DYADIC BMO ON THE BIDISK

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ABSTRACT. We give several new characterizations of the dual of the dyadic Hardy space $H^{1,d}(\mathbb{T}^2)$, the so-called dyadic BMO space in two variables and denoted $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$. These include characterizations in terms of Haar multipliers, in terms of the "symmetrised paraproduct" Λ_b, in terms of the rectangular BMO norms of the iterated "sweeps", and in terms of nested commutators with dyadic martingale transforms. We further explore the connection between $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ and John-Nirenberg type inequalities, and study a scale of rectangular BMO spaces.

1. INTRODUCTION

Throughout the paper $\mathcal{D}$ denotes the set of dyadic intervals in the unit circle $\mathbb{T}$. In the case of the bicircle $\mathbb{T}^2$, $\mathcal{D}_1$ denotes the dyadic intervals in the first, $\mathcal{D}_2$ the dyadic intervals in the second variable. We write $\mathcal{R} = \mathcal{D}_1 \times \mathcal{D}_2$ for the dyadic rectangles, $|I|$ for the length of I and $|R|$ for the area of R. $(h_I)_{I \in \mathcal{D}}$ stands for the Haar basis in $L^2(\mathbb{T})$ and $(h_R)_{R \in \mathcal{R}}$ for the product Haar basis of $L^2(\mathbb{T}^2)$.

Here $h_I(t) = \frac{1}{|R|^{1/2}} \left(\chi_{I^+}(t) - \chi_{I^-}(t) \right)$ for each dyadic interval $I \in \mathcal{D}$, where I^- denotes the left half of I, and I^+ denotes the right half of I. For each dyadic rectangle $R = I \times J \in \mathcal{R}$, h_R is defined by $h_R(s,t) = h_I(s)h_J(t)$.

We denote by $\mathcal{H}_{00}$ the space of all functions in $L^2(\mathbb{T}^2)$ which have a finite expansion in the product Haar basis.

Given $g \in L^2(\mathbb{T})$, we use the notation $g_I = \langle f, h_I \rangle$ and $m_I g = \frac{1}{|I|} \int_I g(t)dt$. Similarly, given $f \in L^2(\mathbb{T}^2)$, we use the notation $f_R = \langle f, h_R \rangle$, $f_I(s) = \langle f(\cdot, s), h_I \rangle$, $m_I f(s) = \frac{1}{|I|} \int_I f(t,s)dt$, $f_J(t) = \langle f(t,\cdot), h_J \rangle$ and $m_J f(t) = \frac{1}{|J|} \int_J f(t,s)ds$. Therefore

$$f(t,s) = \sum_{R \in \mathcal{R}} f_R h_R(t,s) = \sum_{I \in \mathcal{D}} f_I(s)h_I(t) = \sum_{J \in \mathcal{D}} f_J(t)h_J(s).$$

Let $P_I g = (g - m_I g)\chi_I$ for $g \in L^2(\mathbb{T})$. Observe that P_I is the orthogonal projection on the subspace spanned by the Haar functions $h_{I'}$, $I' \in \mathcal{D}$, $I' \subseteq I$. If $g = \sum_{I \in \mathcal{D}} g_I h_I$, then

$$(1) \qquad P_I g = \sum_{I' \in \mathcal{D}, I' \subseteq I} h_{I'} g_{I'}.$$

Similarly, for each measurable set $\Omega \subseteq \mathbb{T}^2$, let P_Ω be the orthogonal projection on the subspace spanned by the Haar functions $h_{R'}$, $R' \in \mathcal{R}$, $R' \subseteq \Omega$. In particular, for each dyadic rectangle $R = I \times J \in \mathcal{R}$ and for $f = \sum_{R' \in \mathcal{R}} h_{R'} f_{R'} \in L^2(\mathbb{T}^2)$, we have $P_R f = P_I \otimes P_J f \sum_{R' \in \mathcal{R}, R' \subseteq R} h_{R'} f_{R'}$.

It is easy to see that for $R \in \mathcal{R}$ and $f \in L^2(\mathbb{T}^2)$,

$$(2) \qquad P_R f = \chi_{I \times J} \left(f - m_I f - m_J f + m_{I \times J} \right) \chi_{I \times J}.$$

Recall that $g \in L^2(\mathbb{T})$ is said to belong to dyadic BMO, to be denoted $\mathrm{BMO}^d(\mathbb{T})$, if

$$\sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I |g(t) - m_I g|^2 dt \right)^{1/2} < \infty.$$

By John-Nirenberg's lemma, this is equivalent to

$$\sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I |g(t) - m_I g|^p dt \right)^{1/p} < \infty$$

for any $1 \leq p < \infty$.

Hence $g \in \mathrm{BMO}^d(\mathbb{T})$ if and only if there exists a constant C such that for all $I \in \mathcal{D}$

$$\sum_{I' \in \mathcal{D}, I' \subseteq I} |g_{I'}|^2 \leq C|I|^{1/2},$$

or equivalently

$$\sup_{I \in \mathcal{D}} \frac{1}{|I|^{1/p}} \|P_I g\|_p < \infty$$

for $1 \leq p < \infty$.

The space BMO appears in many different contexts. We shall use that $\mathrm{BMO}^d(\mathbb{T}) = (H^{1,d}(\mathbb{T}))^*$ where $H^{1,d}$ is defined in terms of the dyadic square functions $\mathcal{S}$, $\mathcal{S}g = \left(\sum_{I \in \mathcal{D}_2} \frac{\chi_I}{|I|} |g_I|^2 \right)^{1/2}$. That is ,

$$H^{1,d}(\mathbb{T}) = \{ g \in L^1(\mathbb{T}) : \mathcal{S}g \in L^1(\mathbb{T}) \}.$$

Using Carleson measures, this gives rise to a description of BMO^d in terms of symbols g for which the dyadic paraproduct π_g,

$$\pi_g(f) = \sum_{I \in \mathcal{D}} g_I m_I f h_I$$

or its adjoint operator Δ_g, $\Delta_g(f) = \sum_{I \in \mathcal{D}} g_I f_I \frac{\chi_I}{|I|}$, is bounded on $L^2(\mathbb{T})$ (or equivalently, on $L^p(\mathbb{T})$ for $1 < p < \infty$).

The situation in two variables it is rather different and much more delicate. One main reason for the difficulties encountered in the multivariable theory is the failure of the naive generalization of the Carleson Embedding Theorem to several variables (see [C], [Fef]). The reader is referred to [ChFef2] for an overview on the theory and an outline of the main differences.

Several new results (e. g. [FS] and [PS]) further exhibit the differences between certain BMO spaces on the polydisk defined by multi-variable versions of the different yet equivalent characterizations of $\mathrm{BMO}(\mathbb{T})$.

A function $f \in L^2(\mathbb{T}^2)$ is said to belong to the rectangular dyadic BMO space, to be denoted $\mathrm{BMO}^d_{\mathrm{rect}}$, if

$$(3) \qquad \sup_{R = I \times J \in \mathcal{R}} \left(\frac{1}{|R|} \int_R |f(t,s) - m_I f(s) - m_J f(t) - m_{I \times J} f|^2 dt ds \right)^{1/2} < \infty.$$

Or equivalently,

$$\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}} = \sup_{R \in \mathcal{R}} \frac{1}{|R|^{1/2}} \|P_R \varphi\|_2.$$

We will also consider a *p*-version of the dyadic rectangular norm for $1 \leq p < \infty$,

$$(4) \qquad \|\varphi\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect,p}}} = \sup_{R \in \mathcal{R}} \frac{1}{|R|^{1/p}} \|P_R \varphi\|_p.$$

Here, $\|\cdot\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect}}} = \|\cdot\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect,2}}}$. In the one-variable case, the corresponding norms are of course all equivalent because of John-Nirenberg's lemma.

Let us start by defining $\mathrm{BMO}^{d}_{prod}(\mathbb{T}^2)$ as the dual of $H^{1,d}(\mathbb{T}^2)$, the space of functions $f \in L^1(\mathbb{T}^2)$ such that $\mathcal{S}(f) \in L^1(\mathbb{T}^2)$, where $\mathcal{S}(f) = (\sum_{R \in \mathcal{R}} |f_R|^2 h_R^2)^{1/2}$.

Although $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}(\mathbb{T}^2)$ cannot be characterized by (3) [Fef], it was shown by Bernard in the dyadic case [Be] and also by Chang and R. Fefferman in a continuous version [ChFef1] that $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}(\mathbb{T}^2)$ can also be described as the space of functions $\varphi \in L^2(\mathbb{T}^2)$ for which there exists $C > 0$ such that

$$(5) \qquad \|\varphi\|_{prod} = \sup_{\Omega \subset \mathbb{T}^2} \frac{1}{|\Omega|^{1/2}} \|P_\Omega \varphi\|_2 < \infty,$$

where the supremum is taken over all measurable sets $\Omega \subseteq \mathbb{T}^2$. This immediately implies $\mathrm{BMO}^{d}_{prod} \subseteq \mathrm{BMO}_{rect,2}$.

The connection between both spaces can be also seen from the description of BMO^{d}_{prod} in terms of the boundedness of the dyadic paraproduct in two variables, defined by $\pi_b^{(1,2)}(f) = \sum_{R \in \mathcal{R}} b_R m_R f h_R$.

It follows from Chang's generalization of the Carleson Embedding Theorem (see [Ch]) that $b \in \mathrm{BMO}^{d}_{prod}$ if and only if the double paraproduct $\pi_b^{(1,2)}$ is bounded on $L^2(\mathbb{T}^2)$. In our paper the following fact will be rather crucial:

$$(6) \qquad \|\varphi\|_{prod} \approx \|\pi_\varphi^{(1,2)}\|.$$

An similar characterization for BMO^{d}_{rect} was proved in [PS], Proposition 3.3.1, namely that $b \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect}}$ if and only if $\pi_b^{(1,2)}$ maps $L^2(\mathbb{T}) \hat{\otimes} L^2(\mathbb{T})$ boundedly into $L^2(\mathbb{T}^2)$, where $L^2(\mathbb{T}) \hat{\otimes} L^2(\mathbb{T})$ stands for the projective tensor product. This also implies that $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}} \subsetneq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect}}$ (see [Fef] for an alternative approach).

We shall try to better understand the difference between both spaces. Two approaches are used to this end. First we observe that John-Nirenberg type inequalities do not hold in BMO^{d}_{rect}, in the sense that the 2-norm in the definition of $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect}}$ cannot be replaced by any other *p*-norm. This solves a question left open in [FS]. Secondly, we analyse the behaviour of the sweep of functions in the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ and in the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect,p}}$ spaces.

Our main new tool will be characterizations of $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ in terms of Haar multipliers. Recall that sequence of functions $(\phi_R)_{R \in \mathcal{R}}$ is called a Haar multiplier (see (23) or [Per]) on $L^p(\mathbb{T}^2)$, if the map $f = \sum_{R \in \mathcal{R}} f_R h_R(t,s) \mapsto \sum_{R \in \mathcal{R}} \phi_R(t,s) f_R h_R(t,s)$ defines a bounded operator on $L^p(\mathbb{T}^2)$.

We shall say that $b \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{mult}}$ if $\{(P_R b)\}_{R \in \mathcal{R}}$ defines a Haar multiplier on $L^2(\mathbb{T}^2)$.

Using the characterization of $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ in terms of dyadic paraproducts, we observe that $b \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ if the operator $\Delta_b = (\pi_{\bar{b}}^{(1,2)})^*$ given by $\Delta_b(f) = \sum_R b_R h_R f_R h_R$ defines a bounded operator on $L^2(\mathbb{T}^2)$. Hence $b \in \mathrm{BMO}^{d}_{prod}$ if and only if $(b_R h_R)_{R \in \mathcal{R}}$ is a Haar multiplier on $L^2(\mathbb{T}^2)$.

On the other hand, letting the Haar multiplier $(P_{R'} b)_{R' \in \mathcal{R}}$ act on h_R, we see that $\frac{1}{|R|^{1/2}} \|P_R b\|_2 \leq \|b\|_{\mathrm{mult}}$, implying that $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{mult}} \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{rect}}$.

We shall get a description of $\mathrm{BMO}^{d}_{\mathrm{mult}}$ in terms of the boundedness of the operator

$$\Lambda_b = \pi_b^{(1,2)} + (\pi_b^{(1,2)})^* + \Delta_{\pi_b} + (\Delta_{\pi_b})^*,$$

where Δ_{π_b} (see Definition 2.4) is an operator combining the one-variable paraproduct π and its adjoint. This will allow us to prove that $\mathrm{BMO}^{d}_{\mathrm{mult}}(\mathbb{T}^2) = \mathrm{BMO}^{d}_{\mathrm{prod}}(\mathbb{T}^2)$ (see Theorem 2.8).

On the other hand, $\mathrm{BMO}^{d}_{\mathrm{rect}}$ can also be described using Λ_b. We show that $\mathrm{BMO}^{d}_{\mathrm{rect}}$ can be characterized in terms of "average boundedness" of Λ_b or in terms of its boundedness from $L^2(\mathbb{T})\hat{\otimes}L^2(\mathbb{T})$ into $L^2(\mathbb{T}^2)$.

The paper is divided into four sections. The first one is devoted to the introduction of the space $\mathrm{BMO}^{d}_{\mathrm{mult}}$ and the proof of some of its properties. We see that $\mathrm{BMO}^{d}_{\mathrm{mult}}$ can be characterized as the space of symbols b for which the operator Λ_b is bounded, and that this space coincides with $\mathrm{BMO}^{d}_{\mathrm{prod}}$.

The second section deals with results on sweep functions. We prove the following formula connecting the boundedness of $\pi_b^{(1,2)}$ and Λ_{S_b} (see Lemma 3.2):

$$(7) \qquad\qquad \pi_b^{(1,2)^*}\pi_b^{(1,2)} = \Lambda_{S_b} + D_b,$$

where D_b is bounded if $b \in \mathrm{BMO}^{d}_{rect,2}$.

This allows us to see that $b \in \mathrm{BMO}^{d}_{\mathrm{prod}}$ if and only if $S_b \in \mathrm{BMO}^{d}_{\mathrm{prod}}$. We also obtain a characterization of $\mathrm{BMO}^{d}_{\mathrm{prod}}$ in terms of nested commutators with dyadic martingale transforms, sharpening a result from [PS].

In the third section, we further use the formula (7) to quantify the difference between the BMO spaces we have considered, and to get a characterization of $\mathrm{BMO}^{d}_{\mathrm{prod}}$ relying only upon the $\mathrm{BMO}^{d}_{\mathrm{rect}}$ norm of the n-fold sweeps.

Finally, in the fourth section, we apply the results from the third part together with interpolation to study the scale of spaces $\mathrm{BMO}^{d}_{\mathrm{rect,p}}$ introduced in (4) and show that these spaces are pairwise distinct. As a corollary, we obtain that $\mathrm{BMO}^{d}_{\mathrm{prod}} \subsetneq \cap_{p\geq 1} \mathrm{BMO}^{d}_{\mathrm{rect,p}}$.

2. BMO via Haar multipliers.

Definition 2.1. *We shall say that $b \in \mathrm{BMO}^{d}_{mult}$ if $\{(P_R b)\}_{R\in\mathcal{R}}$ defines a Haar multiplier on $L^2(\mathbb{T}^2)$, i.e. there exists $C > 0$ such that*

$$\|\sum_{R\in\mathcal{R}} P_R b\, f_R h_R\|_2 \leq C\|f\|_2$$

for all $f \in L^2(\mathbb{T}^2)$. We define $\|b\|_{mult}$ as the norm of the corresponding operator.

Let us start by pointing out some simple facts about this space.

Given $I \in \mathcal{D}$ we write P_I for the operator on $L^2(\mathbb{T})$ given by (1), and $\tilde{P}_I = P_I \otimes \mathrm{id}$ for the corresponding projection on $L^2(\mathbb{T}^2)$,

$$\tilde{P}_I(f)(t,s) = \sum_{I'\in\mathcal{D},I'\subseteq I} h_{I'}(t)f_{I'}(s).$$

Similarly, given $J \in \mathcal{D}_2$, we write $\tilde{P}_J$ for $\mathrm{id} \otimes P_J$.

Of course, $\tilde{P}_I(f)(t,s) = P_I(f(s,\cdot))(t)$ and $P_R f = \tilde{P}_J(\tilde{P}_I f)$ for $R = I \times J$.

Proposition 2.2.

(8)
$$L^\infty(\mathbb{T}^2) \subset \mathrm{BMO}_{mult}^d(\mathbb{T}^2)$$

(9)
$$\mathrm{BMO}^d(\mathbb{T}) \otimes \mathrm{BMO}^d(\mathbb{T}) \subseteq \mathrm{BMO}_{mult}^d(\mathbb{T}^2)$$

Proof. Using (2), one easily obtains the following formula:

$$\sum_{R \in \mathcal{R}} P_R b f_R h_R = fb - \sum_{I \in \mathcal{D}} (m_I b) f_I h_I - \sum_{J \in \mathcal{D}} (m_J b) f_J h_J + \sum_{R \in \mathcal{R}} m_R b f_R h_R.$$

Now (8) follows from this expression together with

$$|| \sum_{J \in \mathcal{D}} (m_J b) f_J h_J ||^2_{L^2(\mathbb{T}^2)} = \sum_{J \in \mathcal{D}} ||m_J b f_J||^2_{L^2(\mathbb{T})},$$

since

$$|| \sum_{J \in \mathcal{D}} (m_J b) f_J h_J ||^2_{L^2(\mathbb{T}^2)} \leq ||b||_\infty \sum_{J \in \mathcal{D}} ||f_J||^2_{L^2(\mathbb{T})} = ||b||^2_\infty ||f||^2_2,$$

and the trivial estimates for the terms bf and $\sum_{R \in \mathcal{R}} m_R b f_R h_R$.

To see (9), note first that for $b_1 \in \mathrm{BMO}^d$ and $f \in L^2(\mathbb{T})$,

(10)
$$\sum_{I \in \mathcal{D}} P_I b_1 f_I h_I = (\pi_{b_1} + \Delta_{b_1}) f = (\pi_{b_1} + (\pi_{\bar{b}_1})^*) f.$$

Therefore $(P_I b_1)_{I \in \mathcal{D}}$ defines a bounded Haar multiplier on $L^2(\mathbb{T})$.

Now let $b(t, s) = b_1(t) b_2(s)$ with $b_1, b_2 \in \mathrm{BMO}^d(\mathbb{T})$. Then $P_R(b) = P_I(b_1) P_J(b_2)$ and therefore

$$\sum_{R \in \mathcal{R}} P_R b f_R h_R = \sum_{I \in \mathcal{D}} P_I b_1 \left(\sum_{J \in \mathcal{D}} P_J b_2 f_J h_J \right)_I h_I.$$

This yields

$$\begin{aligned}
|| \sum_{R \in \mathcal{R}} P_R b f_R h_R ||^2_2 &= \int_{\mathbb{T}^2} || \sum_{I \in \mathcal{D}} P_I b_1(t) \left(\sum_{J \in \mathcal{D}} P_J b_2 f_J h_J \right)_I (s) h_I(t)||^2_2 dt ds \\
&\leq C ||b_1||^2_{\mathrm{BMO}} \int_{\mathbb{T}} \sum_{I \in \mathcal{D}} \left| \sum_{J \in \mathcal{D}} (P_J b_2 f_J h_J)_I(s) \right|^2 ds \\
&\leq C^2 ||b_1||^2_{\mathrm{BMO}} ||b_2||^2_{\mathrm{BMO}} \sum_{I \in \mathcal{D}} | \sum_{J \in \mathcal{D}} |f_{I \times J}|^2
\end{aligned}$$

with some absolute constant $C > 0$. $\qquad\square$

As announced in the introduction, we first relate this space to $\mathrm{BMO}_{\mathrm{prod}}^{\mathrm{d}}$. For this purpose we introduce the dyadic paraproducts in two variables (see [PS]):

Definition 2.3. *Given* $b \in L^2(\mathbb{T}^2)$, *we write*

$$\pi_b^{(1,2)}(f) = \sum_{R \in \mathcal{R}} b_R m_R f h_R$$

and

$$\Delta_b^{(1,2)}(f) = (\pi_{\bar{b}}^{(1,2)})^*(f) = \sum_{R \in \mathcal{R}} b_R f_R \frac{\chi_R}{|R|}.$$

The formula

$$(11) \qquad \langle \pi_b^{(1,2)}(f), g \rangle = \langle f, \Delta_{\bar{b}}^{(1,2)}(g) \rangle = \int_{\mathbb{T}^2} b \left(\sum_{R \in \mathcal{R}} m_R(f) g_R h_R \right) dt ds$$

for $f, g \in \mathcal{H}_{00}$ completely describe the action of the operators $\pi_b^{(1,2)}$ and $\Delta_b^{(1,2)}$.

Let us now define the following mixed operators (see [PS]).

Definition 2.4. *Given* $b \in L^2(\mathbb{T}^2)$, *we define the operators* π_{Δ_b} *and* Δ_{π_b} *by*

$$(12) \qquad \langle \pi_{\Delta_b}(f), g \rangle = \langle f, \Delta_{\pi_{\bar{b}}}(g) \rangle = \int_{\mathbb{T}^2} b \left(\sum_{I \times J \in \mathcal{R}} m_I(f_J) m_J(\bar{g}_I) h_{I \times J} \right) dt ds$$

for $f, g \in \mathcal{H}_{00}$.

We write

$$\Lambda_b = \pi_b^{(1,2)} + \Delta_b^{(1,2)} + \Delta_{\pi_b} + \pi_{\Delta_b}.$$

Clearly we have the following expressions:

$$(13) \qquad \pi_b^{(1,2)}(f)(t,s) = \sum_{I \in \mathcal{D}} \pi_{b_I}(m_I f)(s) h_I(t)$$

$$(14) \qquad \Delta_b^{(1,2)}(f)(t,s) = \sum_{I \in \mathcal{D}} \Delta_{b_I}(f_I)(s) h_I^2(t)$$

$$(15) \qquad \Delta_{\pi_b}(f)(t,s) = \sum_{I \in \mathcal{D}} \pi_{b_I}(f_I)(s) h_I^2(t)$$

$$(16) \qquad \pi_{\Delta_b}(f)(t,s) = \sum_{I \in \mathcal{D}} \Delta_{b_I}(m_I f)(s) h_I(t).$$

Lemma 2.5. *Let* $R = I \times J \in \mathcal{R}$ *and denote* $R_+ = I^+ \times J^+ \cup I^- \times J^-$ *and* $R_- = I^+ \times J^- \cup I^- \times J^+$. *Then*

$$(17) \qquad \pi_b^{(1,2)}(h_R) = (P_{R_+}(b) + P_{R_-}(b)) h_R = (P_{R_+}(b) - P_{R_-}(b)) |R|^{-1/2}$$

Proof. Using that $m_{R'}(h_R) \neq 0$ only if $I' \subsetneq I$ and $J' \subsetneq J$ and that in this case $m_{R'}(h_R) = h_R(x_{R'})$, where $x_{R'} = (t_{I'}, s_{J'})$ is the center of R, we obtain that

$$\pi_b^{(1,2)}(h_R) = \sum_{I' \subsetneq I, J' \subsetneq J} b_{R'} h_R(x_{R'}) h'_R.$$

Observe that $h_R(x_{R'}) = h_R(t,s) = \frac{1}{|R|^{1/2}}$ for $R' \subset R_+$ and $(t,s) \in R^+$. Similarly $h_R(x_{R'}) = h_R(t,s) = -\frac{1}{|R|^{1/2}}$ for $R' \subset R_+$ and $(t,s) \in R^-$. This gives (17). $\qquad \square$

Corollary 2.6. *Let* $b \in L^2(\mathbb{T}^2)$. *Then* $b \in \mathrm{BMO}_{prod}^d$ *if and only if* $(P_{R_+}(b) + P_{R_-}(b)) h_R)_{R \in \mathcal{R}}$ *is a Haar multiplier on* $L^2(\mathbb{T}^2)$.

Lemma 2.7. $\Lambda_b(f) = \sum_{R \in \mathcal{R}} P_R(b) f_R h_R$.

Proof. Note that for $\phi, g \in \mathcal{H}_{00}$, we have

$$(18) \qquad \phi g = \pi_\phi(g) + \Delta_\phi(g) + \pi_g(\phi).$$

As in (10), one obtains

$$(19) \qquad \sum_{J \in \mathcal{D}} P_J(\phi) g_J h_J = \pi_\phi(g) + \Delta_\phi(g).$$

Combining (19) with the formulas in (13) we get

$$\Lambda_b(f)(t,s) = \sum_{I\in\mathcal{D}}(\pi_{b_I}+\Delta_{b_I})(m_If)(s)h_I(t) + \sum_{I\in\mathcal{D}}(\pi_{b_I}+\Delta_{b_I})(f_I)(s)h_I^2(t)$$

$$= \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}P_J(b_I)(s)(m_If)_Jh_J(s)h_I(t) + \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}P_J(b_I)(s)(f_I)_Jh_J(s)h_I^2(t)$$

$$= \sum_{J\in\mathcal{D}}(\sum_{I\in\mathcal{D}}(\tilde{P}_Jb)_I(s)m_I(f_J)h_I(t))h_J(s) + \sum_{J\in\mathcal{D}}(\sum_{I\in\mathcal{D}}(\tilde{P}_Jb)_I(s)(f_J)_Ih_I^2(t))h_J(s)$$

$$= \sum_{J\in\mathcal{D}}\left(\pi_{\tilde{P}_Jb(s,\cdot)}(f_J)\right)(t)h_J(s) + \sum_{J\in\mathcal{D}}\left(\Delta_{\tilde{P}_Jb(s,\cdot)}(f_J)\right)(t)h_J(s)$$

$$= \sum_{I\times J\in\mathcal{R}}\tilde{P}_I(\tilde{P}_J(b))(t,s)f_{I\times J}h_{I\times J}(t,s)$$

$$= \sum_{R\in\mathcal{R}}P_R(b)(t,s)f_Rh_R(t,s).$$

$\square$

We now are ready to prove our characterization of $\text{BMO}^{\text{d}}_{\text{prod}}$ in terms of Haar multipliers.

Theorem 2.8. $\text{BMO}^{\text{d}}_{\text{prod}} = \text{BMO}^{\text{d}}_{\text{mult}}.$

Proof. To see that $\text{BMO}^{\text{d}}_{\text{prod}} \subseteq \text{BMO}^{\text{d}}_{\text{mult}}$, it suffices to see that the boundedness of $\pi^{(1,2)}$ implies the boundedness of Δ_{π_b}. This was proved in [PS], we include here a proof for the sake of completeness.

By (12) and the characterization of $\text{BMO}^{\text{d}}_{\text{prod}}$ as the dual of $H^{1,d}$, the space of functions with integrable square function, we simply need to show that $F = \sum_{I\times J\in\mathcal{R}} m_I(f_J)m_J(g_I)h_{I\times J}(t,s)$ belongs to $H^{1,d}$. Note that

$$\mathcal{S}(F)(t,s) = \left(\sum_{I\times J\in\mathcal{R}}|m_I(f_J)|^2|m_J(g_I)|^2h_{I\times J}^2(t,s)\right)^{1/2}.$$

Therefore

$$\mathcal{S}(F)(t,s) \leq \left(\sum_{J\in\mathcal{D}}\sum_{I\in\mathcal{D}}(g_I)^*(s)^2h_J^2(s)(f_J^*(t))^2h_I^2(t)\right)^{1/2}$$

$$= \left(\sum_{I\in\mathcal{D}}(g_I^*(s))^2h_I^2(t)\right)^{1/2}\left(\sum_{J\in\mathcal{D}}(f_J^*(t))^2h_J^2(s)\right)^{1/2},$$

and hence

$$\int_{\mathbb{T}^2}\mathcal{S}(F)(t,s)dtfs \leq \left(\int_{\mathbb{T}^2}\sum_{I\in\mathcal{D}}(g_I^*(s))^2h_I^2(t)dtds\right)^{1/2}\left(\int_{\mathbb{T}^2}\sum_{J\in\mathcal{D}}(f_J^*(s))^2h_J^2(t)dtds\right)^{1/2}$$

$$= \left(\sum_{I\in\mathcal{D}}\int_{\mathbb{T}^2}(g_I^*(s))^2ds\right)^{1/2}\left(\sum_{J\in\mathcal{D}}\int_{\mathbb{T}^2}(f_J^*(t))^2dt\right)^{1/2}$$

$$\leq C\left(\sum_{I\in\mathcal{D}}||g_I||_2^2\right)^{1/2}\left(\sum_{J\in\mathcal{D}}||f_J||_2^2\right)^{1/2} = C||g||_2||f||_2.$$

To prove the reverse inclusion $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{mult}} \subseteq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$, we shall use the characterization of $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ given in (6).

It is clear that for each measurable set Ω, we have $P_\Omega(b) = P_\Omega(\pi_b^{(1,2)}(\chi_\Omega))$. We shall show now that
$$P_\Omega(\pi_b^{(1,2)}(\chi_\Omega)) = P_\Omega(\Lambda_b(\chi_\Omega)).$$
Let $R \in \mathcal{R}$ and $R \subseteq \Omega$. Then
$$\langle \Delta_b^{(1,2)}(\chi_\Omega), h_R \rangle = \langle \chi_\Omega, \pi_b^{(1,2)}(h_R) \rangle = |R|^{-1/2} \langle \chi_\Omega, P_{R^+}b - P_{R^-}b \rangle = 0.$$
This shows that $P_\Omega(\Delta_b^{(1,2)}(\chi_\Omega)) = 0$.

On the other hand, we also have for $R = I \times J \subseteq \Omega$ that
$$\pi_{\Delta_b}(h_R) = \sum_{I' \subsetneq I} b_{I' \times J} m_{I'}(h_I) \chi_J h_{I'}.$$
Using that $\langle \chi_\Omega, \chi_J h_{I'} \rangle = 0$ for all $I' \subseteq I$, we obtain $P_\Omega(\pi_{\Delta_b}(\chi_\Omega)) = 0$.

Similarly, $P_\Omega(\Delta_{\pi_b}(\chi_\Omega)) = 0$. Finally,
$$\begin{aligned} ||P_\Omega(b)|| &= ||P_\Omega(\pi_b^{(1,2)}(\chi_\Omega))|| = ||P_\Omega(\Lambda_b(\chi_\Omega))|| \\ &\leq ||\Lambda_b(\chi_\Omega))|| \leq ||\Lambda_b|| |\Omega|^{1/2}. \end{aligned}$$
$\square$

As a consequence of Thm 2.8, we can sharpen Thm 7.7.2 from [PS] and characterize $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$ in terms of the boundedness of nested commutators with dyadic martingale transforms. This can be understood as a dyadic analogue of the characterization of the continuous product BMO space $\mathrm{BMO}_{\mathrm{prod}}$ as the space of functions for which the nested commutator
$$[H_1, [H_2, b]] : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$$
is bounded, where H_1 resp. H_2 denote the Hilbert transform in the first resp. second variable on $L^2(\mathbb{T}^2)$. The latter was proved in [FS] and [LF].

Let Σ_1, Σ_2 be the spaces of all sequences of signs indexed by the elements of $\mathcal{D}_1$, $\mathcal{D}_2$, $\Sigma_1 = \{0,1\}^{\mathcal{D}_1}$, $\Sigma_1 = \{0,1\}^{\mathcal{D}_2}$, and let $d\sigma_1$ denote the natural product probability measure on Σ_1, which assigns measure 2^{-n} to each cylindrical set of length n. Let $d\sigma_2$ denote the corresponding measure on Σ_2. Let $\Sigma = \Sigma_1 \times \Sigma_2$, with $d\sigma$ denoting the product measure, and $\mathcal{R} = \mathcal{D}_1 \times \mathcal{D}_2$ as before.

For $\sigma_1 = (\sigma_1(I))_{I \in \mathcal{D}_1} \in \Sigma_1$, $\sigma_2 = (\sigma_2(J))_{J \in \mathcal{D}_2} \in \Sigma_2$, let $T_{\sigma_1}, T_{\sigma_2}$ denote the dyadic martingale transforms
$$T_{\sigma_1} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2), \quad f = \sum_{I \times J \in \mathcal{R}} f_{I \times J} h_{I \times J} \mapsto \sum_{I \times J \in \mathcal{R}} \sigma_1(I) f_{I \times J} h_{I \times J},$$
$$T_{\sigma_2} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2), \quad f = \sum_{I \times J \in \mathcal{R}} f_{I \times J} h_{I \times J} \mapsto \sum_{I \times J \in \mathcal{R}} \sigma_2(J) f_{I \times J} h_{I \times J}.$$

Theorem 2.9. *Let $b \in L^2(\mathbb{T}^2)$. Then the following are equivalent:*
  (i) *$b \in \mathrm{BMO}^{\mathrm{d}}_{\mathrm{prod}}$*
  (ii) *The nested commutators*
$$(20) \qquad [T_{\sigma_1}, [T_{\sigma_2}, b]] : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$$
   are uniformly bounded for all $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$.

(iii) *The nested commutators* $[T_{\sigma_1}, [T_{\sigma_2}, b]] : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ *are bounded on average, in the sense that the map*

$$\Phi_b : L^2(\mathbb{T}^2) \rightarrow L^2(\Sigma_1 \times \Sigma_2 \times \mathbb{T}^2), f \mapsto [T_{\sigma_1}, [T_{\sigma_2}, b]]f$$

is bounded.

In this case, we have

$$(21) \qquad \|b\|_{\mathrm{BMO}^d_{\mathrm{prod}}} \approx \|\Lambda_b\| \leq \|\Phi_b\| \leq \sup_{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2} \|[T_{\sigma_1}, [T_{\sigma_2}, b]]\| \leq 4\|\Lambda_b\|.$$

Proof. We use the ideas of the proofs of Thm 3.4, Cor 4.1 in [GPTV], adapted to the two-variable case, and of Thm 7.7.2 in [PS].
¿From [PS], p 493, we know that

$$[T_{\sigma_1}, [T_{\sigma_2}, b]] = [T_{\sigma_1}, [T_{\sigma_2}, \Lambda_b]].$$

Therefore $\sup_{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2} \|[T_{\sigma_1}, [T_{\sigma_2}, b]]\| \leq 4\|\Lambda_b\|$. The second inequality in (21) is obvious. Finally, for $f \in L^2(\mathbb{T}^2)$ one has

$$(22) \quad \begin{aligned} \|\Phi_b f\|^2 &= \int\int_{\Sigma_1 \times \Sigma_2} \|[T_{\sigma_1}, [T_{\sigma_2}, b]]f\|^2_{L^2(\mathbb{T}^2)} d\sigma_1 d\sigma_2 \\ &= \int\int_{\Sigma_1 \times \Sigma_2} \|\sum_{I \times J \in \mathcal{R}} \sigma_1(I)\sigma_2(J)[\tilde{P}_I, [\tilde{P}_J, b]]f\|^2_{L^2(\mathbb{T}^2)} d\sigma_1 d\sigma_2 \\ &= \sum_{I \times J \in \mathcal{R}} \|[\tilde{P}_I, [\tilde{P}_J, b]]f\|^2_{L^2(\mathbb{T}^2)} \\ &= \sum_{I \times J \in \mathcal{R}} \|[\tilde{P}_I, [\tilde{P}_J, \Lambda_b]]f\|^2_{L^2(\mathbb{T}^2)} \\ &= \sum_{I \times J \in \mathcal{R}} \|(\tilde{P}_I \tilde{P}_J \Lambda_b - \tilde{P}_I \Lambda_b \tilde{P}_J - \tilde{P}_J \Lambda_b \tilde{P}_I + \Lambda_b \tilde{P}_I \tilde{P}_J)f\|^2_{L^2(\mathbb{T}^2)} \\ &\geq \sum_{I \times J \in \mathcal{R}} \|\tilde{P}_I \tilde{P}_J \Lambda_b f\|^2 = \|\Lambda_b f\|^2, \end{aligned}$$

since $\tilde{P}_I \Lambda_b \tilde{P}_I = 0$ and $\tilde{P}_J \Lambda_b \tilde{P}_J = 0$. This proves the first inequality in (21). □

The martingale transformation approach is also interesting in the study of $\mathrm{BMO}^d_{\mathrm{rect}}$. Although Λ_b is in general not bounded for $b \in \mathrm{BMO}^d_{\mathrm{rect}}$, the space $\mathrm{BMO}^d_{\mathrm{rect}}$ can be characterized in terms of "average boundedness" of Λ_b, and also in terms of the boundedness of Λ_b from $L^2(\mathbb{T}) \hat{\otimes} L^2(\mathbb{T})$ into $L^2(\mathbb{T}^2)$. For $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, let $T_\sigma = T_{\sigma_1} T_{\sigma_2} : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$.

Theorem 2.10. *For $\varphi \in \mathcal{H}_{00}$, $\|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}}$ is equal to the norm of the operator*

$$\Psi_\varphi : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2 \times \Sigma), f \mapsto \Lambda_\varphi T_\sigma f.$$

Proof. Let $f \in L^2(\mathbb{T}^2)$ and $\varphi \in \mathcal{H}_{00}$. From Lemma 2.7 we have

$$(23) \qquad\qquad\qquad \Lambda_\varphi f = \sum_{R \in \mathcal{R}} P_R \varphi f_R h_R.$$

Thus

$$\int_\Sigma \int_\mathbb{T} \|\Lambda_\varphi T_\sigma f\|^2 dt d\sigma \;=\; \int_\mathbb{T} \int_\Sigma \left\| \sum_{R \in \mathcal{R}} \sigma(R)(P_R\varphi)(t) f_R h_R(t) \right\|^2 d\sigma dt$$

$$= \int_\mathbb{T} \sum_{R \in \mathcal{R}} |f_R|^2 \frac{\chi_R(t)}{|R|} |(P_R\varphi)(t)|^2 dt$$

$$= \sum_{R \in \mathcal{R}} |f_R|^2 \frac{1}{|R|} \|P_R\varphi\|^2.$$

Thus the operator norm of Ψ_φ is $\sup_{R \in \mathcal{R}} \frac{1}{|R|^{1/2}} \|P_R\varphi\| = \|\varphi\|_{\mathrm{BMO}^d_{\mathrm{rect}}}$. $\qquad\square$

Proposition 2.11. *If $b \in \mathrm{BMO}^d_{rect}$ then Λ_b maps $L^2(\mathbb{T}) \hat{\otimes} L^2(\mathbb{T})$ into $L^2(\mathbb{T}^2)$.*

Proof. Assume $f(t,s) = f_1(t)f_2(s)$ with $\|f_1\| = \|f_2\| = 1$. Then we have

$$\sum_{R \in \mathcal{R}} P_R b\, f_R h_R = \sum_{I \in \mathcal{D}} \tilde{P}_I (\sum_{J \in \mathcal{D}} \tilde{P}_J b (f_2)_J h_J)(s)(f_1)_I h_I(t)$$

Writing $g(t,s) = \sum_{J \in \mathcal{D}} P_J(b(t,\cdot))(s)(f_2)_J h_J(s)$, we obtain

$$\| \sum_{R \in \mathcal{R}} P_R b f_R h_R \|_2^2 = \int_\mathbb{T} \int_\mathbb{T} | \sum_{I \in \mathcal{D}} P_I(g(\cdot,s))(t)(f_1)_I h_I(t)|^2 dt ds.$$

Now let us consider g as a function in t taking values in the Hilbert space $L^2(\mathbb{T})$. Recall that as in the scalar case, the Haar multiplier norm of $(P_I g)_{I \in \mathcal{D}}$ is controlled by the vector $\mathrm{BMO}^d(\mathbb{T})$ norm of g given by

$$\sup_{I \in \mathcal{D}} \frac{1}{|I|^{1/2}} \|P_I g\|_{L^2(\mathbb{T}, L^2(\mathbb{T}))}.$$

Thus

$$\sup_{\|f_1\|=1} \int_\mathbb{T} \| \sum_{I \in \mathcal{D}} P_I(g(\cdot,s))(t)(f_1)_I h_I(t)\|^2_{L^2(\mathbb{T})} dt \le C \sup_I \frac{1}{|I|} \|P_I g\|^2_{L^2(\mathbb{T}, L^2(\mathbb{T}))}.$$

Notice now that

$$P_I(\sum_{J \in \mathcal{D}} P_J(b(t,\cdot))(s)(f_2)_J h_J(s)) = \sum_{J \in \mathcal{D}} P_{I \times J}(b)(t,s)(f_2)_J h_J(s).$$

On the other hand, applying a corresponding argument to the function $(\tilde{P}_I b)(t,s) = \sum_{J \in \mathcal{D}} (P_{I \times J} b)(t,s) h_J(s)$ understood as a function in s which takes values in $L^2(\mathbb{T})$, we obtain for $\|f_2\|_2 = 1$

$$\|P_I g\|^2_{L^2(\mathbb{T}, L^2(\mathbb{T}))} \;=\; \int_{\mathbb{T}^2} | \sum_{J \in \mathcal{D}} P_{I \times J}(b)(t,s)(f_2)_J h_J(s)|^2 dt ds$$

$$= \int_\mathbb{T} \| \sum_{J \in \mathcal{D}} P_{I \times J}(b)(\cdot,s)(f_2)_J h_J(s)\|^2_{L^2(\mathbb{T})} ds$$

$$\le C \sup_J \frac{1}{|J|} \|P_{I \times J}(b)\|^2_{L^2(\mathbb{T}, L^2(\mathbb{T}))} \le C\|I\| \|b\|^2_{\mathrm{BMO}^d_{\mathrm{rect}}}.$$

This finishes the proof of the proposition. $\qquad\square$

3. Sweeps of functions in BMO.

Let us now recall that the (dyadic) sweep of a function $\varphi \in L^2(\mathbb{T}^2)$ is defined as follows:

$$S_\varphi = \sum_{R \in \mathcal{R}} |\varphi_R|^2 \frac{\chi_R}{|R|},$$

i.e. $S_\varphi = \mathcal{S}(\varphi)^2$.

We list some properties of the sweep which will be relevant for our purposes, the proofs of which are elementary and left to the reader.

Proposition 3.1. (i) $S_\varphi(t, s) = \sum_{I \in \mathcal{D}} S_{\varphi_I}(s) \frac{\chi_I(t)}{|I|}$.

(ii) $S_\varphi = \Delta_\varphi^{(1,2)}(\varphi)$.

(iii) $P_\Omega(S\varphi) = P_\Omega(S_{P_\Omega \varphi})$.

(iv) If $p > \frac{1}{2}$ then $\varphi \in L^{2p}(\mathbb{T}^2)$ if and only if $S_\varphi \in L^p(\mathbb{T}^2)$.

(v) If $S_\varphi \in L^\infty$ then $\varphi \in \mathrm{BMO}^d_{prod}$.

(vi) $\|S\varphi\|_2 \le C\|\varphi\|_{\mathrm{BMO}^d_{prod}} \|\varphi\|_2$.

Here it is the basic result relating the boundedness of $\pi_b^{(1,2)}$ and Λ_{S_b}.

Lemma 3.2. Let $b \in \mathcal{H}_{00}$. Then

$$\pi_b^{(1,2)*} \pi_b^{(1,2)} = \Lambda_{S_b} + D_b,$$

where D_b is a linear operator on $L^2(\mathbb{T}^2)$ with $\|D_b\| \le C\|b\|^2_{\mathrm{BMO}^d_{rect}}$, and $C > 0$ is an absolute constant.

Proof. Let $R = I \times J$, $R' = I' \times J' \in \mathcal{R}$.

First, observe that

$$(24) \quad \left\langle \pi_b^{(1,2)*} \pi_b^{(1,2)} h_R, h_{R'} \right\rangle$$

$$= \left\langle \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} h_{I'' \times J''} \, b_{I'' \times J''} \, m_{I'' \times J''}(h_R), \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} h_{I'' \times J''} \, b_{I'' \times J''} \, m_{I'' \times J''}(h_{R'}) \right\rangle$$

$$= \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2, I'' \subsetneq I, J'' \subsetneq J} |b_{I'' \times J''}|^2 \, m_{I''}(h_I) m_{I''}(h_{I'}) m_{J''}(h_J) m_{J''}(h_{J'}).$$

We now do a kind of triangular truncation with respect to the indices I, I', J, J'.

(i) $I \supsetneq I'$, $J \supsetneq J'$.

$$\left\langle \pi_{S_b}^{(1,2)} h_R, h_{R'} \right\rangle = \langle S_b, h_{R'} \rangle m_{R'}(h_R)$$

$$= \left\langle \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} \frac{\chi_{I'' \times J''}}{|I''||J''|} |b_{I'' \times J''}|^2, h_{R'} \right\rangle m_{R'}(h_R)$$

$$= \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} |b_{I'' \times J''}|^2 \, m_{I''}(h_{I'}) \, m_{J''}(h_{J'}) \, m_{I'}(h_I) \, m_{J'}(h_J).$$

This is nonzero only if $I' \subsetneq I$ and $J' \subsetneq J$. In this case, we get contributions only for $I'' \subsetneq I'$ and $J'' \subsetneq J'$, and the expression agrees with (24).

(ii) $I \subsetneq I'$, $J \subsetneq J'$. Observe that

$$\langle \Delta_{S_b}^{(1,2)} h_R, h_{R'} \rangle = \langle h_R, \pi_{S_b}^{(1,2)} h_{R'} \rangle = \langle \pi_{S_b}^{(1,2)} h_{R'}, h_R \rangle.$$

As shown above, this equals $\langle \pi_b^{(1,2)^*} \pi_b^{(1,2)} h_{R'}, h_R \rangle$ if $I' \supsetneq I$ and $J' \supsetneq J$, and is 0 otherwise.

(iii) $I \supsetneq I'$, $J \subsetneq J'$.

$$
\begin{aligned}
\left\langle \pi_{\Delta_{S_b}} h_R, h_{R'} \right\rangle &= \left\langle \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} S_{bI'' \times J''} h_{I''} \frac{\chi_{J''}}{|J''|} m_{I''}(h_{RJ''}), h_{R'} \right\rangle \\
&= \left\langle \sum_{I'' \in \mathcal{D}_1} S_{bI'' \times J} h_{I''} \frac{\chi_J}{|J|} m_{I''}(h_I), h_{R'} \right\rangle \\
&= S_{bI' \times J} m_{I'}(h_J) m_J(h_{I'}) = \langle S_b, h_{I' \times J} \rangle m_{I'}(h_I) m_J(h_{J'}) \\
&= \left\langle \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} \frac{\chi_{I'' \times J''}}{|I''||J''|} |b_{I'' \times J''}|^2, h_{I' \times J} \right\rangle m_{I'}(h_I) m_J(h_{J'}) \\
&= \sum_{I'' \times J'' \in \mathcal{D}_1 \times \mathcal{D}_2} |b_{I'' \times J''}|^2 m_{I''}(h_{I'}) m_{J''}(h_J) m_{I'}(h_I) m_J(h_{J'}).
\end{aligned}
$$

This is nonzero only for $I' \subsetneq I$ and $J' \supsetneq J$. In this case, the sum has only contributions for $I'' \subsetneq I'$ and $J'' \subsetneq J$, and agrees with (24).

(iv) $I' \supsetneq I$ and $J' \subsetneq J$. Note that $\langle \Delta_{\pi_{S_b}} h_R, h_{R'} \rangle = \langle \pi_{\Delta_{S_b}} h_{R'}, h_R \rangle$. As shown above, this is only nonzero for $I' \supsetneq I$ and $J' \subsetneq J$, and agrees with (24) in this case.

(v) $I' = I$ or $J = J'$. Let $f \in L^2(\mathbb{T}^2)$. Then

$$
\begin{aligned}
& \sum_{I \in \mathcal{D}_1} \sum_{J, J' \in \mathcal{D}_2} \langle \pi_b^{(1,2)^*} \pi_b^{(1,2)} h_{I \times J} f_{I \times J}, h_{I \times J'} f_{I \times J'} \rangle \\
&= \sum_{I \in \mathcal{D}_1} \frac{1}{|I|} \sum_{I'' \subsetneq I} \sum_{J'' \in \mathcal{D}_2} |b_{I'' \times J''}|^2 f_{I \times J} f_{I \times J'} m_{J''}(h_J) m_{J''}(h_{J'}) \\
&= \sum_{I \in \mathcal{D}_1} \|\pi_{b^I} f_I\|^2,
\end{aligned}
$$

where for each I, f_I stands for the one-variable function $\sum_{J \in \mathcal{D}_2} h_J f_{I \times J}$, and b^I for the function $\sum_{J \in \mathcal{D}_2} h_J \frac{1}{|I|^{1/2}} \left(\sum_{I'' \subsetneq I} |b_{I'' \times J}|^2 \right)^{1/2}$. It is easy to see that $\|b^I\|_{\mathrm{BMO}^d} \leq \|b\|_{\mathrm{BMO}_{\mathrm{rect}}^d}$ for all $I \in \mathcal{D}_1$. Thus the above sum is bounded by $c \|b\|_{\mathrm{BMO}_{\mathrm{rect}}^d}^2 \|f\|^2$, where c is an absolute constant.

The same estimate holds for the terms corresponding to $J = J'$.

Now we have counted the terms corresponding to $I = I'$, $J = J'$ twice and need to estimate them separately. Let $f \in L^2(\mathbb{T}^2)$. Then

$$
(25) \quad \sum_{I \in \mathcal{D}_1} \sum_{J \in \mathcal{D}_2} \langle \pi_b^{(1,2)^*} \pi_b^{(1,2)} h_{I \times J} f_{I \times J}, h_{I \times J} f_{I \times J} \rangle
$$

$$
= \sum_{I \in \mathcal{D}_1} \sum_{J \in \mathcal{D}_2} \frac{1}{|I||J|} \sum_{I'' \subsetneq I, J'' \subsetneq J} |b_{I'' \times J''}|^2 |f_{I \times J}|^2 \leq \|b\|_{\mathrm{BMO}_{\mathrm{rect}}^d}^2 \|f\|^2.
$$

Defining D_b now by

$$(26) \quad \langle D_b f, f \rangle = \sum_{I \times J, I' \times J' \in \mathcal{R}, I' = I} \langle \pi_b^{(1,2)^*} \pi_b^{(1,2)} h_{I \times J} f_{I \times J}, h_{I' \times J'} f_{I' \times J'} \rangle +$$

$$\sum_{I \times J, I' \times J' \in \mathcal{R}, I' \neq I, J' = J} \langle \pi_b^{(1,2)^*} \pi_b^{(1,2)} h_{I \times J} f_{I \times J}, h_{I' \times J'} f_{I' \times J'} \rangle,$$

we obtain the statement of the lemma. $\qquad\square$

Now we are ready to state the main result of this section.

Theorem 3.3. *Let* $b \in \mathrm{BMO}_{rect,2}^d$. *Then* $b \in \mathrm{BMO}_{prod}^d$ *if and only if* $S_b \in \mathrm{BMO}_{prod}^d$.

Proof. We will first show that there exist $C > 0$ such that

$$(27) \qquad \|S_b\|_{prod} \leq C \|b\|_{prod}^2.$$

Indeed, by Chang's Theorem [Ch], [ChFef2] it is sufficient to show that there exists a constant $C > 0$ with

$$\|P_\Omega S_b\|_2 \leq C \|b\|_{prod}^2 |\Omega|$$

for all $\Omega \subseteq \mathbb{T}^2$ measurable (see (5)). Using Proposition 3.1, we obtain

$$(28) \quad \|P_\Omega S_b\|_2 \;=\; \|P_\Omega S_{P_\Omega b}\|_2 \leq \|S_{P_\Omega b}\|_2 \leq \|P_\Omega b\|_{prod} \|P_\Omega b\|_2 \leq \|b\|_{prod}^2 |\Omega|$$

For the converse, assume that $S_b \in \mathrm{BMO}_{prod}^d$. Then Λ_{S_b} is bounded by Thm 2.8. Now Lemma 3.2 finishes the proof. $\qquad\square$

Remark. The first implication can also be shown with the John-Nirenberg Theorem for product BMO, which was proved in [ChFef1] (for a dyadic version, see [T]).

The sweep can be understood as a bilinear map. For $f, g \in \mathcal{H}_{00}$, let $S_{f,g} = \sum_{R \in \mathcal{R}} \frac{\chi_R}{|R|} f_r g_r$, so $S_f = S_{f,\bar{f}}$.

Corollary 3.4. $S : \mathrm{BMO}_{prod}^d \times \mathrm{BMO}_{prod}^d \to \mathrm{BMO}_{prod}^d$ *is bounded.*

Proof. The Cauchy-Schwarz inequality gives the pointwise inequality $S_{f,g} \leq (S_f)^{1/2}(S_g)^{1/2}$ for $f, g \in \mathcal{H}_{00}$. Let $\Omega \subseteq \mathbb{T}^2$ be measurable. Using an adaption of 3.1(iii), we see that

$$\|P_\Omega S_{f,g}\|_2 = \|P_\Omega S_{P_\Omega f, P_\Omega g}\|_2 \leq \|S_{P_\Omega f, P_\Omega g}\|_2 \leq \|(S_{P_\Omega f})^{1/2}(S_{P_\Omega g})^{1/2}\|_2$$

$$\leq \|S_{P_\Omega f}\|_2^{1/2} \|S_{P_\Omega g}\|_2^{1/2} \leq |\Omega|^{1/2} \|f\|_{prod} \|g\|_{prod}$$

by (28). $\qquad\square$

Another application of Lemma 3.2 yields the following result.

Theorem 3.5. *Let* $\|\cdot\|_*$ *be an positive homogeneous function of degree 1 on* $\mathcal{H}_{00}$ *such that*

(i) *There exists* $c > 0$ *such that* $\|\cdot\|_{\mathrm{BMO}_{rect}^d} \leq c \|\cdot\|_*$
(ii) *There exists* $k > 0$ *such that* $\|S_\varphi\|_* \leq k \|\varphi\|_*^2$.

Then there exists a constant $\tilde{C}$ *such that for all* $\varphi \in \mathcal{H}_{00}$, $\|\varphi\|_{\mathrm{BMO}_{prod}^d} \leq \tilde{C} \|\varphi\|_*$.

Proof. Write

$$\pi_\varphi^* \pi_\varphi = \pi_{S_\varphi}^{(1,2)} + \pi_{S_\varphi}^{(1,2)^*} + \pi_{\Delta_{S_\varphi}} + \pi_{\Delta_{S_\varphi}}^* + D_\varphi,$$

with $\|D_\varphi\| \leq C\|\varphi\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}}^2$ as above.

Let $E_n = \mathrm{span}\{h_{I\times J} : I \in \mathcal{D}_1, J \in \mathcal{D}_2, |I|, |J| \geq 2^{-n}\}$, let P_n be the orthogonal projection onto E_n in $L^2(\mathbb{T}^2)$, and let

$$c(n) = \sup\{\|\pi_\varphi|_{E_n}\|, \|\varphi\|_* \leq 1\}.$$

A trivial estimate shows that $c(n) < \infty$ for each $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, choose $f_n \in E_n$ and $\varphi \in \mathcal{H}_{00}$ with $\|\varphi\|_* = 1$, $\|f_n\| = 1$ and $\|\pi_\varphi f_n\| \geq (1-\varepsilon)c(n)$. Then

$$(29) \quad (1-\varepsilon)^2 c(n)^2 \leq \|\pi_\varphi f_n\|^2 = \langle \pi_\varphi^* \pi_\varphi f_n, f_n \rangle$$

$$= \langle \pi_{S_\varphi}^{(1,2)} f_n, f_n \rangle + \langle \pi_{S_\varphi}^{(1,2)^*} f_n, f_n \rangle + \langle \pi_{\Delta_{S_\varphi}} f_n, f_n \rangle + \langle \pi_{\Delta_{S_\varphi}}^* f_n, f_n \rangle + \langle D_\varphi f_n, f_n \rangle.$$

By definition of $c(n)$, the first two terms can be estimated by $c(n)\|S_\varphi\|_* \leq c(n)k$. For the next two terms, we have to remark that that

$$(30) \quad \langle \pi_{\Delta_{S_\varphi}} f_n, f_n \rangle = \langle \pi_{\Delta_{P_n S_\varphi}} f_n, f_n \rangle \leq \|\pi_{\Delta_{P_n S_\varphi}}\| \leq \tilde{c}\|\pi_{P_n S_\varphi}^{(1,2)}\| \leq \tilde{c}\|\pi_{S_\varphi}^{(1,2)}|_{E_n}\|$$

Here, we use as in the proof of Thm 2.8 that there exists a constant $\tilde{c}$ such that $\|\pi_{\Delta_b}\| \leq \tilde{c}\|\pi_b^{(1,2)}\|$ for all $b \in \mathcal{H}_{00}$ (see [PS], Thm 7.7.2).

The last term is easily controlled by $\langle D_\varphi f_n, f_n \rangle \leq C\|\varphi\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}}^2$. Altogether, we obtain that

$$(31) \quad \|\pi_\varphi^{(1,2)}|_{E_n}\|^2 \leq 4\tilde{c}\|\pi_{S_\varphi}^{(1,2)}|_{E_n}\| + C\|\varphi\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}}^2.$$

With $\|\varphi\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}} \leq c\|\varphi\|_*$ and $\|\pi_{S_\varphi}^{(1,2)}|_{E_n}\| \leq c(n)\|S_\varphi\|_* \leq kc(n)\|\varphi\|_*^2$, it follows that

$$(32) \quad (1-\varepsilon)^2 c(n)^2 \leq 4\tilde{c}kc(n) + c^2 C$$

Thus $c(n) \leq \sqrt{4\tilde{c}^2 k^2 + Cc^2} + 2\tilde{c}k$. With $\tilde{C} = \sqrt{4\tilde{c}^2 k^2 + Cc^2} + 2\tilde{c}k$, it follows that $\|\pi_\varphi\| \leq \tilde{C}\|\varphi\|_*$. $\qquad\square$

We can now characterize BMO_{prod}^d in terms of the $\mathrm{BMO}_{rect,2}^d$-norm.

Theorem 3.6. *Let* $\varphi \in \mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}(\mathbb{T}^2)$. *Then* $\varphi \in \mathrm{BMO}_{prod}^d$ *if and only if* $(\|S_\varphi^{(n)}\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}}^{1/2^n})_{n\in\mathbb{N}}$ *is bounded, where* $S_\varphi^{(n)}$ *is the n-fold sweep of* ϕ, *defined recursively by* $S_\varphi^{(n)} = S_{S_\varphi}^{(n-1)}$.

Proof. By Thm 3.3, we have for each $n \in \mathbb{N}$

$$\|S_\varphi^{(n)}\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}} \leq \|S_\varphi^{(n)}\|_{\mathrm{BMO}_{prod}^d} \leq C \cdot C^2 \cdots C^{2^{n-1}}\|\varphi\|_{\mathrm{BMO}_{prod}^d}^{2^n} \leq C^{2^n}\|\varphi\|_{\mathrm{BMO}_{prod}^d}^{2^n},$$

and consequently

$$\|S_\varphi^{(n)}\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}}^{1/2^n} \leq C\|\varphi\|_{\mathrm{BMO}_{prod}^d}.$$

Conversely, the map $\varphi \mapsto \sup_{n\in\mathbb{N}}\|S_\varphi^{(n)}\|_{\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}}^{1/2^n}$ clearly defines a positive homogeneous function on $\mathcal{H}_{00}$ with satisfies conditions in Theorem 3.5. $\qquad\square$

Another consequence of Theorem 3.5 is

Corollary 3.7. *S does not map* $\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}} \times \mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}$ *boundedly into* $\mathrm{BMO}_{\mathrm{rect}}^{\mathrm{d}}$.

Proof. We know that the $\| \cdot \|_{prod}$ norm cannot be controlled by the $\| \cdot \|_{rect}$ norm. So Condition (ii) in Thm 3.5 cannot hold, and in particular S does not map $\mathrm{BMO}^d_{rect} \times \mathrm{BMO}^d_{rect}$ boundedly into BMO^d_{rect}. $\square$

4. THE SCALE $\mathrm{BMO}^d_{rect,p}$.

Recall that for $1 \leq p < \infty$, a function $\varphi \in L^2(\mathbb{T}^2)$ is said to belong to $\mathrm{BMO}^d_{rect,p}$ if

$$\|\varphi\|_{rect,p} = \sup_{R \in \mathcal{R}} \frac{1}{|R|^{1/p}} \|P_R \varphi\|_p < \infty.$$

Note that $\mathrm{BMO}^d_{rect,p_2} \subseteq \mathrm{BMO}^d_{rect,p_1}$ for $p_1 \leq p_2$.

The reader should also be aware that functions in $\mathrm{BMO}^d_{rect,p}$ are actually in $L^p(\mathbb{T}^2)$, due to the identities $m_I(f) = m_I(P_{I \times \mathbb{T}} f)$ and $m_J(f) = m_I(P_{\mathbb{T} \times J} f)$.

The following proposition characterizes the behaviour of the $\mathrm{BMO}^d_{rect,p}$ norms under the sweep.

Proposition 4.1. *Let $p > \frac{1}{2}$ and let $C_p = \|\mathcal{S}\|_{L^{2p} \to L^{2p}}$. Then*

$$\|S_\varphi\|_{rect,p} \leq 4C_p^2 \|\varphi\|^2_{rect,2p}.$$

Proof. Since $P_R(S_\varphi) = P_R(S_{P_R\varphi})$ and $\|P_R(g)\|_p \leq 4\|g\|_p$, we obtain

$$\|S_\varphi\|_{rect,p} \leq 4 \sup_{R \in \mathcal{R}} \frac{1}{|R|^{1/p}} \|S_{P_R(\varphi)}\|_p \leq 4C_p^2 \sup_{R \in \mathcal{R}} \frac{1}{|R|^{1/p}} \|P_R(\varphi)\|^2_{2p}.$$

This gives the result. $\square$

It is known that $\mathrm{BMO}^d_{prod} \subsetneq \mathrm{BMO}^d_{rect,2}$. Indeed, this is basically the content of Carleson's original counterexample [C] (for the continuous case, see [Fef]). As pointed out in [Fef], the example in [C] implies that $\mathrm{BMO}^d_{rect,2} \not\subseteq L^4(\mathbb{T}^2)$.

We shall improve this by showing that actually $\mathrm{BMO}^d_{prod} \subsetneq \mathrm{BMO}^d_{rect,p}$ for all p. We will show that for any $p_2 > p_1 \geq 1$, $\mathrm{BMO}^d_{rect,p_1} \not\subseteq L^{p_2}(\mathbb{T}^2)$ and therefore in particular $\mathrm{BMO}^d_{rect,p_1} \not\subseteq \mathrm{BMO}^d_{rect,p_2}$. For the case $p_1 = 1$, $p_2 = 2$, this answers a question posed in [FS].

As a corollary, we show that $\mathrm{BMO}^d_{prod} \subsetneq \bigcap_{p \geq 1} \mathrm{BMO}^d_{rect,p}$.

Theorem 4.2. *Let $p \geq 2$. Then $\mathrm{BMO}^d_{prod} \subseteq \mathrm{BMO}^d_{rect,p}$.*
Moreover

$$(33) \qquad \|\varphi\|_{rect,p} \leq C\|\varphi\|^{1-2/p}_{prod} \|\varphi\|^{2/p}_{rect,2}$$

Proof. Let us first show that $\mathrm{BMO}^d_{prod} \subseteq L^p(\mathbb{T}^2)$ and

$$(34) \qquad \|\varphi\|_p \leq C\|\varphi\|^{1-2/p}_{prod} \|\varphi\|^{2/p}_2$$

For $p = 2^k$, $k \in \mathbb{N}$, we shall prove (34) by induction.

It is obvious for $k = 1$. For $k = 2$ we have

$$(35) \qquad \|\varphi\|^2_4 = \|S_\varphi\|_2 = \|\Delta^{(1,2)}_\varphi(\varphi)\|_2 \leq C\|\varphi\|_{prod}\|\varphi\|_2.$$

Assume it holds for $p_k = 2^k$.

$$\|\varphi\|^2_{p_{k+1}} = \|S_\varphi\|_{p_k} \leq C\|S_\varphi\|^{1-2/p_k}_{prod} \|S_\varphi\|^{2/p_k}_2.$$

Now from (27) and (35) we obtain

$$||\varphi||_{p_{k+1}} \leq C||\varphi||_{prod}^{1-2/p_k}||\varphi||_4^{2/p_k} \leq C||\varphi||_{prod}^{1-2/p_{k+1}}||\varphi||_2^{2/p_{k+1}}.$$

Now the general case follows by interpolation.

Given $p > 2$ and $p \neq 2^k$ for any $k \in \mathbb{N}$, find $m \in \mathbb{N}$ such that $2^{m-1} < p < 2^m$. Write $\frac{1}{p} = \frac{1-\theta_m}{2^{m-1}} + \frac{\theta_m}{2^m}$. Now apply the previous case combined with

$$||\varphi||_p \leq ||\varphi||_{2^{m-1}}^{1-\theta_m}||\varphi||_{2^m}^{\theta_m}.$$

Let us use (34) to obtain the desired estimate for the $\mathrm{BMO}_{rect,p}^d$-norm. Given $R \in \mathcal{R}$ we have

$$||P_R\varphi||_p \leq C||P_R\varphi||_{prod}^{1-2/p}||P_R\varphi||_2^{2/p} \leq C||\varphi||_{prod}^{1-2/p}||\varphi||_{rect,2}^{2/p}|R|^{1/p}.$$

This finishes the proof. □

Proposition 4.3. *Let $2 < p$. There exists $\phi \in \mathrm{BMO}_{rect,2}^d \setminus L^p(\mathbb{T}^2)$.*

In particular $\mathrm{BMO}_{prod}^d \subseteq \mathrm{BMO}_{rect,p}^d \subsetneq \mathrm{BMO}_{rect,2}^d$.

Proof. We shall find a sequence φ_N such that $\sup_N ||\varphi_N||_{rect,2} < \infty$ but $\sup_N ||\varphi_N||_p = \infty$. A standard argument then gives the existence of ϕ.

¿From Carleson's construction [C] we know that for each $N \in \mathbb{N}$ there exists a collection of dyadic rectangles Φ_N such that

$$(36) \qquad \sum_{R \in \Phi_N} |R| = 1$$

$$(37) \qquad |\cup_{R \in \Phi_N} R| < \frac{1}{N}$$

$$(38) \qquad \sum_{R \in \Phi_N, R \subseteq R'} |R| \leq C|R'|, \qquad R' \in \mathcal{R}$$

where C is a constant independent of N.

Defining

$$\varphi_N = \sum_{R \in \Phi_N} |R|^{1/2}h_R$$

we have that

$$||\varphi_N||_2 = 1, \qquad ||\varphi_N||_{rect,2} \leq C$$

but, since $supp(\varphi_N) \subset \cup_{R \in \Phi_N}R$,

$$||\varphi_N||_p \geq |\cup_{R \in \Phi_N} R|^{\frac{1}{p}-\frac{1}{2}} \geq N^{\frac{1}{2}-\frac{1}{p}}.$$

□

We now can answer in the negative the above mentioned question of C. Sadosky and S. Ferguson posed in [FS].

Corollary 4.4. *There exists $\phi \in \mathrm{BMO}_{rect,1}^d \setminus \cup_{p>1} L^p(\mathbb{T}^2)$. In particular, for each $p > 1$, $\mathrm{BMO}_{rect,p}^d \subsetneq \mathrm{BMO}_{rect,1}^d$, and the norms $\| \cdot \|_{rect,1}$ and $\| \cdot \|_{rect,p}$ are not equivalent.*

Proof. We use the sequence of functions $(\varphi_n)_{n \in \mathbb{N}}$ with

$$||\varphi_n||_{rect,2} \leq C \text{ and } ||\varphi_n||_p \geq n^{1/2-1/p}$$

for each $n \in \mathbb{N}$, $p > 2$ from Proposition 4.3.

Define $\phi = \sum_{n=1}^{\infty} \frac{1}{n^2} S_{\varphi_{2^n}}$. Then $\phi \in \mathrm{BMO}_{rect,1}$ by Proposition 4.1, but $\|S_{\varphi_{2^n}}\|_p \approx \|\varphi_{2^n}\|_{2p}^2 \geq 2^{n(1-1/p)}$ for each $n \in \mathbb{N}$, $p > 1$ and consequently $\phi \notin \cup_{p>1} L^p(\mathbb{T}^2)$. $\quad\square$

To differentiate the spaces $\mathrm{BMO}_{rect,p}^d$ and BMO_{prod}^d we shall introduce the following coefficients.

Definition 4.5. *Let $E_n = \mathrm{span}\{h_{I \times J} : I \in \mathcal{D}_1, J \in \mathcal{D}_2, |I|, |J| \geq 2^{-n}\}$ and let P_n be the orthogonal projection onto E_n in $L^2(\mathbb{T}^2)$.*

For each $q \geq 1$ and each $n \in \mathbb{N}$

$$c(n,q) = \sup\{\|\pi_\varphi\| : \varphi \in E_n, \|\varphi\|_{rect,q} \leq 1\},$$

and for $p \geq q$,

$$a(n,p,q) = \sup\{\|\varphi\|_{rect,p} : \varphi \in E_n, \|\varphi\|_{rect,q} \leq 1\},$$

We first analyse the behaviour of these constants.

Of course we have

$$(39) \qquad\qquad c(n,p_2) \leq c(n,p_1), \quad p_1 \leq p_2$$

$$(40) \qquad\qquad a(n,p,q_1) \leq a(n,p,q_2), \quad q_1 \leq q_2$$

$$(41) \qquad\qquad a(n,p_2,q) \leq a(n,p_1,q), \quad p_1 \leq p_2.$$

If $p \geq q$, clearly

$$(42) \qquad\qquad c(n,q) \leq a(n,p,q)c(n,p).$$

Let us now extend Therorem 4.2.

Theorem 4.6. *Let $p > q \geq 2$ and $\varphi \in \mathrm{BMO}_{prod}^d$. If $q \leq 2^k \leq p$ for some $k \in \mathbb{N}$ then*

$$(43) \qquad\qquad \|\varphi\|_{rect,p} \leq C_{p,q}\|\varphi\|_{prod}^{1-q/p}\|\varphi\|_{rect,q}^{q/p}.$$

In particular, for $p > q_1 \geq q_2 \geq 2$ we have

$$(44) \qquad\qquad a(n,p,q_2) \leq C_p c(n,q_2)^{1-q_1/p} a(n,q_1,q_2)^{q_1/p}.$$

Proof. We shall see first that

$$(45) \qquad\qquad \|\varphi\|_p \leq C_{p,q}\|\varphi\|_{prod}^{1-\theta}\|\varphi\|_q^\theta$$

for the above values of $\theta = q/p$. We do this in several steps.

First suppose that $q = 2^n$ for some $n \in \mathbb{N}$. Theorem 4.2 gives the case $n = 1$. Assume that the result is true for $n \geq 2$, and let us consider the case $q = 2^{n+1}$.

Let $p > 2^{n+1}$. Applying the induction assumption to S_φ for $p/2$, we get

$$
\begin{aligned}
\|\varphi\|_p^2 &\approx \|S_\varphi\|_{p/2} \\
&\leq C\|S_\varphi\|_{prod}^{1-2^{n+1}/p}\|S_\varphi\|_{2^n}^{2^{n+1}/p} \\
&\leq C\|\varphi\|_{prod}^{2(1-2^{n+1}/p)}\|\varphi\|_{2^{n+1}}^{2^{n+2}/p}.
\end{aligned}
$$

This shows that

$$\|\varphi\|_p \leq C\|\varphi\|_{prod}^{1-2^{n+1}/p}\|\varphi\|_{2^{n+1}}^{2^{n+1}/p}.$$

Let us now proceed to the general case. We may assume that $q < 2^k \leq p$ for some $k \in N$. We can apply the previous case for $n = k$ together with interpolation. Writing $\frac{1}{2^k} = \frac{1-\alpha}{q} + \frac{\alpha}{p}$, we obtain

$$
\begin{aligned}
||\varphi||_p &\leq C||\varphi||_{prod}^{1-2^k/p}||\varphi||_{2^k}^{2^k/p} \\
&\leq C||\varphi||_{prod}^{1-2^k/p}(||\varphi||_q^{1-\alpha}||\varphi||_p^{\alpha})^{2^k/p}
\end{aligned}
$$

Consequently

$$
||\varphi||_p^{1-\alpha 2^k/p} \leq C||\varphi||_{prod}^{1-2^k/p}||\varphi||_q^{(1-\alpha)2^k/p}.
$$

Note that $(1-\alpha)2^k/q = 1 - \alpha 2^k/p$. Hence we get with $\theta = q/p$ that

$$
||\varphi||_p \leq C||\varphi||_{prod}^{1-\theta}||\varphi||_q^{\theta}.
$$

To finish the proof, note that for each $R \in \mathcal{R}$,

$$
\begin{aligned}
||P_R\varphi||_p &\leq C_{p,q}||P_R\varphi||_{prod}^{1-q/p}||P_R\varphi||_q^{q/p} \\
&\leq C_{p,q}||\varphi||_{prod}^{1-q/p}||\varphi||_{rect,q}^{q/p}|R|^{1/p}.
\end{aligned}
$$

$\square$

Let us now establish a further connection between the constants introduced in 4.5.

Theorem 4.7. *There exist $K_1 > 0$ and $K_2 > 0$ such that for all $n \in \mathbb{N}$ and $p \geq 1$*

$$
c^2(n, 2p) \leq K_1 C_p^2 c(n, p) + K_2,
$$

where $C_p = ||\,\mathcal{S}\,||_{L^{2p} \to L^{2p}}$.

Proof. Write

$$
\pi_\varphi^* \pi_\varphi = \Lambda_{S_\varphi} + D_\varphi,
$$

with $||D_\varphi|| \leq C||\varphi||_{\text{BMO}_{\text{rect}}^d}^2$ as above.

For $n \in \mathbb{N}$, $p \geq 1$ and $\varepsilon > 0$, choose $f_n \in L^2(\mathbb{T}^2)$ and $\varphi \in E_n$ with $||\varphi||_{rect,2p} = 1$, $||f_n||_2 = 1$ and $||\pi_\varphi f_n||_2 \geq (1-\varepsilon)c(n, 2p)$. Then

$$
(1-\varepsilon)^2 c(n, 2p)^2 \leq ||\pi_\varphi f_n||_2^2 = \langle \pi_\varphi^* \pi_\varphi f_n, f_n \rangle = \langle \Lambda_{S_\varphi} f_n, f_n \rangle + \langle D_\varphi f_n, f_n \rangle.
$$

Therefore, we obtain that

(46) $(1-\varepsilon)^2 c(n, 2p)^2 \leq ||\Lambda_{S_\varphi}|| + C||\varphi||_{\text{BMO}_{\text{rect}}^d}^2.$

Since $||\varphi||_{rect,2p} = 1$, Proposition 4.1 implies $||S_\varphi||_{rect,p} \leq 4C_p^2$. Therefore, since $||\varphi||_{rect,2} \leq ||\varphi||_{rect,2p}$, it follows that

(47) $(1-\varepsilon)^2 c(n, 2p)^2 \leq 4C_p^2 c(n, p) + C.$

Using (42) we get the second part. $\square$

Corollary 4.8. *Let $p \geq 1$. Then $\text{BMO}_{prod}^d \subsetneq \text{BMO}_{rect,p}^d$.*

Proof. Observe first that Proposition 4.3 implies that

(48) $\lim_{n \to \infty} c(n, 2) = \infty.$

This shows that $\text{BMO}_{prod}^d \subsetneq \text{BMO}_{rect,p}^d$ for any $1 \leq p \leq 2$.

On the other hand, the estimates (42) and (44) in the case $q = q_1 = q_2$ imply that if $p > q \geq 2$ with $q \leq 2^k \leq p$ for some k,

$$(49) \qquad\qquad\qquad c^{q/p}(n,q) \leq Cc(n,p),$$

where C is independent of n.

Hence $\mathrm{BMO}^d_{prod} = \mathrm{BMO}^d_{rect,p}$ for some $p > 2$ would imply $\sup_n c(n,p) < \infty$ and therefore $\sup_n c(n,2) < \infty$, contradicting (48). $\qquad\square$

The particular case $p = 4$ means that a question left open in [PS] can be answered in the negative. There, it was asked whether the condition

$$(50) \quad \|(\pi_b^{(1,2)})^* \pi_b^{(1,2)} h_{I'} f\|^2 = \frac{1}{|I'|} \left\| \sum_{I \times J \in \mathcal{D}_1 \times \mathcal{D}_2, I \subsetneq I'} \frac{\chi_{I \times J}}{|I||J|} |b_{IJ}|^2 m_J f \right\|_{L^2(\mathbb{T}^2)}^2$$
$$\leq C\|f\|_{L^2(\mathbb{T})} \qquad (f \in L^2(\mathbb{T}), I' \in \mathcal{D}_1)$$

((27) and (28) in [PS]) already implies that $b \in \mathrm{BMO}^d_{\mathrm{prod}}$. Note that f here denotes a function in the second variable. We know from Prop 4.1 that $b \in \mathrm{BMO}_{rect,4}$ implies $S_b \in \mathrm{BMO}^d_{\mathrm{rect}}$. By Lemma 3.2,

$$\|(\pi_b^{(1,2)})^* \pi_b^{(1,2)} h_{I'} f\| = \|(\Lambda_{S_b} + D_b) h_{I'} f\|,$$

where D_b is bounded on $L^2(\mathbb{T}^2)$ and Λ_{S_b} maps $L^2(\mathbb{T}) \hat{\otimes} L^2(\mathbb{T})$ boundedly into $L^2(\mathbb{T}^2)$ by Prop 2.11. Thus $b \in \mathrm{BMO}_{rect,4}$ implies (50). This condition is therefore not sufficient for $b \in \mathrm{BMO}^d_{\mathrm{prod}}$.

As pointed out in [PS], this has also consequences for the study of operator-valued Carleson measures, in the sense that a certain vector BMO condition of the sweep of an operator-valued measure does not imply boundedness of the corresponding vector Carleson embedding.

We can further show that even the intersection of all $\mathrm{BMO}^d_{rect,p}$ spaces is still bigger than $\mathrm{BMO}^d_{\mathrm{prod}}$.

Corollary 4.9.
$$\mathrm{BMO}^d_{\mathrm{prod}} \subsetneq \bigcap_{p \geq 1} \mathrm{BMO}^d_{rect,p}.$$

Proof. Obviously $\cap_{p \geq 1} \mathrm{BMO}^d_{rect,p} = \cap_{p \in \mathbb{N}} \mathrm{BMO}^d_{rect,p}$. With the locally convex topology defined by the increasing sequence of seminorms $(\|\cdot\|_{rect,p})_{p \in \mathbb{N}}$, the latter is a metrizable locally convex linear space. Since each of the $\mathrm{BMO}^d_{rect,p}$ is complete in $\|\cdot\|_{rect,p}$, $\cap_{p \in \mathbb{N}} \mathrm{BMO}^d_{rect,p}$ is complete in this topology and therefore a Fréchet space. We know from Thm 4.2 that $\mathrm{BMO}^d_{\mathrm{prod}} \subseteq \bigcap_{p \geq 1} \mathrm{BMO}^d_{rect,p}$, and that the embedding is continuous with respect to the norm topology on $\mathrm{BMO}^d_{\mathrm{prod}}$ and the locally convex topology on $\bigcap_{p \geq 1} \mathrm{BMO}_{rect,p}$. Let us assume towards a contradiction that the embedding is surjective. Then the open mapping theorem implies that the locally convex topology on $\bigcap_{p \geq 1} \mathrm{BMO}^d_{rect,p}$ is normable with $\|\cdot\|_{prod}$ and therefore contains a nonempty open neighbourhood of 0 which is bounded with respect to $\|\cdot\|_{prod}$. Since the family $(\|\cdot\|_{rect,p})_{p \in \mathbb{N}}$ is increasing, this means that there exists $p \in \mathbb{N}$ and $\varepsilon > 0$ such that $\|b\|_{prod} < 1$ whenever $\|b\|_{rect,p} < \varepsilon$, in contradiction to Corollary 4.8. $\qquad\square$

We will now separate the $\mathrm{BMO}^d_{rect,p}$ spaces. Note that for Corollary 4.4 means that

$$(51) \qquad\qquad \lim_{n\to\infty} a(n,p,1) = \infty$$

for all $p > 1$.

Let us see that this holds in general.

Corollary 4.10. *Let $p > q \geq 1$. Then $\mathrm{BMO}^d_{rect,p} \subsetneq \mathrm{BMO}^d_{rect,q}$.*

Proof. We have to show that $\lim_{n\to\infty} a(n,p,q) = \infty$.

It suffices to prove $\lim_{n\to\infty} a(n,q+\varepsilon,q) = \infty$ for sufficiently small ε. For fixed $q > 1$, choose $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $q < q+\varepsilon < 2^k \leq 2q$.

Using Theorem 4.7, (42) and (49), we obtain constants C_1, C_2 and C_3 independent of n such that

$$c^2(n,2q) \leq C_1 c(n,q) \leq C_1 C_2 a(n,q+\varepsilon,q)c(n,q+\varepsilon) \leq C_1 C_2 C_3 a(n,q+\varepsilon,q)c(n,2q)^{\frac{2p}{p+\varepsilon}}.$$

This shows that

$$c(n,2q)^{\frac{2\varepsilon}{p+\varepsilon}} \leq Ca(n,q+\varepsilon,q)$$

where C is independent of n. Now the result follows from Corollary 4.8. $\qquad\square$

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