^{1/(p_n - q)} < \infty$.*

THEOREM 4.4. *Let $(p_n)_n \subseteq]1, \infty)$ and let $1 < p \leq 2$. Then the following statements are equivalent*:

(i) *$\ell(\{p_n\})$ has type p.*

(ii) *$(p_n)_n$ is bounded and $\ell^p \subseteq \ell(\{p_n\})$.*

(iii) *$(p_n)_n$ is bounded and there exist a positive constant K and a sequence $(c_n)_n \in \ell^1$ of nonnegative numbers such that*

$$x^p \geq K(x^{p_n} - c_n)$$

for every $n \in \mathbb{N}$ such that $p_n < p$ and $0 < x \leq 1$.

(iv) *There exists $0 < C < 1$ such that $\sum_{p_n < p} C^{1/(p - p_n)} < \infty$ and $(p_n)_n$ is bounded.*

These theorems follow easily from Lemma 4.1, Corollary 2.2, and Theorems 4.1 and 4.2, but they can also be obtained, in a different way, as consequences of Theorems 3.1 and 3.2 and Corollary 2.2.

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