On Taylor coefficients of entire functions integrable against exponential weights

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Abstract

In this paper we shall analyze the Taylor coefficients of entire functions integrable against $d\mu_p(z) = \frac{p}{2\pi} e^{-|z|^p} \mid z \mid^{p-2} d\sigma(z)$ where $d\sigma$ stands for the Lebesgue measure on the plane and $p \in \mathbb{N}$, as well as the Taylor coefficients of entire functions in some weighted sup-norm spaces.

In this paper we shall analyze the Taylor coefficients of entire functions satisfying some growth estimates. To be more precise, given $p \in \mathbb{N}$, we will deal with the Banach space $B_1(p)$ of entire functions belonging to $L_1(d\mu)$, where $d\mu(z) = \frac{p}{2\pi} e^{-|z|^p} \mid z \mid^{p-2} d\sigma(z)$ and $d\sigma$ stands for the Lebesgue measure on the plane, as well as with the Banach space $H(e^{-|z|^p})(\mathbb{C})$ of those entire functions f such that $\sup_{z\in\mathbb{C}} e^{-|z|^p} \mid f(z) \mid < \infty$. These spaces have been considered in several contexts by different authors. See [1, 6, 7, 8, 9, 10]. The general question we are going to discuss can be stated as follows: given a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $X (:= B_1(p)$ or $H(e^{-|z|^p})(\mathbb{C}))$, what can be said on the Taylor coefficients (a_n)?. Conversely, it is also interesting to ask how a function in X can be recognized by the behaviour of its Taylor coefficients. The paper is organized as follows. In the first section we present a method to describe the boundedness of operators from $B_1(p)$ into a general Banach space X by the fact that the $X-$valued analytic function constructed by the action of the operator on the reproducing kernel K_p belongs to the vector-valued space $H(e^{-|z|^p})(\mathbb{C}; X)$. This will allow to identify the dual space of $B_1(p)$ with the weighted sup-norm space $H(e^{-|z|^p})(\mathbb{C})$. Then we will discuss a Hardy's type inequality for Taylor coefficients of functions in $B_1(p)$. In the second section we give a complete characterization of the Taylor coefficients for lacunary entire functions in both spaces $B_1(p)$ and $H(e^{-|z|^p})(\mathbb{C})$. As an application we obtain a sufficient condition on the Taylor coefficients of a function f in order to enssure that it belongs to $H(e^{-|z|^p})(\mathbb{C})$. In section 3 we find conditions on n_k in order to get the unconditional convergence of $\sum a_k z^{n_k}$ to be equivalent to the absolute convergence of the series.

Let us denote by $H(e^{-|z|^p})_0(\mathbb{C})$ the closed subspace of $H(e^{-|z|^p})(\mathbb{C})$ consisting of those functions f such that $e^{-|z|^p} f(z)$ vanishes at infinity. Since the polynomials are dense in $B_1(p)$ and in $H(e^{-|z|^p})_0(\mathbb{C})$ it is natural to ask whether the Taylor series of a function in those spaces necessarily converges in norm. Such a question was raised by D.J.H. Garling and P. Wojtaszczyk [7] for the space $B_1(2)$, corresponding to those entire functions which are integrable with respect to a gaussian measure, and it was recently solved in the negative by W. Lusky [10] for all the spaces $B_1(p)$ and $H(e^{-|z|^p})_0(\mathbb{C})$. Nevertheless our results in Section 2 show that when restricted to a lacunary sequence n_k, i.e. $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$, we have that (z^{n_k}) is a basic subsequence in $B_1(p)$. The final part of the paper is devoted to give a necessary and also two sufficient conditions in order to ensure the unconditional convergence of a given Taylor series in $H(e^{-|z|^p})_0(\mathbb{C})$.

1 Duality

In this section we present the Banach spaces $B_1(p)$ and $H(e^{-|z|^p})(\mathbb{C})$ and show that $(B_1(p))^* = H(e^{-|z|^p})(\mathbb{C})$. This duality is applied to discuss the sharpness of a Hardy's type inequality for functions in $B_1(p)$. Moreover, as a previous step to get the duality some necessary and sufficient conditions for a function to belong to $H(e^{-|z|^p})(\mathbb{C})$ are given.

Definition 1.1 *Given a continuous and radial weight v on $\mathbb{C}$ and a complex Banach space $(X, \| \cdot \|)$ we define*

(a) $H(v)(\mathbb{C}, X) := \{F : \mathbb{C} \to X$ entire function; $\| F \| := sup \; v(z) \| F(z) \| < \infty\}$,

(b) $H(v)_0(\mathbb{C}, X)$ is the subspace of $H(v)(\mathbb{C}, X)$ consisting of those functions F such that Fv vanishes at infinity.

If X is the field of complex numbers we drop it from the notation and write $H(v)(\mathbb{C})$ or $H(v)_0(\mathbb{C})$. We are interested in weights $v(z) = \exp(- \mid z \mid^p)$, $p \in \mathbb{N}$.

Definition 1.2 *Given a natural number $p \in \mathbb{N}$ we denote by $B_1(p)$ the space of entire functions f such that*

$$\| f \| := \frac{p}{2\pi} \int_{\mathbb{C}} \mid f(z) \mid e^{-|z|^p} \mid z \mid^{p-2} d\sigma(z) < \infty.$$

We write $M_\infty(f, r) := \max\{\mid f(z) \mid : \mid z \mid = r\}$ and $M_1(f, r) := \frac{1}{2\pi} \int_0^{2\pi} \mid f(re^{it}) \mid dt$. Then, for every $f \in B_1(p)$, we have $\| f \| = \int_0^\infty M_1(f, r) e^{-r^p} p r^{p-1} dr$.

Lemma 1.1 *(a) Let v be a continuous and radial weight on $\mathbb{C}$ such that the polynomials are contained in $H(v)_0(\mathbb{C})$. Then the polynomials are dense in $H(v)_0(\mathbb{C})$.*

(b) For every $p \in \mathbb{N}$, the polynomials are dense in $B_1(p)$.

Proof: A proof of part (a) can be found in [3, 1.5(a)]. To prove (b) proceed as in [7, Proposition 5]. ■

Let us first remark that $\varphi_p(r) = r^n e^{-r^p}$ is an increasing function in $[0, (\frac{n}{p})^{\frac{1}{p}}]$ and decreasing in $[(\frac{n}{p})^{\frac{1}{p}}, +\infty[$. What shows that $u_n(z) = z^n$ satisfies $\| u_n \|_{H(e^{-|z|^p})(\mathbb{C})} = (\frac{n}{p})^{\frac{n}{p}} e^{-\frac{n}{p}}$. This, using the trivial estimate $| b_n | R^n \leq M_\infty(g, R)$, also allows to say that if $g(z) = \sum b_n z^n \in H(e^{-|z|^p})(\mathbb{C})$ then

$$\sup_{n \in \mathbb{N}} \frac{| b_n | \Gamma(\frac{n}{p} + 1)}{\sqrt{n+1}} \leq C \| g \|_{H(e^{-|z|^p})} . \tag{1.1}$$

Let us start by mentioning a simple condition on (b_n) which implies that $g \in H(e^{-|z|^p})(\mathbb{C})$.

Lemma 1.2 *(a) Let $p \in \mathbb{N}$ and let (b_n) be a sequence such that $\sup_{n \in \mathbb{N}} | b_n | \Gamma(\frac{n}{p} + 1) < \infty$. Then $g(z) = \sum b_n z^n \in H(e^{-|z|^p})(\mathbb{C})$.*

(b) If $\lim_{n \to \infty} | b_n | \Gamma(\frac{n}{p} + 1) = 0$ then $g(z) = \sum b_n z^n \in H(e^{-|z|^p})_0(\mathbb{C})$.

Proof: To see (a) it suffices to show that

$$\sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\frac{n}{p} + 1)} \leq C e^{r^p}$$

for every $r > 0$. For each $n \in \mathbb{N}$ write $n = pk + j$, $k \in \mathbb{N}$ and $j = 0, 1, \ldots p - 1$, and decompose the sum as follows

$$\sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\frac{n}{p} + 1)} = \sum_{j=0}^{p-1} \varphi_j(r^p)$$

where

$$\varphi_j(t) = \sum_{k=0}^{\infty} \frac{t^{k + \frac{j}{p}}}{\Gamma(k + \frac{j}{p} + 1)}.$$

Since $\varphi_j'(t) = \frac{j}{p\Gamma(\frac{j}{p}+1)} t^{\frac{j}{p} - 1} + \varphi_j(t)$ we have

$$\varphi_j(t) = e^t(\varphi_j(0) + \frac{j}{p\Gamma(\frac{j}{p} + 1)} \int_0^t e^{-s} s^{\frac{j}{p} - 1} ds) \leq e^t(\varphi_j(0) + 1).$$

Adding the values for $j = 0, 1, \ldots p - 1$ we get

$$\sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\frac{n}{p} + 1)} = \sum_{j=0}^{p-1} \varphi_j(r^p) \leq (1 + p) e^{r^p}.$$

(b) Since $H(e^{-|z|^p})_0(\mathbb{C})$ is a closed subspace of $H(e^{-|z|^p})(\mathbb{C})$ it suffices to show that $g = \lim_{N\to\infty} \sum_{k=0}^{N} b_k u_k$ in $H(e^{-|z|^p})(\mathbb{C})$. But this follows from

$$\begin{aligned}(\sum_{k=N+1}^{\infty} |b_k| r^k) e^{-r^p} &\leq \big(\sup_{k>N} |b_k| \Gamma(\frac{k}{p}+1)\big)\big(\sum_{k=N+1}^{\infty} \frac{r^k}{\Gamma(\frac{k}{p}+1)}\big) e^{-r^p} \\ &\leq C\big(\sup_{k>N} |b_k| \Gamma(\frac{k}{p}+1)\big). \blacksquare\end{aligned}$$

Let us now find some necessary condition for a function to belong to $H(e^{-|z|^p})(\mathbb{C})$.

Lemma 1.3 *Let (α_n) be a sequence of positive real numbers. If $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ belongs to $H(e^{-|z|^p})(\mathbb{C})$ then*

$$\sup_{m\in\mathbb{N}} \frac{1}{m} \sum_{n=0}^{m} \alpha_n \Gamma(\frac{n}{p}+1) < \infty.$$

Proof: Since $\alpha_n \geq 0$ then we are assuming that $\sum_{n=0}^{\infty} \alpha_n r^{\frac{n}{p}} \leq Ce^r$ for every $r > 0$. Hence, multiplying by e^{-ar} $(a > 1)$ and integrating over $(0, \infty)$ we get

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{a^{\frac{n}{p}+1}} \Gamma(\frac{n}{p}+1) \leq \frac{C}{a-1}.$$

For $m \in \mathbb{N}$ take $a = \frac{m+1}{m}$ and then

$$(1 - \frac{1}{m+1})^{\frac{m}{p}+1} \sum_{n=0}^{m} \alpha_n \Gamma(\frac{n}{p}+1) \leq \sum_{n=0}^{\infty} \alpha_n \Gamma(\frac{n}{p}+1)(1 - \frac{1}{m+1})^{\frac{n}{p}+1} \leq Cm.$$

Using that $\lim_{m\to\infty}(1 - \frac{1}{m+1})^{\frac{m}{p}+1} = e^{-\frac{1}{p}}$ we finish the proof. $\blacksquare$

In order to get the duality between $B_1(p)$ and $H(e^{-|z|^p})(\mathbb{C})$ let us first give a natural pairing on these spaces. If $f \in B_1(p)$ and $g \in H(e^{-|z|^p})(\mathbb{C})$ we can define

$$< f, g >= \frac{p}{2\pi} \int_{\mathbb{C}} f(\overline{\omega}) g(\omega) e^{-2|\omega|^p} \mid \omega \mid^{p-2} d\sigma(\omega).$$

Clearly $|< f, g >| \leq \| f \|_{B_1(p)} \| g \|_{H(e^{-|z|^p})(\mathbb{C})}$. Observe that $< u_n, g >= b_n \frac{\Gamma(\frac{2n}{p}+1)}{2^{\frac{2n}{p}+1}}$ for $g(z) = \sum b_n z^n$.

This leads to the consideration of the following function $K_p(z) = \sum_{n=0}^{\infty} \frac{2^{\frac{2n}{p}+1}}{\Gamma(\frac{2n}{p}+1)} z^n$.

Let us denote by $K_p(z,\omega)=K_p(z\overline{\omega})$. Then

$$g(z)=\frac{p}{2\pi}\int_{\mathbb{C}}K_p(z,\omega)g(\omega)e^{-2|\omega|^p}\mid\omega\mid^{p-2}d\sigma(\omega)$$

for every polynomial g.

We also write

$$K_p(z)=\sum_{n=0}^{\infty}\frac{2^{\frac{2n}{p}+1}}{\Gamma(\frac{2n}{p}+1)}z^n u_n$$

as a function taking values in $B_1(p)$ (note that this series is absolutely convergent in $B_1(p)$ because $\parallel u_n\parallel_{B_1(p)}=\Gamma(\frac{n}{p}+1)$ and $\frac{2^{\frac{2n}{p}+1}|z|^n}{\Gamma(\frac{2n}{p}+1)}\Gamma(\frac{n}{p}+1)\simeq\frac{|z|^n\sqrt{n}}{\Gamma(\frac{n}{p}+1)}$).

In order to get estimates on the norm $\parallel K_p(z)\parallel_{B_1(p)}$ as $\mid z\mid$ goes to ∞ we first need the following Lemma.

Lemma 1.4 *Let $p\in\mathbb{N}$ and let $f(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\frac{2n}{p}+1)}$. There exists $C_p>0$ such that*

$$M_1(f,r)\leq C_p\sum_{n=0}^{\infty}\frac{r^n}{\Gamma(\frac{2n}{p}+1)\sqrt{n+1}}$$

for all $r>0$.

Proof: As in Lemma 1.2 let us write $n=kp+j$ for $k\in\mathbb{N}$ and $j=0,1,\dots p-1$. Then $f(z)=\sum_{j=0}^{p-1}z^jf_j(z)$ where $f_j(z)=\sum_{k=0}^{\infty}\frac{z^{pk}}{\Gamma(2k+\frac{2j}{p}+1)}$. Now, let us rewrite f_j as follows

$$\begin{aligned}f_j(z)&=\sum_{k=0}^{\infty}\frac{z^{pk}}{\Gamma(2k+1)\Gamma(\frac{2j}{p})}B(2k+1,\frac{2j}{p})\\&=\frac{1}{\Gamma(\frac{2j}{p})}\sum_{k=0}^{\infty}\frac{z^{pk}}{\Gamma(2k+1)}\int_0^1x^{2k}(1-x)^{\frac{2j}{p}-1}dx\\&=\frac{1}{\Gamma(\frac{2j}{p})}\int_0^1(\sum_{k=0}^{\infty}\frac{(z^px^2)^k}{(2k)!})(1-x)^{\frac{2j}{p}-1}dx.\end{aligned}$$

Therefore

$$\begin{aligned}M_1(f_j,r)&\leq\frac{1}{\Gamma(\frac{2j}{p})}\int_0^1(\int_0^{2\pi}\mid cosh(xr^{\frac{p}{2}}e^{i\frac{p}{2}\theta})\mid\frac{d\theta}{2\pi})(1-x)^{\frac{2j}{p}-1}dx\\&\leq\frac{1}{\Gamma(\frac{2j}{p})}\int_0^1(\int_0^{2\pi}cosh(xr^{\frac{p}{2}}cos\frac{p\theta}{2}\frac{d\theta}{2\pi})(1-x)^{\frac{2j}{p}-1}dx\\&\leq\frac{1}{p\Gamma(\frac{2j}{p})}\sum_{k=0}^{\infty}\frac{r^{pk}}{(2k)!}(\int_0^1x^{2k}(1-x)^{\frac{2j}{p}-1}dx)\int_{-\pi p}^{\pi p}(cost)^{2k}\frac{dt}{2\pi}.\end{aligned}$$

Using that $\int_{-\pi}^{\pi}(cost)^{2n}\frac{dt}{2\pi}=2^{-n}\begin{pmatrix}2n\\n\end{pmatrix}$ we have

$$M_1(f_j,r)\leq\sum_{k=0}^{\infty}\frac{r^{pk}}{(k!)^22^{2k}}\frac{B(2k,\frac{2j}{p})}{\Gamma(\frac{2j}{p})}=\sum_{k=0}^{\infty}\frac{r^{pk}\Gamma(2k+1)}{(k!)^22^{2k}\Gamma(2k+\frac{2j}{p}+1)}.$$

Adding all the values of j we get

$$M_1(f,r) \leq \sum_{j=0}^{p-1} r^j M_1(f_j,r) \leq C_p \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(\frac{2n}{p}+1)\sqrt{[\frac{n}{p}]+1}}. \ \blacksquare$$

Lemma 1.5 *Let* $p \in \mathbb{N}$ *and* $K_p(z) = \sum_{n=0}^{\infty} \frac{2^{\frac{2n}{p}+1}}{\Gamma(\frac{2n}{p}+1)} z^n u_n$. *Then* $K_p \in H(e^{-|z|^p})(\mathbb{C}, B_1(p))$.

Proof: Using Lemma 1.4 we have

$$M_1(K_p(z),r) \leq C_p \sum_{n=0}^{\infty} \frac{2^{\frac{2n}{p}+1} \mid z \mid^n r^n}{\Gamma(\frac{2n}{p}+1)\sqrt{n+1}}.$$

Now, integrating over $(0,\infty)$ with the measure $e^{-r^p} p r^{p-1} dr$ and applying Lemma 1.2 we get

$$\| K_p(z) \|_{B_1(p)} \leq C \sum_{n=0}^{\infty} \frac{2^{\frac{2n}{p}+1} \mid z \mid^n}{\Gamma(\frac{2n}{p}+1)\sqrt{n+1}} \Gamma(\frac{n}{p}+1) \leq C \sum_{n=0}^{\infty} \frac{\mid z \mid^n}{\Gamma(\frac{n}{p}+1)} \leq C e^{|z|^p}. \ \blacksquare$$

Theorem 1.1 *Let* X *be a Banach space,* $p \in \mathbb{N}$. *Let* T *be a bounded operator from* $B_1(p)$ *into* X. *Then* $F(z) = T(K_p(z)) \in H(e^{-|z|^p})(\mathbb{C}, X)$ *and* $\| F \| \leq C_p \| T \|$.
Conversely, given $F \in H(e^{-|z|^p})(\mathbb{C}, X)$, *then*

$$T(f) = \int_{\mathbb{C}} F(z) f(\overline{z}) e^{-2|z|^p} \mid z \mid^{p-2} d\sigma(z)$$

defines a bounded operator from $B_1(p)$ *into* X *and* $\| T \| \leq \| F \|$. *Moreover,* $T(K_p(z)) = \frac{2\pi}{p} F(z)$.

Proof: The first statement follows from the boundedness of T and Lemma 1.5. The converse follows since $F(z)f(\overline{z})$ is a X-valued continuous function and $\| F(z) \| \| f(\overline{z}) \| \leq \| F \| \| f(\overline{z}) \| e^{|z|^p}$. Hence the Bochner integral exists and

$$\| T(f) \| \leq \frac{2\pi}{p} \| F \|_{H(e^{-|z|^p},X)} \| f \|_{B_1(p)} .$$

$\blacksquare$

Corollary 1.1 *Let* $p \in \mathbb{N}$. *Then* $(B_1(p))^* = H(e^{-|z|^p})(\mathbb{C})$ *with equivalent norms under the pairing* $< . >$.

We also give a direct proof of the other duality.

Theorem 1.2 *Let $p \in \mathbb{N}$. Then $(H(e^{-|z|^p})_0(\mathbb{C}))^* = B_1(p)$ with equivalent norms under the pairing $< . >$.*

Proof: Define $T : B_1(p) \to (H(e^{-|z|^p})_0(\mathbb{C}))^*$ given by

$$< T(f), g >= \frac{p}{2\pi} \int_{\mathbb{C}} g(z) f(\overline{z}) e^{-2|z|^p} \mid z \mid^{p-2} d\sigma(z).$$

Clearly T is well defined and bounded with $\| T \| \leq 1$. Since $< T(u_n), g >= \frac{b_n}{2^{\frac{2n}{p}+1}} \Gamma(\frac{2n}{p}+1)$ for $g(z) = \sum b_n z^n$ then T is injective. To see that T is surjective let us take $\phi \in (H(e^{-|z|^p})_0(\mathbb{C}))^*$ and, by Hahn-Banach, find a bounded measure ν such that $\phi(g) = \int g(z) e^{-|z|^p} d\nu(z)$ for any $g \in H(e^{-|z|^p})_0(\mathbb{C})$. Define now $f(z) = \int_{\mathbb{C}} K_p(\omega, \overline{z}) e^{-|\omega|^p} d\nu(\omega)$. We shall see that $f \in B_1(p)$. Indeed,

$$\begin{aligned} \int_{\mathbb{C}} \mid f(z) \mid e^{-|z|^p} \mid z \mid^{p-2} d\sigma(z) &\leq \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \mid K_p(\omega, \overline{z}) \mid e^{-|\omega|^p} d\nu(\omega) \right) e^{-|z|^p} \mid z \mid^{p-2} d\sigma(z) \\ &= \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \mid K_p(\omega, \overline{z}) \mid e^{-|z|^p} \mid z \mid^{p-2} d\sigma(z) \right) e^{-|\omega|^p} d\nu(\omega). \end{aligned}$$

Now, to get $\| f \|_{B_1(p)} \leq C \| \nu \|$ we apply Lemma 1.5. On the other hand, for any polynomial g we have

$$\begin{aligned} < T(f), g > &= \tfrac{p}{2\pi} \int_{\mathbb{C}} g(z) f(\overline{z}) e^{-2|z|^p} \mid z \mid^{p-2} d\sigma(z) \\ &= \tfrac{p}{2\pi} \int_{\mathbb{C}} \left(\int_{\mathbb{C}} K_p(\omega, z) e^{-|\omega|^p} d\nu(\omega) \right) g(z) e^{-2|z|^p} \mid z \mid^{p-2} d\sigma(z) \\ &= \int g(\omega) e^{-|\omega|^p} d\nu(\omega) = \phi(g). \end{aligned}$$

Using, finally, that the polynomials are dense in $H(e^{-|z|^p})_0(\mathbb{C})$ the proof is complete. ■

Lusky [10] showed that, for every $p > 0$ there are functions $f \in H(e^{-|z|^p})_0(\mathbb{C})$ whose Taylor series do not converge in norm. The duality results in this section have been used in [10] to prove that the same conclusion holds for the spaces $B_1(p)$, $p \in \mathbb{N}$. The vector-valued duality (theorem 1.1) will be applied in the last section to find a necessary condition for the unconditional convergence of a given Taylor series in $H(e^{-|z|^p})_0(\mathbb{C})$.

To finish this section we present a Hardy's type inequality for functions in $B_1(p)$.

Let us start by noticing that, using the monotonicity of $M_1(f, r)$, one has

$$|a_n| r^{\frac{n}{p}} e^{-r} \leq \int_r^\infty M_1(f, s^{\frac{1}{p}}) e^{-s} ds.$$

Hence taking $r = \frac{n}{p}$ we have that if $f \in B_1(p)$ then $\lim_{n \to \infty} \frac{|a_n| \Gamma(\frac{n}{p}+1)}{\sqrt{n}} = 0.$

On the other hand, applying Hardy inequality for Hardy spaces (see [5]) we have

$$\sum_{n=0}^{\infty} \frac{\mid a_n \mid R^n}{n+1} \le CM_1(f,R)$$

for $f(z) = \sum a_n z^n$. Therefore

$$\sum_{n=0}^{\infty} \frac{\mid a_n \mid}{n+1} \Gamma(\frac{n}{p}+1) \le C \parallel f \parallel_{B_1(p)} .$$

This is far to being sharp as the following theorem shows.

Theorem 1.3 *Let $p \in \mathbb{N}$.*

(a) There exists a constant $C_p > 0$ such that

$$\sum_{n=0}^{\infty} \frac{\mid a_n \mid}{\sqrt{n+1}} \Gamma(\frac{n}{p}+1) \le C_p \parallel f \parallel_{B_1(p)}$$

where $f(z) = \sum a_n z^n$.

(b) Let (α_n) be a sequence of non negative real numbers such that there exists a constant $C_p > 0$ such that for $f(z) = \sum a_n z^n$

$$\sum_{n=0}^{\infty} \mid a_n \mid \Gamma(\frac{n}{p}+1)\alpha_n \le C_p \parallel f \parallel_{B_1(p)} .$$

Then

$$\sup_{m \in \mathbb{N}} \frac{1}{m} \sum_{n=1}^{m} \alpha_n \sqrt{n} < \infty.$$

Proof: (a) Let $f(z) = \sum a_n z^n$, then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\mid a_n \mid}{\sqrt{n+1}} \Gamma(\frac{n}{p}+1) &\le C \sum_{n=0}^{\infty} \mid a_n \mid (\frac{n}{p})^{\frac{n}{p}} e^{-\frac{n}{p}} \\ &\le C \sum_{n=0}^{\infty} \int_{\frac{n}{p}}^{\frac{n+1}{p}} \mid a_n \mid s^{\frac{n}{p}} e^{-s} ds \\ &\le C \sum_{n=0}^{\infty} \int_{\frac{n}{p}}^{\frac{n+1}{p}} M_1(f, s^{\frac{1}{p}}) e^{-s} ds = C \parallel f \parallel_{B_1(p)} .\end{aligned}$$

(b) From duality we have that $g(z) = \sum_{n=0}^{\infty} \alpha_n \frac{2^{\frac{2n}{p}+1}\Gamma(\frac{n}{p}+1)}{\Gamma(\frac{2n}{p}+1)} z^n \in H(e^{-|z|^p})(\mathbb{C})$. Now the conclusion follows from lemma 1.3 and the Stirling's formula. ∎

Remarks:

(i) Note that, from duality, the inequality

$$\sum_{n=0}^{\infty} \frac{| a_n |}{\sqrt{n+1}} \Gamma(\frac{n}{p}+1) \leq C_p \parallel f \parallel_{B_1(p)}$$

is equivalent to Lemma 1.2.

(ii) Note that part (b) means that the previous inequality is sharp in the following sense: For $\alpha_n = \frac{1}{n^\beta}$ the best exponent is $\beta = \frac{1}{2}$.

2 Lacunary entire functions. Applications.

We now get some inequalities holding for lacunary entire functions in $B_1(p)$ and in $H(e^{-|z|^p})(\mathbb{C})$. As an application we will present a sufficient condition on the Taylor coefficients of an entire function f in order to ensure that it belongs to $H(e^{-|z|^p})_{(0)}(\mathbb{C})$.

First we need the following lemmas. The second one will be also applied in the next section.

Lemma 2.1 *There exist C_1, $C_2 > 0$ such that, for every $p > 0$,*

$$C_1\Gamma(p+1) \leq \int_p^{p+\sqrt{p}} r^p e^{-r} dr \leq C_2\Gamma(p+1).$$

Proof: Recall that $\varphi_p(r) = r^p e^{-r}$ increases in $(0,p)$ and decreases in (p,∞). Hence

$$(p+\sqrt{p})^p e^{-(p+\sqrt{p})}\sqrt{p} \leq \int_{p+\sqrt{p}}^{p} r^p e^{-r} dr \leq p^p e^{-p}\sqrt{p}.$$

Now the result follows from Stirling's formula and the fact

$$\lim_{p\to\infty} \frac{(p+\sqrt{p})^p e^{-(p+\sqrt{p})}\sqrt{p}}{\Gamma(p+1)} = \sqrt{\frac{1}{2\pi e}}. \ \blacksquare$$

Lemma 2.2 *Let $0 < q \leq 1$, $\alpha_k \geq 0$ and $\beta_k > 0$. Assume that there exists $m \in \mathbb{N}$ such that*

$$\lim_{k\to\infty} \inf \frac{\beta_{k+m} - \beta_k}{\sqrt{\beta_k}} > \frac{1}{\sqrt{q}}.$$

Then there exists $0 < C < 1$ such that

$$C \sum_{k=0}^{\infty} \alpha_k^q \Gamma(\beta_k q + 1) \leq \int_0^\infty (\sum_{k=0}^{\infty} \alpha_k s^{\beta_k})^q e^{-s} ds \leq \sum_{k=0}^{\infty} \alpha_k^q \Gamma(\beta_k q + 1).$$

Proof: Since $0 < q \leq 1$ we have

$$\int_0^\infty (\sum_{k=0}^\infty \alpha_k s^{\beta_k})^q e^{-s} ds \leq \int_0^\infty (\sum_{k=0}^\infty \alpha_k^q s^{\beta_k q}) e^{-s} ds = \sum_{k=0}^\infty \alpha_k^q \Gamma(\beta_k q + 1).$$

On the other hand, the assumption implies that there exists k_0 such that $q\beta_{k+m} \geq q\beta_k + \sqrt{q\beta_k}$ for $k \geq k_0$. Now, using Lemma 4.2,

$$\begin{aligned}
\sum_{k=k_0}^\infty \alpha_k^q \Gamma(\beta_k q + 1) &\leq C \sum_{k=k_0}^\infty \alpha_k^q \int_{q\beta_k}^{q\beta_k + \sqrt{q\beta_k}} r^{q\beta_k} e^{-r} dr \\
&\leq C \sum_{k=k_0}^\infty \int_{q\beta_k}^{q\beta_{k+m}} (\sum_{l=0}^\infty \alpha_l r^{\beta_l})^q e^{-r} dr \\
&= C \sum_{k=k_0}^\infty \sum_{j=k}^{k+m-1} \int_{q\beta_j}^{q\beta_{j+1}} (\sum_{l=0}^\infty \alpha_l r^{\beta_l})^q e^{-r} dr \\
&\leq Cm \sum_{j=k_0}^\infty \int_{q\beta_j}^{q\beta_{j+1}} (\sum_{l=0}^\infty \alpha_l r^{\beta_l})^q e^{-r} dr \\
&= Cm \int_{q\beta_{k_0}}^\infty (\sum_{l=0}^\infty \alpha_l r^{\beta_l})^q e^{-r} dr.
\end{aligned}$$

Since

$$\sum_{k=0}^{k_0} \alpha_k^q \Gamma(\beta_k q + 1) \leq (k_0 + 1) \int_0^\infty (\sum_{k=0}^\infty \alpha_k s^{\beta_k})^q e^{-s} ds$$

we have the desired result. ■

Similar conditions to the ones imposed in the above lemma appeared in [4]. The next theorem should be compared with [4, theorem 8].

Let us denote by $V_n = \frac{u_n}{\Gamma(\frac{n}{p}+1)}$ the normalized sequence in $B_1(p)$.

Theorem 2.1 *Let (n_k) be a sequence such that there exists $\lambda > 1$ for which $\frac{n_{k+1}}{n_k} \geq \lambda > 1$. Then there exist $0 < A_p, B_p < \infty$ (depending only on λ, p) such that*

(*a*) $$A_p \sum_{k=0}^\infty | a_k | \leq \| \sum_{k=0}^\infty a_k V_{n_k} \|_{B_1(p)} \leq \sum_{k=0}^\infty | a_k |,$$

(*b*) $$\sum_{k=0}^\infty | a_{n_k} | \Gamma(\frac{n_k}{p} + 1) \leq B_p \| \sum_{n=0}^\infty a_n z^n \|_{B_1(p)} .$$

Proof: To prove (a) recall that, from Kintchine's inequalities for lacunary systems (see [13] or [11]) we have

$$M_1(g,r) \simeq (\sum_{k=0}^{\infty} \mid b_k \mid^2 r^{2n_k})^{\frac{1}{2}}$$

if $g(z) = \sum b_k z^{n_k}$. Then (a) follows from Lemma 2.2 applied to $\beta_k = \frac{2n_k}{p}$, $q = \frac{1}{2}$ and $m = 1$, because $\beta_{k+1} - \beta_k \geq (\lambda - 1)\beta_k$ gives $\lim_{k\to\infty} \frac{\beta_{k+1}-\beta_k}{\sqrt{\beta_k}} = \infty$.

To get (b) use Paley's inequality, instead to Kintchine's (see [5, page 104]) to have

$$(\sum_{k=0}^{\infty} \mid a_{n_k} \mid^2 r^{2n_k})^{\frac{1}{2}} \leq CM_1(f,r)$$

for $f(z) = \sum a_n z^n$, and apply a similar argument. ∎

Remark 2.1 It is well-known that $B_1(p)$ is isomorphic to l^1 (see [7] for the case $B_1(2)$ and [6] together with the duality provided by Theorem 1.2 for the general case). Note that Theorem 2.1 provides a projection into a subspace isomorphic to l^1.

Let us denote be $W_n = (\frac{n}{p})^{-\frac{n}{p}} e^{\frac{n}{p}} u_n$ the normalized sequence in $H(e^{-|z|^p})(\mathbb{C})$.

Theorem 2.2 *Let (n_k) be a sequence such that there exists $\lambda > 1$ for which $\frac{n_{k+1}}{n_k} \geq \lambda > 1$.*

(a) There exist $0 < C_p < \infty$ (depending only on λ, p) such that

$$\sup_{k\in\mathbb{N}} \mid a_k \mid \leq \| \sum_{k=0}^{\infty} a_k W_{n_k} \|_{H(e^{-|z|^p})} \leq C_p \sup_{k\in\mathbb{N}} \mid a_k \mid .$$

(b) $\sum_{k=0}^{\infty} a_k W_{n_k} \in H(e^{-|z|^p})_0(\mathbb{C})$ *if and only if* $(a_k) \in c_0$.

Proof:

(a) The first estimate follows from (1.1).

To see the second one, use duality combined with $\frac{\Gamma(\frac{2n}{p}+1)}{\Gamma(\frac{n}{p}+1)2^{\frac{2n}{p}+1}} \approx \frac{\Gamma(\frac{n}{p}+1)}{\sqrt{n}}$ and (b) in Theorem 2.1.

If $f(z) = \sum_{n=0}^{\infty} b_n z^n$ then

$$\begin{aligned} \mid < f, \textstyle\sum_{k=0}^{N} a_k W_{n_k} > \mid \quad &= \mid \sum_{k=0}^{N} a_k b_{n_k} \frac{\Gamma(\frac{2n_k}{p}+1)}{2^{\frac{2n_k}{p}+1}} (\frac{n_k}{p})^{-\frac{n_k}{p}} e^{\frac{n_k}{p}} \mid \\ &\leq C \sum_{k=0}^{N} |a_k||b_{n_k}|\Gamma(\frac{n_k}{p}+1) \\ &\leq C(\sup_{k\in\mathbb{N}} |a_k|)||f||_{B_1(p)}. \end{aligned}$$

It follows that

$$|\sum_{k=0}^{N} a_k W_{n_k}(z)|e^{-|z|^p} \leq C(\sup_{k\in\mathbb{N}} |a_k|)$$

for every $N \in \mathbb{N}$ and $z \in \mathbb{C}$. Consequently

$$\sup_{z\in\mathbb{C}} |\sum_{k=0}^{\infty} a_k W_{n_k}(z)|e^{-|z|^p} \leq C(\sup_{k\in\mathbb{N}} |a_k|).$$

(b) If $\lim_{k\to\infty} |a_k| = 0$ then, arguing as in (a) we get

$$||\sum_{k=N}^{\infty} a_k W_{n_k}||_{H(e^{-|z|^p})} \leq C(\sup_{k\geq N} |a_k|).$$

Hence $g = \lim_{N\to\infty} \sum_{k=0}^{N} a_k W_{n_k} \in H(e^{-|z|^p})_0(\mathbb{C})$.

Conversely, if $g \in H(e^{-|z|^p})_0(\mathbb{C})$ then g is limit of the sequence of Cèsaro means of its Taylor series (see [3]). Hence, given $\varepsilon > 0$, there exists a polynomial $h(z) = \sum_{k=0}^{N} c_k W_{n_k}$ with $||g-h||_{H(e^{-|z|^p})} \leq \varepsilon$. Applying (a) to the function $g-h$ we get $\sup_{k>N} |a_k| \leq \varepsilon$. ■

Theorem 2.3 *Let* $I_k = [2^k, 2^{k+1}) \cap \mathbb{N}$.

(a) If $\sup_{k\in\mathbb{N}} \sum_{n\in I_k} \frac{|a_n|\Gamma(\frac{n}{p}+1)}{\sqrt{n}} < \infty$ *then* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(e^{-|z|^p})(\mathbb{C})$.

(b) If $\lim_{k\to\infty} \sum_{n\in I_k} \frac{|a_n|\Gamma(\frac{n}{p}+1)}{\sqrt{n}} = 0$ *then* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(e^{-|z|^p})_0(\mathbb{C})$.

Proof:

(a) Take $g(z) = \sum_{n=0}^{\infty} b_n z^n \in B_1(p)$. Then

$$\begin{aligned} |\sum_{n=1}^{\infty} a_n b_n \frac{\Gamma(\frac{2n}{p}+1)}{2^{\frac{2n}{p}+1}}| &\leq \sum_{k=0}^{\infty} \sum_{n\in I_{2k}} |b_n|\Gamma(\frac{n}{p}+1) \frac{|a_n|\Gamma(\frac{2n}{p}+1)}{\Gamma(\frac{n}{p}+1)2^{\frac{2n}{p}+1}} \\ &+ \sum_{k=0}^{\infty} \sum_{n\in I_{2k+1}} |b_n|\Gamma(\frac{n}{p}+1) \frac{|a_n|\Gamma(\frac{2n}{p}+1)}{\Gamma(\frac{n}{p}+1)2^{\frac{2n}{p}+1}}. \end{aligned}$$

Now let n_k and m_k given by

$$|b_{n_k}|\Gamma(\frac{n_k}{p}+1) = \sup_{n\in I_{2k}} |b_n|\Gamma(\frac{n}{p}+1),$$

$$|b_{m_k}|\Gamma(\frac{m_k}{p}+1) = \sup_{n\in I_{2k+1}} |b_n|\Gamma(\frac{n}{p}+1).$$

Since m_k and n_k are 2-lacunary sequences, applying (b) in Theorem 2.1, we have

$$\begin{aligned}|\sum_{n=1}^{\infty} a_n b_n \frac{\Gamma(\frac{2n}{p}+1)}{2^{\frac{2n}{p}+1}}| &\leq C\sum_{k=0}^{\infty}|b_{n_k}|\Gamma(\frac{n_k}{p}+1)\sum_{n\in I_{2k}}\frac{|a_n|\Gamma(\frac{n}{p}+1)}{\sqrt{n}} \\ &+C\sum_{k=0}^{\infty}|b_{m_k}|\Gamma(\frac{m_k}{p}+1)\sum_{n\in I_{2k+1}}\frac{|a_n|\Gamma(\frac{n}{p}+1)}{\sqrt{n}} \\ &\leq C\sup_{k\in\mathbb{N}}\sum_{n\in I_{2k}}|a_n|\frac{\Gamma(\frac{n}{p}+1)}{\sqrt{n}}\sum_{k=0}^{\infty}|b_{n_k}|\Gamma(\frac{n_k}{p}+1) \\ &+C\sup_{k\in\mathbb{N}}\sum_{n\in I_{2k+1}}|a_n|\frac{\Gamma(\frac{n}{p}+1)}{\sqrt{n}}\sum_{k=0}^{\infty}|b_{m_k}|\Gamma(\frac{m_k}{p}+1) \\ &\leq C\sup_{k\in\mathbb{N}}\sum_{n\in I_k}|a_n|\frac{\Gamma(\frac{n}{p}+1)}{\sqrt{n}}||g||_{B_1(p)}\end{aligned}$$

Hence the result follows now from duality.
(b) The previous argument actually shows that

$$||\sum_{n=2^k}^{\infty} a_n z^n||_{H(e^{-|z|^p})} \leq C\sup_{l\geq k}\sum_{n\in I_l}|a_n|\frac{\Gamma(\frac{n}{p}+1)}{\sqrt{n}}.$$

Hence it follows the desired result. ■

3 Unconditional convergence of Taylor series

It follows from theorems 2.1 and 2.2 that the Taylor series of every function f in the closed subspace generated by (z^{n_k}) in $B_1(p)$ or $H(e^{-|z|^p})_0(\mathbb{C})$ is absolutely convergent in case $\liminf \frac{n_{k+1}}{n_k} > 1$. The next theorem gives some subspaces of $B_1(p)$ for which the unconditional convergence of a Taylor series is equivalent to its absolute convergence. To finish the paper we present some necessary or sufficient conditions in order to ensure the unconditional convergence of a given Taylor series in $H(e^{-|z|^p})_0(\mathbb{C})$.

Theorem 3.1 *Let $\alpha \geq 2$ and $n_k = [k^\alpha]$, $k \in \mathbb{N}$. Then the series $\sum_{k=1}^{\infty} a_k V_{n_k}$ converges unconditionally in $B_1(p)$ if and only if it converges absolutely, i.e. $\sum_{k=1}^{\infty} | a_k | < \infty$.*

Proof: Let $f = \sum_{k=1}^{\infty} a_k V_{n_k}$ an unconditionally convergent series in $B_1(p)$. Denoting by $(r_n(t))$ the sequence of Rademacher functions we define $f_t(z) = \sum_{k=0}^{\infty} r_{n_k}(t) a_k V_{n_k}$. Since $\sup_{t\in[0,1]} \| f_t \|_{B_1(p)} < \infty$ then, using Fubini and Kintchine's inequality, we have

$$\begin{aligned}\int_0^\infty (\sum_{k=0}^{\infty} \frac{| a_k |^2 \, r^{\frac{2n_k}{p}}}{\Gamma(\frac{n_k}{p}+1)^2})^{\frac{1}{2}} e^{-r} dr &\simeq \int_0^\infty (\int_0^1 M_1(f_t, r) dt) e^{-r^p} p r^{p-1} dr \\ &= \int_0^1 \| f_t \|_{B_1(p)} \, dt < \infty.\end{aligned}$$

Then, taking $\beta_k = \frac{2n_k}{p}$, $q = \frac{1}{2}$ and $m \in \mathbb{N}$, we have

$$\beta_{k+m} - \beta_k = \frac{2}{p}([(k+m)^\alpha] - [k^\alpha]) \geq \frac{2}{p}((k+m)^\alpha - k^\alpha - 1) \geq \frac{2}{p}(\alpha m k^{\alpha-1} - 1).$$

Therefore if $\alpha > 2$ we can take $m = 1$ and then $\lim_{k\to\infty} \inf \frac{\beta_{k+m}-\beta_k}{\sqrt{\beta_k}} = \infty$.

For $\alpha = 2$ we can choose $m > \frac{\sqrt{p}}{2}$ and then $\lim_{k\to\infty} \inf \frac{\beta_{k+m}-\beta_k}{\sqrt{\beta_k}} > \sqrt{2}$ and Lemma 2.2 can be applied again to get $\sum_{k=1}^{\infty} | a_k | < \infty$. ■

Let us now give some results regarding the unconditional convergence in $H(e^{-|z|^p})_0(\mathbb{C})$.

Proposition 3.1 *Let (b_n) be a sequence such that* $\lim_{n\to\infty} \sup \sqrt{n+1} \mid b_n \mid = 0$. *Then $\sum_{n=0}^{\infty} b_n W_n$ converges unconditionally in $H(e^{-|z|^p})_0(\mathbb{C})$.*

Proof: Given any sequence (ϵ_n) with $\epsilon_n = \pm 1$ and $N \leq M$ we have that

$$\begin{aligned}\| \textstyle\sum_{n=N}^{M} \epsilon_n b_n W_n \|_{H(e^{-|z|^p})_0(\mathbb{C})} &\leq \sup_{|z|<1} e^{-|z|^p} \textstyle\sum_{n=N}^{M} \frac{|b_n|\sqrt{n+1}|z|^n}{\Gamma(\frac{n}{p}+1)} \\ &\leq (\sup_{n\geq N} \sqrt{n+1} \mid b_n \mid)(\textstyle\sum_{n=1}^{\infty} \frac{|z|^n}{\Gamma(\frac{n}{p}+1)}) e^{-|z|^p}.\end{aligned}$$

Now, from Lemma 1.2 and the assumption follows that $\sum_{n=1}^{\infty} \epsilon_n b_n W_n$ converges in $H(e^{-|z|^p})_0(\mathbb{C})$. ■

Proposition 3.2 *Let (b_n) be a sequence such that, if $I_k = \left[2^k, 2^{k+1}\right) \cap \mathbb{N}$,*

$$\lim_{k\to\infty} \sum_{n\in I_k} | b_n | = 0.$$

Then $\sum_{n=0}^{\infty} b_n W_n$ converges unconditionally in $H(e^{-|z|^p})_0(\mathbb{C})$.

Proof: Recall that $\sum_{n=0}^{\infty} b_n W_n$ is unconditionally convergent in $H(e^{-|z|^p})_0(\mathbb{C})$ if and only if $T(f) = (\langle f, W_n\rangle\, b_n)$ is a compact operator from $B_1(p)$ into l^1 (see [11]). Hence, it suffices to see that, denoting by $T_N(f) = (\langle f, W_n\rangle\, b_n)_{n\geq N}$, we have $\| T_N \| \to 0$ as $N \to \infty$.

We fix $k_0 \in \mathbb{N}$ and $N \geq 2^{2k_0}$. Given now $\epsilon > 0$, there exists $f = \sum_{n=1}^{\infty} a_n z^n \in B_1(p)$ such that $\| f \|_{B_1(p)} = 1$ and

$$\| T_N \| < \| T_N f \| + \epsilon \leq C \sum_{n=2^{2k_0}}^{\infty} | a_n || b_n | \Gamma(\frac{n}{p} + 1) + \epsilon.$$

Let us split this sum as follows

$$\sum_{k=k_0}^{\infty} \sum_{n\in I_{2k}} | a_n || b_n | \Gamma(\frac{n}{p} + 1) + \sum_{k=k_0}^{\infty} \sum_{n\in I_{2k+1}} | a_n || b_n | \Gamma(\frac{n}{p} + 1)$$

Take $n_k \in I_{2k}$ such that $| a_{n_k} | \Gamma(\frac{n_k}{p} + 1) = \sup_{n\in I_{2k}} | a_n | \Gamma(\frac{n}{p} + 1)$ and $n'_k \in I_{2k+1}$ such that $| a_{n'_k} | \Gamma(\frac{n'_k}{p} + 1) = \sup_{n\in I_{2k+1}} | a_n | \Gamma(\frac{n}{p} + 1)$. Observe that $\frac{n_{k+1}}{n_k} \geq 2$ and $\frac{n'_{k+1}}{n'_k} \geq 2$ and then from (b) theorem 2.1 we can say that

$$\| T_N \| < C \sup_{l\geq 2k_0} (\sum_{n\in I_l} | b_n |) + \epsilon.$$

Applying now the assumption we finish the proof. ∎

Proposition 3.3 *Let* $\sum_{n=0}^{\infty} b_n W_n$ *be an unconditionally convergent series in* $H(e^{-|z|^p})_0(\mathbb{C})$. *Then one has*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k} \sqrt{n+1}\, | b_n | = 0.$$

Proof: Using the compactness of the operator $T : B_1(p) \to l^1$ given by $T(f) = (\langle f, W_n\rangle\, b_n)$ and Lemma 1.5 we easily deduce that

$$\lim_{|z|\to\infty} e^{-|z|^p} \| T(K_p(z)) \|_1 = 0.$$

Since $T(K_p(z)) = ((\frac{n}{p})^{-\frac{n}{p}} e^{\frac{n}{p}} z^n b_n)$ we have

$$\lim_{|z|\to\infty} e^{-|z|^p} \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\Gamma(\frac{n}{p}+1)} | z^n || b_n | = 0.$$

Therefore, given $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\Gamma(\frac{n}{p}+1)} \mid b_n \mid r^{\frac{n}{p}} \leq \epsilon e^{-r}$$

for every $r \geq \frac{n_\epsilon}{2p}$. Multiplying by e^{-ar}, $1 < a < 2$, and integrating over $(\frac{n_\epsilon}{ap}, \infty)$ one has

$$\sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\Gamma(\frac{n}{p}+1)} \frac{\mid b_n \mid}{a^{\frac{n}{p}+1}} \int_{\frac{n_\epsilon}{p}}^{\infty} t^{\frac{n}{p}} e^{-t} dt \leq \epsilon \frac{1-e^{-r_\epsilon}}{a-1}$$

where $r_\epsilon = (a-1)\frac{n_\epsilon}{ap}$. Hence, using Lemma 2.1

$$\sum_{n=n_\epsilon}^{\infty} \sqrt{n+1} \frac{\mid b_n \mid}{a^{\frac{n}{p}+1}} \leq C \sum_{n=n_\epsilon}^{\infty} \frac{\sqrt{n+1}}{\Gamma(\frac{n}{p}+1)} \frac{\mid b_n \mid}{a^{\frac{n}{p}+1}} \int_{\frac{n}{p}}^{\frac{n}{p}+\sqrt{\frac{n}{p}}} t^{\frac{n}{p}} e^{-tr} dt \leq C \frac{\epsilon}{a-1} e^{-r_\epsilon}.$$

In particular, if $k \in \mathbb{N}$ satisfies $k \geq n_\epsilon$ we have, for $\frac{1}{a} = 1 - \frac{1}{k}$,

$$\sum_{n=n_\epsilon}^{k} \sqrt{n+1} \mid b_n \mid (1-\frac{1}{k})^{\frac{n}{p}} \leq C\epsilon k.$$

Therefore

$$(1-\frac{1}{k})^{\frac{k}{p}} \sum_{n=n_\epsilon}^{k} \sqrt{n+1} \mid b_n \mid \leq C\epsilon k.$$

On the other hand, since $G(z) := \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\Gamma(\frac{n}{p}+1)} \mid b_n \mid z^n$ belongs to $H(e^{-|z|^p})(\mathbb{C})$ we can apply Lemma 1.3 to obtain

$$\frac{1}{n_\epsilon} \sum_{n=0}^{n_\epsilon} \sqrt{n+1} \mid b_n \mid \leq C$$

for some constant C not depending on ϵ and

$$\frac{1}{k} \sum_{n=0}^{k} \sqrt{n+1} \mid b_n \mid \leq \frac{n_\epsilon}{k} C + (1-\frac{1}{k})^{-\frac{k}{p}} C\epsilon$$

showing what we wanted. ∎

Remark. The identity $\lim_{k\to\infty} \frac{1}{k} \sum_{n=0}^{k} \sqrt{n+1} \mid b_n \mid = 0$ is equivalent to

$$\lim_{k\to\infty} 2^{-\frac{k}{2}} \sum_{n \in I_k} \mid b_n \mid = 0.$$

Consequently the Proposition 3.3 can be regarded as a partial converse of Propositions 3.1 and 3.2.

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References

[1] Anderson, J.M.; Duncan, J.; 'Duals of Banach spaces of entire functions', *Glasgow Math. J.* **32** (1990), 215-220.

[2] Bargmann, V.; 'On a Hilbert space of analytic functions and an associated integral transform', *Comm. Pure Appl. Math.* **14** (1961), 187-214.

[3] Bierstedt, K.D.; Bonet, J.; Galbis, A.; 'Weighted spaces of holomorphic functions on balanced domains', *Michigan Math. J.* **40** (1993), 271-297.

[4] Dyakonov, K.M.; 'Moment problems for bounded functions', *Comm. in Analysis and Geometry* **2**,n^0 3, (1994), 533-562.

[5] Duren, P.; *Theory of H^p-spaces.* Academic Press, New York, 1970

[6] Galbis, A.; 'Weighted Banach spaces of entire functions', *Archiv Math.* **62** (1994), 58-64.

[7] Garling, D.J.H.; Wojtaszczyk, P.; 'Some Bargmann spaces of analytic functions', *Lecture Notes in Pure and Applied Math. "Function Spces. The second conference."* **172** Marcel Dekker (1995), 123-138.

[8] Grochenig,K.; Walnut, D.; 'A Riesz basis for Bargmann-Fock space related to sampling and interpolation', *Archiv. fur Math.* **30** (1992), 283-395.

[9] Janson, S.; Peetre, J.; Rochberg, R.; 'Hankel forms and the Fock space', *Revista Mat. Iberoamericana* **3** (1987), 61-138.

[10] Lusky, W.; 'On the Fourier series of unbounded harmonic functions', *Preprint.*

[11] Wojtaszczyk, P.; *Banach spaces for analysts.* Cambrigde Univ. Press, New York, 1991

[12] Zhu, K.; 'Duality of Bloch spaces and norm convergence of Taylor series', *Michigan Math. J.* **38** (1991), 89-101.

[13] Zygmund, A.; *Trigonometric series .* Cambrigde Univ. Press, New York, 1959