A note on Carleson measures for Hardy spaces

Oscar Blasco[(*)] **and Hans Jarchow**

Abstract We investigate Carleson measures μ on $\overline{\mathbb{D}}$ where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, along with functional analytic properties of the formal identitiy of the Hardy space $H^p(\mathbb{D})$ into the Lebesgue space $L^q(\mu)$, for any previously fixed $0<p,q<\infty$. Our corresponding characterizations do not only extend the classical results for measures concentrated on $\mathbb{D}$ but also provide different proofs for the latter ones. Among the applications are generalizations to formal identities as above of several results which have been known for composition operators only.

1. Introduction

We are going to work on the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane, its closure $\overline{\mathbb{D}}$ and the unit circle $\mathbb{T}=\partial\mathbb{D}$.

In the sequel, $dm\equiv dt/2\pi$ will be the Haar measure on the Borel subsets of $\mathbb{T}$. Given a Borel set $B\subseteq\mathbb{T}$, we shall often write $|B|$ instead of $m(B)$. The Lebesgue spaces $L^p(m)$ will also be denoted $L^p(\mathbb{T})$, $0<p\leq\infty$. The canonical norm (p-norm if $0<p<1$) on $L^p(\mathbb{T})$ is $\|\cdot\|_p$.

We denote by $\mathcal{H}(\mathbb{D})$ the space of all analytic functions $\mathbb{D}\to\mathbb{C}$. This is a Fréchet space with respect to the topology of local uniform convergence (uniform convergence on compact subsets of $\mathbb{D}$). By Montel's Theorem, bounded subsets of $\mathcal{H}(\mathbb{D})$ are relatively compact.

Let $f:\mathbb{D}\to\mathbb{C}$ be analytic. For each $0<r<1$, $f_r:\overline{\mathbb{D}}\to\mathbb{C}:z\mapsto f(rz)$ is continuous, analytic on $\mathbb{D}$, and $M_p(f,r):=\|f_r\|_p<\infty$ for all $0<p\leq\infty$. The classical *Hardy space* $H^p(\mathbb{D})$ consists of all analytic functions $f:\mathbb{D}\to\mathbb{C}$ such that $\|f\|_{H^p}:=\sup_{r<1}M_p(f,r)$ is finite. Again, we get a Banach space if $1\leq p<\infty$, and a p-Banach space if $0<p<1$. The usual Banach space of bounded analytic functions will be denoted by $H^\infty(\mathbb{D})$. If $0<q<p<\infty$, then $H^\infty(\mathbb{D})\hookrightarrow H^p(\mathbb{D})\hookrightarrow H^q(\mathbb{D})$ continuously with 'norm' one.

Recall that if f is in $H^p(\mathbb{D})$, for $0<p\leq\infty$, then $f^*(e^{it})=\lim\limits_{r\to1}f_r(e^{it})$ exists m-a.e. on $\mathbb{T}$ (Fatou's Theorem). Moreover, an element f^* of $L^p(\mathbb{T})$ is generated in this way, and $f\mapsto f^*$ defines

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an isometric embedding $H^p(\mathbb{D})\to L^p(\mathbb{T})$. Its range, $H^p(\mathbb{T})$, is the closure (weak*-closure if $p=\infty$) of the set of polynomials in $L^p(\mathbb{T})$. This leads to the identification of $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$, and to the use of H^p as a common symbol.

We find it justified, however, to be more explicit and look at $f\mapsto(f,f^*)$ as an isometric isomorphism of $H^p(\mathbb{D})$ onto the "diagonal" of $H^p(\mathbb{D})\oplus_\infty H^p(\mathbb{T})$. Accordingly, we denote by $f^\bullet$ the pair (f,f^*), as well as any measurable function $F:\overline{\mathbb{D}}\to\mathbb{C}$ such that $F|_{\mathbb{D}}=f$ and $F(e^{it})=f^*(e^{it})$ m-a.e. on $\mathbb{T}$. We may thus say that $f^\bullet$ is obtained by juxtaposition of f and (any representative of) f^*.

We will investigate *Carleson measures* on $\overline{\mathbb{D}}$, i.e. finite, positive Borel measures μ such that, for given values of $0<p,q<\infty$, the formal identity $J_\mu:H^p\to L^q(\mu):f\mapsto f$ exists. By the closed graph theorem, J_μ is then a bounded operator.

Let us start by two examples of such a situation. The first one is due to L. Carleson (see [4], [5]): Let $0<p<\infty$ and let (z_n) be a uniformly separated sequence in $\mathbb{D}$, that is, $\prod_{j=1,j\neq k}^{\infty}\frac{|z_k-z_j|}{|1-\bar z_j z_k|}\geq\delta$ holds for some $\delta>0$ and all $k\in\mathbb{N}$. Then

$$\Big(\sum_{n=1}^{\infty}(1-|z_n|)|f(z_n)|^p\Big)^{1/p}\ \leq\ C\,\|f\|_p \tag{1}$$

for all $f\in H^p(\mathbb{D})$.

The second is the Féjer-Riesz inequality (see [4], page 46): if $0<p<\infty$ then

$$\Big(\int_{-1}^{1}|f(x)|^p dx\Big)^{1/p}\ \leq\ C\,\|f\|_p \tag{2}$$

for all $f\in H^p(\mathbb{D})$.

Each of these examples furnishes us with a measure μ on $\mathbb{D}$ such that the formal identity $J_\mu:H^p(\mathbb{D})\to L^p(\mu):f\to f$ exists as a bounded operator. Such measures are characterized by a celebrated theorem due to L. Carleson. In order to state it we need some notation.

Let $\mathcal{I}$ be the collection of half-open intervals in $\mathbb{T}$ of the form $I=\{e^{it}:\theta_1\leq t<\theta_2\}$ where $0\leq\theta_1<\theta_2<2\pi$. With each $0\neq z\in\mathbb{D}$ we associate the interval $I(z)\in\mathcal{I}$ such that $|I(z)|=1-|z|$ and $z/|z|$ is the center of $I(z)$. Let $S(z)$ be the half-open *Carleson box* over $I(z)$ which has z on its 'inner arc'; we suppose this inner arc and the boundary part 'to the right' to belong to $S(z)$. For convenience, let us put $I(0)=\mathbb{T}$ and $S(0)=\mathbb{D}$.

Given $I\in\mathcal{I}$, we will also write $S(I)$ for the Carleson box $S(z_I)$ where $z_I\in\mathbb{D}$ is such that $1-|z_I|=|I|$ and $\zeta_I=z_I/|z_I|$ is the center of I.

Theorem 1.1. (see [4], [5]) *Let μ be a finite, positive measure on $\mathbb{D}$ and let $0<p<\infty$. Then $J_\mu : H^p(\mathbb{D}) \to L^p(\mu)$ exists if and only if*

$$\mu_{\mathbb{D}}(S(z)) \leq C|I(z)| \qquad \forall\, 0 \neq z \in \mathbb{D}.$$

Several classical inequalities are equivalent to the existence of J_μ as an operator $H^p(\mathbb{D}) \to L^q(\mu)$ when $p \leq q$.

An example is the following result of Hardy-Littlewood (see [4], page 87): if $0<p<q<\infty$ then

$$\Big(\int_{\mathbb{D}} (1-|z|)^{(q/p)-2} |f(z)|^q \, dA(z)\Big)^{1/q} \leq C\|f\|_p \tag{3}$$

for all $f \in H^p(\mathbb{D})$.

Another example concerns the Bergman space B^2 which consists of all $f \in \mathcal{H}(\mathbb{D})$ such that $\int_{\mathbb{D}} |f(z)|^2 \, dA(z)$ is finite. Here we use $dA(z)$ to denote normalized area measure $(dxdy)/\pi$ on the Borel subsets of $\mathbb{D}$. It is readily seen, using e.g. Hardy's inequality (see [4], page 48), that $H^1(\mathbb{D})$ embeds boundedly into B^2:

$$\Big(\int_{\mathbb{D}} |f(z)|^2 \, dA(z)\Big)^{1/2} \leq \|f\|_1 \tag{4}$$

for all $f \in H^2(\mathbb{D})$.

More generally, if we associate with each $\alpha > -1$ the probability measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ on $\mathbb{D}$, and if we take $0<p\leq q<\infty$, then $H^p(\mathbb{D})$ embeds boundedly into the 'weighted Bergman space' $B^q_\alpha := L^q(dA_\alpha) \cap \mathcal{H}(\mathbb{D})$ if and only if $\alpha+2 \geq q/p$. See C. Horowitz [6] and also [10].

P. Duren extended Carleson's Theorem 1.1 to the range $0<p<q<\infty$.

Theorem 1.2. (see [4], page 163) *Let μ be a finite, positive measure on $\mathbb{D}$ and let $0<p<q<\infty$. Then $J_\mu : H^p(\mathbb{D}) \to L^q(\mu)$ exists if and only if*

$$\mu_{\mathbb{D}}(S(z)) \leq C|I(z)|^{q/p} \qquad \forall\, 0 \neq z \in \mathbb{D}\ .$$

Now we turn to the case $0<q<p<\infty$. Recall that

$$(1-|z|^2)^{1/p} |f(z)| \leq \|f\|_p \tag{5}$$

for any $0<p<\infty$, $f \in H^p(\mathbb{D})$ and $z \in \mathbb{D}$. Thus, if $p>1$ and $g \in L^1(dA)$, then the measure $d\mu(z) = (1-|z|)^{1/p} g(z)\, dA(z)$ provides an example where $J_\mu : H^p(\mathbb{D}) \to L^1(\mu)$ exists and is bounded. Moreover, if $p>q$, then H^p embeds into $B^q_\alpha \subseteq L^q(A_\alpha)$ if and only if $\alpha+1 > q/p$; see again [6] and [10]. The complete characterization of measures of this type is due to I.V. Videnskii [14] and D.H. Luecking [11]. Let us denote by $\Gamma(\zeta)$ the *Stolz domain* generated by $\zeta \in \mathbb{T}$, i.e. the interior of the convex hull of $\{\zeta\} \cup (\alpha\overline{\mathbb{D}})$; here $0<\alpha<1$ is arbitrary, but fixed.

Theorem 1.3. (see [11], Theorem C) *Let μ be a finite, positive measure on $\mathbb{D}$ and let $0<q<p<\infty$. Then J_μ maps $H^p(\mathbb{D})$ into $L^q(\mu)$ if and only if the function*

$$\zeta \;\mapsto\; \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2} \quad \textit{belongs to} \quad L^{p/(p-q)}(\mathbb{T}) \;.$$

As we will see in this paper, it is actually important not to limit considerations to Carleson measures which are defined only on $\mathbb{D}$ but to take measures into account which live on all of the closed disk $\overline{\mathbb{D}}$. For instance, if $1/q=1/p+1/s$, then any measure concentrated on $\mathbb{T}$ which is of the form $d\mu=g\,dm$ with $g\in L^s(\mathbb{T})$ gives rise to a bounded embedding from $H^p(\mathbb{D})$ into $L^q(\mu)$.

Composition operators provide further examples. Let $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic function and let $0<p,q<\infty$ be arbitrary. The problem of characterizing those 'symbols' φ for which $C_\varphi : f\mapsto f\circ\varphi$ acts boundedly from $H^p(\mathbb{D})$ to $H^q(\mathbb{D})$, that is for which

$$\Big(\int_{\mathbb{T}} |f(\varphi(e^{i\theta}))|^q\, d\theta\Big)^{1/q} \;\le\; C\,\|f\|_p$$

holds for all $f\in H^p(\mathbb{D})$, can be rephrased in terms of image measures defined on $\overline{\mathbb{D}}$. Indeed, a Borel measure m_φ on $\overline{\mathbb{D}}$ is given by

$$m_\varphi(B) \;:=\; m((\varphi^*)^{-1}(B))$$

and

$$\int_{\overline{\mathbb{D}}} |f^\bullet|^q\, dm_\varphi \;=\; \int_{\mathbb{T}} |f^\bullet\circ\varphi^*|^q\, dm \;=\; \int_{\mathbb{T}} |(f\circ\varphi)^*|\, dm \;=\; \|C_\varphi f\|_{H^q(\mathbb{D})}$$

holds for all $f\in H^q(\mathbb{D})$. Thus $J_{m_\varphi}: H^p(\mathbb{D})\to L^q(m_\varphi)$ exists and is bounded if and only if C_φ maps $H^p(\mathbb{D})$ boundedly into $H^q(\mathbb{D})$.

In what follows we are going to present proofs of versions of Theorems 1.1, 1.2 and 1.3 which are valid for measures defined on $\overline{\mathbb{D}}$. We settle the case $p<q$ first and use this in turn to pass the case $p=q$. Our approach for the case $p<q$ relies upon the Hardy-Littlewood inequality (3) and is thus quite different from the classical one where arguments from the proof of Carleson's theorem (case $p=q$) are used to derive Duren's theorem (case $p<q$).

In Section 3 we are going to investigate compactness properties of Carleson measures μ on $\overline{\mathbb{D}}$ such that the *Carleson embedding* $J_\mu : H^p(\mathbb{D})\to L^q(\mu)$ exists and is compact, completely continuous, or weakly compact. Among others, we will see that several results which are known for composition operators on Hardy spaces can actually be extended to such embeddings.

In this paper, constants will generally be denoted by C; sometimes we will use indices. As a consequence, the value of $C\ldots$ may change at each occurrence, e.g. from line to line in chains of inequalities.

2. (p,q)-Carleson measures on $\overline{\mathbb{D}}$

Let us agree that, from now on, all measures on $\overline{\mathbb{D}}$, $\mathbb{D}$, or $\mathbb{T}$ will be finite, positive Borel measures. Let $0<p,q<\infty$. A measure μ on $\overline{\mathbb{D}}$ is called a (p,q)-*Carleson measure* if $f\mapsto f^{\bullet}$ defines a (linear, bounded) operator

$$J_\mu : H^p(\mathbb{D}) \to L^q(\mu) .$$

First we observe that this notion only depends on the ratio p/q.

Lemma 2.1. *Let $0<p,s,q<\infty$. A measure μ on $\overline{\mathbb{D}}$ is a (p,q)-Carleson measure if and only if it is an (sp,sq)-Carleson measure.*

Proof. It suffices to check one implication. Assume $J_\mu : H^p(\mathbb{D})\to L^q(\mu)$ is defined. Given $f\in H^{ps}(\mathbb{D})$ we write $f=bf_1$ where $f_1(z)\neq 0$ for all $z\in\mathbb{D}$ and b is the Blaschke product defined by the zeros of f. Consider $g=f_1^s\in\mathcal{H}(\mathbb{D})$. Then $\|g\|_p=\|f\|_{ps}$, hence

$$\int_{\overline{\mathbb{D}}} |f(z)|^{qs}\,d\mu(z) \le \int_{\overline{\mathbb{D}}} |g(z)|^{q}\,d\mu(z) \le C\,\|g\|_p^q = C\,\|f\|_{ps}^q . \qquad \square$$

We start by presenting a version of Theorem 1.3 for measures on $\overline{\mathbb{D}}$. The proof uses ideas from [11]. For each $0\neq z\in\mathbb{D}$, we denote by $\tilde{I}(z)$ the interval $\{\zeta\in\mathbb{T} : z\in\Gamma(\zeta)\}$. Clearly $z/|z|$ is the center of $\tilde{I}(z)$ and $|\tilde{I}(z)|\approx 1-|z|$. Let $P_z(\zeta)=\dfrac{1-|z|^2}{|1-z\bar{\zeta}|^2}$ be the Poisson kernel. It is easy to see that $\dfrac{\chi_{I(z)}}{1-|z|^2}\le C\,P_z$ and $\dfrac{\chi_{\tilde{I}(z)}}{1-|z|^2}\le C\,P_z$.

Given any measure μ on $\overline{\mathbb{D}}$, we denote by $\mu_{\mathbb{D}}$ and $\mu_{\mathbb{T}}$ its restrictions to the Borel subsets of $\mathbb{D}$ and $\mathbb{T}$, respectively. If μ is a (p,q)-Carleson measure then $J_{\mu_{\mathbb{D}}} : H^p(\mathbb{D})\to L^q(\mu_{\mathbb{D}}) : f\mapsto f$ and $J_{\mu_{\mathbb{T}}} : H^p(\mathbb{T})\to L^q(\mu_{\mathbb{T}}) : f^*\mapsto f^*$ are well-defined operators.

Theorem 2.2. *Let μ be a measure on $\overline{\mathbb{D}}$, $0<q<p<\infty$ and $s=p/(p-q)$. For μ to be a (p,q)-Carleson measure on $\overline{\mathbb{D}}$ it is necessary and sufficient that $\zeta\mapsto\displaystyle\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|^2}$ is a function in $L^s(\mathbb{T})$ and that $\mu_{\mathbb{T}}=F\,dm$ holds for some $F\in L^s(\mathbb{T})$.*

Proof. If μ is a (p,q)-Carleson measure on $\overline{\mathbb{D}}$ then, by Lemma 2.1, it is also a $(p/q,1)$-Carleson measure. We claim that $\zeta\mapsto\displaystyle\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|^2}$ defines a member of $L^s(\mathbb{T})$.

Since s is the conjugate exponent for p/q, it suffices to show that, given any $0\le g\in L^{p/q}(\mathbb{T})$ of norm one,

$$\int_{\mathbb{T}} g(\zeta)\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|^2}\,dm(\zeta) \le C . \tag{6}$$

Observe that

$$\int_{\mathbb{T}} g(\zeta)\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|^2}\,dm(\zeta) = \int_{\mathbb{D}}\Big(\int_{\tilde{I}(z)} g(\zeta)\,dm(\zeta)\Big)\frac{d\mu(z)}{1-|z|^2} .$$

Using that $\dfrac{1}{|\tilde{I}(z)|} \leq C P_z(\zeta)$ for $\zeta \in \tilde{I}(z)$ we get

$$\int_{\mathbb{D}} \Big(\int_{\tilde{I}(z)} g(\zeta)\, dm(\zeta) \Big) \frac{d\mu(z)}{1-|z|^2} \;\leq\; C \int_{\mathbb{D}} P(g)(z)\, d\mu(z) \;;$$

here $P(g)$ is the harmonic extension of g. Since $p/q > 1$ there is an $f \in H^{p/q}(\mathbb{D})$ such that $P(g) = \mathrm{Re}(f)$ and $\|f\|_{p/q} \leq C_p \|g\|_{p/q}$. This proves (6).

It is easy to see that $\mu_{\mathbb{T}} = F\, dm$ holds for some $F \in L^s(\mathbb{T})$. In fact, if $f \in \mathcal{H}(\mathbb{D}) \cap C(\mathbb{T})$ then $|\int_{\mathbb{T}} f\, d\mu_T| \leq \int_{\mathbb{T}} |f|\, d\mu_{\mathbb{T}} \leq C\, \|f\|_{p/q}$. Thus $\mu_{\mathbb{T}} \in (H^{p/q})^* = H^s$ and so $\mu_{\mathbb{T}} = F\, dm$, even for some $F \in H^s(\mathbb{T})$.

Suppose conversely that the function $\zeta \mapsto \displaystyle\int_{\Gamma(\zeta)} \frac{d\nu(z)}{1-|z|^2}$ belongs to $L^s(\mathbb{T})$ and that there is an $F \in L^s(\mathbb{T})$ such that $\mu_{\mathbb{T}} = F\, dm$. Then

$$\int_{\mathbb{T}} |f(\zeta)|^q\, d\mu(\zeta) \;=\; \int_{\mathbb{T}} |f(\zeta)|^q F(\zeta)\, dm(\zeta) \;\leq\; \|f\|_p^q\, \|F\|_s^q$$

and

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^q d\mu(z) &\approx \int_{\mathbb{D}} \Big(\int_{\tilde{I}(z)} dm(\zeta) \Big) |f(z)|^q \frac{d\mu(z)}{1-|z|^2} \\ &= \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)|^q \frac{d\mu(z)}{1-|z|^2}\, dm(\zeta) \\ &\leq C \int_{\mathbb{T}} (P(f)^*(\zeta))^q \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2}\, dm(\zeta) \;; \end{aligned}$$

here $P(f)^*$ is the maximal function given by $P(f)^*(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|$. By Hölder's inequality,

$$\int_{\mathbb{D}} |f(z)|^q\, d\mu(z) \;\leq\; C \Big(\int_{\mathbb{T}} (P(f)^*(\zeta))^p\, dm(\zeta) \Big)^{q/p} \int_{\mathbb{T}} \Big(\int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|^2})^s\, dm(\zeta) \Big)^{1/s} .$$

The proof is completed by recalling that $P(f)^* \in L^p$ if $f \in H^p$. □

Now we pass to the case $p \leq q$. The following contains one implication of the $\overline{\mathbb{D}}$-versions of Theorems 1.1 and 1.2.

Proposition 2.3. *Let μ be a measure on $\overline{\mathbb{D}}$ and let $0 < p \leq q < \infty$. If μ is (p,q)-Carleson measure then*

$$\max\{\mu_{\mathbb{D}}(S(z)), \mu_{\mathbb{T}}(I(z))\} \;\leq\; C\, |I(z)|^{q/p} \qquad \forall\; 0 \neq z \in \mathbb{D} \;.$$

Proof. Take $z \in \mathbb{D}$ and consider $f(w) = \dfrac{1}{(1-\bar{z}w)^{2/p}}$. Then $\|f\|_p = \dfrac{1}{(1-|z|^2)^{1/p}}$ and by assumption

$$\int_{\mathbb{D}} \frac{1}{|1-\bar{z}w|^{2q/p}}\, d\mu(w) \;\leq\; C\, \frac{1}{(1-|z|^2)^{q/p}} \;.$$

Notice that $w\in S(z)\cup I(z)$ implies $\left|w-\dfrac{z}{|z|}\right|\le C_1(1-|z|)$ and so $|w-z|\le 2C_1(1-|z|)$. Also, $|1-\bar{z}w|\le C_2\,(1-|z|^2)$. Hence

$$\frac{\mu_{\mathbb{D}}(S(z))+\mu_{\mathbb{T}}(I(z))}{(1-|z|^2)^{2q/p}} \;\le\; C_3\int_{\overline{\mathbb{D}}}\frac{1}{|1-\bar{z}w|^{2q/p}}\,d\mu(w) \;\le\; C\,\frac{1}{(1-|z|^2)^{q/p}}\;. \qquad \square$$

To obtain the complete version of Duren's Theorem 1.2 for measures on $\overline{\mathbb{D}}$, we use a characterization of Carleson measures in terms of the Poisson kernel and a theorem due to Hardy-Littlewood. The following lemma is a modification of Lemma 3.3 in [5] (see also [1] for a proof which covers even more general situations).

Lemma 2.4. *Let μ be a measure on $\overline{\mathbb{D}}$ and $0<\alpha<\beta$. Then*

$$\max\{\mu_{\mathbb{D}}(S(z)),\mu_{\mathbb{T}}(I(z))\} \;\le\; C\,|I(z)|^{\alpha} \qquad \forall\; 0\neq z\in\mathbb{D}$$

if and only if

$$\sup_{|z|<1}\int_{\overline{\mathbb{D}}}\frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\,d\mu(w) \;<\; \infty\;.$$

Proof. Suppose that the supremum exists and fix $0\neq z\in\mathbb{D}$. Recall that $|1-\bar{w}z|\le C\,(1-|z|^2)$ for $w\in S(z)\cup I(z)$. It follows that

$$\frac{\mu_{\mathbb{D}}(S(z))}{(1-|z|^2)^{\alpha}}+\frac{\mu_{\mathbb{T}}(I(z))}{(1-|z|^2)^{\alpha}} \;\le\; C\int_{\overline{\mathbb{D}}}\frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\,d\mu(w)\;.$$

Assume now that $\max\{\mu_{\mathbb{D}}(S(z)),\mu_{\mathbb{T}}(I(z))\}\le C\,|I(z)|^{\alpha}$ holds for all $0\neq z\in\mathbb{D}$. In order to verify the boundedness of our integrals it suffices to look at z's which are 'big', e.g. $|z|>3/4$. Fix such a z and consider the sets

$$E_n \;=\; \left\{w\in\overline{\mathbb{D}}:\left|w-\frac{z}{|z|}\right|<2^n(1-|z|)\right\} \qquad (n=0,1,2,\ldots)\;.$$

These sets form an increasing sequence for which a smallest integer $n(z)$ exists such that $E_n=\overline{\mathbb{D}}$ if $n>n(z)$. We put $z_n=0$ if $n>n(z)$. For each $1\le n\le n(z)$ there is a $z_n\in D$ (a multiple of $z/|z|$) such that $E_n\subseteq S(z_n)\cup I(z_n)$ and $1-|z_n|\le 2^n\,(1-|z|)$. Thanks to our assumption

$$\mu(E_n) \;\le\; C\,2^{n\alpha}(1-|z|)^{\alpha}$$

for all n. On the other hand, using that $1-|z|\le|1-\bar{w}z|$ and that $|1-\bar{w}z|\ge C\,2^n(1-|z|^2)$ for $w\notin E_n$, we get

$$\begin{aligned}\int_{\overline{\mathbb{D}}}\frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\,d\mu(w) &\le \int_{E_0}\frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\,d\mu(w) + \sum_{n=1}^{n(z)}\int_{E_n\setminus E_{n-1}}\frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\,d\mu(w)\\ &\le C\,\frac{\mu(E_0)}{(1-|z|^2)^{\alpha}} + C\sum_{n=1}^{n(z)}\frac{\mu(E_n\setminus E_{n-1})}{2^{\beta n}(1-|z|^2)^{\alpha}}\\ &\le C + C\sum_{n=1}^{\infty}\frac{1}{2^{n(\beta-\alpha)}} \;<\; \infty\;. \qquad \square\end{aligned}$$

Theorem 2.5. *Let $0<p<q<\infty$. A measure μ on $\overline{\mathbb{D}}$ is a (p,q)-Carleson measure if and only if $\mu_{\mathbb{T}}=0$ and there exists a $C>0$ such that*

$$\mu_{\mathbb{D}}(S(z)) \;\le\; C\,|I(z)|^{q/p} \qquad \forall\; 0\neq z\in\mathbb{D}\ .$$

Proof. Assume that μ is a (p,q)-Carleson measure. Then the condition on $\mu_{\mathbb{D}}$ follows from Proposition 2.3. To verify $\mu_{\mathbb{T}}=0$ we observe that since $p/q<1$ and since every open set $\Omega\subseteq\mathbb{T}$ is the union of countably many disjoint intervals $I(z)$, we may conclude that $\mu_{\mathbb{T}}(\Omega)^{p/q}\le C\,|\Omega|$. By the regularity of the involved measures, we even get $\mu_{\mathbb{T}}(B)^{p/q}\le C\,|B|$ for all Borel sets $B\subseteq\mathbb{T}$. In particular, $\mu_{\mathbb{T}}\ll m$ and so $d\mu_{\mathbb{T}}=F\,dm$ for some $F\in L^1(m)$. Now we apply the Lebesgue differentiation theorem to get

$$F(\zeta) \;=\; \lim_{|I|\to 0,\zeta\in I}\frac{1}{|I|}\int_I F\,dm \;\le\; \lim_{|I|\to 0,\zeta\in I}|I|^{(q/p)-1} \;=\; 0$$

m-a.e., whence $\mu_{\mathbb{T}}=0$.

Suppose now that conversely $\mu_{\mathbb{D}}(S(z))\le C\,|I(z)|^{q/p}$ $\forall 0\neq z\in\mathbb{D}$ and that $\mu_{\mathbb{T}}=0$. Accordingly, we can identify μ and $\mu_{\mathbb{D}}$. By Lemma 2.1, boundedness of $J_\mu : H^p(\mathbb{D})\to L^q(\mu)$ is equivalent to boundedness of $J_\mu : H^{p/q}(\mathbb{D})\to L^1(\mu)$, and from Lemma 2.4 we know that, for any $\beta>q/p$,

$$\sup_{|z|<1}\int_{\mathbb{D}}\frac{(1-|z|^2)^{\beta-(q/p)}}{|1-\bar{w}z|^{\beta}}\,d\mu(z) \;<\; \infty\ .$$

We use the following reproducing formula for $\beta\ge 2$ (see [15], p.53): if $z\in\mathbb{D}$ then

$$f(z) \;=\; C_\beta\int_{\mathbb{D}}\frac{(1-|w|^2)^{\beta-2}f(w)}{(1-\bar{w}z)^{\beta}}\,dA(w)$$

for appropriate analytic functions f. In particular, if $f\in H^{p/q}(\mathbb{D})$ and $\beta>\max\{q/p,2\}$ then

$$\begin{aligned}
\int_{\overline{\mathbb{D}}}|f(z)|d\mu(z) \;&\le\; C_\beta\int_{\overline{\mathbb{D}}}\int_{\mathbb{D}}\frac{(1-|w|^2)^{\beta-2}|f(w)|}{|1-\bar{w}z|^{\beta}}\,dA(w)d\mu(z)\\
&=\; C_\beta\int_{\mathbb{D}}\Big(\int_{\overline{\mathbb{D}}}\frac{d\mu(z)}{|1-\bar{w}z|^{\beta}}\Big)(1-|w|^2)^{\beta-2}|f(w)|dA(w)\\
&\le\; C\int_{\mathbb{D}}|f(w)|(1-|w|)^{(q/p)-2}dA(w)\ .
\end{aligned}$$

Now appeal to (3) to obtain $\int_{\mathbb{D}}|f(z)|\,d\mu(z)\le C\,\|f\|_{p/q}$. The proof is complete. □

Finally we proceed to the $\overline{\mathbb{D}}$-version of Carleson's theorem 1.1.

Theorem 2.6. *Let $0<p<\infty$. A measure μ on $\overline{\mathbb{D}}$ is a (p,p)-Carleson measure if and only if there exists $F\in L^\infty(m)$ such that $\mu_{\mathbb{T}}=F\,dm$ and $\mu_{\mathbb{D}}(S(z))\le C\,|I(z)|$ for all $0\neq z\in\mathbb{D}$ and some constant $C>0$.*

Proof. The direct implication is Proposition 2.3. As in the proof of Theorem 2.5 we obtain $\mu_{\mathbb{T}}=F\,dm$ for some $F\ge 0$ in $L^1(m)$. Writing $\int_B F\,dm=\mu_{\mathbb{T}}(B)\le C\,|B|=C\int_B dm$ for Borel sets $B\subseteq\mathbb{T}$, we see that $F\le C$ m-a.e.

Assume next that $\mu=\mu_{\mathbb{D}}+F\,dm$ for some $F\in L^\infty(m)$ and that $\mu_{\mathbb{D}}(S(z))\le C|I(z)|\ \forall 0\ne z\in\mathbb{D}$. We shall prove that $J_\mu: H^2(\mathbb{D})\to L^2(\mu)$ is bounded, equivalently, that $J_\mu: H^2(\mathbb{D})\to L^1(g\mu)$ is bounded uniformly in the functions $g\ge 0$ from the unit ball of $L^2(\mu)$. Given such a g, put $d\nu=g\,d\mu$. By Therorem 2.2, we need to show that $\zeta\mapsto\int_{\Gamma(\zeta)}\frac{g(z)\,d\mu(z)}{1-|z|^2}$ is a member of $L^2(\mathbb{T})$ and that $\nu_{\mathbb{T}}=G\,dm$ for some $G\in L^2(\mathbb{T})$.

On the one hand, $\int_{\mathbb{T}}|g(\zeta)|^2 d\mu_T(\zeta)\le\int_{\overline{\mathbb{D}}}|g(w)|^2\,d\mu(w)<\infty$ so that $G(\zeta)=g(\zeta)F(\zeta)$ defines a member of $L^2(\mathbb{T})$.

On the other hand, if $0\le h\in L^2(\mathbb{T})$ then

$$\int_{\mathbb{T}}h(\zeta)\Big(\int_{\Gamma(\zeta)}\frac{g(z)\,d\mu(z)}{1-|z|^2}\Big)\,dm(\zeta)\ =\ \int_{\mathbb{D}}\Big(\int_{\tilde I(z)}h(\zeta)dm(\zeta)\Big)\frac{g(z)\,d\mu(z)}{1-|z|^2}\ .$$

Writing $H(z)=\frac{1}{|I(z)|}\int_{\tilde I(z)}h(\zeta)\,dm(\zeta)$ we have

$$\int_{\mathbb{T}}h(\zeta)\Big(\int_{\Gamma(\zeta)}\frac{g(z)\,d\mu(z)}{1-|z|^2}\Big)\,dm(\zeta)\ \approx\ \int_{\mathbb{D}}H(z)\,g(z)\,d\mu(z)\ .$$

All we need to show then is that $h\mapsto H$ defines a bounded operator $L^2(\mathbb{T})\to L^2(\nu)$.

Define $\mathcal{M}H(\zeta)=\sup_{z\in\Gamma(\zeta)}H(z)$ and note that if $H(z)>\lambda$ then $\mathcal{M}H(\zeta)>\lambda$ for $\zeta\in\tilde I(z)$. Represent $\{\mathcal{M}H(\zeta)>\lambda\}$ as a disjoint union of intervals $I_j\in\mathcal{I}$ and denote by S_j the corresponding Carleson boxes. Put $A_j=\{z\in\overline{\mathbb{D}}:\tilde I(z)\subseteq I_j\}$ and note that $A_j\subseteq\tilde S_j$ where $\tilde S_j$ is an appropriate dilated copy of S_j. Now the assumption $\mu_{\mathbb{D}}(S_j)\le C|I_j|$ yields

$$\mu_{\mathbb{D}}\{H(z)>\lambda\}\ \le\ \sum_j\mu_{\mathbb{D}}(A_j)\ \le\ C\sum_j|I_j|\ \le\ C|\{\mathcal{M}H(\zeta)>\lambda\}|\ . \tag{7}$$

Since $\mathcal{M}H(\zeta)\le C\,(P(h))^*(\zeta)=C\sup_{z\in\Gamma(\zeta)}P(h)(z)$ we obtain the desired result, using a Hardy-Littlewood weak type (1,1)-estimate and interpolation with the trival boundedness from $L^\infty(\mathbb{T})$ to $L^\infty(\mu_{\mathbb{D}})$. Details are e.g. in J.B. Garnett's book [5]. □

3. Compactness of Carleson measures

Let X and Y be quasi-Banach spaces, X with a separating dual (e.g., $X=H^p$, $0<p\le\infty$). An operator $u:X\to Y$ is called *completely continuous* if $\lim_n\|ux_n\|_Y=0$ holds for every weak null sequence (x_n) in X.

Compact operators are completely continuous, and completely continuous operators with domain e.g. a reflexive Banach space are compact. On the other hand, there are infinite dimensional Banach spaces whose identity is completely continuous, the most prominent example being the sequence space ℓ^1; see e.g. [3], page 6.

We say that a measure μ on $\overline{\mathbb{D}}$ is a *compact* (p,q)*-Carleson measure* if the formal identity $J_\mu: H^p(\mathbb{D})\to L^q(\mu)$ exists as a compact operator.

Proposition 3.1. *Let $0<p,q<\infty$. If J_μ is completely continuous (in particular if μ is a compact (p,q)-Carleson measure on $\overline{\mathbb{D}}$) then $\mu_{\mathbb{T}}=0$.*

PROOF. The sequence (z^n) of monomials is a weak null sequence in $H^p(\mathbb{D})$ so that $\lim_n \int_{\overline{\mathbb{D}}} |z^n|^q \, d\mu = 0$ by hypothesis. But $\mu(\mathbb{T}) = \int_{\mathbb{T}} |z^n|^q \, d\mu$ for all n, whence $\mu(\mathbb{T})=0$. □

As in the case boundedness, compactness of Carleson measures only depends on p/q.

Proposition 3.2. *Let $0<p,q,r<\infty$. A measure μ on $\overline{\mathbb{D}}$ is a compact (p,q)-Carleson measure if and only if it is a compact (rp,rq)-Carleson measure.*

PROOF. If $J_\mu : H^p(\mathbb{D}) \to L^q(\mu)$ is compact then $\mu_{\mathbb{T}}=0$, as we have just seen. Let (f_n) be a bounded sequence in $H^{rp}(\mathbb{D})$. By Montel's Theorem and Fatou's Lemma, there is no loss of generality to assume that (f_n) converges to 0 locally uniformly. For each n write $f_n = b_n g_n$ where b_n is a Blaschke product and $g_n \in H^{rp}$ has no zeros. Again, we may assume that (b_n) and (g_n) converge locally uniformly to some $b \in H^\infty(\mathbb{D})$ and $g \in H^{rp}(\mathbb{D})$, respectively. A well-known theorem of Hurwitz tells us that g either has no zeros or vanishes identically. But since $b(z)\,g(z)=0 \ \forall z \in \mathbb{D}$ and since the b_n's are Blaschke products, the second alternative applies.

Now (g_n^r) is a bounded sequence in $H^p(\mathbb{D})$ which converges pointwise to zero. By hypothesis, $\lim_k \|g_{n_k}^r\|_{L^q(\mu)} = 0$ for suitably chosen n_k, and so $\lim_k \|f_{n_k}\|_{L^{rq}(\mu)} = 0$. □

Lemma 2.4 and Theorem 2.5 have "little o" counterparts.

Lemma 3.3. *Let μ be a measure on $\overline{\mathbb{D}}$ and let $0<\alpha<\beta$. Then*

$$\lim_{I \in \mathcal{I}, |I| \to 0} \frac{\max\{\mu_{\mathbb{D}}(S(I)), \mu_{\mathbb{T}}(I)\}}{|I|^\alpha} \;=\; 0$$

if and only if

$$\lim_{|z| \to 1} \int_{\overline{\mathbb{D}}} \frac{(1-|z|^q)^{\beta-\alpha}}{|1-\bar{w}z|^\beta} \, d\mu(w) \;=\; 0 \;.$$

PROOF. The "if" implication is obtained as in Lemma 2.4:

$$\frac{\mu_{\mathbb{D}}(S(I))}{|I|^\alpha} + \frac{\mu_{\mathbb{T}}(I)}{|I|^\alpha} \;\le\; C \int_{\overline{\mathbb{D}}} \frac{(1-|z_I|^2)^{\beta-\alpha}}{|1-\bar{w}z_I|^\beta} \, d\mu(w) \qquad \forall\, I \in \mathcal{I} \;.$$

As for the converse, take any $\varepsilon>0$ and choose $\delta>0$ such that

$$\max\{\mu_{\mathbb{D}}(S(z)), \mu_{\mathbb{T}}(I(z))\} \;\le\; \varepsilon\, |I(z)|^\alpha$$

whenever if $|I(z)|<\delta$. As in the proof of 2.4, fix $|z|>3/4$ and consider the sets

$$E_n \;=\; \left\{ w \in \overline{\mathbb{D}} : \left| w - \frac{z}{|z|} \right| < 2^n (1-|z|) \right\} \;, \quad n=0,1,2,\ldots \;.$$

Recall that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\, d\mu(w) \;\leq\; C\, \frac{\mu(E_0)}{(1-|z|^2)^{\alpha}} \;+\; C \sum_{n=1}^{\infty} \frac{\mu(E_n \setminus E_{n-1})}{2^{\beta n}(1-|z|^2)^{\alpha}}$$

and that, independent of z, the series is majorized by a convergent series. Accordingly, there is an n_0 such that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\, d\mu(w) \;\leq\; C\, \frac{\mu(E_0)}{(1-|z|^2)^{\alpha}} \;+\; C \sum_{n=1}^{n_0} \frac{\mu(E_n \setminus E_{n-1})}{2^{\beta n}(1-|z|^2)^{\alpha}} \;+\; \varepsilon\ .$$

Now take any $0<\delta'<\delta 2^{-n_0}$. For $1-|z|<\delta'$ and $n\geq n_0$, our assumption yields

$$\mu(E_n) \;\leq\; \varepsilon\, 2^{n\alpha}\, (1-|z|)^{\alpha}\ ,$$

so that, whenever $|z|<1-\delta'$,

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\beta-\alpha}}{|1-\bar{w}z|^{\beta}}\, d\mu(w) \;\leq\; C\varepsilon\ . \qquad \square$$

Theorem 3.4. *Let $0<p\leq q<\infty$. A measure μ on $\overline{\mathbb{D}}$ is a compact (p,q)-Carleson measure if and only if $\mu_{\mathbb{T}}=0$ and*

$$\lim_{|I|\to 0} \frac{\mu_{\mathbb{D}}(S(I))}{|I|^{q/p}} \;=\; 0\ .$$

Proof. Assume μ on $\overline{\mathbb{D}}$ is a compact (p,q)-Carleson measure. By Proposition 3.1 one gets $\mu_{\mathbb{T}}=0$. Select an increasing sequence (r_n) of positive numbers with limit 1 and define unit vectors $f_n\in H^p(\mathbb{D})$ by $f_n(w)=\dfrac{(1-r_n^2)^{1/p}}{(1-r_n w)^{2/p}}$, $w\in\mathbb{D}$, $n\in\mathbb{N}$. By assumption, some subsequence $(f_{r_{n_k}})_k$ converges in $L^q(\mu)$. But (f_n) converges pointwise to zero so that

$$\lim_{k\to\infty} \int_{\mathbb{D}} \frac{(1-r_{n_k}^2)^{q/p}}{|1-r_{n_k} w|^{2q/p}}\, d\mu(w) \;=\; 0\ .$$

The direct implication follows easily by invoking Lemma 3.3.

Assume conversely that $\mu_{\mathbb{T}}=0$ and $\displaystyle\lim_{I\in\mathcal{I},|I|\to 0} \frac{\mu_{\mathbb{D}}(S(I))}{|I|^{q/p}}=0$. By Theorem 2.5, μ is a (p,q)-Carleson measure.

First we settle case $p<q$. By Propositon 3.2 we are done once we have shown that $J_{\mu_{\mathbb{D}}}: H^{p/q}(\mathbb{D})\to L^1(\mu_{\mathbb{D}})$ is compact. To this end, we fix $\beta>\max\{q/p,2\}$ and use Lemma 3.3 to find, given any $\varepsilon>0$, a $\delta>0$ such that if $1-|w|<\delta$ then

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta-(q/p)}}{|1-\bar{w}z|^{\beta}}\, d\mu(z) \;<\; \varepsilon\ .$$

Repeating an argument from the proof of Theorem 2.5 we get, for any $f\in H^{p/q}(\mathbb{D})$,

$$\begin{aligned}\int_{\mathbb{D}}|f(z)|d\mu(z) &\leq C\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|w|)^{\beta-2}|f(w)|}{|1-\bar{w}z|^{\beta}}dA(w)d\mu(z)\\ &\leq C\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{d\mu(z)}{|1-\bar{w}z|^{\beta}}\Big)(1-|w|)^{\beta-2}|f(w)|dA(w)\\ &\leq C\int_{|w|\leq 1-\delta}|f(w)|(1-|w|)^{(q/p)-2}dA(w)\\ &\quad + C\varepsilon\int_{|w|>1-\delta}|f(w)|(1-|w|)^{(q/p)-2}dA(w)\\ &\leq C\int_{|w|\leq 1-\delta}|f(w)|(1-|w|)^{(q/p)-2}dA(w) + C\varepsilon\|f\|_{p/q}\ .\end{aligned}$$

Let now (f_n) be a bounded sequence in $H^{p/q}(\mathbb{D})$. Again we can assume, by Montel's Theorem and Fatou's Lemma, that (f_n) converges to zero locally uniformly. In combination with the previous estimates this finishes the proof when $p<q$.

Now we pass to the case $p=q$. By Proposition 3.2 it is enough to deal with $p=q=2$ and to show that $\lim_{n\to\infty}\int_{\mathbb{D}}|f_n(z)|^2\,d\mu(z)=0$ holds for any weak null sequence (f_n) in H^2.

For $0<r<1$, write $d\mu_r=\chi_{\{|z|\leq r\}}d\mu$. Then $d\mu-d\mu_r=\chi_{\{r<|z|<1\}}\,d\mu$ and

$$\int_{\mathbb{D}}|f_n(z)|^2\,d\mu(z) = \int_{\mathbb{D}}|f_n(z)|^2\,d\mu_r(z) + \int_{\mathbb{D}}|f_n(z)|^2\,(d\mu-d\mu_r)(z)\ .$$

Apply Theorem 2.6 to the (2,2)-Carleson measure $\mu-\mu_r$ to get

$$\Big(\int_{\mathbb{D}}|f_n(z)|^2\,(d\mu-d\mu_r)(z)\Big)^{1/2} \leq C\sup_{I\in\mathcal{I}}\frac{(\mu-\mu_r)S(I)}{|I|}\|f_n\|_2\ .$$

Observe now that, given $I\in\mathcal{I}$ and $0<r<1$ such that $1-r<|I|$,

$$S(I)\cap\{z:r<|z|<1\} = \bigcup_{k=1}^{N(I)}S(I_k)$$

where $N(I)\approx\frac{|I|}{1-r}$ and the I_k are pairwise disjoint intervals of equal length $1-r$ (except perhaps one interval of length $\leq 1-r$). Note that

$$\frac{(\mu-\mu_r)(S(I))}{|I|} = \frac{\mu(S(I)\cap\{z:r<|z|<1\})}{|I|} \leq \sup_{|J|\leq 1-r}\frac{\mu(S(J))}{|J|}\ .$$

Since (f_n) is bounded in H^2, the assumption shows that, given $\varepsilon>0$, there exists an $r_0<1$ such that for any $r_0<r<1$

$$\int_{\mathbb{D}}|f_n(z)|^2\,(d\mu-d\mu_r)(z) < \varepsilon/2\ .$$

Since (f_n) converges locally uniformly to zero, there exists an n_0 such that

$$\int_{\mathbb{D}}|f_n(z)|^2\,d\mu_{r_0}(z) < \varepsilon/2$$

for $n\geq n_0$. The proof is complete. □

Corollary 3.5. *If $0<p\le r<q<\infty$ then every (p,q)-Carleson measure on $\overline{\mathbb{D}}$ is a compact (p,r)-Carleson measure.*

Next we take a closer look at complete continuity of Carleson embeddings. We start by a general result:

Theorem 3.6. *Let μ be a (p,q)-Carleson measure on $\overline{\mathbb{D}}$, $0<p,q<\infty$. Then $\mu_{\mathbb{T}}=0$ if and only if $J_\mu: H^{p/q}(\mathbb{D})\to L^1(\mu)$ is completely continuous.*

Proof. If $\mu_{\mathbb{T}}=0$ then $\mu=\mu_{\mathbb{D}}$ and J_μ is just the map $H^{p/q}(\mathbb{D})\to L^1(\mu_{\mathbb{D}}): f\mapsto f$. Let (f_n) be a weak null sequence in $H^{p/q}(\mathbb{D})$. By continuity of point evaluations, $\lim_n f_n(z)=0\ \forall z\in\mathbb{D}$. Also, (f_n) is uniformly integrable in $L^1(\mu_{\mathbb{D}})$: given $\varepsilon>0$ there is a $\delta>0$ such that if $B\subseteq\mathbb{D}$ is any Borel set with $\mu(B)<\delta$ then $\int_B |f_n|\,d\mu<\varepsilon$ for all n. But $f_n\to 0$ pointwise, so that Egorov's Theorem provides us with a Borel set $B\subseteq\mathbb{D}$ such that $\mu(B)<\delta$ and $\lim_n f_n(z)=0$ uniformly on $\mathbb{D}\setminus B$. Accordingly, there is an $n_\varepsilon\in\mathbb{N}$ such that $\int_{\mathbb{D}\setminus B}|f_n|\,d\mu<\varepsilon$ for $n\ge n_\varepsilon$. We conclude that $\|f_n\|_{L^1(\mu)}<2\varepsilon$ for all $n\ge n_\varepsilon$: (f_n) is a null sequence in the Banach space $L^1(\mu)$. We have shown that $J_\mu: H^{p/q}(\mathbb{D})\to L^1(\mu)$ is completely continuous.

Conversely, if $J_\mu: H^{p/q}(\mathbb{D})\to L^1(\mu)$ is completely continuous then, by Proposition 3.1, $\mu_{\mathbb{T}}=0$. □

When $p\ne q$, then complete continuity of Carleson embeddings depends heavily on the sign of $p-q$.

Corollary 3.7. *Let $0<q<p<\infty$ and μ a (p,q)-Carleson measure on $\overline{\mathbb{D}}$. The following statements are equivalent:*

(i) $J_\mu: H^p(\mathbb{D})\to L^q(\mu)$ *is completely continuous.*

(ii) $\mu_{\mathbb{T}}=0$.

(iii) μ *is a compact (p,q)-Carleson measure.*

Proof. (i)$\Rightarrow$(ii) follows from Proposition 3.1.

(ii)$\Rightarrow$(iii) It follows from Theorem 3.6 that $J_\mu: H^{p/q}(\mathbb{D})\to L^1(\mu)$ is completely continuous and so compact, since $p/q>1$. Hence $J_\mu: H^p(\mathbb{D})\to L^q(\mu)$ is compact by Proposition 3.2.

(ii)$\Rightarrow$(iii) is again obvious. □

Let now $1<s\le\infty$, and let $i_s: H^s(\mathbb{D})\to H^1(\mathbb{D})$ be the formal identity. Combining the previous results we get

Corollary 3.8. *If μ is a $(1,1)$-Carleson measure on $\overline{\mathbb{D}}$ then the following statements are equivalent:*

(i) $\mu_{\mathbb{T}}=0$.

(ii) $J_\mu: H^1(\mathbb{D})\to L^1(\mu)$ *is completely continuous.*

(iii) $J_\mu\circ i_s: H^s(\mathbb{D})\to L^1(\mu)$ *is compact for all $1<s\le\infty$.*

(iv) $J_\mu \circ i_s : H^s(\mathbb{D}) \to L^1(\mu)$ *is compact for some* $1 < s \leq \infty$.

PROOF. (i)$\Rightarrow$(ii) follows from Theorem 3.7. (ii)$\Rightarrow$(iii) holds because i_s is weakly compact, and (iii)$\Rightarrow$(iv) is trivial. It is also easy to settle (iv)$\Rightarrow$(i): (z^n) is bounded in $H^\infty(\mathbb{D})$ and so has, by compactness of $J_\mu \circ i_\infty$, a subsequence (z^{n_k}) which converges in $L^1(\mu)$. The limit must be zero since (z^n) is weak null in H^1. By Lebesgue dominated convergence theorem, $\lim_n \int_{\mathbb{D}} |z^{n_k}|\, d\mu = 0$, so that the claim follows since $\mu(\mathbb{T}) = \int_{\overline{\mathbb{D}}} |z^{n_k}|\, d\mu - \int_D |z^{n_k}|\, d\mu$ for all k. □

In a special case, the equivalence of (i) and (ii) was used by J. Cima and A. Matheson (see [2]) to show:

Corollay 3.9. *Let* $C_\varphi : H^1 \to H^1$ *be a composition operator defined by a non-constant analytic symbol* $\varphi : \mathbb{D} \to \mathbb{D}$. *Then* C_φ *is completely continuous if and only if the set* $E_\varphi = \{\zeta \in \mathbb{T} : |\varphi^*(\zeta)| = 1\}$ *of 'contact points' has* m*-measure zero.*

PROOF. Recall a result of J.V. Ryff (see [12]) (see also H. Hunziker [7], [8]): if f is in $H^1(\mathbb{D})$, then $(f \circ \varphi)^* = f^\bullet \circ \varphi^*$ m-a.e. Consider the Borel measure $m_\varphi := m((\varphi^*)^{-1}(\cdot))$ on $\overline{\mathbb{D}}$. We know that m_φ is a (1,1)-Carleson measure.

Note that C_φ and J_{m_φ} are injective since φ is non-constant. It follows that $C_\varphi f \mapsto J_{m_\varphi} f$ extends to an isometric isomorphism of the closure of the range of C_φ onto the closure of the range of J_{m_φ}. Consequently, J_{m_φ} is completely continuous iff C_φ is. By Propositon 3.6, we are done. □

We now study the complete continuity of $J_\mu : H^p(\mathbb{D}) \to L^q(\mu)$ in the case $p < q$. The case $p > 1$ is covered by Theorem 3.4 so that we can now restrict to $0 < p \leq 1$.

First note that a combination of Theorems 3.6 and 2.5 gives the following:

Corollary 3.10. *Let* $0 < p < 1$ *and* μ *is a* $(p,1)$*-Carleson measure on* $\overline{\mathbb{D}}$ *then the Carleson embedding* $J_\mu : H^p(\mathbb{D}) \to L^1(\mu)$ *is completely continuous.*

For the remaining case we make use of the an extension of Corollary 3.8 from [9]: Let X be any quasi-Banach space with a separating dual. An operator $u : H^1(\mathbb{D}) \to X$ is completely continuous if and only if, for some (and then all) $1 < s \leq \infty$, the composition $u \circ i_s : H^s(\mathbb{D}) \to X$ is compact. As before, i_s is the formal identity $H^s(\mathbb{D}) \to H^1(\mathbb{D})$.

Corollary 3.11. *If* $1 < q < \infty$ *and* μ *is a* $(1,q)$*-Carleson measure on* $\overline{\mathbb{D}}$*, then the Carleson embedding* $J_\mu : H^1(\mathbb{D}) \to L^q(\mu)$ *is completely continuous.*

PROOF. By Corollary 3.5, $J_\mu : H^1(\mathbb{D}) \to L^1(\mu)$ is compact, and by Proposition 3.2, the same is true for $J_\mu : H^q(\mathbb{D}) \to L^q(\mu)$. Now apply the quoted result. □

It was shown by D. Sarason in 1991 (see [13]) that a composition operator $C_\varphi : H^1 \to H^1$ is compact iff it is weakly compact. This generalizes to Carleson measures as well.

Theorem 3.12. *Let $0<p\leq 1$, and let μ be measure on $\overline{\mathbb{D}}$ such that $J_\mu : H^p(\mathbb{D}) \to L^1(\mu)$ is a weakly compact operator. Then J_μ is compact.*

PROOF. The strategy is the same as in [13], but we do not need to employ duality.

We first show that $\mu_{\mathbb{T}}=0$. It suffices to look at $p=1$ since, for $0<p<1$, $\mu_{\mathbb{T}}=0$ is an automatic consequence of Theorem 2.5. To reach our goal, we use Proposition 2.3 and write $\mu_{\mathbb{T}}=F\,dm$ with $F\in L^\infty(m)$, $F\geq 0$. We will prove that $F=0$ m-a.e.

Note that each interval $I\in\mathcal{I}$ has the form $I=I(z_I)$ where $z_I\in\mathbb{D}$ is such that $|I|=1-|z_I|$ and $z_I/|z_I|$ is the center of I. The continuous function

$$g_I : \overline{\mathbb{D}} \;\to\; \mathbb{C} : z \;\mapsto\; \frac{1-|z_I|^2}{(1-\overline{z_I}z)^2}$$

is analytic on $\mathbb{D}$ and has H^1-norm one. A little geometry reveals that $|g_I(\zeta)| \geq \dfrac{1}{2|I|}$ for all $\zeta\in I$ and (sufficiently small) $|I|$ so that $\dfrac{1}{|I|}\displaystyle\int_I F\,dm \;\leq\; 2\|F\|_\infty \int_I |g_I|\,dm$. Thanks to our hypothesis, the g_I's form an uniformly integrable family in $L^1(\mu)$. Hence, given $\varepsilon>0$, there is a $\delta>0$ such that if $B\subseteq\overline{\mathbb{D}}$ is a Borel set and $\mu(B)<\delta$, then $\int_B |g_I|\,d\mu<\varepsilon$ for all intervals I. In particular, if $C\geq 1$ is such that $\|F\|_\infty\leq C$ then $\mu_{\mathbb{T}}(I)=\int_I F\,dm\leq C|I|<\delta$ for all I with $|I|<\delta/C$, and so

$$\frac{1}{|I|}\int_I F\,dm \;\leq\; 2\int_I |g_I|\,dm \;\leq\; 2\varepsilon\ .$$

This proves $\displaystyle\lim_{|I|\to 0}\frac{1}{|I|}\int_I F\,dm = 0$. By Lebesgue's Differentiation Theorem, $F=0$ m-a.e.

Compactness is now obtained by repeating an argument from the proof of Theorem 3.6. Let in fact (f_n) be a bounded sequence in $H^p(\mathbb{D})$. Again it suffices to look at a bounded sequence (f_n) in $H^p(\mathbb{D})$ which converges pointwise to zero. By hypothesis and since $\mu_{\mathbb{T}}=0$, (f_n) is uniformly integrable in $L^1(\mu)=L^1(\mu_{\mathbb{D}})$. But since $f_n\to 0$ pointwise on $\mathbb{D}$, Egorov's Theorem leads us to $\lim_n \|f_n\|_{L^1(\mu)}=0$. □

Corollary 3.13. *Let $0<p\leq q<\infty$ and μ a measure on $\overline{\mathbb{D}}$. Then μ is a compact (p,q)-Carleson measure if and only if $J_\mu : H^{p/q}(\mathbb{D})\to L^1(\mu)$ is weakly compact.*

PROOF. This is clear by Theorem 3.12 and Proposition 3.5. □

Each $F\in L^\infty(m)$ gives rise to a multiplication operator $M_F : L^1(m)\to L^1(m) : f\mapsto fF$. The above argument shows that M_F can only be weakly compact if $F=0$ m-a.e. In fact, just look at the functions $g_I := \dfrac{\chi_I}{|I|}$, $I\in\mathcal{I}$. – By duality, the conclusion is also valid for multiplication operators $M_F : L^\infty(m)\to L^\infty(m)$.

Several related results and generalizations are available. For example, if X is a subspace of $L^1(m)$ which contains a weak null sequence of unimodular functions, then complete continuity of $M_F : X \to L^1(m)$ obviously implies $F=0$ m-a.e. This applies to any orthonormal sequence in $L^2(m)$ which consists of unimodular functions (Rademacher functions, trigonometric functions, ...).

References

[1] O. Blasco, Operators on weighted Bergman spaces and applications. Duke Math. J. **66**, (1992), 443-467.

[2] J. Cima, A. Matheson, Completely continuous composition operators. Trans. Amer. Math. Soc. **344** (1994), 849-856.

[3] J. Diestel, H. Jarchow, A. Tonge, *Absolutely Summing Operators.* Cambrigde Univ. Press, Cambrigde, 1995.

[4] P. Duren, *H^p Spaces.* Academic Press, New York, 1970.

[5] J.B. Garnett, *Bounded Analytic Functions.* Academic Press, New York, 1981.

[6] C. Horowitz, Zeros of functions in the Bergman spaces. Duke Math. J. **41** (1974), 693-710.

[7] H. Hunziker, *Kompositionsoperatoren auf klassischen Hardyräumen.* Dissertation Universität Zürich 1989.

[8] H. Hunziker, H. Jarchow, Composition operators which improve integrability. Math. Nachr. **152** (1991), 83-99.

[9] H. Jarchow, Compactness properties of composition operators. Rend. Circ. Mat. Palermo, Suppl. **56** (1998), 91-97.

[10] H. Jarchow, U. Kollbrunner, Carleson embeddings for weighted Bergman spaces. Contemp. Math. **328** (2003), 217-236.

[11] D.H. Luecking, Embedding derivatives of Hardy spaces into Lebesgue spaces. Proc. London Math. Soc. **63** (1991), 565-619.

[12] J.V. Ryff, Subordinate H^p functions. Duke Math. J. **33** (1966), 347-354.

[13] D. Sarason, Weak compactness of holomorphic composition operators on H^1. Lecture Notes in Math. **1511** (1992), 75-79.

[14] I.V. Videnskii, On an analogue of Carleson measures. Soviet Math. Dokl. **37** (1988), 186-190.

[15] K. Zhu, *Operator Theory in Function Spaces.* Marcel Dekker, Inc., New York, 1990.

Oscar Blasco de la Cruz, Departamento de Análisis Matemático,
Universidad de Valencia, E 46100 (Burjassot) Valencia / Spain
e-mail: oscar.blasco@uv.es

Hans Jarchow, Institut für Mathematik, Universität Zürich
Winterthurerstr. 190, CH 8057 Zürich / Switzerland
e-mail: jarchow@math.unizh.ch